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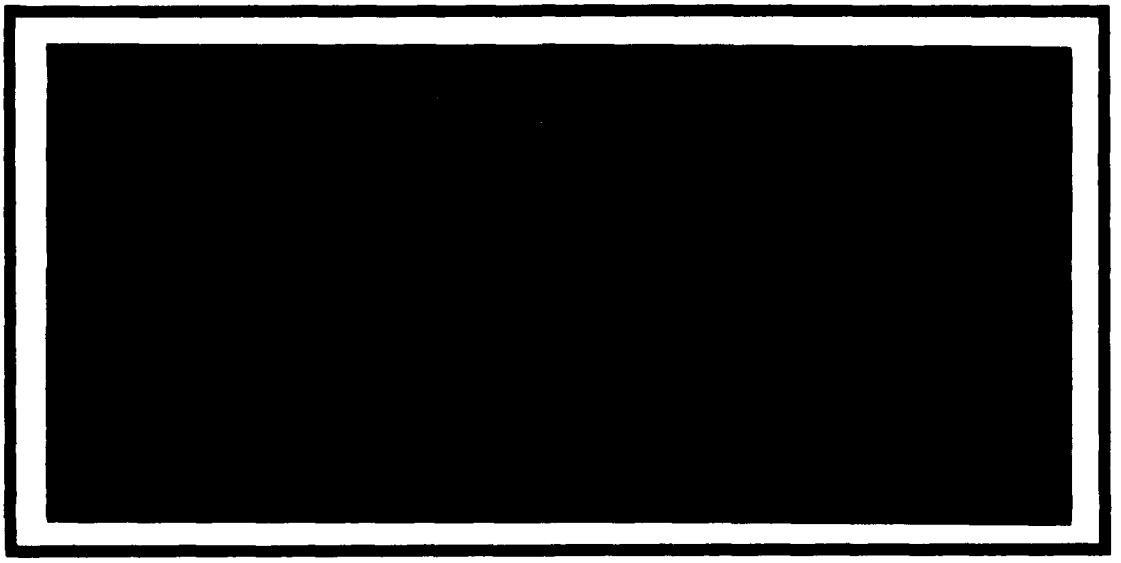

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A POSTERIORI ERROR ESTIMATES FOR PARAMETRIZED NONLINEAR EQUATIONS

by

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A Posteriori Error Estimates for Parametrized Nonlinear Equations*

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Abstract: *A new approach to the construction of a posteriori error estimates for finite element solutions of multi-parameter nonlinear problems is presented. The estimates are derived from local, element-by-element solutions of linearizations of the problems; they turn out to be very effective, computationally rather inexpensive, and insensitive to the choice of the local coordinate system on the solution manifold.*

1. Introduction

Frequently, in practical computations in engineering and science, the aim is to obtain results which are sufficiently accurate and reliable to allow for a decision about the physical system under study. A posteriori error estimates play a very important role in achieving this aim. Such estimates are needed not only for judging the reliability of the computed results but also for controlling adaptive processes designed to achieve desired error tolerances at minimal cost or best possible solutions within allowable cost ranges.

Many structural mechanics problems are modelled by parameter dependent nonlinear equations. The parameters may characterize, for instance, load points and load directions, material properties, geometrical data, etc. In general, the set of all solutions forms a differentiable manifold in the space of the state variables and parameters. This is often called the equilibrium surface of the structure and its form and characteristic features can provide considerable insight into the behavior and stability properties of the structure. Not surprisingly, the quality assessment and control of approximate solutions of such parameterized nonlinear problems is much more difficult and expensive than that of linear problems. In particular, the parameter-dependence causes the discretization error to become a local concept which depends on the choice of the local coordinate system on the equilibrium manifold, [17].

In this paper, we present a new approach for computing a posteriori error estimates of finite element solutions of nonlinear equations involving

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several real parameters. The approach is insensitive to the choice of the local coordinate system on the solution manifold. Moreover, it has already shown itself to be highly effective and computationally inexpensive.

Many error estimations for nonlinear problems work with linearizations of the given equations (see e.g. [16] [19]) and apply known a posteriori error estimators for linear problems to approximate the solution-norms of these linearizations. So far, these techniques have been very limited in the permissible choices of the local coordinate systems on the solution manifold. Another approach consists in the defect-based estimations proposed, for instance, in [10], [11], [12], [13]. However, their widespread applications appears to have been prevented by a rather high computational cost.

Our approach here combines both approaches. It is based on the use of linearizations of the original problem, but determines the required norms of their solution by solving the linearized equations only locally, element-by-element. In essence, the defect-correction is performed locally on the linearized problem. As a result, the computational cost is reduced to very acceptable levels that are comparable to the cost of typical a posteriori estimates for linear problems. Moreover, the resulting estimates vary smoothly on the solution manifold of the given problem and apply to large classes of discretizations and error norms.

In Section 2, we discuss briefly some properties of solution manifolds of parameterized nonlinear problems and their discretizations. Then Section 3 presents the general construction of the error estimates and Section 4 concerns algorithmic aspects. Finally, some results of numerical experiments for a model problem are given in Section 5.

2. Solution Manifolds and Their Discretizations

As noted before, equilibrium problems for many physical systems are modelled by parameter dependent nonlinear equations

$$F(x) = 0, \quad x = (z, \lambda) \tag{2.1}$$

where z represents a state variable and λ a vector of parameters. More specifically, suppose that the nonlinear mapping F satisfies the condition

> **(Fcon) :** *$F : S \subset X \mapsto Y$ is a Fredholm mapping of class C^r, $r \geq 1$ and index $p \geq 1$ from an open subset $S \subset X$ of a real Banach space X into another such space Y.*

Then it is well-known that the set of all regular solutions,

$$M = \{x;\ x \in S,\ F(x) = 0,\ DF(x)X = Y\} \tag{2.2}$$

is either empty or a p-dimensional C^r-manifold in X without boundary. We assume always that $M \neq \emptyset$

For the numerical analysis of the solution manifold (2.2), we need a computationally feasible scheme for fixing local coordinate systems at a given point $x_0 \in M$. For this suppose that a splitting

$$X = T \oplus W, \ \dim T = p \tag{2.3}$$

of X has been chosen such that

$$W \cap \ker DF(x_0) = \{0\}. \tag{2.4}$$

Evidently (Fcon) and (2.4) together imply that the partial derivative $D_W F(x_0)$ is an isomorphism between W and Y. Now the following result (see [17]) shows some neighborhood of the origin of T to be diffeomorphic to a (relative) neighborhood of x_0 in M:

Theorem 2.1: *Under the condition (Fcon), suppose that at $x_0 \in M$, a splitting (2.3), (2.4) has been given. Then there exist an open ball $B = B(0,r) \subset T$ centered at $0 \in T$, an open neighborhood $U \subset X$ of x_0, and a unique C^r-function $\eta : B \to W$ such that $\eta(0) = 0$ and the local coordinate mapping*

$$\Phi : B \subset T \mapsto X, \ \Phi(x) = x(t) = x_0 + t + \eta(t), \quad \forall t \in B \tag{2.5}$$

is a C^r-diffeomorphism from B onto $M \cap U$.

A point $x_0 \in M$ is called a foldpoint with respect to the splitting (2.3) if the condition (2.4) is violated; that is, if (2.3) does not induce a local coordinate system. Note that the finite-dimensional subspace $T = \ker DF(x_0)$ always admits a splitting (2.3) for which (2.4) holds.

In applications the equation (2.1) usually represents some boundary value problem and, as indicated in (2.1), we have a natural parameter splitting $X = Z \oplus \Lambda$, $\dim \Lambda = p$ of X into a state space Z and p-dimensional parameter space Λ. Evidently, this natural parameter splitting may be considered for the definition of a local coordinate system. We will assume that there is at least one point $x \in M$ where this is possible; that is, where $Z \cap \ker DF(x) = \{0\}$. Then, as noted earlier, Z and Y are isomorphic, and accordingly there exists an operator $Q \in L(X,Y)$ such that $\ker Q = \Lambda$ and the restriction $Q_0 = Q|_Z \in L(Z,Y)$ is an isomorphism.

For the computation, it is necessary to replace (2.1) by some finite dimensional approximating equation. Since the parameter space Λ is already finite dimensional, only the state space Z has to be discretized. We follow the approach in [17] in defining a discretization of (2.1).

Let $\{\Pi_h\}$ be a family of finite-rank projections $\Pi_h \in L(Z)$ with range spaces $Z_h = \Pi_h Z$, indexed by a positive real number $h > 0$. Correspondingly we introduce the extensions $\hat{\Pi}_h \in L(X)$ defined by $\hat{\Pi}_h x = \Pi_h z + \lambda$

for $x = z + \lambda \in X$ which have the ranges $X_h = Z_h \oplus \Lambda$. With the earlier isomorphism Q_0 between Z and Y define now the projections $P_h \in L(Y)$, $P_h = Q_0 \Pi_h (Q_0)^{-1}$ with the ranges $Y_h = P_h Y = Q_0 Z_h$. Then the desired approximate equations are given by

$$F_h(x_h) = 0, \ x_h = z_h + \lambda \in X_h \tag{2.6}$$

where

$$F_h : S_h \subset X_h \mapsto Y_h, \ F_h(x) = P_h F(x), \ \forall x \in S_h = S \cap X_h. \tag{2.7}$$

Hence, the corresponding discrete solution manifold is

$$M_h = \{x; x \in S_h, \ F_h(x) = 0, \ DF_h(x) X_h = Y_h\}. \tag{2.8}$$

For the convergence theory additional assumptions about these discretizations are needed the first of which will be the following consistency condition:

$$\lim_{h \to 0} P_h y = y, \quad \forall y \in Y. \tag{2.9}$$

Clearly, any comparison of the solution manifolds M and M_h must be done locally. Conceptually, we expect that for sufficiently small h the local coordinate system at a point $x_0 \in M$ established by Theorem 2.1 also constitutes a local coordinate system at the corresponding approximate solution $x_h \in M_h$. For this to be correct we require some stability condition.

For the analysis of the approximate problems, the discrete operator (2.7) can be extended to all of $S \subset X$ as follows:

$$\hat{F}_h : S \mapsto Y, \ \hat{F}_h(x) = (I_Y - P_h) Q x + P_h F(x), \quad x \in S. \tag{2.10}$$

Then the mentioned stability condition at $x_0 \in M$ assumes the form:

$$\|D\hat{F}_h(x_0) w\|_Y \geq \gamma \|w\|_W, \quad \forall w \in W, \text{ and sufficiently small } h > 0, \tag{2.11}$$

where $\gamma > 0$ is a positive constant independent of w and h. As shown in [17] this condition is relatively easy to verify in many practical situations. The existence and convergence of solutions of the approximate problems (2.6) and the a priori error estimates between M and M_h can now be summarized in the form of the following theorem which was proved in [17].

Theorem 2.2: *(i) Under the condition (Fcon) consider the discretized problem (2.6) and assume that the consistency condition (2.9) holds. At a given point $x_0 \in M$, suppose that the splitting (2.3), (2.4) induces the local C^r-coordinate-mapping (2.5) and that the stability condition (2.11) is satisfied. Then for all sufficiently small $h > 0$ there exists a unique point*

$x_h \in M_h$ *and we have* $\lim_{h\to 0} x_h = x_0$. *Moreover, the splitting* $X_h = T \oplus W_h$ *with* $W_h = W \cap X_h$ *defines at* $x_h \in M_h$ *a local* C^r*-coordinate-mapping* $\Phi_h : B \subset T \mapsto X_h$. *(ii) If, in addition,* DF *is Lipschitz continuous on bounded sets, then there exists a closed ball* $\bar{B}(0, r_0) \subset B$, *such that for all sufficiently small* $h > 0$ *the estimate*

$$\|x_0(t) - x_h(t)\|_X \leq C\|(I_Y - P_h)Qx_0(t)\|_Y, \quad \forall t \in \bar{B}(0, r_0),$$

holds with a constant C *that is independent of* h.

3. A Posteriori Error Estimates

For finite element discretizations of various linear problems the theory and application of reliable a posteriori error estimates has advanced considerably in recent years, we refer only to [3] and the two proceedings [1], [5] where many other references are given. These results are now also being extended to nonlinear problems, see e.g. [2], [4], [16], [8], [19].

In [2] and [4] it has been shown that effective a posteriori error estimates and adaptive procedures can be constructed and incorporated into a general continuation process for tracing paths on the equilibrium surface. But the applicability of these results is somewhat restricted due to the relatively high computational cost of the estimates. In [16] the estimates use a linearized form of the problem and hence can be computed about as rapidly as in the case of linear problems. However, these estimates were based on a fixed local coordinate system, such as that induced by the natural coordinate splitting $X = Z \oplus \Lambda$. Hence, by definition they are not valid at any foldpoint of that local coordinate system, and in fact, as already simple examples indicate, the estimates may become unduly large near any such point.

As noted earlier, our new construction of a posteriori error estimates is also utilizes local linearizations of the given nonlinear mapping. For the relation between the solutions of the nonlinear and these linearized problems, we present first a simple result based on tools from the theory of Newton's method (see e.g.[14], [15])

Theorem 3.1: *Let* $X, \hat{X}$ *be real Banach spaces and* $G : S \mapsto \hat{X}$ *a* C^r*-map,* $r \geq 1$, *on the open subset* $S \subset X$ *such that* DG *is Lipschitz continuous on bounded subsets. Consider any* $x_0 \in S$ *where* $G(x_0) = 0$ *and* $DG(x_0) \in \mathrm{Isom}(X, \hat{X})$. *Then there exists a closed ball* $\bar{B} = \bar{B}(x_0, r) \subset S$, $r > 0$, *such that for any* $x \in \bar{B}$ *the linearized problem*

$$G(x) + DG(x)w = 0 \tag{3.1}$$

has a unique solution $w = w(x) \in X$ *and*

$$\|x_0 - x\| = (1 + \mathrm{O}(\|x - x_0\|))\|w(x)\|, \quad \text{as } x \to x_0, \ x \in \bar{B}. \tag{3.2}$$

Proof: There exists $r_0 > 0$ such that $\bar{B}(x_0, r_0) \subset S$ and $DG(x)$ is an isomorphism for each x in this ball. Let γ be the Lipschitz constant of DG on the ball and with $\beta = \|DG(x_0)^{-1}\|$ and $\alpha = \beta\gamma$ set $r = \min(r_0, 1/(2\alpha))$. Then for all $x \in \bar{B} = \bar{B}(x_0, r)$ the standard perturbation lemma ensures that $DG(x) \in \mathrm{Isom}(X, \hat{X})$ and

$$\|DG(x)^{-1}\| \leq \frac{\beta}{1 - \alpha\|x - x_0\|} \leq 2\beta, \ \forall x \in \bar{B}.$$

Hence, $w(x) = -DG(x)^{-1}G(x)$ is uniquely defined for $x \in \bar{B}$ and with $x_1 = x + w(x)$ we have

$$\|x_1 - x_0\| \leq \|DG(x)^{-1}\|\|G(x_0) - G(x) - DG(x)(x_0 - x)\| \leq \alpha\|x - x_0\|^2,$$

Thus

$$\|w(x)\| \leq \|x - x_0\| + \|x_0 - x_1\| \leq (1 + \alpha\|x - x_0\|)\|x - x_0\|$$

together with

$$\|x - x_0\| \leq \|x - x_1\| + \|x_1 - x_0\| \leq \|w(x)\| + \alpha\|x - x_0\|^2$$

shows that

$$\frac{1}{1 + \alpha\|x - x_0\|}\|w(x)\| \leq \|x - x_0\| \leq \frac{1}{1 - \alpha\|x - x_0\|}\|w(x)\|$$

which proves (3.2).

Now suppose that we are in the setting of both parts of Theorem 2.2. In other words, consider the problem (2.1) where the mapping F satisfies the condition (Fcon) and DF is Lipschitz continous on bounded sets. Suppose that a discretization has been introduced for which the consistency condition (2.9) holds. At the currently given point $x_0 \in M$ we choose a splitting $X = T \oplus W$, $\dim T = p$ for which x_0 is not a foldpoint; that is, for which (2.4) is satisfied. Let $\tau \in L(X)$ denote the natural projection of X onto T along W. For the mapping

$$G : S \subset X \mapsto Y \times T, \ G(x) = (F(x), \tau(x - x_0)), \ \forall x \in S, \qquad (3.3)$$

$DG(x_0)v = 0$ is equivalent with $v \in ker DF(x_0) \cap W$ and thus $v = 0$ and $DG(x_0)$ is injective. By (Fcon) the mapping is also surjective, and hence we find that $DG(x_0) \in \mathrm{Isom}(X, Y \times T)$ which means that Theorem 3.1 is applicable to G.

Since $x_0 \in M$ represents here the desired solution of the original equations (2.1), we assume accordingly that x is the corresponding solution x_h of

the discretized equations (2.6) for which, by definition $x_0 - x_h \in W$. Thus by Theorem 3.1, the problem of estimating the error $x_0 - x_h$ is asymptotically equivalent to solving the constrained linearized problem

$$F(x_h) + DF(x_h)w = 0, \quad \tau(w) = 0. \tag{3.4}$$

Of course, (3.4) is still an infinite dimensional problem which can only be solved approximately. For this we use the same discretization as in the computation of x_h. In other words, we approximate (3.4) by the constrained discretized problem

$$F_h(x_h) + DF_h(x_h)w_h = 0, \quad \tau(\hat{\Pi}_h w_h) = 0, \tag{3.5}$$

where the second equation evidently requires that $w_h \in W_h$ with $W_h = X_h \cap W$. Because of $F_h(x_h) = 0$, the solution of (3.5) is $w_h = 0$. Hence the norm $\|w\|$ of the solution w of (3.4) is exactly the error arising in the discretization (3.5) of (3.4). In other words, by applying one of the known, linear a posteriori error estimates to (3.5) we obtain asymptotically valid estimates of the discretization error $\|w\|$ and therefore, by Theorem 3.1, also of the error $\|x_0 - x_h\|$.

This was the approach taken in [16] and [17]. However, note that (3.4) and (3.5) incorporate constraints involving the complement W of the local coordinate space T. The available a posteriori error estimates cannot handle arbitrary choices of such spaces W; in fact, many of them work only when W is the given state space Z. Clearly, it is important to allow for a flexible choice of the local coordinate system during the computation of points on the solution manifold of (2.1). In other words, this approach is severely limitated by that requirement.

Since the discretized problems (3.5) only produce the trivial approximation $w_h = 0$ of the solution w of the infinite-dimensional linearized problem (3.4), the question arises how to obtain better approximations of w. Evidently, the difficulty arises when we construct (3.5) by means of the same discretization used in the computation of x_h. In other words, we should work with more accurate discretizations of (3.4). This is the basic concept of the defect correction principle. For matrix equations that principle is also called the method of iterative improvements and was first described by Wilkinson [21]. Extensions to ordinary differential equations were proposed in [20] and since then the principle has been used extensively in many settings, see e.g., [6], [7], [9], [10], [11], [12] , [13].

We shall not use the full iterative form of the defect correction principle, but work instead only with one specific improved discretization of (3.4):

$$F_{\tilde{h}}(x_h) + DF_{\tilde{h}}(x_h)w_{\tilde{h}} = 0, \quad w_{\tilde{h}} \in W_{\tilde{h}}, \tag{3.6}$$

where $W_h \subset W_{\bar{h}} \subset W$. In other words, the system (3.6) has more degrees of freedom than used in the computation of x_h.

For ease of notation we assume that the spaces W_h increase monotonically when h decreases. Then the convergence of the solutions $w_{\bar{h}}$ of (3.6) to that of (3.4) for $\bar{h} \to 0$ may be formulated in the form of the following result:

Theorem 3.2: *Suppose that all conditions of Theorem 2.2 hold. Then there is some sufficiently small ε such that for all $0 < \bar{h} < h < \varepsilon$, the solutions w and $w_{\bar{h}}$ of (3.4) and (3.6), respectively, exist and that*

$$\lim_{\bar{h} \to 0} w_{\bar{h}} = w. \tag{3.7}$$

Proof: Since W and Y are isomorphic, we may choose some $A \in \mathrm{Isom}(Y, W)$. Then, by construction of the local coordinate system, $DF(x)A$ is an isomorphism in some ball $B_\rho = \bar{B}(x_0, \rho) \subset S$. Moreover, by the Lipschitz continuity of DF on bounded sets, we may choose $\rho > 0$ small enough such that the stability condition (2.11) holds for all $x \in B_\rho$ and $0 < h < \varepsilon$ with a suitably small $\varepsilon > 0$; in other words, that

$$\|D\hat{F}_h(x)Ay\|_Y \geq \gamma_0 \|y\|_Y, \quad \forall y \in Y \tag{3.8}$$

with some $\gamma_0 > 0$ that is independent of x and h. Finally, by shrinking, if necessary, ρ and ε further, we may assume that for all $0 < h < \varepsilon$ the approximate solution $x_h \in M_h$ of Theorem 2.2 exists and that $x_h \in B_\rho$.

Clearly (3.8) implies that $D\hat{F}_h(x)A \in \mathrm{Isom}(Y)$ whence for any $y_1 \in Y$ there is a unique $y_2 \in Y$ for which $D\hat{F}_h(x)Ay_2 = y_1$. For $y_1 \in Y_h$ it follows that

$$QAy_2 = P_h QAy_2 - P_h DF(x)Ay_2 + y_1 \in Y_h$$

and therefore $Ay_2 \in X_h$ and $(I_Y - P_h)QAy_2 = 0$; that is, $P_h DF(x)Ay_2 = y_1$. Hence we obtain from (3.8) that

$$\|DF_h(x)Ay\|_Y \geq \gamma_0 \|y\|_Y, \quad \forall y \in Y_h.$$

In other words, $(DF_h(x)A)^{-1} \in L(Y_h)$ exists and is uniformly bounded by $\delta = 1/\gamma_0$ for all $x \in B_\rho$ and $0 < h < \varepsilon$.

Now let h with $0 < h < \varepsilon$ be fixed and choose any $\bar{h}$ such that $0 < \bar{h} < h$. By construction of B_ρ we know that $DF(x_h)A$ is an isomorphism and hence that (3.4) has a unique solution $w = Ay \in W$. Moreover, the inverse $(DF_{\bar{h}}(x_h)A)^{-1} \in L(Y_{\bar{h}})$ exists and thus also (3.6) has a unique solution $w_{\bar{h}} = Ay_{\bar{h}}, y_{\bar{h}} \in Y_h$. By subtraction we find from (3.4) and (3.6) that

$$\|w_{\bar{h}} - w\| \leq \delta(\|(I_Y - P_{\bar{h}})DF(x_h)w\| + \|(I_Y - P_{\bar{h}})F(x_h)\|)$$

which by the consistency condition (2.9) implies that (3.7) holds.

4. Algorithm

As shown in the previous section, for the a posteriori estimation of the discretization error of the computed point $x_h \in M_h$ we should solve the more accurate equation (3.6). Obviously, if $X_{\bar{h}} = X_h$ then the solution is $w_h = 0$; hence for any reasonable approximation of the discretization error w, we require $X_{\bar{h}}$ to be a sufficiently larger subspace than X_h. In practice, the dimension of X_h is already expected to be large and thus a full solution of the equation (3.6) will be more expensive than the computation of x_h itself. This constitutes a severe restriction for the applicability of the defect-correction approach.

Without any further assumptions about the discretization process for (2.1) there appears to be little chance of resolving this problem. However, if we suppose that (2.1) is a differential equation to which a finite element approximation can be applied, then it turns out that an estimate of $\|w_{\bar{h}}\|$ can be obtained by solving (3.6) locally on each element. A detailed theory of this approximation process would exceed the space limitations for this paper and will be given elsewhere. Instead, we shall sketch here only the general computational procedure.

We consider a nonlinear problem in a generic weak form requiring the determination of $z \in Z$ such that for given $\lambda \in \Lambda$

$$b(z, \lambda, v) = g(\lambda, v), \quad \forall v \in V \tag{4.1}$$

where Z and V are real Hilbert spaces and Λ is a p-dimensional inner-product parameter-space. In (4.1), b stands for a form on $Z \times \Lambda \times V$ which is nonlinear in the first two variables but linear in the third one, and g is a functional on $\Lambda \times V$ which is linear in v but may be nonlinear in the parameter-variable. We introduce also the Hilbert space $X = Z \times \Lambda$ with the usual inner product for product spaces denoted by $\langle .,. \rangle$.

The discretizations are now specified by the choice of finite dimensional linear subspaces $Z_h \subset Z$ and $V_h \subset V$ and the corresponding orthogonal projections $\Pi_h \in L(Z)$ onto Z_h. In other words, the resulting (Galerkin-type) discretization (2.6) becomes the problem

$$\begin{aligned} &\text{determine } z_h \in Z_h \text{ such that} \\ &\quad b(z_h, \lambda, v_h) = g(\lambda, v_h), \quad v_h \in V_h. \end{aligned} \tag{4.2}$$

If z_h^j, $j = 1, 2, \ldots, m_h$ and v_h^j, $j = 1, 2, \ldots, m_h$ denote bases of Z_h and V_h, respectively, then (4.2) assumes the usual form

$$b(\sum_{j=1}^{m_h} \zeta_j z_h^j, \lambda, v_h^k) = g(\lambda, v_h^k), \quad k = 1, 2, \ldots, m_h \tag{4.3}$$

of a system of m_h nonlinear equations in $m_h + p$ variables.

At any solution $x = (z, \lambda)$ of (4.1) consider now a local coordinate system defined by a p-dimensional subspace $T \subset X$ together with its orthogonal complement $W = T^{\perp}$ in X. Under appropriate smoothness assumptions on b and g the linearized problem (3.4) at a computed approximation $x_h \in X_h = Z_h \oplus \Lambda$ of x then becomes

$$\begin{aligned} &\text{determine } u \in Z,\ \mu \in \Lambda \text{ such that} \\ &a_h(u, \mu, v) = f(v), \quad \forall v \in V, \\ &\langle t_k, (u, \mu) \rangle = 0, \quad k = 1, \ldots, p \end{aligned} \tag{4.4}$$

where $t_k \in X,\ k = 1, 2, \ldots, p$ represents a basis of T and

$$\begin{aligned} a_h(u, \mu, v) &= D_z b(z_h, \lambda_h, v)u + D_\lambda b(z_h, \lambda_h, v)\mu - D_\lambda g(\lambda_h, v)\mu \\ f(v) &= -b(z_h, \lambda_h, v) + g(\lambda_h, v) \end{aligned} \tag{4.5}$$

As we saw, we have to solve (4.4) with a discretization induced by some larger subspaces $Z_{\tilde{h}} = Z_h \oplus Z_h^c$ and $V_{\tilde{h}} = V_h \oplus V_h^c$ where Z_h^c and V_h^c denote here certain complements of Z_h and V_h in $Z_{\tilde{h}}$ and $V_{\tilde{h}}$,resp ectively. Hence, the resulting discretization is

$$\begin{aligned} &\text{determine } u_h \in Z_h,\ u_h^c \in Z_h^c,\ \mu \in \Lambda \text{ such that} \\ &a_h(u_h, \mu, v_h) = f(v_h), \quad \forall v_h \in V_h, && (4.6a) \\ &a_h(u_h^c, \mu, v_h) = 0, \quad \forall v_h \in V_h, && (4.6b) \\ &a_h(u_h + u_h^c, \mu, v_h^c) = f(v_h^c), \quad \forall v_h^c \in V_h^c, && (4.6c) \\ &\langle t_k, (u_h, \mu) \rangle = 0, \quad k = 1, \ldots, p && (4.6d) \\ &\langle t_k, (u_h^c, \mu) \rangle = 0, \quad k = 1, \ldots, p && (4.6e) \end{aligned}$$

But (4.6a) and (4.6d) constitute exactly the discretization of (4.4) defined by the restriction $u_h \in Z_h$ and $v_h \in V_h$, for which we found the solution to be zero. Hence, it remains only to solve the reduced problem

$$\begin{aligned} &\text{determine } u_h^c \in Z_h^c,\ \mu \in \Lambda \text{ such that} \\ &a_h(u_h^c, \mu, v_h^c) = f(v_h^c), \quad \forall v_h^c \in V_h^c, && (4.7a) \\ &a_h(u_h^c, \mu, v_h) = 0, \quad \forall v_h \in V_h, && (4.7b) \\ &\langle t_k, (u_h^c, \mu) \rangle = 0, \quad k = 1, \ldots, p && (4.7c) \end{aligned}$$

where (4.7b) represents a boundary condition for u_h^c.

For the practical application we have to choose a suitable space T. Since the computed point x_h belongs to the manifold M_h of the discretized problem (4.3) it is natural to select T as the null-space of the Jacobian

of(4.2); that is, as the subspace of X_h corresponding to the tangent vector of M_h at x_h. If Λ is one-dimensional and M_h is computed by a standard continuation process then a normalized basis vector of T is usually available at each computed point. Analogously, in the multi-parameter case, if a triangulation of M_h is being computed (see e.g. [18]) then again orthonormal "tangent-bases" t_h^j, $j = 1, \ldots, p$ are readily available at the computed points on the manifold.

Let Ω_j^h be a typical element of the mesh on the underlying domain Ω of the original problem (4.1) corresponding to the finite element space Z_h. In order to obtain $Z_{\bar{h}} \supset Z_h$, we may subdivide Ω_j^h into sub-elements or increase the order of the element or both, in other words we may use either a local h-version, p-version, or h-p-version of the finite element method. This has the advantage of being applicable to large classes of error norms for u. Of course, for the solution of (4.7a,b,c) it now will become necessary to "extend" the computed point x_h and its corresponding "tangent"-basis t_h^j, $j = 1, \ldots, p$ to $X_{\bar{h}}$ by suitable interpolation. The norm of the computed local solution on Ω_j^h is called the local error indicator on that element; that is,

$$\eta_j = \|u_h^c\|_{\Omega_j^h}, \tag{4.8}$$

and the error estimate for the computed point x_h is the sum

$$\varepsilon = (\sum_j \eta_j^2)^{1/2}, \tag{4.9}$$

over the error-indicators of all elements of Ω.

Altogether, for the computation of the error estimate at any computed point of the solution manifold M_h of the discreterized problem we have now the following algorithm:

1. Let $x_h \in X_h$ on M_h be the current solution of (4.2). Select the tangent space $T_h = span\{t_h^1, \ldots, t_h^p\}$ of M_h at x_h to define the local coordinate system.
2. Loop over all elements Ω_j^h of the domain Ω
   - 2.1 Subdivide the element Ω_j^h into sub-elements corresponding to the desired choice of a finite dimensional subspace $Z_{\bar{h}}^j$ on Ω_j^h.
   - 2.2 On the refined element solve the local linearized finite element problem (4.7a,b,c) for $(u_{\bar{h}}^c, \mu) \in X_{\bar{h}}^j = Z_{\bar{h}}^j \oplus \Lambda$. This involves the interpolation of x_h and $t_1^h, \cdots, t_p^h$ from X_h^j into $X_{\bar{h}}^j$.
   - 2.3 Compute the local error indicator (4.8) for the element.
3. Compute the desired a posteriori estimate (4.9) for the current point $x_h \in M_h$.

5. Numerical Example

As a model problem for numerical experiments we consider the two-dimensional nonlinear boundary value problem

$$-\Delta z = \lambda e^z, \quad z = z(x,y), \quad \forall (x,y) \in \Omega = (0,1) \times (0,1), \qquad (5.1)$$
$$z = 0 \quad \text{on} \quad \partial\Omega.$$

A weak formulation is

$$\int_\Omega (z_x v_x + z_y v_y - \lambda e^z v)\, dx\, dy = 0, \quad \forall v \in H_0^1(\Omega). \qquad (5.2)$$

and we assume that $\lambda \in R^1$ which means that the solutions of (5.2) are expected to form a one-dimensional manifold M.

For the discretization we use a uniform mesh of sixteen bi-quadratic elements on Ω. For the computation of the one-dimensional solution manifold M_h of the discretized problem a continuation process (PITCON) is applied starting from $(z_h^0, \lambda_h^0) = (0,0)$ and our aim is to determine a posteriori estimates of the error between M and M_h at all computed solutions $(z_h, \lambda_h) \in M_h$.

In line with (4.4), the linearized problem at (z, λ) has the form

$$\int_\Omega [u_x v_x + u_y v_y - (\lambda u + \mu) e^z v]\, dx\, dy$$
$$= -\int_\Omega (z_x v_x + z_y v_y - \lambda e^z v)\, dx\, dy, \quad \forall v \in H_0^1(\Omega). \qquad (5.3)$$

As before the point (z, λ) stands here always for one of the computed points $(z_h, \lambda_h) \in M_h$. We apply now the algorithm of Section 4. and approximate the error $\|u\|$ by solving (5.3) locally on a larger finite element subspace $Z_{\tilde{h}}$ of $H_0^1(\Omega)$ together with the auxiliary condition

$$\langle t_h, u \rangle = 0, \qquad (5.4)$$

Here, as noted in Section 4., we choose t_h as a normalized tangent vector on M_h at (z_h, λ_h). Such a tangent vector is available at each step of the continuation process and hence the equation (5.4) will involve little additional computational cost.

For the local solution each one of the 16 elements of Ω were divided into $(k+1)^2$ biquadratic sub-elements with $k = 1, 2, 3, 4$. The resulting error estimates are shown in Table 5.1 where $\|u_{\tilde{h}_k}\|$ denotes the computed error norms for the four cases of k. The computations are very cost-effective, since each local problem involves only a fixed number of degrees of freedom depending on the value of k. If $\|u_{\tilde{h}_4}\|$ is taken as the exact error then the effectivity index of the estimates is surprisingly good even for $k = 1$.

The table also shows that, as expected, the estimated errors vary smoothly along the solution path M_h and show no sudden increases near the limit point $\lambda = 6.804524$.

As mentioned earlier, if the natural coordinate system induced by the parameter space Λ is chosen then we expect the resulting error estimates to become unduly large near the limit point. This is indeed the case as the last column of Table 5.2 shows. At the same time, it should be noted that the computational cost of the two approaches are practically identical.

Table 5.1

step	λ	$\|u_{\bar{h}_1}\|$	$\|u_{\bar{h}_2}\|$	$\|u_{\bar{h}_3}\|$	$\|u_{\bar{h}_4}\|$
1	0.118710	0.000516	0.000538	0.000542	0.000544
2	0.473781	0.002040	0.002128	0.002146	0.002152
3	0.946226	0.004010	0.004184	0.004220	0.004231
4	1.415882	0.005898	0.006155	0.006208	0.006225
5	1.882269	0.007695	0.008031	0.008101	0.008123
6	2.344791	0.009389	0.009801	0.009887	0.009914
7	2.785973	0.010913	0.011395	0.011495	0.011527
8	3.220584	0.012313	0.012861	0.012976	0.013011
9	3.647333	0.013578	0.014188	0.014315	0.014355
10	4.064580	0.014694	0.015361	0.015500	0.015544
11	4.470212	0.015648	0.016368	0.016519	0.016555
12	4.861489	0.016433	0.017200	0.017362	0.017413
13	5.234830	0.017050	0.017861	0.018032	0.018087
14	5.585562	0.017522	0.018373	0.018554	0.018611
15	5.907655	0.017914	0.018805	0.018994	0.019054
16	6.193552	0.018368	0.019300	0.019499	0.019562
17	6.434373	0.019144	0.020128	0.020337	0.020404
18	6.620862	0.020640	0.021692	0.021916	0.021986
19	6.745397	0.023317	0.024468	0.024710	0.024787
20	6.804524	0.027539	0.028832	0.029100	0.029185
21	6.800451	0.033461	0.034945	0.035250	0.035345
22	6.740221	0.041066	0.042795	0.043146	0.043255
23	6.633252	0.050283	0.052305	0.052712	0.052838
24	6.489009	0.061065	0.063424	0.063896	0.064041
25	6.328924	0.072480	0.075190	0.075729	0.075894
26	6.144920	0.085598	0.088703	0.089318	0.089506
27	5.941748	0.100500	0.104046	0.104745	0.104959
28	5.723487	0.117289	0.121324	0.122117	0.122359
29	5.493577	0.136117	0.140694	0.141591	0.141865
30	5.254879	0.157154	0.162338	0.163350	0.163659

Table 5.2

step	λ	$\|u_{h_4}\|$	$\|u\|$
15	5.907655	0.019054	0.022905
16	6.193552	0.019562	0.024439
17	6.434373	0.020404	0.027792
18	6.620862	0.021986	0.041992
19	6.745397	0.024787	0.159595
20	6.804524	0.029185	0.176171
21	6.800451	0.035345	0.086353
22	6.740221	0.043255	0.069906
23	6.633252	0.052838	0.065171
24	6.489009	0.064041	0.065099
25	6.328924	0.075894	0.067672

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