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Evaluation of Integrals and Sums Involving $[\sin(Mx)/\sin(x)]^n$

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Preface

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The response of equispaced arrays, either linear, planar, or volumetric, to distributed spatial fields, typically encounters integrals which involve the kernel sin(Mx)/sin(x) or its square. Since this kernel oscillates rather fast with x for large M and does not decay with x, numerical integration of such functions can be very time consuming. By resorting to Parseval's theorem, such integrals can be significantly simplified, requiring only the Fourier transform of the complementary part of the integrand. This procedure is investigated and applied to several typical examples; programs for the examples are also included.

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LIST OF SYMBOLS

M	integer, number of elements in line array
g(t)	arbitrary function of t, (1)
G(ω)	Fourier transform of g(t), (1)
V	integral of product of g(t) $h^*(t)$, (2)
H(ω)	Fourier transform of h(t), (2)
γ	argument of sine functions, (3)
$h_1(t)$	ratio of sine functions, (3)
prime	sum with prime denotes every other term, (3)
$H_1(\omega)$	Fourier transform of $h_1(t)$, (4)
V_1	general integral (5)
$h_2(t)$	square of ratio of sine functions, (7)
$H_2(\omega)$	Fourier transform of $h_2(t)$, (8)
V_2	general integral (9)
$\phi(m)$	autocorrelation of weights, (12B)
sub r	real part, (13B)
V_{1a}	integral (18)
V_{2a}	integral (22)
V_{1b}	integral (27)
V_{2b}	integral (28)
V_{1c}	integral (33)
V_{2c}	integral (34)
Δ	increment in t, (41),(A-1)
$\delta_\Delta(t)$	impulse train, (41)
⊛	convolution, (42)
$S_N(M,k)$	discrete sine ratio, (52)

INT	integer part, (55)
V_5	general integral (63)
$\psi(p)$	autocorrelation of $\phi(m)$, (61)
$g(n\Delta)$	samples of $g(t)$, (A-1)
$\underline{G}(\omega)$	approximation to $G(\omega)$, (A-1)
N	size of FFT, (A-6)
$g_c(n\Delta)$	collapsed version of $g(n\Delta)$, (A-7)
$\tilde{g}(n\Delta)$	phase modulated $g(n\Delta)$, (A-11)
$\tilde{g}_c(n\Delta)$	collapsed version of $\tilde{g}(n\Delta)$, (A-12)

TR 8689

EVALUATION OF INTEGRALS AND SUMS INVOLVING $[\sin(Mx)/\sin(x)]^n$

INTRODUCTION

The response of an equiweighted equispaced line array to a distributed field involves the kernel $\sin(Mx)/\sin(x)$ or its square, depending on whether the voltage or power response, respectively, is of interest [1;2]. Numerical evaluation of such integrals can be very time consuming for two reasons: this kernel oscillates quickly with x for large M, and it does not decay with x. This necessitates fine sampling in x and large integration regions, both of which can lead to a significant computational burden, especially for two-dimensional or three-dimensional arrays. The object of this report is to give an alternative numerical procedure that can be very advantageous in some cases, and, in fact, leads to closed forms for some examples.

The procedure is also applied to summations involving the same kernel. Its utility depends on the rate of decay of the complementary part of the original integrand, as compared with the Fourier transform of this component. In any event, an alternative is presented for the user to consider in any numerical investigation.

GENERAL APPROACH

For arbitrary function g(t), define its Fourier transform as

$$G(\omega) = \int dt \exp(-i\omega t)\, g(t) \ . \qquad (1)$$

(Integrals without limits are over the range of nonzero integrand.) Then Parseval's theorem states that the following two alternative integrals are equal:

$$V = \int dt\, g(t)\, h^*(t) = \frac{1}{2\pi} \int d\omega\, G(\omega)\, H^*(\omega) \ . \qquad (2)$$

Here, $H(\omega)$ is the Fourier transform of h(t). Now, if $H(\omega)$ takes on a noticeably simpler form than h(t), then the second integral in (2) can offer an attractive alternative to the first integral in (2). That will indeed be the case here.

CASE 1

For integer $M \geq 1$ and constant $\gamma > 0$, consider the special choice of h(t) as

$$h_1(t) = \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{n=0}^{M-1} \exp[i\gamma t(2n+1-M)] =$$

$$= \sum_{m=1-M}^{M-1}{}' \exp(i\gamma t m) \ , \qquad (3)$$

where the prime on the latter sum denotes skipping every other term. Then the Fourier transform, according to (1), is

$$H_1(\omega) = 2\pi \sum_{m=1-M}^{M-1}{}' \delta(\omega - \gamma m) \ . \qquad (4)$$

Substitution of (3) and (4) in (2) yields

$$V_1 = \int dt \ g(t) \ \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1}{}' G(\gamma m) \ . \qquad (5)$$

This result indicates that if $G(\omega)$, the Fourier transform of g(t), can be evaluated, then the t integral in (5) is given by a finite sum of equispaced samples of $G(\omega)$ at increment 2γ. The (complex) function g(t) in (5) is arbitrary, except that the integral must converge. When $G(\omega)$ cannot be analytically evaluated, then proper application of a fast Fourier transform procedure to g(t) can be tailored to yield precisely the equispaced samples required for the right-hand side of (5); this technique is detailed in appendix A.

If function $G(\omega)$ is even in ω, then (5) simplifies to

$$V_1 = \begin{cases} 2 \sum_{m=1}^{M-1}{}' G(\gamma m) & \text{for } M = 2,4,6,\ldots \\ G(0) + 2 \sum_{m=2}^{M-1}{}' G(\gamma m) & \text{for } M = 1,3,5,\ldots \end{cases} . \qquad (6)$$

A program for (6) is given in appendix B.

CASE 2

For integer $M \geq 1$ and constant $\gamma > 0$, consider the alternative special choice of $h(t)$ as

$$h_2(t) = \left[\frac{\sin(M\gamma t)}{\sin(\gamma t)}\right]^2 = \sum_{n,k=0}^{M-1}{}' \exp[i\gamma t(2n-2k)] = \qquad (7A)$$

$$= \sum_{m=1-M}^{M-1}{}' (M - |m|) \exp(i2\gamma tm) , \qquad (7B)$$

where we used (3). There is no prime on the latter summation because all terms from 1-M to M-1 are to be included. The Fourier transform of $h_2(t)$ is

$$H_2(\omega) = 2\pi \sum_{m=1-M}^{M-1}{}' (M - |m|) \delta(\omega - 2\gamma m) . \qquad (8)$$

The use of (7A) and (8) in (2) yields

$$V_2 \equiv \int dt\ g(t)\ \left[\frac{\sin(M\gamma t)}{\sin(\gamma t)}\right]^2 = \sum_{m=1-M}^{M-1} (M - |m|)\ G(2\gamma m)\ . \qquad (9)$$

Again, the integral of interest is given by a finite sum of samples of the Fourier transform of g(t), also at increment 2γ in ω. The fast Fourier transform technique discussed in appendix A is relevant here also. If $G(\omega)$ is even in ω, then we can express (9) as

$$V_2 = M\ G(0) + 2 \sum_{m=1}^{M-1} (M - m)\ G(2\gamma m) \quad \text{for all } M \geq 1\ . \qquad (10)$$

A program for (10) is given in appendix B.

CASE 3

For arbitrary weights $\{w_m\}$ and frequencies $\{\gamma_m\}$, with

$$h_3(t) = \sum_m w_m \exp(i\gamma_m t)\ , \qquad (11A)$$

then we have a generalization of (3), with

$$H_3(\omega) = 2\pi \sum_m w_m\ \delta(\omega - \gamma_m)\ . \qquad (11B)$$

(Summations without limits are over the range of nonzero summand.) Use of these expressions in general result (2) yields

$$V_3 \equiv \int dt\ g(t) \sum_m w_m^*\ \exp(-i\gamma_m t) = \sum_m w_m^*\ G(\gamma_m)\ . \qquad (11C)$$

Again, the Fourier transform of g(t) is required, but now at general arguments $\{\gamma_m\}$.

CASE 4

Function $h_2(t)$ in (7) is a special case of the weighted array power response

$$h_4(t) = \left| \sum_k w_k \exp(-i2\gamma tk) \right|^2 = \sum_m \phi(m) \exp(-i2\gamma tm) , \quad (12A)$$

where $\phi(m)$ is the autocorrelation of the weights:

$$\phi(m) = \sum_k w_k \, w^*_{k-m} = \phi^*(-m) . \quad (12B)$$

The integral in (9) is then generalized to

$$V_4 \equiv \int dt \; g(t) \; h_4^*(t) = \int dt \; g(t) \left| \sum_k w_k \exp(-i2\gamma tk) \right|^2 =$$

$$= \sum_m \phi(m) \; G(2\gamma m) , \quad (13A)$$

upon use of (12A), where g(t) can be complex and nonsymmetric. Thus, integral V_4 requires the autocorrelation of weights $\{w_k\}$ and the Fourier transform of g(t) for its evaluation. The earlier result in (9) corresponds to weights $w_k = 1$ for $1 \leq k \leq M$.

When function g(t) is real (but possibly nonsymmetric) and the weights are real, (13A) can be simplified to

$$V_4 = \phi(0)\, G_r(0) + 2 \sum_{m \geq 1} \phi(m)\, G_r(2\gamma m) \ , \qquad (13B)$$

where $G_r(\omega)$ is the real part of Fourier transform $G(\omega)$ in (1).

A program for (13B) is given in appendix B.

EXAMPLES

EXAMPLE A

The first example of interest is

$$g_a(t) = \frac{1}{(t-\mu)^2 + \beta^2} , \quad \beta > 0 . \tag{14}$$

Its Fourier transform is

$$G_a(\omega) = \frac{\pi}{\beta} \exp(-i\mu\omega - \beta|\omega|) , \tag{15}$$

for which the real part is

$$G_{ar}(\omega) = \frac{\pi}{\beta} \cos(\mu\omega) \exp(-\beta|\omega|) . \tag{16}$$

Since integral (5) is obviously real for example (14), we obtain

$$V_{1a} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1}{}' G_{ar}(\gamma m) . \tag{17}$$

Substitution of (16) in (17) yields the closed form result

$$V_{1a} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(M\gamma t)}{\sin(\gamma t)} =$$

$$= \frac{2\pi}{\beta D}\left[E_{M+3}\, C_{M-1} - E_{M+1}\, C_{M+1} + \begin{cases} C_1\, E_1\, (1 - E_2) & \text{for M even} \\ \frac{1}{2}\,(1 - E_4) & \text{for M odd} \end{cases}\right] , \tag{18}$$

where

$$E_m = \exp(-\beta\gamma m) , \quad C_m = \cos(\mu\gamma m) , \quad D = 1 - 2\, E_2\, C_2 + E_4 . \tag{19}$$

A program for (18) and (19) follows; it is written in BASIC for the Hewlett Packard 9000 computer.

```
 10  INPUT M,Beta,Gamma,Mu        ! Beta > 0,  Gamma > 0
 20  B=Beta*Gamma
 30  C=Mu*Gamma
 40  E=EXP(-B*2)
 50  IF (M MODULO 2)=1 THEN 80
 60  F=COS(C)*SQR(E)*(1-E)
 70  GOTO 90
 80  F=.5-.5*E*E
 90  A=E*COS(C*(M-1))-COS(C*(M+1))
100  A=A*EXP(-B*(M+1))+F
110  V1a=A*2*PI/(Beta*(1-2*E*COS(C*2)+E*E))
120  PRINT V1a
130  END
```

When we instead substitute (14) and (16) in (9), there follows

$$V_{2a} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[\frac{\sin(M\gamma t)}{\sin(\gamma t)}\right]^2 =$$

$$= \frac{\pi}{\beta} \sum_{m=1-M}^{M-1} (M - |m|) \cos(2\mu\gamma m) \exp(-2\beta\gamma|m|) \ . \qquad (20)$$

This finite sum can be written in compact form by use of [3; 0.113]. Namely, define here

$$E = \exp(-2\beta\gamma) \ , \quad C = \cos(2\mu\gamma) \ , \quad S = \sin(2\mu\gamma) \ ,$$
$$E_M = \exp(-2\beta\gamma M), \quad C_M = \cos(2\mu\gamma M), \quad S_M = \sin(2\mu\gamma M),$$
$$A = 1 - E^2 \ , \quad B = 1 + E^2 \ , \quad D = B - 2\,E\,C. \qquad (21)$$

Then

$$V_{2a} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[\frac{\sin(M\gamma t)}{\sin(\gamma t)}\right]^2 =$$

$$= \frac{2\pi}{\beta D}\left[\frac{M}{2} A - \frac{E}{D}\left((C B - 2 E)(1 - E_M C_M) + S A E_M S_M\right)\right] . \quad (22)$$

A program for (21) and (22) follows.

```
 10  INPUT M,Beta,Gamma,Mu        ! Beta > 0,  Gamma > 0
 20  Tb=2*Beta*Gamma
 30  Tm=2*Mu*Gamma
 40  E=EXP(-Tb)
 50  A=E*E
 60  B=1+A
 70  A=1-A
 80  C=COS(Tm)
 90  D=B-2*E*C
100  Em=EXP(-Tb*M)
110  T=(C*B-2*E)*(1-Em*COS(Tm*M))
120  T=T+SIN(Tm)*A*Em*SIN(Tm*M)
130  T=.5*M*A-T*E/D
140  V2a=T*2*PI/(Beta*D)
150  PRINT V2a
160  END
```

EXAMPLE B

The next example to be considered is

$$g_b(t) = \frac{1}{(t-\mu)^2 + \beta^2} \frac{\sin(\alpha t)}{\alpha t} , \quad \beta > 0 , \quad \alpha > 0 . \quad (23)$$

Since $g_b(t)$ is a product of two functions, its Fourier transform $G_b(\omega)$ is given by a convolution of the individual transforms. The Fourier transform of the first term in (23) has already been encountered in (15), and the Fourier transform of the second term in (23) is a rectangle located on interval $(-\alpha, \alpha)$ in ω. Therefore, $G_b(\omega)$ is given by convolution

$$G_b(\omega) = \frac{\pi}{2\alpha\beta} \int_{\omega-\alpha}^{\omega+\alpha} du \, \exp(-i\mu u - \beta|u|) \, . \qquad (24)$$

Since $g_b(t)$ in (23) is real, we need only evaluate the real part, $G_{br}(\omega)$, of $G_b(\omega)$. With the aid of auxiliary variables

$$\begin{aligned}
&C_\omega = \cos(\mu\omega), \; S_\omega = \sin(\mu\omega), \; \underline{C}_\omega = \cosh(\beta\omega), \; \underline{S}_\omega = \sinh(\beta\omega), \\
&C_\alpha = \cos(\mu\alpha), \; S_\alpha = \sin(\mu\alpha), \; \underline{C}_\alpha = \cosh(\beta\alpha), \; \underline{S}_\alpha = \sinh(\beta\alpha), \\
&B_1 = \underline{C}_\omega \, C_\omega \, (\beta \, C_\alpha - \mu \, S_\alpha) + \underline{S}_\omega \, S_\omega \, (\mu \, C_\alpha + \beta \, S_\alpha), \\
&B_2 = \underline{S}_\alpha \, C_\alpha \, (\beta \, C_\omega - \mu \, S_\omega) + \underline{C}_\alpha \, S_\alpha \, (\mu \, C_\omega + \beta \, S_\omega), \qquad (25)
\end{aligned}$$

we find that $G_{br}(\omega)$ is given by

$$G_{br}(\omega) = \frac{\pi}{\alpha\beta(\beta^2+\mu^2)} \begin{cases} \beta - \exp(-\beta\alpha) \, B_1 & \text{for } 0 \le \omega \le \alpha \\ \exp(-\beta\omega) \, B_2 & \text{for } \alpha \le \omega \end{cases} . \qquad (26)$$

To complete the description, we observe that $G_{br}(\omega)$ is even in ω because $g_b(t)$ is real. A program for $G_{br}(\omega)$ follows, where we have made the following identifications: W ≡ ω, A ≡ α, B ≡ β, U ≡ μ.

```
10  DEF FNGbr(W,A,B,U)
20  Wa=ABS(W)
30  F=PI/(A*B*(B*B+U*U))
40  Ea=EXP(-B*A)
50  Ew=EXP(-B*Wa)
60  Ca=COS(U*A)
70  Cw=COS(U*Wa)
80  Sa=SIN(U*A)
90  Sw=SIN(U*Wa)
100 IF Wa<A THEN 150
110 Ra=1./Ea
120 T=(Ra-Ea)*Ca*(B*Cw-U*Sw)
130 B2=.5*(T+(Ra+Ea)*Sa*(U*Cw+B*Sw))
140 RETURN F*Ew*B2
150 Rw=1./Ew
160 T=(Rw+Ew)*Cw*(B*Ca-U*Sa)
170 B1=.5*(T+(Rw-Ew)*Sw*(U*Ca+B*Sa))
180 RETURN F*(B-Ea*B1)
190 FNEND
```

If we now employ (23) in (5), we obtain

$$V_{1b} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(\alpha t)}{\alpha t} \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1}{}' G_{br}(\gamma m) \ , \qquad (27)$$

where $G_{br}(\omega)$ is given by (25), (26), and its even property. Since there is a break in the analytic form for $G_{br}(\omega)$ at $\omega = \pm\alpha$, it is not reasonable to perform the summation in (27) in closed form; those terms in (27) for $\gamma|m| \leq \alpha$ utilize the upper line of (26), while those for $\gamma|m| \geq \alpha$ utilize the lower line of (26). However, since $G_{br}(\omega)$ is even in ω, the simplification in (6) is applicable.

Instead, when (23) is substituted in (9), there follows

$$V_{2b} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(\alpha t)}{\alpha t} \left[\frac{\sin(M\gamma t)}{\sin(\gamma t)}\right]^2 = \sum_{m=1-M}^{M-1} (M - |m|)\, G_{br}(2\gamma m), \qquad (28)$$

where $G_{br}(\omega)$ is given by (25) and (26). Again, the break in form of $G_{br}(\omega)$ at $\omega = \pm\alpha$ precludes a closed form result for the summation in (28); also, the simplification in (10) is immediately applicable to (28).

EXAMPLE C

The final example is

$$g_c(t) = \frac{1}{(t-\mu)^2 + \beta^2} \left[\frac{\sin(\alpha t)}{\alpha t}\right]^2 , \quad \beta > 0 , \quad \alpha > 0 . \qquad (29)$$

The Fourier transform of the second term in (29) is a triangle located on interval $(-2\alpha, 2\alpha)$ in ω. Therefore, $G_c(\omega)$ is given by convolution

$$G_c(\omega) = \frac{\pi}{2\alpha\beta} \int_{\omega-2\alpha}^{\omega+2\alpha} du \, \exp(-i\mu u - \beta|u|) \left(1 - \frac{|\omega-u|}{2\alpha}\right) . \qquad (30)$$

Because $g_c(t)$ is real, only the real part of (30) is needed. This tedious calculation has been carried through, with the following result; define auxiliary variables

$$R = \beta^2-\mu^2, \quad I = 2\beta\mu, \quad D = \beta^2+\mu^2, \quad E_\omega = \exp(-\beta\omega), \quad E_\alpha = \exp(-2\beta\alpha),$$
$$\underline{C}_\alpha = \cosh(2\beta\alpha), \quad \underline{S}_\alpha = \sinh(2\beta\alpha), \quad C_\alpha = \cos(2\mu\alpha), \quad S_\alpha = \sin(2\mu\alpha) ,$$
$$\underline{C}_\omega = \cosh(\beta\omega) , \quad \underline{S}_\omega = \sinh(\beta\omega) , \quad C_\omega = \cos(\mu\omega) , \quad S_\omega = \sin(\mu\omega) ,$$
$$C_1 = \underline{C}_\omega C_\omega (R C_\alpha - I S_\alpha) + \underline{S}_\omega S_\omega (R S_\alpha + I C_\alpha) ,$$
$$C_2 = \underline{C}_\alpha C_\alpha (R C_\omega - I S_\omega) + \underline{S}_\alpha S_\alpha (R S_\omega + I C_\omega) ,$$
$$C = R C_\omega - I S_\omega . \qquad (31)$$

Then we find that

$$G_{cc}(\omega) = \frac{\pi}{2\alpha^2\beta D^2} \begin{cases} D\beta(2\alpha - \omega) - E_\omega C + E_\alpha C_1 & \text{for } 0 \le \omega \le 2\alpha \\ - E_\omega C + E_\omega C_2 & \text{for } 2\alpha \le \omega \end{cases} \qquad (32)$$

Also, $G_{cr}(\omega)$ is even in ω. A program for $G_{cr}(\omega)$ is listed below, where W ≡ ω, A ≡ α, B ≡ β, U ≡ μ.

```
 10  DEF FNGcr(W,A,B,U)      !  A > 0 ,  B > 0
 20  Wa=ABS(W)
 30  Tb=2.*A*B
 40  Tu=2.*A*U
 50  Bw=B*Wa
 60  Uw=U*Wa
 70  B2=B*B
 80  U2=U*U
 90  R=B2-U2
100  I=2.*B*U
110  D=B2+U2
120  Ew=EXP(-Bw)
130  Ea=EXP(-Tb)
140  Ca=COS(Tu)
150  Sa=SIN(Tu)
160  Cw=COS(Uw)
170  Sw=SIN(Uw)
180  C=R*Cw-I*Sw
190  IF Wa<2.*A THEN 250
200  Ra=1./Ea
210  C2=.5*(Ra+Ea)*Ca*C
220  C2=.5*(Ra-Ea)*Sa*(R*Sw+I*Cw)+C2
230  T=Ew*(C2-C)
240  GOTO 290
250  Rw=1./Ew
260  C1=.5*(Rw+Ew)*Cw*(R*Ca-I*Sa)
270  C1=.5*(Rw-Ew)*Sw*(R*Sa+I*Ca)+C1
280  T=D*(Tb-Bw)-Ew*C+Ea*C1
290  RETURN PI*T/(Tb*A*D*D)
300  FNEND
```

We now substitute (29) into (5) and get

$$V_{1c} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[\frac{\sin(\alpha t)}{\alpha t}\right]^2 \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1}{}' G_{cr}(\gamma m) \ , \quad (33)$$

where $G_{cr}(\omega)$ is given by (31), (32), and its even character. The break in form in (32) at $\omega = \pm 2\alpha$ precludes a closed form for the sum in (33). However, (6) is still applicable.

When (29) is utilized in (9), there follows

$$V_{2c} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[\frac{\sin(\alpha t)}{\alpha t}\right]^2 \left[\frac{\sin(M\gamma t)}{\sin(\gamma t)}\right]^2 =$$

$$= \sum_{m=1-M}^{M-1} (M - |m|)\, G_{cr}(2\gamma m) \ . \qquad (34)$$

Equation (10) may also be employed here.

SPECIAL CASES

If we set M = 1 in (17), there follows

$$\int \frac{dt}{(t-\mu)^2 + \beta^2} = G_{ar}(0) = \frac{\pi}{\beta} \ , \qquad (35)$$

where we used (16). The same case in (27) yields

$$\int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(\alpha t)}{\alpha t} = G_{br}(0) =$$

$$= \frac{\pi}{\alpha\beta(\beta^2+\mu^2)} \{\beta - \exp(-\beta\alpha)\, [\beta \cos(\mu\alpha) - \mu \sin(\mu\alpha)]\} \ , \qquad (36)$$

upon use of (26) and (25). Finally, from (33),

$$\int \frac{dt}{(t-\mu)^2 + \beta^2} \left[\frac{\sin(\alpha t)}{\alpha t}\right]^2 = G_{cr}(0) =$$

$$= \frac{\pi}{2\alpha^2\beta(\beta^2+\mu^2)^2} \left(2\alpha\beta(\beta^2 + \mu^2) - R + E_\alpha (R\, C_\alpha - I\, S_\alpha)\right) , \quad (37)$$

using (32) and (31).

APPLICATION TO SUMS

In this section, it is more convenient to use Parseval's theorem (2) in the form

$$V = \int dt\ g(t)\ h^*(t) = \int df\ G(f)\ H^*(f)\ , \tag{38}$$

where Fourier transform

$$G(f) = \int dt\ \exp(-i2\pi ft)\ g(t)\ . \tag{39}$$

Now, we take as our candidate h(t) function,

$$h(t) = p(t)\ \Delta\delta_\Delta(t)\ , \tag{40}$$

where $\delta_\Delta(t)$ is the infinite impulse train

$$\delta_\Delta(t) = \sum_k \delta(t - k\Delta)\ . \tag{41}$$

The Fourier transform of h(t) is then

$$H(f) = P(f) \circledast \delta_{1/\Delta}(f) = \sum_k P\left(f - \frac{k}{\Delta}\right)\ , \tag{42}$$

where P(f) is the Fourier transform of p(t), $\circledast$ denotes convolution, and we have utilized the fact that the Fourier transform of impulse train $\Delta\delta_\Delta(t)$ is another impulse train, $\delta_{1/\Delta}(f)$.

Substitution of (40) and (42) in (38) yields

$$V = \Delta \sum_k g(k\Delta)\, p^*(k\Delta) = \sum_k \int df\, G(f)\, P^*\left(f - \frac{k}{\Delta}\right) . \quad (43)$$

For general p(t) and P(f), this will not be a useful relation, since the right-hand side of (43) is an infinite sum of integrals. However, we will be interested here only in the special cases of

$$p(t) = \left[\frac{\sin(M\gamma t)}{\sin(\gamma t)}\right]^n , \quad n \text{ integer} . \quad (44)$$

CASE n = 0

For n = 0, the above relations specialize to

$$p(t) = 1 , \quad P(f) = \delta(f) ,$$

$$H(f) = \sum_k \delta\left(f - \frac{k}{\Delta}\right) ,$$

$$V_o = \Delta \sum_k g(k\Delta) = \sum_k G\left(\frac{k}{\Delta}\right) . \quad (45)$$

This is a discrete version of Parseval's theorem. Although one infinite sum has been traded for another, we can now choose that alternative that has the most rapidly decaying (and/or easily computed) summand for numerical evaluation.

CASE n = 1

Now we have, via (3),

$$p(t) = \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1}{}' \exp(i\gamma tm) \ . \tag{46}$$

There follows

$$P(f) = \sum_{m=1-M}^{M-1}{}' \delta\left(f - \frac{\gamma m}{2\pi}\right) \ ,$$

$$H(f) = \sum_{k} \sum_{m=1-M}^{M-1}{}' \delta\left(f - \frac{k}{\Delta} - \frac{\gamma m}{2\pi}\right) \ ,$$

$$V_1 = \Delta \sum_{k} g(k\Delta) \frac{\sin(M\gamma\Delta k)}{\sin(\gamma\Delta k)} = \tag{47}$$

$$= \sum_{k} \sum_{m=1-M}^{M-1}{}' G\left(\frac{k}{\Delta} + \frac{\gamma m}{2\pi}\right) \ . \tag{48}$$

Again, we have an alternative infinite sum (48) that hopefully decays faster than the original sum (47). The sin(Mx)/sin(x) term does not help convergence in (47) because this term never decays for large x. Although (48) is a double sum, the summation on m only contains M terms; the utility of (48) depends heavily on the asymptotic decay of G(f) for large f.

CASE n = 2

With the aid of (7), we now find

$$p(t) = \left[\frac{\sin(M\gamma t)}{\sin(\gamma t)}\right]^2 = \sum_{m=1-M}^{M-1} (M - |m|) \exp(i2\gamma tm) ,$$

$$P(f) = \sum_{m=1-M}^{M-1} (M - |m|) \delta\left(f - \frac{\gamma m}{\pi}\right) ,$$

$$H(f) = \sum_{k} \sum_{m=1-M}^{M-1} (M - |m|) \delta\left(f - \frac{k}{\Delta} - \frac{\gamma m}{\pi}\right) ,$$

$$V_2 = \Delta \sum_{k} g(k\Delta) \left[\frac{\sin(M\gamma\Delta k)}{\sin(\gamma\Delta k)}\right]^2 = \qquad (49)$$

$$= \sum_{k} \sum_{m=1-M}^{M-1} (M - |m|) G\left(\frac{k}{\Delta} + \frac{\gamma m}{\pi}\right) . \qquad (50)$$

EXAMPLE

Consider, as in (14) and (16),

$$g_a(t) = \frac{1}{(t-\mu)^2 + \beta^2} ,$$

$$G_{ar}(f) = \frac{\pi}{\beta} \cos(2\pi\mu f) \exp(-2\pi\beta|f|) . \qquad (51)$$

The summations in (47) and (49) are very slowly decaying, leading to difficulty in attaining accurate results. The alternatives in

(48) and (50), on the other hand, have exponential decay and can be evaluated quite accurately. The additional examples given earlier in (23)-(26) and in (29)-(32), along with the corresponding programs, lend reasonable alternatives to some otherwise lengthy numerical calculations.

SOME RELATED SUMS

Here, we collect a few closed form results for sums involving the sin(Mx)/sin(x) kernel. For ease of notation, define

$$S_N(M,k) = \frac{\sin(Mk\pi/N)}{\sin(k\pi/N)} \quad . \tag{52}$$

Observe that

$$S_N(M,k) = \begin{cases} M & \text{for } k = 0,\ \pm 2N,\ \pm 4N,\ldots \\ M(-1)^{M-1} & \text{for } k = \pm N,\ \pm 3N,\ldots \end{cases} \quad . \tag{53}$$

Then, we find the sum over one interval to be

$$\sum_{k=0}^{N-1} S_N(M,k) = \begin{cases} M & \text{for M even} \\ N(1 + 2J) & \text{for M odd} \end{cases} \quad , \tag{54}$$

where

$$J = \text{INT}\left(\frac{M-1}{2N}\right) \quad . \tag{55}$$

The sum over a double interval is

$$\sum_{k=0}^{2N-1} S_N(M,k) = \begin{cases} 0 & \text{for } M = 0,2,4,\ldots\ldots \\ 2N & \text{for } M = 1,3,\ldots,2N-1 \end{cases} . \qquad (56)$$

The correlation on the second variable of S_N is

$$\sum_{k=0}^{N-1} S_N(M,k)\ S_N(M,k+j) = N\ S_N(M,j) \quad \text{for } 0 \le M \le N \text{ and all } j. \qquad (57)$$

Finally, the correlation on the first variable is

$$\sum_{k=0}^{N-1} S_N(M,k)\ S_N(M+2L,k) = M(M + 2L) +$$

$$+ \begin{cases} \underline{M}(N - \underline{M} - 2\underline{L}) & \text{for } 0 \le \underline{M} + \underline{L} \le N \\ N(3\underline{M} + 2\underline{L} - 2N) - \underline{M}(\underline{M} + 2\underline{L}) & \text{for } N < \underline{M} + \underline{L} \end{cases} , \qquad (58)$$

for all M, L, N, where

$$\underline{M} = M \text{ MOD } N , \qquad \underline{L} = L \text{ MOD } N . \qquad (59)$$

SUMMARY

Extensions to integrals involving $[\sin(Mx)/\sin(x)]^n$ for $n > 2$ are possible, based upon the results presented here. For example, starting from (12A) for arbitrary weights, we could consider

$$h_5(t) = h_4^2(t) = \sum_p \psi(p) \exp(-i2\gamma tp) , \qquad (60)$$

where

$$\psi(p) = \sum_m \phi(m) \phi^*(m-p) \qquad (61)$$

is the autocorrelation of sequence $\{\phi(m)\}$ defined in (12B). Therefore, Fourier transform

$$H_5(\omega) = 2\pi \sum_p \psi(p) \delta(\omega + 2\gamma p) , \qquad (62)$$

giving rise to

$$V_5 \equiv \int dt\ g(t)\ h_5(t) = \sum_p \psi(p)\ G(2\gamma p) . \qquad (63)$$

The case of equal weights $\{w_k\}$ in (12A) now corresponds to $n = 4$ in the sine function ratio above, and $\psi(p)$ is the autocorrelation of a triangular sequence.

The evaluation of integrals and sums involving the term $[\sin(Mx)/\sin(x)]^n$ can often be simplified by the use of Parseval's theorem because this term has a Fourier transform which is a finite sum of delta functions. Major effort can then be concentrated on getting the Fourier transform of the complementary part of the integrand. This procedure has been applied here to several examples which arise in evaluation of the response of equispaced arrays to distributed spatial fields. For more complicated fields, a fast Fourier transform procedure combined with the above result leads to a very efficient method of integral evaluations. Applications of this procedure have been made in [5].

APPENDIX A - USE OF FAST FOURIER TRANSFORM

The summations for V_1 and V_2 in (5) and (9), respectively, require the evaluation of the Fourier transform of g(t), namely $G(\omega)$, at equispaced increment 2γ. But this latter function can be approximated by means of the trapezoidal rule according to

$$G(\omega) = \int dt\, \exp(-i\omega t)\, g(t) =$$

$$\simeq \Delta \sum_n \exp(-i\omega\Delta n)\, g(n\Delta) \equiv \underline{G}(\omega) = \sum_n G\left(\omega - n\frac{2\pi}{\Delta}\right) , \qquad (A-1)$$

where Δ is the sampling increment in t. The latter summation in (A-1) indicates aliasing lobes separated by $2\pi/\Delta$ on the ω axis. In order to control aliasing, we must choose Δ small enough, say $\Delta < \Delta_o$. Then samples of approximation $\underline{G}(\omega)$ in (A-1) at multiples of 2γ are given by

$$\underline{G}(2\gamma m) = \Delta \sum_n \exp(-i2\gamma\Delta mn)\, g(n\Delta) . \qquad (A-2)$$

Now since Δ is arbitrary, except for upper limit Δ_o, choose

$$\Delta = \frac{\pi}{N\gamma} , \qquad (A-3)$$

where N is an integer and γ is the prescribed increment in ω. In order that Δ be less than Δ_o, we must take integer

$$N > \frac{\pi}{\gamma\Delta_o} . \qquad (A-4)$$

Use of (A-3) in (A-2) gives

$$\underline{G}(2\gamma m) = \Delta \sum_{n} \exp(-i2\pi mn/N)\, g(n\Delta) = \qquad \text{(A-5)}$$

$$= \Delta \sum_{n=0}^{N-1} \exp(-i2\pi mn/N)\, g_c(n\Delta)\ , \qquad \text{(A-6)}$$

where "collapsed" sequence [4; pages 4-5]

$$g_c(n\Delta) \equiv \sum_{k} g(n\Delta + kN\Delta) \quad \text{for } 0 \le n \le N-1\ . \qquad \text{(A-7)}$$

The manipulation from (A-5) to (A-6) is exact; it avoids truncation error normally associated with functions g(t) which decay slowly with t. The sum on k in (A-7) must be carried out (for each n) until negligible values for g are encountered for both positive as well as negative values of k.

Equation (A-6) indicates that values of $\underline{G}(2\gamma m)$ for m = 0 to N - 1 are available by an N-point fast Fourier transform when N is a power of 2. Values for negative m are available in location m mod N. In order to get all the desired values of $\underline{G}(2\gamma m)$ required for (9), without aliasing, we also require that N/2 > M. Thus, the final condition on integer N is

$$N > \max\left(\frac{\pi}{\gamma\Delta_o},\ 2M\right)\ . \qquad \text{(A-8)}$$

For the case of (5), where the increment on ω is 2γ, but starting at $\omega = \gamma$, we return to (A-1) to find that

$$\underline{G}(\gamma + 2\gamma m) = \Delta \sum_{n} \exp(-i\gamma\Delta n - i2\gamma\Delta mn)\, g(n\Delta)\ . \qquad \text{(A-9)}$$

The same choice of Δ in (A-3) now yields

$$\underline{G}(\gamma + 2\gamma m) = \Delta \sum_{n} \exp(-i2\pi mn/N) \exp(-i\pi n/N)\, g(n\Delta) \ . \quad \text{(A-10)}$$

This result is identical to (A-5) except that $g(n\Delta)$ must be replaced by

$$\exp(-i\pi n/N)\, g(n\Delta) \equiv \tilde{g}(n\Delta) \ . \quad \text{(A-11)}$$

Calculation of the collapsed version of $\tilde{g}$ is eased by the observation that

$$\tilde{g}_c(n\Delta) \equiv \sum_{k} \tilde{g}(n\Delta + kN\Delta) =$$

$$= \sum_{k} \exp(-i\pi(n + kN)/N)\, g(n\Delta + kN\Delta) =$$

$$= \exp(-i\pi n/N) \sum_{k} (-1)^k\, g(n\Delta + kN\Delta) \quad \text{for } 0 \le n \le N - 1 \ , \quad \text{(A-12)}$$

thereby leading exactly to

$$\underline{G}(\gamma + 2\gamma m) = \Delta \sum_{n=0}^{N-1} \exp(-i2\pi mn/N)\, \tilde{g}_c(n\Delta) \ . \quad \text{(A-13)}$$

The leading phase factor in (A-12) only needs to be evaluated at N different values, and the sum in (A-12) requires differencing of "adjacent" samples of g spaced by $N\Delta$, rather than the straight summation previously adequate for (A-6) and (A-7). Condition (A-8) applies here as well.

APPENDIX B – PROGRAMS FOR (6), (10), AND (13B)

Table B-1. Program for (6)

```
 10  M=7                 !  > 0
 20  Gamma=1.31          !  > 0
 30  DOUBLE M,Ms         !  INTEGERS
 40  S=0.
 50  IF (M MODULO 2)=1 THEN 110
 60  FOR Ms=1 TO M-1 STEP 2
 70  S=S+FNG(Gamma*Ms)
 80  NEXT Ms
 90  V1=2.*S
100  GOTO 150
110  FOR Ms=2 TO M-1 STEP 2
120  S=S+FNG(Gamma*Ms)
130  NEXT Ms
140  V1=FNG(0.)+2.*S
150  PRINT M,Gamma,V1
160  END
170  !
180  DEF FNG(W)
```

Table B-2. Program for (10)

```
 10  M=6                 !  > 0
 20  Gamma=.71           !  > 0
 30  DOUBLE M,Ms         !  INTEGERS
 40  G2=2.*Gamma
 50  S=0.
 60  FOR Ms=1 TO M-1
 70  S=S+(M-Ms)*FNG(G2*Ms)
 80  NEXT Ms
 90  V2=M*FNG(0.)+2.*S
100  PRINT M,Gamma,V2
110  END
120  !
130  DEF FNG(W)
```

Table B-3. Program for (13B)

```
 10  M=9                          !  > 0
 20  Gamma=.79                    !  > 0
 30  DOUBLE M,Ms,Ks               !  INTEGERS
 40  DIM W(100)
 50  REDIM W(1:M)
 60  CALL Weights(M,W(*))         !  REAL WEIGHTS
 70  G2=2.*Gamma
 80  S=0.
 90  FOR Ms=1 TO M-1
100  Phi=0.
110  FOR Ks=Ms+1 TO M
120  Phi=Phi+W(Ks)*W(Ks-Ms)       !  CORRELATION OF WEIGHTS
130  NEXT Ks
140  S=S+Phi*FNGr(G2*Ms)
150  NEXT Ms
160  Phi=0.
170  FOR Ks=1 TO M
180  Phi=Phi+W(Ks)*W(Ks)
190  NEXT Ks
200  V4=Phi*FNGr(0.)+2.*S
210  PRINT M,Gamma,V4
220  END
230  !
240  SUB Weights(DOUBLE M,REAL W(*))
250  DOUBLE Ks                    !  INTEGER
260  T=2.*PI/M
270  FOR Ks=1 TO M
280  D=Ks-.5
290  W(Ks)=1.                     !  FLAT WEIGHTS
300  W(Ks)=.5-.5*COS(T*D)         !  HANN WEIGHTS
310  W(Ks)=.54-.46*COS(T*D)       !  HAMMING WEIGHTS
320  NEXT Ks
330  MAT W=W/SUM(W)               !  NORMALIZATION
340  SUBEND
350  !
360  DEF FNGr(W)
```

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