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Orderings of N-Tuples

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SPECIAL CASE

PROOF

We begin with the special case of n-tuples which sum to a given value, k, and build up to the general case. Let Z^+ denote the positive integers and let $S(n, k) = \{\alpha = (a_1, \ldots, a_n), a_i \in Z^+ s.t. \sum a_1 = k\}$ ordered lexicographically. It is well known that $|S(n, k)|$ is simply the binomial coefficient $(k - l)$ choose $(n - l)$, i.e.,

$$|S(n,k)| = \binom{k-1}{n-1} . \tag{1}$$

Now given an $\alpha \in S(n, k)$, can we calculate its position in the ordering? Certainly $(1, 1, \ldots, k + 1 - n)$ is first and $(k + 1 - n, 1, \ldots, 1)$ is last. Let the order function on $S(n, k)$ be $f_{n,k}$: $S(n, k) \to \{1, 2, \ldots, |S(n,k)|\}$.

LEMMA 1.1

Let $S(n, k)$ be the set of n-tuples of positive integers which sum to k, ordered lexicographically. Let $\alpha = (a_1, \ldots, a_n)$ and define $\sigma_j(\alpha) = \sum_{i=j}^{n} a_i$. Then the position of α is given by

$$f_{n,k}(a) = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1} + (a) - 1}{n-j} . \tag{2}$$

PROOF

Consider a fixed $\alpha = (a_1, \ldots, a_n)$ in $S(n, k)$. Let $\beta = b_1, \ldots, b_n)$, $T_j = \{\beta \in S(n, k) | b_1 = a_1, \ldots, b_{j-1} = a_{j-1}, b_j > a_j\}$ and let $T = U T_j$, $j = 1, \ldots$ n. Then certainly $T = \{\beta \in S(n, k) | \beta > \alpha\}$ and $T_i \cap T_j = \phi$ if $i \neq j$, so we have $|T| = \sum_{1 \leq j < n} |T_j|$ (since $T_n = \phi$). But $\{(T_j = (a_1, \ldots, a_{j-1}, a_j + g_1, g_2, \ldots g_{n-j+1}) | g = (g_1, \ldots g_{n-j+1}) \in S(n - j + 1, k - a_1 - \ldots - a_j)\}$. Hence,

$$|T_j| = \binom{k - a_1 - \ldots - a_j - 1}{n-j} = \binom{\sigma_{j+1} + (a) - 1}{n-j} . \tag{3}$$

Therefore, we can conclude that $f_{n,k}(\alpha) = |S(n, k)| - |\{b \in S(n, k) \; s.t. \; \beta > \alpha\}|$

$$= \binom{k-1}{n-1} - |T| = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} |T_j| = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1} + (a) - 1}{n-j} .$$

qed

N-TUPLES OF POSITIVE INTEGERS

Now let $Z^n = Z^+ \times \dots \times Z^+$ be the set of all n-tuples of positive integers. Note that $Z^n = U_{n \le k}$. Let Z^n be ordered by $S(n, n) < S(n, n+1) < S(n, n+2) < \dots$ where the $S(n, k)$ are ordered lexicographically as before. Let $f_n : Z^n \to Z$ be the order function for this space.

THEOREM 1.1

Let Z^n be the set of all n-tuples of positive integers ordered as above. Let $\alpha = (a_1, \dots, a_n) \in Z^n$ and define $\sigma_j(\alpha) = \sum_{i=j}^{n} a_i$. The the position of α is given by

$$f_n(a) = \binom{\sigma_j(a)}{n} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a) - 1}{n-j} . \tag{4}$$

PROOF

By ordering on Z^n and the lemma we have

$$f_n(a) = \sum_{j=1}^{\sigma_1(a)-1} |S(n,k)| + f_{n,\sigma_1(a)}(a) = \sum_{j=1}^{\sigma_1(a)-1} \binom{j-1}{n-1} + f_{n,\sigma_1(a)}(a)$$

$$= \binom{\sigma_1(a) - 1}{n} + f_{n,\sigma_1(a)}(a) = \binom{\sigma_1(a) - 1}{n} + \binom{\sigma_1(a) - 1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a) - 1}{n-j}$$

$$= \binom{\sigma_1(a)}{n} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a) - 1}{n-j} \quad \left[\text{using the basic combinatorial identity } \binom{a+1}{b} = \binom{a}{b} + \binom{a}{b} \right] .$$

qed

COROLLARIES

Next I give two corollaries of lemma 1.1.

COROLLARY 3.1

Let $\boldsymbol{S}(k) = \boldsymbol{U}_{1 \le n \le k}\ \boldsymbol{S}(n, k)$ ordered by $\boldsymbol{S}(1, k) < \dots \boldsymbol{S}(k, k)$ where the $\boldsymbol{S}(n, k)$ are ordered lexicographically as before. Then the lexicographic order function, $f^{(k)}$, of $\alpha = (a_1, \dots, a_n) \in \boldsymbol{S}(k)$ is given by

$$f^{(k)}(a) = \sum_{\lambda=1}^{n} \binom{k-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} . \tag{5}$$

PROOF

Using lemma 1.1, we have

$$f^{(k)}(a) = \sum_{\lambda=1}^{n-1} |\boldsymbol{S}(n, k)| + f_{n,k}(a) = \sum_{\lambda=1}^{n-1} \binom{k-1}{\lambda-1} + \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}$$

$$= \sum_{\lambda=1}^{n} \binom{k-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}$$

qed

COROLLARY 2

Let $\boldsymbol{Z}^* = \boldsymbol{U}_{1 \le n < \infty}\ \boldsymbol{Z}^n = \{\alpha = (a_1, \dots, a_n),\ a_i \in \boldsymbol{Z}^+\}$ ordered by $\boldsymbol{S}(1) < \boldsymbol{S}(2) < \dots$ with the $\boldsymbol{S}(\mathrm{k})$ defined and ordered as above. Then the order of function, f^*, of $\alpha = (a_1, \dots, a_n) \in \boldsymbol{Z}^*$ is given by

$$f^*(a) = 2^{\sigma_1(a)-1} + \sum_{\lambda=1}^{n} \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} . \tag{6}$$

PROOF

Since $|S(m)| = \sum_{j=1}^{m} |S(j, m)| = \sum_{j=1}^{m} \binom{m-1}{j-1} = 2^{(m-1)}$, we have, by the ordering on Z^*,

$$f^*(a) = \sum_{\lambda=1}^{\sigma_1(a)-1} |S(m)| + f^{\sigma_1(a)-1}(a) = \sum_{m=1}^{\sigma_1(a)-1} 2^{m-1} + \sum_{\lambda=1}^{n} \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}$$

$$= 2^{\sigma_1(a)-1} + \sum_{\lambda=1}^{n} \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} .$$

qed

INVERSE FUNCTION

Let us consider the inverse function, $f_n^{-1}: Z \rightarrow Z^n$. There does not appear to be a closed form solution, but it is readily computable. Using this, one can easily implement an algorithm to calculate $\phi_{k,n}: Z^k \rightarrow Z^n$ by $\phi_{k,n} = f_n^{-1} f_k$. Now suppose p in Z^+. To compute $f_n^{-1}(p) = (a_1, \ldots, a_n)$ first find the smallest k_1 *s.t.* $\binom{k_1}{n} \geq p$. Then we find successively largest k_1, i = 2 , ... , n, *s.t.* $\binom{k_1}{n} - \sum_{j=1}^{i-1} \binom{k_{j+1}}{n-j} \geq p$. For the nth case, this expression will be an equality. Then $k_i - \sigma_1(f_n^{-1}(p))$ for $i = 1, \ldots, n$ and given all the σs, one easily finds $(a_1, \ldots, a_n)$.

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13 ABSTRACT *(Maximum 200 words)*

The author defines several orderings on n-tuples of positive integers and shows how to compute the position of a given n-tuple with respect to these orderings. In addition, an algorithm is shown to compute the inverse function.

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