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13. ABSTRACT (Maximum 200 words)

The research completed on this project has been reported in a number of papers which have been (or will be) published in technical journals and proceedings of conferences. A complete list of publications is given in Appendix A. Lectures and presentations of material completed on this project are listed in Appendix B.

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DYNAMIC EFFECTS ON FRACTURE

FINAL REPORT

Approved for public release; distribution unlimited.

by

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1. Research Objectives

The research work on this project has been concerned with dynamic effects on fracture. Two main areas have been investigated: "high-rate loads on bodies containing cracks," and "fast fracture and crack arrest."

Dynamic effects become important if the external loads give rise to propagating mechanical disturbances (as for impact loads and explosive charges) which can strike a crack and cause crack propagation. Spalling is an example of a fracture phenomenon caused by the rapid application of loads. Dynamic effects become also important if a crack propagates very rapidly, so that rapid motions are generated in the solid. Even though in most cases fracture studies should focus on the prevention of crack propagation, cracks sometimes become unstable despite the best attempts for prevention. Both for brittle and ductile materials, it is then important to know whether high speeds of crack propagation can be expected, what the dynamic interaction with the geometry of the body will be, and most importantly how a rapidly propagating crack can be arrested.

A summary of the research objectivs of the completed project is

DYNAMIC EFFECTS ON FRACTURE

I. HIGH-RATE LOADS ON BODIES CONTAINING CRACKS.

I-A: Computation of elastodynamic stress intensity factors.

I-B: Influence of plastic yielding on the dynamic stress field near a crack tip.

I-C: Conditions for crack propagation under dynamic loading conditions.

II. FAST FRACTURE AND CRACK ARREST.

II-A: Computation of elastodynamic stress intensity factors for rapidly propagating cracks.

II-B: Analysis of plastic deformations near a rapidly propagating crack tip.

II-C: Application of fracture criteria to compute crack-tip speeds.

II-D: Arrest of rapidly propagating cracks.

II-E: Crack forking and kinking of rapidly propagating cracks.

The work has evolved through four stages which were stated in the original proposal. These stages are

Stage 1: Development of analytical and numerical methods to analyze dynamic effects on crack-tip fields.

Stage 2: Comparison of crack-tip fields for linear elasticity and various constitutive models of elasto-plastic and visco-plastic material behavior.

Stage 3: Comparison of various fracture criteria.

Stage 4: Application of fracture criteria and the formulation of conditions for initiation of crack propagation, continued crack propagation and crack arrest, as well as for crack kinking and crack forking under dynamic conditions.

From the point of view of the Air Force, the technological significance and relevance of the completed research derives from the fact that it is at the present time either technically or economically not feasible to manufacture so that cracks never occur in factory-new structures, or after a period of service. It is, therefore, necessary to be able to estimate the stability of flaws, the propagating characteristics for fast fracture and the arrest mechanisms for rapidly propagating cracks, under a variety of loading conditions which should include conditions generated by impact loads and explosive charges.

2. Summary of Completed Research

The research completed on this Project has been reported in a number of papers, which have been (or will be) published in Technical

Journals and Proceedings of Conferences. A complete list of publications is given in Appendix A. Lectures and presentations of material completed on this Project are listed in Appendix B.

The research results are briefly summarized as follows.

2.1 Effects of Plasticity on Near-Tip Fields

Under rapid loading conditions the mass density of a material affects the fields of stress and deformation near a crack tip. For such dynamic problems plastic deformation in the immediate vicinity of a crack tip was investigated in [1]* for a stationary crack. Deformation theory was employed for the first phase of the loading when the fields are increasing monotonically with time. The general character of the near-tip fields was analyzed both with respect to its variation with time and with polar angle. The non-linear near-tip fields were related to the linearly elastic far-field by means of a path-independent integral.

In [2], deformation theory was applied to a thin plate containing a through-crack. The in-plane normal stresses were assumed to be tensile, but with a non-uniform variation through the thickness of the plate, as a result of bending and a dominant extension. The nonlinear near-tip fields (which are singular) have been analyzed asymptotically on the basis of plate theory. It was found that the angular variations of the near-tip fields are just the same as for generalized plane stress. Assuming small-scale yielding a path-independent integral, which is valid in a region close to the crack edge, was used to connect the nonlinear near-tip fields with the corresponding singular parts of the linear fields. It was shown that the nonlinear behavior near the crack edge significantly affects the through-the-thickness variations of the near-tip fields. The singular parts of the in-plane stresses tend to become more uniform through the thickness of the plate with stronger strain hardening.

*Numbers in brackets refer to the list of publications in Appendix A.

For dynamic fracture problems plastic deformation in the immediate vicinity of a crack-tip was investigated in [3]. Both stationary and propagating crack tips were considered. For a stationary crack tip, deformation theory was employed for the first phase of the loading when the fields are increasing monotonically with time. The general character of the near-tip fields was analyzed both with respect to its variation with time and with polar angle. The non-linear near-tip fields were related to the linearly elastic far-field by means of a path-independent integral. In the second part of the paper we considered rapidly propagating cracks. The near-tip fields for various models of material behavior were discussed. In particular, some earlier work by Achenbach and Kanninen for a rapidly propagating Mode-III crack, in a material which displays strain hardening was reviewed. In the last part of the paper the fields near a rapidly propagating crack-tip in an elastic perfectly-plastic material was considered for the case that inertia terms are of importance. The system of governing equations in the plastic region was presented and shown to be hyperbolic in nature. As a first approximation the steady-state case with respect to the moving crack-tip was considered and an asymptotic analysis of the near-tip field was carried out.

A dynamic analysis of fast fracture and crack arrest based on the Dugdale model was presented in [7]. Numerical approaches by the finite difference method have been considered in [13] and [18].

2.2 Computation of Elastodynamic Stress-Intensity Factors

The effect of proximity of a boundary on elastodynamic stress intensity factors has been investigated in [4] for a surface-breaking crack, and in [14] for a subsurface crack parallel to the surface of a half-space. In recent work we have also considered a subsurface crack which is oriented under an arbitrary angle with the surface of a half-space, see [16].

The configurations that have been considered, are two dimensional with deformations in plane strain. Systems of coupled singular integral equations for the Mode-I and Mode-II dislocation densities have been derived. These equations have been solved numerically for the cases of time-harmonic uniform tension and uniform shear applied at the surface of the half-space. The ratios of elastodynamic to eleastostatic stress intensity factors have been computed. The results display the dependence on the frequency and on the ratio d/a, where d is the distance from the upper crack tip to the free surface and a is the crack length. For small angles of inclination with the free surface, and for small values of d/a, time-harmonic excitations of the body may induce quite strong resonance vibrations of the layer between the crack and the free surface. Such resonance vibrations give rise to substantial increases in both the Mode-I and Mode-II stress intensity factors. These resonance effects were investigated in some detail for the parallel crack in [14].

A three-dimensional stress analysis problem for a surface-breaking crack was considered in [17]. A half-space containing a surface-breaking crack of uniform depth, oriented normal to the surface, was subjected to three-dimensional dynamic loading. The elastodynamic stress analysis problem was decomposed into two problems which are symmetric and antisymmetric, respectively, relative to the plane of the crack. The formulation of these problems was subsequently reduced to singular integral equations for the corresponding dislocation densities, namely, a single integral equation for the symmetric problem and a set of coupled integral equations for the antisymmetric problem. The systems of integral equations were solved numerically. The dislocation densities directly yield the stress-intensity factors. As an example an applied stress field with harmonic variation along the crack length and exponential decay with crack depth was considered. The dependence of the stress intensity factors on the frequency and on the wavelength in the crack direction has been displayed by numerical results.

2.3 Fast Fracture and Crack Arrest in an Elastic-Viscoplastic Material.

In this work, we have investigated both the effects of plastic deformation near a propagating crack tip and dynamic effects due to high crack-tip speeds. The constitutive equations that have been employed define an elastic viscoplastic material. The constitutive model, which was proposed by Bodner and Partom does not require the statement of a separate yield criterion, nor is it necessary to consider loading and unloading separately. Plastic deformations always exist, but they are negligibly small when the material behavior should be essentially elastic.

The geometry that has been considered is a two-dimensional one of a thick strip which contains a rapidly propagating semi-infinite crack in its center-plane. The faces of the strip were subjected to uniform in-plane displacements, so that the crack propagates in Mode-I. A steady-state situation relative to the moving crack tip has been assumed. The plastic deformations near the crack tip, the residual plastic strains in the wake of the crack tip and other field variables have been obtained directly from the complete solution without the assumption of small scale yielding.

Detailed results have been given in [5], [6] and [11]. The method of solution is numerical and involves an iterative procedure which is continued until a steady state solution is reached, in which the equation of motion, the flow rules and the boundary conditions are satisfied simultaneously. The method is based on a finite difference procedure which is unconditionally stable and has a second order accuracy.

In the special case of a perfectly elastic strip the problem possesses an analytical solution which has been employed to check the accuracy of the numerical method, and excellent agreement has been obtained.

Typical effects due to viscoplastic constitutive behavior have been studied by comparisons with the corresponding elastic fields. The effects of high crack-tip speeds, which are directly related to the strain-rate

dependence of the material, have been studied by comparisons of solutions for three crack-tip velocities. The influence of the inertia term in the governing equations has been studied by comparisons with the corresponding quasi-static solutions. In particular, the dependence on the crack-tip speed of the plastic zone in the vicinity of the crack tip, the level of plastic straining, the amount of dissipative plastic work and the crack-opening displacements have been examined.

In [9] and [12] the transient problem of deceleration and arrest of a rapidly propagating crack tip has been investigated for a crack which initially propagates in an elastic solid but then enters a region of viscoplastic material properties. Of particular interest in the dynamic arrest problem is the plastic strain just ahead of the arresting crack tip. This strain has been computed as a function of time. Within the context of a critical strain criterion it has been assumed that the crack tip will arrest if this strain does not exceed a certain critical value.

2.4 Crack-Kinking under Stress-Wave Loading

When a stress pulse strikes a crack, the crack may be induced to propagate, but not necessarily in it's own plane. In earlier work, it has been attempted to explain kinking of a crack at finite kinking angles. In work completed on this project, [10] and [15], we have reconsidered the two-dimensional configuration of an initially stationary crack which kinks under an angle $\kappa\pi$ with it's original plane. We have discussed earlier results for Mode III, and we have proposed a way to approximate the elastodynamic stress intensity factors for the Mixed Mode I-II case.

The tip of the kinked crack is assumed to propagate at a constant velocity c_F, and kinking is initiated at an angle $\kappa\pi$ at the instant that an incident disturbance first strikes the original crack tip. These two assumptions render the solution self-similar. Three cases have been considered corresponding to incidence of either an anti-plane transverse wave, an in-plane transverse wave or a longitudinal wave.

The elastodynamic stress intensity factors have been computed as functions of the crack tip speed, c_F, the kinking angle, $\kappa\pi$, and the angle of wave incidence $\alpha\pi$. For a given angle of incidence, the elastodynamic stress intensity factors have been used to compute the corresponding energy fluxes into the propagating crack tip.

Mode-III problems of the kind formulated in [10] can be solved rigorously. Results for Mode-III kinking under an arbitrary angle were given earlier. The corresponding mixed Mode I-II problems have, however, as yet eluded a rigorous analytical solution.

For the mixed Mode I-II problems we have proposed a simple approximation to the solution of the superposition problem. The approximation is based on an observation from the exact Mode-III solution of the analogous superposition problem, that for an important range of kinking angles the elastodynamic stress intensity factor of the kinked crack is affected more by the dependence on κ of the loading on the new crack faces than by the wedge geometry at the original crack tip. This observation then suggests that in first approximation we may ignore the wedge geometry altogether, and we may compute elastodynamic stress intensity factors by considering a crack propagating in its own plane but where the new crack faces are subjected to tractions corresponding to the kinking crack.

For the Mode-III case the approximation of the elastodynamic stress intensity factor for the kinked crack can be checked by comparisons with exact results. The range of kinking angles $\kappa\pi$ for which the approximation gives good results turns out to be surprisingly large. For the mixed Mode I-II case comparisons with the numerical results have been carried out, and satisfactory agreement has been obtained. No rigorous mathematical proof of the approximation's validity is given in this paper. It has, however, been

shown recently that the results correspond to the first order terms in a perturbation procedure for small kinking angle [19].

The results of this work suggested that the approximation could be used to analyze crack kinking at gradually increasing angles and at time-varying crack tip speeds. The details of that analysis have been completed [20]. Publication [20] also includes a study of the actual conditions for crack kinking by the use of the criterion of the balance of rates of energies.

2.5 Asymptotic Form of Dynamic Crack-Tip Fields According to the Bodner-Partom Constitutive Equations

Bodner and Partom[†] have proposed a set of constitutive equations for an elastic-viscoplastic work-hardening material. These equations have the convenient property that no separate specification of a yield criterion is required, nor is it necessary to consider loading and unloading separately. Within the context of the Bodner-Partom model both elastic and inelastic deformations are present at all stages of loading and unloading, but the plastic deformations are very small when the material behavior should be essentially elastic.

In the final stages of this project we obtained the order of the singularities in stress and deformation near a rapidly propagating crack tip in a Bodner-Partom material. The crack propagates under Mode-I conditions, and the effects of material inertia are taken into account. The near-tip fields are analyzed by the asymptotic method presented by Achenbach, Kanninen and Popelar°.

The results, which are presented in Appendix C, show that the near-tip plastic strain rates are bounded, while, just as for a linearly elastic material, the total strains and the stresses display square-root singularities.

[†] S.R. Bodner and Y. Partom, "Constitutive Equations for Elastic-Viscoplastic Strain-Hardening Materials", J. Appl. Mech., 42, p. 385, 1975.

° J. D. Achenbach, M. F. Kanninen and C. H. Popelar, "Crack-Tip Fields for Fast Fracture of an Elastic-Plastic Material", J. Mech. Phys. Solids, 29, p. 211, 1981.

3. Interactions

Lectures and presentations on research carried out on this Project have been listed in Appendix B.

4. Professional Personnel

J. D. Achenbach - Principal Investigator

Contributing to the research efforts for various periods of time:

J. Aboudi	-	Visiting Scholar
J. Dempsey	-	Post-Doctoral Research Associate
P. Burgers	-	Post-Doctoral Research Associate
Y. C. Angel	-	Post-Doctoral Research Associate
N. Nishimura	-	Post-Doctoral Research Associate
L. M. Keer	-	Cooperating Faculty
T. Bubenik	-	Research Assistant
D. Mendelsohn	-	Research Assistant
V. Dunayevsky	-	Research Assistant
H. K. Chung	-	Research Assistant
M. K. Kuo	-	Research Assistant
W. Lin	-	Research Assistant

5. Graduate Degrees

T. Bubenik: Ph.D. Theoretical and Applied Mechanics, 1979
now at Exxon Production Res. Co., Houston, TX

D. Mendelsohn: Ph.D. Theoretical and Applied Mechanics, 1980
now at Ohio State University

V. Dunayevsky: Ph.D. Applied Mathematics, 1981
now at Technology Center, Sohio Petroleum Company, Dallas, TX

H. K. Chung Ph.D. Civil Engineering, 1982
now employed by Consulting Firm in New Jersey.

M. K. Kuo Ph.D. Civil Engineering, 1984
now at National Taiwan University

Appendix A: Publications

1. J.D. Achenbach, "Dynamic Crack-Tip Fields According to Deformation Theory," J. of Applied Mechanics, 46, pp. 707-708, 1979.

2. J.D. Achenbach and T. Bubenik, "Near-Tip Fields in a Cracked Plate According to Deformation Theory", Proceedings International Conference on Fracture Mechanics in Engineering Application (G.C. Sih, editor), Sijthoff & Noordhoff Inc., pp. 67-80, 1979.

3. J.D. Achenbach, P. Burgers and V. Dunayevsky, "Near-Tip Plastic Deformations in Dynamic Fracture Problems", Proceedings Symposium on Nonlinear and Dynamic Fracture (N. Perrone and S.N. Atluri, eds.), AMD Vol. 35, American Society of Mechanical Engineers, pp. 105-124, 1979.

4. J.D. Achenbach, L.M. Keer and D. Mendelsohn, "Elastodynamic Analysis of an Edge Crack," J. of Applied Mechanics, 47, pp. 556-581, 1980.

5. J. Aboudi and J.D. Achenbach, "Fast Fracture of a Strip of Viscoplastic Work-Hardening Material", Proc. Int. Conf. on Analytical and Experimental Fracture Mechanics, Rome, June 23-27, 1980, Noordhoff, 1981.

6. J. Aboudi and J.D. Achenbach, "Rapid Mode-III Crack Propagation in a Strip of Viscoplastic Work-Hardening Material", Int. J. of Solids and Structures, 17, pp. 879-890, 1981.

7. J.D. Achenbach and A. Neimitz, "Fast Fracture and Crack Arrest According to the Dugdale Model", Engineering Fracture Mechanics, 14, pp. 385-395, 1981.

8. J.D.Achenbach and R.J. Brind, "Elastodynamic Stress-Intensity Factors for a Crack Near a Free Surface", J. of Applied Mechanics, 48, pp. 539-542, 1981.

9. J.D. Achenbach and J. Aboudi, "Transition from Brittle to Ductile Fracture for a Rapidly Propagating Crack", ASME Preprint 81-PVP-6.

10. J. P. Dempsey, M.-K. Kuo and J. D. Achenbach, "Mode-III Crack Kinking under Stress-Wave Loading," WAVE MOTION, 4, pp. 181-190, 1982.
11. J. Aboudi and J. D. Achenbach, "Numerical Analysis of Fast Mode-I Fracture of a Strip of Viscoplastic Work-Hardening Material", Int. J. Fracture, 21, pp. 133-148, 1983.
12. J. Aboudi and J. D. Achenbach, "Arrest of Fast Mode-I Fracture in an Elastic-Viscoplastic Transition Zone," Engineering Fracture Mechanics, 18, pp. 109-119, 1983.
13. H. K. Chung and J. D. Achenbach, "Crack Propagation and Arrest under Dynamic Loading Conditions in an Elastic-Plastic Strip," Proc. 29th Sagamore Army Materials Research Conference, edited by John Mescall, July 19-23, 1982, Lake Placid, NY, pp. 251-264.
14. L. M. Keer, W. Lin and J. D. Achenbach, "Resonance Effects for a Crack near a Free Surface," J. Applied Mechanics, 51, pp. 65-70, 1984.
15. J. D. Achenbach, M. K. Kuo and J.P. Dempsey, "Mode III and Mixed Mode I-II Crack Kinking under Stress-Wave Loading," Int. J. Solids and Structures, 20, pp. 395-410, 1984.
16. W. Lin, L. M. Keer and J. D. Achenbach, "Dynamic Stress Intensity Factors for an Inclined Subsurface Crack," J. Applied Mechanics, 106, pp. 773-779, 1984.
17. Y. C. Angel and J. D. Achenbach, "Stress-Intensity Factors for 3-D Dynamic Loading of a Cracked Half-Space," J. of Elasticity, 15, pp. 89-102, 1985.
18. Z. G. Xu, H. K. Chung and J. D. Achenbach, "Numerical Analysis of Fast Fracture and Crack Arrest in a Sheet of Elastic Perfectly-Plastic Material," Theoretical and Applied Fracture Mechanics, 2, pp. 235-245, 1984.
19. M. K. Kuo and J. D. Achenbach, "Perturbation Method to Analyze the Elastodynamic Field near a Kinked Crack," Int. J. Solids and Structures, 21, pp. 272-278, 1985.
20. J. D. Achenbach and M. K. Kuo, "Conditions for Crack-Kinking under Stress-Wave Loading," Engineering Fracture Mechanics, 22, pp. 165-180, 1985.
21. J. D. Achenbach, Z. L. Li and N. Nishimura, "Dynamic Fields Generated by Rapid Crack Growth," Int. J. Fracture, 27, pp. 215-227, 1985.

Appendix B: Lectures and Presentations

"Near-Tip Fields in a Cracked Plate According to Deformation Theory," International Conference on Fracture Mechanics in Engineering Application, Bangalore, India, March 26-30, 1979.

"Some Observations on Plastic Deformations near a Rapidly Propagating Crack Tip," 16th Annual Meeting Society of Engineering Science, Sept. 5-7, 1979.

"Near-Tip Plastic Deformations in Dynamic Fracture Problems," Symposium on Nonlinear and Dynamic Fracture, Winter Annual Meeting, ASME, Dec. 2-7, 1979.

"Small-Scale Yielding at the Tip of a Through Crack in a Shell," 3rd National Congress on Pressure Vessel and Piping Technology, June 25-29, 1979.

"Fast Fracture of a Strip of Viscoplastic Work-Hardening Material," International Conference on Analytical and Experimental Fracture Mechanics, June 23-27, 1980, Rome, Italy.

"Crack-Tip Plasticity for Rapid Crack Propagation," 5th Int. Conference on Fracture Mechanics, 1981, Cannes, France.

"Transition from Brittle to Ductile Fracture for a Rapidly Propagating Crack," 1981 ASME PVP Conference, June 21-25, 1981, Denver, CO.

"Plasticity Effects in Dynamic Crack Propagation," 18th Annual Meeting Society of Engineering Science, Brown, Sept. 2-4, 1981.

"Continuum Scale Elastic-Plastic Dynamic Crack Propagation," Meeting on Deformation, Fracture, Wear and Nondestructive Evaluation of Materials," American Physical Society, New Orleans, November 23-24, 1981.

"Plastic Deformations in Fast Fracture," Lawrence Livermore National Laboratory, March 18, 1982.

"Plastic Deformation near a Rapidly Propagating Crack Tip," 29th Sagamore Army Materials Research Conference, July 19-23, 1982.

"Plastic Deformation near a Rapidly Propagating Crack Tip," Workshop on Dynamic Fracture, California Institute of Technology, Feb. 17-18, 1983.

"Plastic Deformations in Fast Fracture," Colloquium Lecture, University of Minnesota, Dept. of Aerospace Engineering, April 22, 1983.

"Crack Growth under Plane Stress Conditions in an Elastic Perfectly-Plastic Material," Sixteenth National Symposium on Fracture Mechanics, August 14-15, 1983, Battelle Columbus Labs, Columbus, Ohio.

"Plastic Deformations near a Rapidly Propagating Crack Tip," Eshelby Memorial Symposium, Sheffield April 2-5, 1984, International Union of Theoretical and Applied Mechanics.

Appendix C: Near-Tip Fields According to the Bodner-Partom Model

C.1 Governing Equations

In the usual manner the total rate of strain is expressed as the superposition of elastic (reversible) and plastic (irreversible) components:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{e}_{ij} + \dot{\varepsilon}^{p}_{ij} , \qquad i,j = 1,2,3 \tag{1}$$

The stress rates are related to the elastic strain rates by Hooke's law:

$$\dot{\sigma}_{ij} = \lambda(\dot{\varepsilon}_{kk} - \dot{\varepsilon}^{p}_{kk})\delta_{ij} + 2\mu(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^{p}_{ij}) \tag{2}$$

For plane strain and plane stress, substitution of $\dot{\sigma}_{ij}$ into

$$\sigma_{ij,j} = \rho\ddot{u}_{i} , \tag{3}$$

where ρ is the mass density, yields

$$E\dot{u} - P\dot{\varepsilon}^{p} = 0 . \tag{4}$$

In (4) the elastic operator E is defined as

$$E = \begin{bmatrix} \mu\nabla^2 + (\hat{\lambda}+\mu)\dfrac{\partial^2}{\partial x_1^2} - \rho(\ddot{\ }) & (\hat{\lambda}+\mu)\dfrac{\partial^2}{\partial x_1 \partial x_2} \\ (\hat{\lambda}+\mu)\dfrac{\partial^2}{\partial x_1 \partial x_2} & \mu\nabla^2 + (\hat{\lambda}+\mu)\dfrac{\partial^2}{\partial x_2^2} - \rho(\ddot{\ }) \end{bmatrix} \tag{5}$$

while the plastic operator P is

$$P = \begin{bmatrix} (\hat{\lambda}\omega + 2\mu)\dfrac{\partial}{\partial x_1} & \hat{\lambda}\omega\dfrac{\partial}{\partial x_1} & 2\mu\dfrac{\partial}{\partial x_2} \\ \hat{\lambda}\omega\dfrac{\partial}{\partial x_2} & (\hat{\lambda}\omega + 2\mu)\dfrac{\partial}{\partial x_2} & 2\mu\dfrac{\partial}{\partial x_1} \end{bmatrix} \tag{6}$$

The vectors $\dot{\underset{\sim}{u}}$ and $\dot{\underset{\sim}{\varepsilon}}^p$ are

$$\dot{\underset{\sim}{u}} = \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}, \quad \text{and} \quad \dot{\underset{\sim}{\varepsilon}}^p = \begin{pmatrix} \dot{\varepsilon}^p_{11} \\ \dot{\varepsilon}^p_{22} \\ \dot{\varepsilon}^p_{12} \end{pmatrix} \tag{7a,b}$$

Also,

$$\text{plane strain :} \quad \hat{\lambda} = \frac{2\mu\nu}{1-2\nu}, \quad \omega = 0 \tag{8}$$

$$\text{plane stress :} \quad \hat{\lambda} = \frac{2\mu\nu}{1-\nu}, \quad \omega = 1 \tag{9}$$

where ν is Poisson's ratio.

It is assumed that the plastic deformations are incompressible $\dot{\varepsilon}^p_{kk} = 0$, k=1,2,3 and that the Prandtl-Reuss flow law holds. Thus

$$\dot{\varepsilon}^p_{ij} = \dot{e}^p_{ij} = \Lambda s_{ij} \tag{10}$$

where s_{ij} and $\dot{e}^p_{ij}$ denote the deviators of the stress tensor and the plastic strain-rate tensor, respectively. Equation (10) can be squared to yield Λ in the form

$$\Lambda^2 = D_2^p / J_2 \tag{11}$$

Here

$$D_2^p = \frac{1}{2} \dot{e}^p_{ij} \dot{e}^p_{ij}, \text{ and } J_2 = \frac{1}{2} s_{ij} s_{ij} \tag{12a,b}$$

are the second invariants of the plastic strain-rate deviator and the stress deviator, respectively.

Following Ref.[1] we take the following relation between D_2^p and J_2:

$$D_2^p = D_o^2 \exp\left[-(A^2/J_2)^n\right] , \tag{13}$$

where D_o, A and n are material constants.

Now let us consider a crack tip moving with crack tip speed v(t), and let the $x_1x_2x_3$ system be a moving coordinate system centered at the crack tip. The x_3-axis is parallel to the crack front, and x_1 points in the direction of crack growth. Relative to the moving coordinate system we also define polar coordinates r,θ with θ = 0 coinciding with the positive x_1-direction. The geometry is shown in Fig. 1. Relative to the moving coordinates we have

$$(\dot{}) = \frac{\partial}{\partial t} - v(t)\,\frac{\partial}{\partial x_1} \tag{14}$$

$$(\ddot{}) = \frac{\partial^2}{\partial t^2} - \dot{v}(t)\,\frac{\partial}{\partial x_1} - 2v(t)\,\frac{\partial^2}{\partial t \partial x_1} + [v(t)]^2\,\frac{\partial^2}{\partial x_1^2} \tag{15}$$

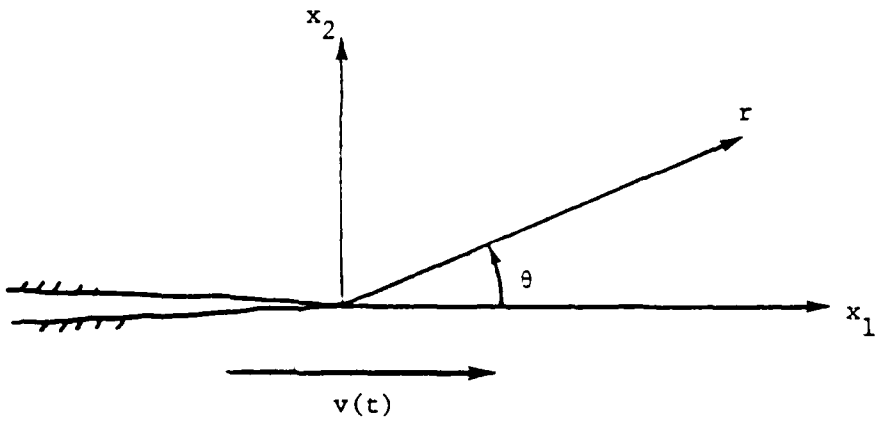

Fig. 1: Propagating Crack Tip

C.2 Dynamic Crack-Tip Fields

It follows from (10)-(13) that

$$\dot{\varepsilon}^{p}_{ij} = D_o \exp[(-A^2/J_2)^n/2] s_{ij}/(J_2)^{\frac{1}{2}} \tag{16}$$

Even if s_{ij} is singular, the singularity of s_{ij} as $r \to 0$ will be cancelled by the singularity of $(J_2)^{\frac{1}{2}}$ in the term $s_{ij}/(J_2)^{\frac{1}{2}}$. The term $\exp[(-A^2/J_2)^n/2]$ is bounded, whether J_2 is singular or not, since $n > 0$ (in Ref.[1], n is taken as $n = 1$). Hence, Eq.(16) implies that the plastic strain-rate is bounded as $r \to 0$.

To investigate the nature of the singularity in the elastic strains and stresses, we consider

$$\dot{\underset{\sim}{u}}(r,\vartheta,t) = r^{s_1} \dot{\underset{\sim}{U}}^{(1)}(\theta,t) + r^{s_2} \dot{\underset{\sim}{U}}^{(2)}(\vartheta,t) + \text{---} , \tag{17}$$

where $-1 < s_1 < s_2 < \text{---}$, and

$$\dot{\underset{\sim}{u}} = (\dot{u}_1, \dot{u}_2)^T \tag{18}$$

$$\dot{\underset{\sim}{U}}^{(m)} = (\dot{U}_1^{(m)}, \dot{U}_2^{(m)})^T, \quad m = 1,2 \tag{19}$$

The lower bound for s_1 follows from the boundedness of $\underset{\sim}{u}$ at the tip.

Also

$$\dot{\underset{\sim}{\varepsilon}}^{p}(r,\vartheta,t) = r^{q_1} \dot{\underset{\sim}{E}}^{(1)}(\theta,t) + r^{q_2} \dot{\underset{\sim}{E}}^{(2)}(\vartheta,t) + \text{---} , \tag{20}$$

where $q_1 < q_2 < \text{---}$, and

$$\dot{\underset{\sim}{\varepsilon}}^{p} = (\dot{\varepsilon}^{p}_{11}, \dot{\varepsilon}^{p}_{22}, \dot{\varepsilon}^{p}_{12})^T \tag{21}$$

$$\dot{\underset{\sim}{E}}^{(m)} = (\dot{E}_1^{(m)}, \dot{E}_2^{(m)}, \dot{E}_3^{(m)})^T, \quad m = 1,2 \tag{22}$$

As noted above, $\dot{\underset{\sim}{\varepsilon}}^p$ is bounded, and hence

$$q_1 = 0 \tag{23}$$

Substitution of (17) and (20) into Eq.(4) yields

$$r^{s_1-2} E_1 \dot{\underset{\sim}{U}}^{(1)} + r^{s_2-2} E_2 \dot{\underset{\sim}{U}}^{(2)} + \cdots - r^{-1} P_1 \dot{\underset{\sim}{E}}^{(1)} - r^{q_2-1} P_2 \dot{\underset{\sim}{E}}^{(2)} = 0 \tag{24}$$

In Eq.(24), the operators E_1 and E_2 are just the same as those for the linearly elastic problem, which can be obtained by substituting (17) in $\dot{E\underset{\sim}{U}}$, and collecting terms of orders r^{s_1-2} and r^{s_2-2}, respectively, while P_1 and P_2 follow by substituting (20) in $P\,\dot{\underset{\sim}{\varepsilon}}^p$, and collecting terms of orders r^{-1} and r^{q_2-1}. It is noted that we have used the result $q_1 = 0$ in identifying the term of order r^{-1}.

Now let us consider the stresses corresponding to the expansions given by Eqs.(17) and (20). It follows from Eq.(2) that

$$\dot{\sigma}_{ij} = \dot{\sigma}_{ij}^{(e)} - \dot{\sigma}_{ij}^{(p)} , \tag{25}$$

where for i,j = 1,2:

$$\dot{\sigma}_{ij}^{(e)} = \hat{\lambda}(\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22})\delta_{ij} + 2\mu\dot{\varepsilon}_{ij} \tag{26}$$

$$\dot{\sigma}_{ij}^{(p)} = \hat{\lambda}(\dot{\varepsilon}_{11}^p + \dot{\varepsilon}_{22}^p)\omega\delta_{ij} + 2\mu\dot{\varepsilon}_{ij}^p \tag{27}$$

In (26) and (27), the constants $\hat{\lambda}$ and ω are defined by Eqs.(8) and (9) for plane strain and plane stress, respectively. Substitution of (17) and (20) into (26) and (27) yields

$$\dot{\sigma}_{22}^{(e)} = r^{s_1-1}\{\hat{\lambda}[F_1(\theta,t) + F_2(\theta,t)] + 2\mu F_2(\theta,t)\} + r^{s_2-1}\{\hat{\lambda}[G_1(\theta,t) + G_2(\theta,t)] + 2\mu G_2(\theta,t)\} + \cdots \tag{28}$$

$$\dot{\sigma}_{22}^{(p)} = 2\mu[\dot{E}_2^{(1)}(\theta,t) + --] + \hat{\lambda}[\dot{E}_1^{(1)}(\theta,t) + \dot{E}_2^{(1)}(\theta,t) + ---]\omega \tag{29}$$

$$\dot{\sigma}_{21}^{(e)} = \mu r^{s_1-1} T_1(\theta,t) + \mu r^{s_2-1} T_2(\theta,t) + --- \tag{30}$$

$$\dot{\sigma}_{21}^{(p)} = 2\mu[\dot{E}_3^{(1)}(\theta,t) + ---] \tag{31}$$

where

$$F_1(\theta,t) = s_1\cos\theta\ \dot{U}_1^{(1)}(\theta,t) - \sin\theta\ \frac{\partial}{\partial\theta}\dot{U}_1^{(1)}(\theta,t) \tag{32a}$$

$$F_2(\theta,t) = s_1\sin\theta\ \dot{U}_2^{(1)}(\theta,t) + \cos\theta\ \frac{\partial}{\partial\theta}\dot{U}_2^{(1)}(\theta,t) \tag{32b}$$

$$G_1(\theta,t) = s_2\cos\theta\ \dot{U}_1^{(2)}(\theta,t) - \sin\theta\ \frac{\partial}{\partial\theta}\dot{U}_1^{(2)}(\theta,t) \tag{32c}$$

$$G_2(\theta,t) = s_2\sin\theta\ \dot{U}_2^{(2)}(\theta,t) + \cos\theta\ \frac{\partial}{\partial\theta}\dot{U}_2^{(2)}(\theta,t) \tag{32d}$$

Also

$$T_1(\theta,t) = s_1[\sin\theta\ \dot{U}_1^{(1)}(\theta,t) + \cos\theta\ \dot{U}_2^{(1)}(\theta,t)] + \cos\theta\ \frac{\partial}{\partial\theta}\dot{U}_1^{(1)}(\theta,t) - \sin\theta\ \frac{\partial}{\partial\theta}\dot{U}_2^{(1)}(\theta,t) \tag{33a}$$

$$T_2(\theta,t) = s_2[\sin\theta\ \dot{U}_1^{(2)}(\theta,t) + \cos\theta\ \dot{U}_2^{(2)}(\theta,t)] + \cos\theta\ \frac{\partial}{\partial\theta}\dot{U}_1^{(2)}(\theta,t) - \sin\theta\ \frac{\partial}{\partial\theta}\dot{U}_2^{(2)}(\theta,t) \tag{33b}$$

For the propagating crack the conditions are

$$\theta = 0\ ,\ r > 0: \quad \dot{u}_2 = 0\ ,\ \dot{\sigma}_{21} = 0 \tag{34}$$

$$\theta = \pi\ ,\ r > 0: \quad \dot{\sigma}_{22} = 0\ ,\ \dot{\sigma}_{21} = 0 \tag{35}$$

Following the general method of Achenbach, Kanninen and Popelar [2], we now construct eigenvalue problems for the determination of s_1, $\dot{\underset{\sim}{U}}^{(1)}$, s_2, $\dot{\underset{\sim}{U}}^{(2)}$, $\dot{\underset{\sim}{E}}^{(1)}$ and q_2, $\dot{\underset{\sim}{E}}^{(2)}$, by collecting the highest order singularities in (24) and (34), (35). The four terms in Eq.(24) are of orders r^{s_1-2}, r^{s_2-2}, r^{-1} and r^{q_2-1}, respectively. Since

$s_1 < s_2$, the first term is more singular than the second, and since $1 < q_2$, the third is more singular than the fourth. At the outset it is, however, not evident whether the term of order r^{s_1-2} is more singular than the term of order r^{-1}.

Suppose $s_1-2 \geq -1$, then $s_1 \geq 1$. In this case we would, however, have no singularity at all for the strain. Hence we must have $s_1 - 2 < -1$, or $s_1 < 1$. Then, Eq.(24) implies

$$r^{s_1-2} \underset{\sim}{E}_1 \dot{\underset{\sim}{U}}^{(1)} = 0 \quad , \tag{36}$$

or

$$\underset{\sim}{E}_1 \dot{\underset{\sim}{U}}^{(1)} = 0 \quad . \tag{37}$$

The corresponding terms in (34) - (35) yield

$$\dot{U}_2^{(1)}(0,t) = 0 \quad , \qquad T_1(0,t) = 0 \tag{38a,b}$$

$$\hat{\lambda}[F_1(\pi,t) + F_2(\pi,t)] + 2\mu F_2(\pi,t) = 0 \tag{39}$$

$$T_1(\pi,t) = 0 \tag{40}$$

Equations (37) - (40) define an "eigenvalue" problem for $\dot{\underset{\sim}{U}}^{(1)}$ and s_1. This eigenvalue problem is, however, exactly the same as for dynamic crack propagation in a linearly elastic material, which was considered in Refs.[3] and [4]. We can immediately conclude that

$$s_1 = -\frac{1}{2} \tag{41}$$

It is of interest to consider the higher order terms in Eq.(24), since it is conceivable that one of them may give rise to a singular strain, or at least to a singular strain rate. After the term r^{s_1-2}, either r^{s_2-2} or r^{-1} may be the most singular term in Eq.(24). We must consider the following three possibilities

$$s_2-2 < -1, \quad s_2-2 > -1, \text{ or } s_2-2 = -1 \tag{42}$$

Suppose $s_2-2 < -1$, then Eq.(24) implies

$$E_2\dot{\underset{\sim}{U}}^{(2)} = 0 \tag{43}$$

This equation must be supplemented by

$$\dot{U}_2^{(2)}(0,t) = 0 , \qquad T_2(0,t) = 0 \tag{44a,b}$$

$$\hat{\lambda}[G_1(\pi,t) + G_2(\tau,t)] + 2\mu G_2(\tau,t) = 0 \tag{45}$$

$$T_2(\pi,t) = 0 \tag{46}$$

Equations (43)-(46) are, however, completely equivalent to the equations for the next term in the linearly elastic problems. Hence

$$s_2 = \frac{1}{2} , \tag{47}$$

which satisfies the condition $s_2 - 2 < -1$.

Now if we assume $s_2 - 2 > -1$, then, it follows from Eq.(24) that

$$\dot{\underline{E}}^{(1)}(\theta,t) = 0 \tag{48}$$

Since the plastic strain rates actually depend on the stresses, as follows from Eq.(16), and since singular stresses do actually give rise to bounded plastic strain rates, Eq.(48) can be discarded. Hence $s_2 - 2 > -1$ is not acceptable.

Finally we consider the case $s_2 - 2 = -1$, or $s_2 = 1$. There is no reason to ignore this case. Both the strain and the strain rate corresponding to $s_2 = 1$ will, however, be bounded.

In conclusion, for the dynamic problem the lowest order displacement rate terms are

$$\dot{\underline{u}} = r^{-1/2}\dot{\underline{U}}^{(1)}(\theta,t) + r^{1/2}\dot{\underline{U}}^{(2)}(\theta,t) + \cdots \tag{49a}$$

or

$$\underline{u} = r^{1/2}\underline{\tilde{U}}^{(1)}(\theta,t) + r^{3/2}\underline{\tilde{U}}^{(2)}(\theta,t) + \cdots \tag{49b}$$

Thus, we have shown that the two most dominant terms of the near-tip particle velocity are of orders $r^{-1/2}$ and $r^{1/2}$. A more careful approach is needed to determine the higher order terms. For the stationary crack the procedure to determine the higher-order terms is discussed in the next Section. Following the method of Section C.3 for the propagating crack we find

$$\dot{\underline{u}} = r^{-1/2}\,\dot{\underline{U}}^{(1)}(\vartheta,t) + r^{1/2}\,\dot{\underline{U}}^{(2)}(\vartheta,t) + r\log r\,\dot{\underline{U}}^{(3)}(\vartheta,t) + r\dot{\underline{U}}^{(4)}(\vartheta,t) + \cdots \tag{49.c}$$

or

$$\underline{u} = r^{1/2}\underline{\tilde{U}}^{(1)}(\vartheta,t) + r^{3/2}\underline{\tilde{U}}^{(2)}(\vartheta,t) + r^{2}\log r\,\underline{\tilde{U}}^{(3)}(\vartheta,t) + r^{2}\underline{\tilde{U}}^{(4)}(\vartheta,t) + \cdots \tag{49d}$$

It should be noted that there is no essential difference between the dynamic and the quasi-static result for the propagating crack. The quasi-static formulation follows by removing the operator $\rho(\ddot{\ })$ from E, Eq.(5). This would, however not affect the result. There is, however, a significant difference with the field near a stationary crack tip, as shown in the next section.

C.3 Field Near a Stationary Crack Tip

The formulation of Section C.2 reduces to the one for a stationary crack tip by setting $v(t) \equiv 0$. The $x_1x_2x_3$-system then becomes a stationary system, and the time derivatives given by Eqs.(14) and (15) reduce to

$$(\dot{}) = \frac{\partial}{\partial t}, \quad (\ddot{}) = \frac{\partial^2}{\partial t^2} \tag{50a,b}$$

We first follow the same argument as in the previous section for the dynamic propagating crack case, i.e. we assume that the asymptotic expression in (17) holds for the stationary crack problem. Substitution of Eqs.(17) and (20) into (4) now yields

$$r^{s_1-2} E_1 \dot{\underline{U}}(1) + r^{s_2-2} E_2 \dot{\underline{U}}(2) - r^{-1} P_1 \dot{\underline{E}}(1) - r^{q_2-1} P_2 \dot{\underline{E}}(2) = 0 \tag{51}$$

The four terms in (51) are now of orders r^{s_1-2}, r^{s_2-2}, r^{-1} and r^{q_2-1}, respectively.

Suppose $s_1-2 \geq -1$, then $s_1 \geq 1$. In this case we would not have a singularity in the strain. Hence we must have $s_1-2 < -1$, or $s_1 < 1$. This implies however that the terms multiplying r^{s_1-2} must vanish identically, which leads to the formulation given by Eqs.(37)-(40), but where $(\ddot{})$ in Eq.(5) is defined by Eq.(50b). It follows that

$$s_1 = \frac{1}{2} \tag{52}$$

With regard to the value of s_2 we have the following alternatives:

$$s_2-2 < -1, \quad s_2-2 > -1, \quad \text{or } s_2-2 = -1 \tag{53}$$

If $s_2-2 < -1$, the terms containing s_2 must vanish, which leads to the formulation given by Eqs.(43)-(46), and thus $s_2 = 3/2$. This result violates, however, the assumption $s_2-2 < -1$, and it does, therefore, not apply. If, on the other hand, $s_2-2 > -1$, then $\dot{\underline{E}}^{(1)}(\theta,t) = 0$. As discussed before this result is not consistent with a bounded plastic strain rate in the presence of singular stresses. Hence we discard the possibility of $s_2-2 > -1$. This leaves $s_2-2 = -1$, or $s_2 = 1$. For this case Eq.(51) yields

$$E_2\dot{\underline{U}}^{(2)} - P_1\dot{\underline{E}}^{(1)} = 0 \tag{54}$$

For the stationary crack one would conclude at this point,

$$\underline{u} = r^{1/2}\underline{U}^{(1)}(\theta,t) + r\underline{U}^{(2)}(\theta,t) + \cdots \tag{55}$$

A more careful study shows, however, that the corresponding homogeneous equation of Eq.(54) has nontrivial solutions. In the Mode I case, for example, the homogeneous equation

$$\Delta^*[r\,\dot{\underline{U}}^{(2)}(\theta,t)] = 0 \tag{56}$$

where Δ^* is defined as

$$\Delta^* = (\lambda+\mu)\nabla\nabla\cdot + \mu\nabla^2 \ , \tag{57}$$

with boundary conditions

$$\theta = \pi: \quad T[r\,\dot{\underline{U}}^{(2)}(\theta,t)] = 0 \tag{58}$$

$$\theta = 0: \quad \underline{i}_1 \cdot T[r\,\dot{\underline{U}}^{(2)}(\theta,t)] = 0 \tag{59a}$$

$$\underline{i}_2 \cdot [r\,\dot{\underline{U}}^{(2)}(\theta,t)] = 0 \tag{59b}$$

where $\underline{i}_j$ are unit vectors, has a solution corresponding to a uniaxial tensile field in the x_1-direction. Hence the asymptotic expression based on (17) becomes ambiguous.

A modification of the asymptotic expression of (55) to include a term of the form, $r\log r\,\dot{\underline{U}}^{(3)}(\theta,t)$ resolves the problem. It is found that

$$\Delta^*[r\,\dot{\underline{U}}^{(2)}(\theta,t)] = \Delta^*[r\log r\,\dot{\underline{U}}^{(3)}(\theta,t)] + \underline{\nabla}\cdot[\dot{\underline{E}}^{(1)}(\theta,t)] \tag{60}$$

with boundary conditions appropriate to Mode I does have a unique solution within a term which would give rise to a uniform contribution to r_{11}.

In conclusion, for the stationary crack we have

$$\underline{u} = r^{1/2}\underline{U}^{(1)}(\theta,t) + r\log r\,\underline{U}^{(3)}(\theta,t) + r\,\underline{U}^{(2)}(\theta,t) + \cdots \tag{61}$$

which shows a difference in the second term as compared to Eq.(49).

C.4 References

1. S. R. Bodner and Y. Partom, "Constitutive Equations for Elastic-Viscoplastic Strain-Hardening Materials", J. Appl. Mech., 42, p. 385, 1975.

2. J.D. Achenbach, M.F. Kanninen and C.H. Popelar, "Crack-Tip Fields for Fast Fracture of an Elastic-Plastic Material", J. Mech. Phys. Solids, 29, p. 211, 1981.

3. J.D. Achenbach and Z.P. Bazant, "Elastodynamic Near-Tip Stress and Displacement Fields for Rapidly Propagating Cracks in Orthotropic Materials," J. Appl. Mech., 42, p. 183, 1975.

4. J.D. Achenbach, "Wave Propagation, Elastodynamic Stress-Intensity Factors and Fracture", Theoretical and Applied Mechanics (W.T. Koiter, ed.), Proc. 14th IUTAM Congress, North-Holland Publ. Co., Amsterdam, 1976.