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# Orderings of N-Tuples

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## SPECIAL CASE

### PROOF

We begin with the special case of n-tuples which sum to a given value,  $k$ , and build up to the general case. Let  $\mathbf{Z}^+$  denote the positive integers and let  $\mathbf{S}(n, k) = \{\alpha = (a_1, \dots, a_n), a_i \in \mathbf{Z}^+ \text{ s.t. } \sum a_i = k\}$  ordered lexicographically. It is well known that  $|\mathbf{S}(n, k)|$  is simply the binomial coefficient  $\binom{k-1}{n-1}$  choose  $(n-1)$ , i.e.,

$$|\mathbf{S}(n, k)| = \binom{k-1}{n-1}. \quad (1)$$

Now given an  $\alpha \in \mathbf{S}(n, k)$ , can we calculate its position in the ordering? Certainly  $(1, 1, \dots, k+1-n)$  is first and  $(k+1-n, 1, \dots, 1)$  is last. Let the order function on  $\mathbf{S}(n, k)$  be  $f_{n,k}: \mathbf{S}(n, k) \rightarrow \{1, 2, \dots, |\mathbf{S}(n, k)|\}$ .

### LEMMA 1.1

Let  $\mathbf{S}(n, k)$  be the set of n-tuples of positive integers which sum to  $k$ , ordered lexicographically. Let  $\alpha = (a_1, \dots, a_n)$  and define  $\sigma_j(\alpha) = \sum_{i=j}^n a_i$ . Then the position of  $\alpha$  is given by

$$f_{n,k}(\alpha) = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1} + \binom{a}{n-j} - 1}{n-j}. \quad (2)$$

### PROOF

Consider a fixed  $\alpha = (a_1, \dots, a_n)$  in  $\mathbf{S}(n, k)$ . Let  $\beta = (b_1, \dots, b_n)$ ,  $T_j = \{\beta \in \mathbf{S}(n, k) \mid b_1 = a_1, \dots, b_{j-1} = a_{j-1}, b_j > a_j\}$  and let  $T = \bigcup_{j=1}^n T_j$ ,  $j = 1, \dots, n$ . Then certainly  $T = \{\beta \in \mathbf{S}(n, k) \mid \beta > \alpha\}$  and  $T_i \cap T_j = \emptyset$  if  $i \neq j$ , so we have  $|T| = \sum_{1 \leq j < n} |T_j|$  (since  $T_n = \emptyset$ ). But  $\{(T_j = (a_1, \dots, a_{j-1}, a_j + g_1, g_2, \dots, g_{n-j+1}) \mid g = (g_1, \dots, g_{n-j+1}) \in \mathbf{S}(n-j+1, k - a_1 - \dots - a_j)\}$ . Hence,

$$|T_j| = \binom{k - a_1 - \dots - a_j - 1}{n-j} = \binom{\sigma_{j+1} + \binom{a}{n-j} - 1}{n-j}. \quad (3)$$

Therefore, we can conclude that  $f_{n,k}(\alpha) = |\mathbf{S}(n, k)| - |\{b \in \mathbf{S}(n, k) \text{ s.t. } \beta > \alpha\}|$

$$= \binom{k-1}{n-1} - |T| = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} |T_j| = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1} + \binom{a}{n-j} - 1}{n-j}.$$

*qed*

## N-TUPLES OF POSITIVE INTEGERS

Now let  $\mathbf{Z}^n = \mathbf{Z}^+ \times \dots \times \mathbf{Z}^+$  be the set of all n-tuples of positive integers. Note that  $\mathbf{Z}^n = \cup_{n \leq k}$ . Let  $\mathbf{Z}^n$  be ordered by  $\mathbf{S}(n, n) < \mathbf{S}(n, n+1) < \mathbf{S}(n, n+2) < \dots$  where the  $\mathbf{S}(n, k)$  are ordered lexicographically as before. Let  $f_n : \mathbf{Z}^n \rightarrow \mathbf{Z}$  be the order function for this space.

### THEOREM 1.1

Let  $\mathbf{Z}^n$  be the set of all n-tuples of positive integers ordered as above. Let  $\alpha = (a_1, \dots, a_n) \in \mathbf{Z}^n$  and define  $\sigma_j(\alpha) = \sum_{i=j}^n a_i$ . The position of  $\alpha$  is given by

$$f_n(\alpha) = \binom{\sigma_j(\alpha)}{n} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j}. \quad (4)$$

### PROOF

By ordering on  $\mathbf{Z}^n$  and the lemma we have

$$\begin{aligned} f_n(\alpha) &= \sum_{j=1}^{\sigma_1(\alpha)-1} |\mathbf{S}(n, k)| + f_{n, \sigma_1(\alpha)}(\alpha) = \sum_{j=1}^{\sigma_1(\alpha)-1} \binom{j-1}{n-1} + f_{n, \sigma_1(\alpha)}(\alpha) \\ &= \binom{\sigma_1(\alpha)-1}{n} + f_{n, \sigma_1(\alpha)}(\alpha) = \binom{\sigma_1(\alpha)-1}{n} + \binom{\sigma_1(\alpha)-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha)-1}{n-j} \\ &= \binom{\sigma_1(\alpha)}{n} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha)-1}{n-j} \left[ \begin{array}{l} \text{using the basic combinatorial identity} \\ \binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1} \end{array} \right]. \end{aligned}$$

*qed*

## COROLLARIES

Next I give two corollaries of lemma 1.1.

### COROLLARY 3.1

Let  $S(k) = \bigcup_{1 \leq n \leq k} S(n, k)$  ordered by  $S(1, k) < \dots < S(k, k)$  where the  $S(n, k)$  are ordered lexicographically as before. Then the lexicographic order function,  $f^{(k)}$ , of  $\alpha = (a_1, \dots, a_n) \in S(k)$  is given by

$$f^{(k)}(\alpha) = \sum_{\lambda=1}^n \binom{k-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j}. \quad (5)$$

### PROOF

Using lemma 1.1, we have

$$\begin{aligned} f^{(k)}(\alpha) &= \sum_{\lambda=1}^{n-1} |S(n, k)| + f_{n,k}(\alpha) = \sum_{\lambda=1}^{n-1} \binom{k-1}{\lambda-1} + \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j} \\ &= \sum_{\lambda=1}^n \binom{k-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j} \end{aligned}$$

*qed*

### COROLLARY 2

Let  $Z^* = \bigcup_{1 \leq n < \infty} Z^n = \{\alpha = (a_1, \dots, a_n), a_i \in \mathbf{Z}^+\}$  ordered by  $S(1) < S(2) < \dots$  with the  $S(k)$  defined and ordered as above. Then the order of function,  $f^*$ , of  $\alpha = (a_1, \dots, a_n) \in Z^*$  is given by

$$f^*(\alpha) = 2^{\sigma_1(\alpha)-1} + \sum_{\lambda=1}^n \binom{\sigma_\lambda(\alpha) - 1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(\alpha) - 1}{n-j}. \quad (6)$$

## PROOF

Since  $|S(m)| = \sum_{j=1}^m |S(j, m)| = \sum_{j=1}^m \binom{m-1}{j-1} = 2^{(m-1)}$ , we have, by the ordering on  $Z^*$ ,

$$\begin{aligned} f^*(a) &= \sum_{\lambda=1}^{\sigma_1(a)-1} |S(m)| + f^{\sigma_1(a)-1}(a) = \sum_{m=1}^{\sigma_1(a)-1} 2^{m-1} + \sum_{\lambda=1}^n \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} \\ &= 2^{\sigma_1(a)-1} + \sum_{\lambda=1}^n \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} . \end{aligned}$$

qed

## INVERSE FUNCTION

Let us consider the inverse function,  $f_n^{-1}: Z \rightarrow Z^n$ . There does not appear to be a closed form solution, but it is readily computable. Using this, one can easily implement an algorithm to calculate  $\phi_{k,n}: Z^k \rightarrow Z^n$  by  $\phi_{k,n} = f_n^{-1} f_k$ . Now suppose  $p$  in  $Z^+$ . To compute  $f_n^{-1}(p) = (a_1, \dots, a_n)$  first find the

smallest  $k_1$  s.t.  $\binom{k_1}{n} \geq p$ . Then we find successively largest  $k_i, i = 2, \dots, n$ , s.t.  $\binom{k_i}{n} - \sum_{j=1}^{i-1} \binom{k_{j+1}}{n-j} \geq p$ .

For the  $n^{\text{th}}$  case, this expression will be an equality. Then  $k_i - \sigma_i(f_n^{-1}(p))$  for  $i = 1, \dots, n$  and given all the  $\sigma$ s, one easily finds  $(a_1, \dots, a_n)$ .

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