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# Canonical Eigenvalues

## Part I. General Theory

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## INTRODUCTION

A formulation of the eigenvalue problem, here referred to as the canonical eigenvalue method or canonical eigenvalue theory, was developed to facilitate analysis of mode coupling effects in double ducts in underwater acoustics. This canonical eigenvalue method can also be used if eigenvalue curves as a function of some variable parameter, such as frequency, are required to analyze some acoustic phenomenon. Also, if one or more eigenvalues are required at many values of such a parameter, as for instance in broadband simulations, the canonical eigenvalue method is a useful eigenvalue computation technique.

This report is one of a series of reports that explain the canonical eigenvalue method and demonstrate its application to various acoustic ducts, including some double ducts.

The chief objective of this report is to present the general theory of canonical eigenvalues. In the approach taken here, the eigenvalue equation is couched in purely mathematical terms, involving two dimensionless variables and a set of dimensionless parameters related to the sound-speed profile. The physical quantities of interest, such as frequency and phase and group velocity, are generated after the eigenvalues have been determined. This approach has a number of advantages over customary approaches, as will be discussed in detail.

The basic concept of canonical eigenvalues was introduced in Ref. 1, where the eigenvalue equation for three simple ducts was expressed in terms of one mathematical variable. For an unbounded refractive duct, the equation also involves one dimensionless profile parameter. The extension of the canonical eigenvalue approach to more complicated profiles was suggested and the first steps of the process were outlined in Ref. 1, which also contains a brief history of how the canonical approach came about.

Section 1 extends the basic concept of Ref. 1 into a definitive treatment of the general theory of canonical eigenvalues. The theory is developed so that it may be applied to propagation in any multi-layered profile in which the squared index of refraction is linear in each layer and is piecewise continuous. The method is extended to include improvements in the treatment of group velocity and the determination of normalization coefficients for the eigenvalues. Further generalizations of the method are outlined, as well as its application to complex eigenvalues or complex parameters in the sound-speed profile.

Section 2 presents an outline of the steps necessary to evaluate acoustic fields by the canonical approach. Ref. 2 is next in this series of reports on double-duct propagation and associated mode theory and applies the theory of canonical eigenvalues to the class of one-layer bounded profiles, as discussed in detail at the end of Section 2.

## SECTION 1. CANONICAL EIGENVALUE APPROACH

This section presents the general theory of the canonical eigenvalue approach.

### BASIC EQUATIONS

The basic canonical eigenvalue equations are reviewed in this section. The derivation of these equations, together with additional explanatory material, may be found in Ref. 1.

The sound-speed profile is layered in depth and there is no dependence on range. The sound speed in each layer of the profile is expressed as

$$[C_i/C(Z)]^2 = 1 - 2\gamma_i(Z - Z_i)/C_i \quad (1)$$

where  $C_i$ ,  $Z_i$ , and  $\gamma_i$  are the sound speed, depth, and sound-speed gradient, respectively, at the top of layer  $i$ . The canonical eigenvalue method is limited here to the case of continuous sound speed at layer interfaces; i. e., the sound speed at the top of layer  $i$  and at the bottom of layer  $i-1$  is the same, with a value of  $C_i$ .

The unnormalized mode functions are given by

$$F_i(Z) = D_i Ai(-\zeta_i) + E_i Bi(-\zeta_i) . \quad (2)$$

For the profile of Eq. 1,  $F_i(Z)$  satisfies Airy's differential equation. Thus, the solution is a linear combination of the Airy functions  $Ai$  and  $Bi$ , with  $D_i$  and  $E_i$  being coefficients independent of depth.

The argument of the Airy functions is given by

$$\zeta_i(Z) = [a_i^3(Z - Z_i) + \omega^2(C_i^2 - C_p^2)]/a_i^2 \quad (3)$$

where  $\omega = 2\pi f$ ,  $C_p$  is the mode phase velocity, and  $a_i$  is defined by

$$a_i^3 = -2\gamma_i \omega^2/C_i^3 . \quad (4)$$

The sound-speed gradient at the bottom of layer  $i$  is given by

$$\gamma_{i0} = C_{i+1}^3 \gamma_i/C_i^3 . \quad (5)$$

Equation 3 may be expressed in terms of  $C$  rather than  $Z$  by

$$\zeta_i(C) = [1 - (C/C_p)^2] (C_i/C)^2 |\gamma_i|^{-2/3} \pi^{2/3} f^{2/3} . \quad (6)$$

In order to satisfy the boundary and interface conditions, the eigenvalue equation will involve Airy functions with arguments which are the negative of Eq. 6 as evaluated at boundaries and interfaces. Equation 6 leads to

$$\zeta_{i1} = \zeta_i(Z_i) = \zeta_i(C_i) = [1 - (C_i/C_p)^2] \pi^{2/3} f^{2/3} |\gamma_i|^{-2/3} \quad (7)$$

for the upper interface and to

$$\zeta_{i0} = \zeta_i(Z_{i+1}) = \zeta_i(C_{i+1}) = [1 - (C_{i+1}/C_p)^2] \pi^{2/3} f^{2/3} |\gamma_{i0}|^{-2/3} \quad (8)$$

for the lower interface.

We next define a dimensionless profile parameter  $\rho_i$  as given by

$$\rho_i = (-\gamma_{i0}/\gamma_{i+1})^{1/3} . \quad (9)$$

It follows then that

$$\zeta_{i+1,1} = \rho_i^2 \zeta_{i0} \quad (10)$$

and

$$a_{i+1} = -\rho_i^{-1} a_i . \quad (11)$$

The introduction of  $\rho_i$  is a customary procedure in most normal-mode approaches to the profile of Eq. 1 because Eq. 10 and 11 simplify the eigenvalue equation.

All the equations thus far can be applied to the customary approach to generating eigenvalues. This customary approach consists of solving the eigenvalue equation to determine phase velocity as a function of frequency for some given set of profile parameters. For some fixed desired frequency, one evaluates the  $\zeta_{i1}$  and  $\zeta_{i0}$  of Eq. 7 and 8, using a first estimate of phase velocity  $C_p$ . One then evaluates the eigenvalue equation for these values of  $\zeta_{i1}$  and  $\zeta_{i0}$  and iterates on  $C_p$  until the eigenvalue equation is solved. This iteration is repeated for each mode. Then the entire process is repeated for each desired frequency to eventually yield the phase velocity vs. frequency curves for each mode.

At this point, we take a different tack. We introduce the dimensionless mathematical variables  $x$  and  $y$  as defined by

$$x = \zeta_{11} \quad (12)$$

and

$$y = \zeta_{10} . \quad (13)$$

Here  $x$  is the negative of the Airy function argument as evaluated at the top of layer 1, i.e., at the upper boundary of the duct; whereas  $y$  is the negative of the Airy function argument as evaluated at the bottom of layer 1. As we shall demonstrate in the next subsection of Section 1, the arguments of all the Airy functions in the eigenvalue equation, i.e.,  $-\zeta_{i1}$  and  $-\zeta_{i0}$ , can be written as linear functions of  $x$  and  $y$  with dimensionless coefficients that are independent of  $f$  and  $C_p$ . The canonical eigenvalues are the curves of  $y$  vs  $x$  which satisfy the eigenvalue equation for a fixed set of these dimensionless coefficients.

Assume now that we have the  $x$ ,  $y$  solutions to the eigenvalue equation. The next task is to associate these mathematical variables with the physical variables  $f$  and  $C_p$ . We first derive an expression for the frequency. Consider Eq. 7 and 8 as evaluated for  $i = 1$ ; i.e., the left sides are  $x$  and  $y$ , respectively. We eliminate  $C_p$  from these two expressions and solve for  $f$  to obtain

$$f = (y-x)^{3/2} (C_1^2 C_2^{-2} - 1)^{-3/2} |\gamma_1| \pi^{-1} . \quad (14)$$

The requirement that the frequency be real is an aid in the interpretation of the canonical eigenvalues. This requirement is

$$y > x \rightarrow C_1 > C_2 \quad (15)$$

and

$$y < x \rightarrow C_1 < C_2 . \quad (16)$$

We shall refer to a plot of  $|y-x|^{3/2}$  vs.  $x$  as a canonical frequency plot, which is independent of profile parameters. The frequency may be obtained from such a plot by the scale factor  $|C_1^2 C_2^{-2} - 1|^{-3/2} |\gamma_1| \pi^{-1}$ , as evaluated for the desired parameters of the first layer of the profile. Once the frequency of Eq. 14 has been evaluated, the phase velocity may be evaluated as a function of frequency from the expression

$$C_p = C_1 [1 - f^{2/3} \pi^{-2/3} |\gamma_1|^{2/3} x]^{-1/2} . \quad (17)$$

Equation 17 is obtained by solving Eq. 7 for  $C_p$  for  $i = 1$ . At first sight, it would appear that  $f$  and  $C_p$  only depend on the profile parameters of the first layer. We note, however, that  $f$  or  $C_p$  depend on  $x$  and  $y$  or  $x$ , respectively. These variables in turn depend on all of the profile parameters as contained in the eigenvalue equation. Thus,  $f$  and  $C_p$  are explicit functions of the first layer parameters and are implicit functions of the other profile parameters.

## FORMULATION FOR ARBITRARY MULTILAYER PROFILES

The major purpose of this subsection is to demonstrate how the canonical eigenvalue equation can be formulated for an arbitrary piecewise continuous profile of the form of Eq. 1. The only problem to be solved is to obtain a suitable form for the  $\zeta_{i1}$  and  $\zeta_{i0}$  of Eq. 7 and 8. These are the negatives of the various Airy function arguments that appear in the eigenvalue equation. Other features of the eigenvalue equation, such as the particular boundary conditions and the values of  $\rho_i$  of Eq. 9 that appear as coefficients of the Airy functions, pose no problem in the formulation.

Our goal is to express  $\zeta_{i1}$  and  $\zeta_{i0}$  as linear functions of  $x$  and  $y$  with coefficients that are independent of  $f$  and  $C_p$ . Let us assume that we have expressions for  $\zeta_{i1}$  and  $\zeta_{i0}$  and wish to derive the expression for  $\zeta_{i+1,1}$  and  $\zeta_{i+1,0}$  in terms of  $\zeta_{i1}$  and  $\zeta_{i0}$ . The solution for  $\zeta_{i+1,1}$  is given by Eq. 10, which is used in all eigenvalue approaches that introduce the dimensionless parameter  $\rho_i$ . Our task is to determine the coefficients  $M_{i+1}$  and  $N_{i+1}$  such that

$$\zeta_{i+1,0} = M_{i+1} \zeta_{i1} + N_{i+1} \zeta_{i0} . \quad (18)$$

Substitutions from Eq. 7 and 8 lead to

$$\begin{aligned} [1 - (C_{i+2}/C_p)^2] \pi^{2/3} f^{2/3} |\gamma_{i+1,0}|^{-2/3} = \\ M_{i+1} [1 - (C_i/C_p)^2] \pi^{2/3} f^{2/3} |\gamma_i|^{-2/3} + N_{i+1} [1 - (C_{i+1}/C_p)^2] \pi^{2/3} f^{2/3} |\gamma_{i0}|^{-2/3} . \end{aligned} \quad (19)$$

We see immediately that the form of Eq. 18 must be homogeneous in  $\zeta_{i1}$  and  $\zeta_{i0}$ , for the addition of a constant term to Eq. 18 would violate the requirement of independence of frequency. Any constant term would need to contain  $f^{2/3}$  as a factor.

For the homogeneous equation,  $\pi^{2/3} f^{2/3}$  cancels out of Eq. 19 as a common factor. Since  $M_{i+1}$  and  $N_{i+1}$  are to be independent of  $C_p$ , Eq. 19 leads to

$$|\gamma_{i+1,0}|^{-2/3} = M_{i+1} |\gamma_i|^{-2/3} + N_{i+1} |\gamma_{i0}|^{-2/3} \quad (20)$$

and

$$C_{i+2}^2 |\gamma_{i+1,0}|^{-2/3} = M_{i+1} C_i^2 |\gamma_i|^{-2/3} + N_{i+1} C_{i+1}^2 |\gamma_{i0}|^{-2/3} . \quad (21)$$

Equation 20 is obtained by equating terms in Eq. 19 that do not contain  $C_p$ . Equation 21 equates these terms of Eq. 19 that contain  $(1/C_p)^2$  as a common factor that has now been canceled.

From Eq. 5, we may obtain

$$|\gamma_{i0}|^{-2/3} = |\gamma_i|^{-2/3} C_i^2/C_{i+1}^2 \quad (22)$$

From Eq. 9, we obtain

$$|\gamma_{i+1}|^{-2/3} = \rho_i^2 |\gamma_{i0}|^{-2/3} \quad (23)$$

From Eq. 22 and 23, we obtain

$$|\gamma_{i+1,0}|^{-2/3} = \rho_i^2 |\gamma_i|^{-2/3} C_i^2/C_{i+2}^2 \quad (24)$$

We next substitute Eq. 24 and Eq. 22 into Eq. 20 and cancel common factor  $|\gamma_i|^{-2/3}$  to obtain

$$\rho_i^2 C_i^2/C_{i+2}^2 = M_{i+1} + N_{i+1} C_i^2/C_{i+1}^2 \quad (25)$$

We substitute Eq. 24 and Eq. 22 into Eq. 21 and cancel common factors  $C_i^2 |\gamma_i|^{-2/3}$  to obtain

$$\rho_i^2 = M_{i+1} + N_{i+1} \quad (26)$$

Thus

$$M_{i+1} = \rho_i^2 - N_{i+1} \quad (27)$$

We next substitute Eq. 27 into Eq. 25 to obtain

$$N_{i+1} = \rho_i^2 [(C_i/C_{i+2})^2 - 1] / [(C_i/C_{i+1})^2 - 1] \quad (28)$$

We note that  $M_{i+1}$  and  $N_{i+1}$  do not depend upon  $f$  or  $C_p$  and depend only on functions of dimensionless ratios of sound speed and gradients.

The analysis is simplified if we introduce the parameter  $S_{i+1}$  as given by

$$S_{i+1} = [(C_i/C_{i+2})^2 - 1] / [(C_i/C_{i+1})^2 - 1] \quad (29)$$

It follows from Eq. 28 and 27 that

$$N_{i+1} = \rho_i^2 S_{i+1} \quad (30)$$

and

$$M_{i+1} = \rho_i^2 (1 - S_{i+1}) \quad (31)$$

Equations 10 and 18 allow us to determine all values of  $\zeta_{i1}$  and  $\zeta_{i0}$  in terms of  $x$  and  $y$  by a recursion process. We introduce the coefficients  $P_{i1}$ ,  $P_{i0}$ ,  $Q_{i1}$ , and  $Q_{i0}$  as defined by

$$\zeta_{i1} = P_{i1} x + Q_{i1} y \quad (32)$$

and

$$\zeta_{i0} = P_{i0} x + Q_{i0} y \quad (33)$$

The coefficients are given by



$$P_{11} = 1 ; Q_{11} = 0 \quad (34)$$

$$P_{10} = 0 ; Q_{10} = 1 \quad (35)$$

$$P_{21} = 0 ; Q_{21} = \rho_1^2 \quad (36)$$

$$P_{20} = \rho_1^2(1 - S_2) ; Q_{20} = \rho_1^2 S_2 \quad (37)$$

$$P_{31} = \rho_2^2 \rho_1^2(1 - S_2) ; Q_{31} = \rho_2^2 \rho_1^2 S_2 \quad (38)$$

and

$$P_{30} = \rho_2^2 \rho_1^2 S_3(1 - S_2) ; Q_{30} = \rho_2^2 \rho_1^2 [1 - S_3(1 - S_2)] . \quad (39)$$

Equations 34 and 35 follow from Eq. 12 and 13, respectively. Equation 36 follows from Eq. 10. Equation 37 follows from Eq. 18, 30, and 31. Equation 38 follows from Eq. 10 and 37. Equation 39 follows from Eq. 18, 36, and 37. This process can be continued for as many layers as desired.

This procedure assumes that the profile has both upper and lower boundaries. The same scheme works when the deepest layer is an infinite half space. When the deepest layer is bounded, one requires both  $\zeta_{i1}$  and  $\zeta_{i0}$ . However, when the deepest layer is unbounded, only  $\zeta_{i1}$  is required.

The procedure must be modified slightly when the upper layer is an infinite half space. Here we associate  $x$  and  $y$  with upper and lower arguments of the second layer. Thus

$$P_{21} = 1 ; Q_{21} = 0 \quad (40)$$

and

$$P_{20} = 0 ; Q_{20} = 1 . \quad (41)$$

Higher order terms may be obtained from Eq. 36 to 39 by increasing the subscripts by 1. It follows also that

$$P_{10} = \rho_1^{-2} \text{ and } Q_{10} = 0 . \quad (42)$$

This example points out that  $x$  and  $y$  do not have to be associated with layer 1. Indeed, we may associate  $x$  and  $y$  with any layer, e.g.

$$x = \zeta_{i1} \quad (43)$$

and

$$y = \zeta_{i0} . \quad (44)$$

We may obtain higher order terms by Eq. 34 to 39 by increasing the subscripts by  $i-1$ . We may obtain lower order terms by running the recursion process down rather than up. Equation 10 and 18, respectively, may be manipulated to yield

$$\zeta_{i-1,0} = \rho_{i-1}^{-2} \zeta_{i1} \quad (45)$$

and

$$\zeta_{i-1,1} = M_i^{-1}(\zeta_{i0} - N_i \zeta_{i-1,0}) . \quad (46)$$

We illustrate the use of Eq. 43 and 44 with layer 1 having an upper bound rather than the infinite half space of our previous example. Thus, Eq. 40 and 41 hold. Equation 45 leads to Eq. 42, while Eq. 46 leads to

$$P_{11} = -\rho_1^{-2}(1-S_2)^{-1} S_2; Q_{11} = \rho_1^{-2}(1-S_2)^{-1} . \quad (47)$$

If one defines  $x$  and  $y$  by Eq. 43 and 44,  $C_1$ ,  $C_2$ , and  $\gamma_1$  in Eq. 14 must be replaced by  $C_i$ ,  $C_{i+1}$ , and  $\gamma_i$ . Similarly, 1 must be replaced by  $i$  in Eq. 17. The definition of  $x$  and  $y$  in terms of layer 1 is indicated for a profile consisting of a few layers. However, there might be some advantage to selecting a different reference layer for complicated multilayer profiles. For example, consider a major refractive duct with a minor surface duct above. In this case, we would recommend that  $x$  and  $y$  be referred to the lower branch of the major duct. With this reference, the effects of the minor duct will be to change the  $x,y$  solution slightly from that of the major duct alone. In contrast, the  $x,y$  solution couched in terms of the first layer will bear relatively little relation to the solution for the surface duct alone.

We next examine two different implementations of the canonical eigenvalue method. The simplest implementation is to take a specific set of profile parameters, e.g.,  $C_i$  and  $\gamma_i$ . For a general profile consisting of  $n$  interfaces and  $m$  boundaries, there are  $2n+m+1$  profile parameters. We then proceed to evaluate the coefficients of Eq. 36 to 39, plus higher order terms necessary, for the given profile parameters.

This method is simple but makes no use of the canonical nature of the method. The coefficients of Eq. 36 to 39 depend only on the parameters  $\rho_i$  and  $S_i$ . For a profile with  $n$  interfaces and  $m$  boundaries, there are  $n$  values of  $\rho_i$  and  $n+m-2$  values of  $S_i$ . Suppose now that we specify a set of  $2n+m-2$  numerical values of  $\rho_i$  and  $S_i$ , evaluate the coefficients of Eq. 36 to 39, and solve the eigenvalue equation. (In passing, we note that we must specify  $\rho_i$  rather than  $\rho_i^2$  as would be possible in Eq. 36 to 39 because  $\rho_i$  appears in the eigenvalue equation as a coefficient of some of the Airy functions.) We then have the solution for arbitrary values of  $C_1$ ,  $C_2$ , and  $\gamma_1$  for the profile class. The canonical profile class is determined by the set of  $\rho_i$  and  $S_i$  and the boundary conditions. Once we select a set of values for  $C_1$ ,  $C_2$ , and  $\gamma_1$ , the remaining  $2n+m-2$  profile parameters for the specific profile can be generated from the set of  $\rho_i$  and  $S_i$ .

This generation is accomplished by recursion. Equation 29 may be solved to yield  $C_{i+2}$  as given by

$$C_{i+2} = C_i \{ [(C_i/C_{i+1})^2 - 1] S_{i+1} + 1 \}^{-1/2} . \quad (48)$$

We first determine  $C_3$  in terms of the given  $C_1$ ,  $C_2$ , and  $S_2$  and use Eq. 48 recursively to generate all values of  $C_j$ . Equation 9 may be solved for  $\gamma_{i+1}$  to yield

$$\gamma_{i+1} = -\gamma_{i0} \rho_i^{-3} . \quad (49)$$

We initially use Eq. 5 to generate  $\gamma_{10}$  in terms of the given  $C_1$ ,  $C_2$ , and  $\gamma_1$ . We then use Eq. 49 to solve for  $\gamma_2$ . We then alternately use Eq. 5 and 49 to generate  $\gamma_{i+1}$  from the determined  $\gamma_i$  and  $C_i$  and  $C_{i+1}$  as previously established with the use of Eq. 48.

It is of interest to compare the canonical eigenvalue approach with the customary approach in terms of free variables, dependent (constrained) variables, and parameters. We will define parameters as those quantities which are fixed in the solution of the eigenvalue equation. We will define free variables as those that we may choose independently of the eigenvalue equation. Consider the general profile of  $n$  interfaces and  $m$  layers. The eigenvalue equation in the customary approach contains the

profile parameters plus the frequency, for a total of  $2n+m+2$  parameters and one dependent variable,  $C_p$ , as constrained by the eigenvalue equation.

In the canonical approach, we add the variables  $x$  and  $y$ . Thus, we must account for a total of  $2n+m+5$  values, which may be either free variables, dependent variables, or parameters. The parameters in this approach are given by the  $2n+m-2$  values of  $\rho_i$  and  $S_j$ . The free variables are  $C_1$ ,  $C_2$ ,  $\gamma_1$ , and  $x$ . The dependent variables are  $y$ ,  $f$ , and  $C_p$  as determined respectively by the constraints of the canonical eigenvalue equation, by Eq. 14, and by Eq. 17. This accounts for the requisite number of  $n+m+5$  variables.

The significance of the canonical eigenvalue method is that the number of parameters in the eigenvalue equation has been reduced by four as compared to the customary approach. The four parameters that have been removed are  $C_1$ ,  $C_2$ ,  $\gamma_1$ , and  $f$ . We indicate the mathematical features that allow these to be removed. Equation 29 contains three sound speeds. This allows one to treat the canonical eigenvalue equation as independent of  $C_1$  and  $C_2$ . Here,  $S_2$  in the eigenvalue equation determines  $C_3$  in Eq. 48 for any choice of  $C_1$  and  $C_2$ . Similarly,  $\gamma_1$  is independent because it is the only gradient which does not appear in the definition of the  $\rho_i$  of the canonical eigenvalue equation. The parameter  $f$  is removed from the eigenvalue equation by making it satisfy the constraint of Eq. 14 in terms of the removed parameters  $C_1$ ,  $C_2$ , and  $\gamma_1$ , and the independent variable  $x$ .

Consider now the one-layer bounded profile. The four parameters appearing in the customary eigenvalue equation are reduced to zero. In this case, the canonical eigenvalues are independent of all profile parameters. The solutions apply to all profiles of the class, where the class is defined as a one-layer profile with specified free, rigid, or combination boundary conditions.

Consider next the three simple profiles treated in Ref. 1. Two of the three profiles were one-layer surface ducts bounded above by a free or rigid boundary. For these profiles, the three parameters of the customary approach were reduced to zero in the canonical eigenvalue equation. For the unbounded two-layer refractive profile, the four parameters of the customary approach are reduced to one in the canonical approach, i.e., the parameter  $\rho_i$ . These three simple profiles are special cases in which either  $x$  or  $y$  alone but not both appear in the eigenvalue equation. Here, Eq. 14 does not apply as the frequency is independent of  $x$  or  $y$  and the phase velocity for all frequencies may be generated from Eq. 17 or its  $y$  equivalent.

The two-layer bounded duct is the simplest profile which exhibits the phenomenon of double-duct propagation. As discussed in Ref. 1, we obtained numerical examples of this phenomenon but were unable to cope with it analytically because of the large number of parameters in the customary approach. The canonical eigenvalue approach reduces the six parameters ( $C_1$ ,  $C_2$ ,  $C_3$ ,  $\gamma_1$ ,  $\gamma_2$ , and  $f$ ) for the two-layer bounded profile to two, viz.,  $\rho_i$  and  $S_2$ . Our original goal in the process of developing the canonical eigenvalue method was to facilitate the theoretical analysis of the double-duct propagation. It appears now that this goal has been met, but the degree of success remains to be demonstrated.

In the case of many profile layers, the advantage of the canonical formulation in reducing the number of parameters is minimal. For example, in the case of a four-layer profile, the reduction from 10 to 6 parameters is not particularly helpful. However, we believe that the canonical formulation may still have advantages over the customary formulation, even when the solution involves the numerical evaluation of the eigenvalue equation in matrix form. We believe that the solution of the eigenvalue equation in  $x$ ,  $y$  space will be more predictable than the solution in  $C_p$  and  $f$  space.

We will now determine if there are any restrictions on the choice of the set of  $\rho_i$  and  $S_j$  in the canonical eigenvalue equations. We will first do the analysis for the two-layer case.

As will be illustrated in more detail later, the eigenvalue solutions and profile class are divided into Case A as given by Eq. 16 and Case B as given by Eq. 15. It is convenient to treat these separately. Consider first Case A, with  $\rho_1$  positive. Here there are two possibilities. The first is

$$C_1 \leq C_3 < C_2 . \quad (50)$$

From Eq. 29, we determine that when  $C_3 = C_2$ ,  $S_2 = 1$ , and when  $C_3 = C_1$ ,  $S_2 = 0$ . Thus, the allowable values of  $S_2$  under the conditions of Eq. 50 are

$$0 \leq S_2 < 1 . \quad (51)$$

The second possibility is

$$0 \leq C_3 \leq C_1 < C_2 . \quad (52)$$

The counterpart of Eq. 52 is

$$-\infty \leq S_2 \leq 0 . \quad (53)$$

Consider next Case A, with  $\rho_1$  negative. Here the only possibility is

$$C_1 \leq C_2 < C_3 < \infty . \quad (54)$$

The allowable values of  $S_2$  are

$$1 < S_2 < \infty . \quad (55)$$

Consider now Case B. The counterparts of Eq. 50, 52, and 54 are, respectively

$$C_2 < C_3 \leq C_1 \quad (56)$$

$$C_2 < C_1 \leq C_3 < \infty \quad (57)$$

and

$$0 \leq C_3 \leq C_2 < C_1 . \quad (58)$$

However, the restrictions on  $S_2$  are the same as for Case A; i.e., Eq. 51, 53, and 55, respectively, apply here.

We see then that the only restriction on  $\rho_1$  and  $S_2$  is that  $S_2$  must be greater than 1 for  $\rho_1 < 0$  and less than 1 for  $\rho_1 > 0$ . The analysis is similar for  $\rho_i$  and  $S_{i+1}$ . Here, the restriction is that  $S_{i+1}$  must be greater than 1 for  $\rho_i < 0$  and less than 1 for  $\rho_i > 0$ . These represent very mild restrictions. The case of  $\rho_i$  negative is not of great theoretical interest since it merely changes the degree of slope. The case of  $\rho_i$  positive reverses the sign of the slope and sets up a double-duct configuration. Values of  $S_{i+1}$  between 1 and -1 will be of most interest since values of  $S_{i+1} < -1$  may correspond to unrealistically low values of sound speed.

For any values of  $S_{i+1}$ , both Case A and Case B solutions are possible. The canonical eigenvalue curves are divided into two sets—those for Case A and Case B profiles. This works fine for bounded profiles. However, there is a problem when the deepest profile layer is unbounded. As discussed in Ref. 1, the canonical eigenvalue approach was first used on a two-layer refractive duct bounded above by a free surface, with the second layer unbounded. The canonical eigenvalue curves were valid for

Case B, but not for Case A, where the half space has a negative rather than positive gradient. The problem is that the solution for a positive-gradient half space involves the Airy function  $Ai$  whereas that of a negative gradient half space involves the modified Hankel function  $h_2$ . Thus, for unbounded ducts, the canonical eigenvalue equation must take two separate forms—one for Case A and one for Case B.

For an unbounded duct with a negative gradient, the phase velocity has a complex component which results in attenuated modes associated with leakage out of the channel. In such a case, the eigenvalues  $x$  and  $y$  will be complex. We will now consider complex eigenvalues to see what particular problems are involved. From Eq. 7 and 8 and the fact that  $C_p$  is the only complex term in the right side, it follows that

$$Im \zeta_{i1} = Im \zeta_{i0} . \quad (59)$$

In particular, it follows that

$$Im x = Im y . \quad (60)$$

Equation 60 also results in a real frequency in Eq. 14.

We have not proved that Eq. 60 holds, but we have inferred it from the relationship between the physical and mathematical variables. There must be something about the mathematical form of the canonical eigenvalue equation that leads to Eq. 60. This analysis is beyond the scope of this report and is best left until specific examples of complex eigenvalues are considered. We can demonstrate that Eq. 59 holds if we assume that Eq. 60 holds. We assume that Eq. 59 holds for  $i$  and need to demonstrate that it holds for  $i+1$ . From Eq. 18 and Eq. 26, it follows that

$$Im \zeta_{i+1,0} = (M_{i+1} + N_{i+1}) Im \zeta_{i0} = \rho_i^2 Im \zeta_{i0} . \quad (61)$$

From Eq. 10, it follows that

$$Im \zeta_{i+1,1} = \rho_i^2 Im \rho_{i0} . \quad (62)$$

and we have demonstrated that Eq. 59 holds for  $i+1$ . In passing, it is of interest to note that the imaginary component of the Airy function arguments only involves  $Im x$  and the set of  $\rho_i^2$ .

Other than the derivation of Eq. 60, we see no particular obstacles to the application of the canonical approach for complex eigenvalues. The eigenvalue curves will consist of plots of  $Re y$  vs  $Re x$  and  $Im y$  vs  $Re x$ . These should suffice when  $Im y$  is small relative to  $Re y$ , which is the usual case.

We have just discussed a case where the profile parameters are real but the eigenvalues of both the customary and canonical approach are complex. We now point out an application where the profile parameters are complex. As discussed in Ref. 3, one of the techniques for modeling bottom loss is to introduce a sedimentary bottom layer with complex values of  $C_i$  and  $\gamma_i$  in Eq. 1. Here, complex parameters are introduced into the customary eigenvalue equation, with the result that  $C_p$  has an additional complex component due to attenuation in the bottom sediment.

The canonical approach can simplify this analysis if we define  $x$  and  $y$  as associated with the sedimentary bottom layer. We thus eliminate the effect of the complex parameters on the canonical eigenvalue equation. The effect of the complex parameters is contained in the equivalent of Eq. 17 for the sedimentary layer. One can then make a parametric study of how complex  $C_i$  and  $\gamma_i$  for the sedimentary layer affect the model attenuation for the profile class. In the case of a bounded profile,  $x$  in Eq. 17 can be real. In the case of an unbounded profile,  $x$  will have an imaginary argument that also

contributes to the attenuation. In the canonical approach, the attenuation associated with complex  $x$  is clearly separated from that caused by complex profile parameters. By contrast, in the customary eigenvalue approach, the two attenuation factors are scrambled together, yielding only the combined complex component of  $C_p$ .

There are several details to be worked out before this application can be implemented. We would like the values of  $\rho_i$  and  $S_i$  to be real, although it is not entirely clear that this can be done when some of the profile parameters are complex. We also need to treat the case of a sound-speed discontinuity at the water-sediment interface. This will introduce some complication, which needs to be solved. Finally, the presence of complex parameters in Eq. 14 will result in an imaginary component in the frequency which needs to be interpreted.

## GROUP VELOCITY AND EIGENFUNCTIONS

In this subsection, we investigate group velocity and eigenfunctions to determine how they can be formulated in the canonical approach.

Consider first the group velocity. Equation 33 of Ref. 1 gives an expression for the group velocity in terms of the phase velocity as

$$C_g = C_p [1 - f(dC_p/df)/C_p]^{-1}. \quad (63)$$

We now develop an expression for  $C_g$ , using the canonical approach. Our first step is to square Eq. 17, transpose the bracketed term to the left side, differentiate with respect to  $f$ , and simplify to obtain

$$\begin{aligned} 2(dC_p/df) C_1^2 C_p^{-1} + 2C_p^2 f^{-1} [1 - C_1/C_p]^2 3^{-1} \\ - C_p^2 f^{-2/3} \pi^{-2/3} |\gamma_1|^{2/3} (dx/df) = 0. \end{aligned} \quad (64)$$

It follows that

$$\begin{aligned} -f(dC_p/df) C_p^{-1} = -(C_p/C_1)^2 [1 - (C_1/C_p)^2] 3^{-1} \\ + (C_p/C_1)^2 2^{-1} f^{1/3} \pi^{-2/3} |\gamma_1|^{2/3} (dx/df). \end{aligned} \quad (65)$$

We next take the 2/3 power of Eq. 14 and differentiate with respect to  $f$  to obtain

$$2f^{-1/3} 3^{-1} = -L^{-1} (dx/df) (C_1^2 C_2^2 - 1)^{-1} |\gamma_1|^{2/3} \pi^{-2/3} \quad (66)$$

where

$$L = (1 - dy/dx)^{-1}. \quad (67)$$

We solve Eq. 66 for  $dx/df$ , substitute into Eq. 65, and simplify to obtain

$$-f(dC_p/df) C_p^{-1} = -(C_p/C_1)^2 [1 - (C_1/C_p)^2] 3^{-1} + (C_p/C_1)^2 3^{-1} L (C_1^2 C_2^2 - 1) \quad (68)$$

Equation 63 may now be written as

$$C_g = C_p [1 + (C_p^2 - C_1^2)/3C_1^2 + C_p^2 (C_2^2 - C_1^2) L/3]^{-1}. \quad (69)$$

The application of Eq. 69 starts with the evaluation of  $L$ , which we refer to as the canonical group velocity factor. When the eigenvalue curves are generated, we also generate plots of  $L$  vs  $x$ .

Equation 14 then gives the frequency for  $x$  and the desired profile parameters. Equation 17 generates the phase velocity and we can then evaluate Eq. 69, plotting it as a function of frequency.

One cannot appreciate the elegance of Eq. 69 unless one has evaluated group velocity for the conventional approach. As discussed in Ref. 4, this can be done by involved matrix methods. For profiles with many layers,  $dy/dx$  will need to be done by a similar matrix method. However, the formulation should be somewhat easier to implement than the conventional approach. In the case of the simple ducts of Ref. 1, where  $y$  or  $x$  is missing, the last term of Eq. 69 is suppressed. Reference 1 demonstrates that Eq. 69 is the same result that we would get from the ray theory result

$$C_g = \bar{R} / \bar{T} \quad (70)$$

where  $\bar{R}$  and  $\bar{T}$  are the cycle range and travel time, respectively.

In the case of eigenfunctions, the same general procedures apply here as in the conventional approach. Here, the depth dependence of Eq. 3 may be written as

$$\zeta_i(Z) = \zeta_{i1} + a_i(Z - Z_i) . \quad (71)$$

Not very much simplification here. The  $\zeta_{i1}$  depends on the profile class. However,  $a_i$  depends on the particular profile.

It is now convenient to introduce some new notation. We define the following:

$$F_i(-\zeta_{i1}) = D_i A_i(-\zeta_{i1}) + E_i B_i(-\zeta_{i1}) , \quad (72)$$

$$F_i(-\zeta_{i0}) = D_i A_i(-\zeta_{i0}) + E_i B_i(-\zeta_{i0}) , \quad (73)$$

$$F_i' = dF/d\zeta , \quad (74)$$

$$F_i'(-\zeta_{i1}) = -D_i A_i'(-\zeta_{i1}) - E_i B_i'(-\zeta_{i1}) , \quad (75)$$

and

$$F_i'(-\zeta_{i0}) = -D_i A_i'(-\zeta_{i0}) - E_i B_i'(-\zeta_{i0}) . \quad (76)$$

We note that the continuity conditions at the interface result in

$$F_{i+1}(-\zeta_{i+1,1}) = F_i(-\zeta_{i0}) \quad (77)$$

and

$$F'_{i+1}(-\zeta_{i+1,1}) = -\rho_i F_i'(-\zeta_{i0}) . \quad (78)$$

Our next step is to evaluate the coefficients  $D_i$  and  $E_i$  of Eq. 2. Let us assume that we know  $D_i$  and  $E_i$  and wish to evaluate  $D_{i+1}$  and  $E_{i+1}$ . Equations 77 and 78 represent a pair of linear equations in  $D_{i+1}$  and  $E_{i+1}$  in which the right side is known. The solution is

$$D_{i+1} = \pi [F_i(-\zeta_{i0}) B_i'(-\rho_i^2 \zeta_{i0}) - \rho_i F_i'(-\zeta_{i0}) B_i(-\rho_i^2 \zeta_{i0})] \quad (79)$$

and

$$E_{i+1} = -\pi [F_i(-\zeta_{i0}) A_i'(-\rho_i^2 \zeta_{i0}) - \rho_i F_i'(-\zeta_{i0}) A_i(-\rho_i^2 \zeta_{i0})] . \quad (80)$$

The  $\pi$  factor in Eq. 79 and 80 is the reciprocal of the Wronskian of Airy's differential equation.

We start the process at the ocean surface. For a free surface

$$D_1 = Bi(-\zeta_{11}) \text{ and } E_1 = -Ai(-\zeta_{11}) \quad (81)$$

satisfies the boundary condition. The counterpart of Eq. 81 for a rigid surface is

$$D_1 = Bi'(-\zeta_{11}) \text{ and } E_1 = -Ai'(-\zeta_{11}) . \quad (82)$$

We initially use Eq. 81 or 82 to evaluate  $F_1(-\zeta_{10})$  and  $F_1'(-\zeta_{10})$ . We then use Eq. 79 and 80 to evaluate  $D_2$  and  $E_2$ . We then use these to evaluate  $F_2(-\zeta_{20})$  and  $F_2'(-\zeta_{20})$  and use Eq. 79 and 80 to evaluate  $D_3$  and  $E_3$ . This process is repeated until  $D_i$  and  $E_i$  are evaluated for the last profile layer.

In passing, we note that the above procedure satisfies all boundary and interface conditions except the last one. In the case of a boundary at the bottom of layer  $I$ , this procedure does not satisfy the bottom boundary condition, because  $D_I$  and  $E_I$  are already determined. Similarly, in the case of an unbounded layer with positive gradient at the last interface, one condition will not be satisfied. What happens is that the eigenvalue equation itself guarantees the satisfaction of this condition.

The procedure outlined in Eq. 77 to 82 is essentially the same as for the customary approach. There is one interesting feature. The evaluation only involves quantities which are available from the canonical eigenvalue solution. What this means is that the coefficients  $D_i$  and  $E_i$  can be specified for the entire profile class.

We now turn to the problem of normalization. The eigenvalue equation ensures a solution to the homogeneous equations which satisfies the boundary and interface conditions. However, this solution is not unique, since any constant times the solution is also a solution. A unique multiplicative constant is determined by a normalization procedure. We define the normalization factor  $D_n$  as

$$D_n = \sum_{i=1}^I \int_{Z_1}^{Z_{i+1}} F_i^2(Z) dZ . \quad (83)$$

The normalized-mode depth functions are given by

$$U_n(Z_0) U_n(Z) = F_n(Z_0) F_n(Z) / D_n . \quad (84)$$

Here  $n$  is the mode number,  $Z_0$  is the source depth, and  $Z$  is the receiver depth. The  $F_n(Z_0)$  or  $F_n(Z)$  of Eq. 84 is the  $F_i$  of Eq. 2 for the layer in which the source or receiver is located, as evaluated for mode  $n$ .

We start the solution Eq. 83 by noting that Eq. 2 is a solution of Airy's differential equation as given by

$$d^2F/d\zeta^2 - (-\zeta)F = 0 . \quad (85)$$

We next note that

$$F^2(-\zeta) = d[\zeta F^2 + (dF/d\zeta)^2] / d\zeta . \quad (86)$$



We obtain  $F^2$  if we carry out the derivative in the right side of Eq. 86 and make use of Eq. 85. It follows then that

$$\int_{z_i}^{z_{i+1}} F_i^2(Z) dZ = a_i^{-1} [\zeta F^2 + (dF_i/d\zeta)^2] \Big|_{z_i}^{z_{i+1}} \quad (87)$$

Equation 87 makes use of the relationship obtained from Eq. 3 of

$$d\zeta/dZ = a_i \quad (88)$$

Our goal here is not only to solve for  $D_n$  but to cast it into the form

$$D_n = a_1^{-1} D_c \quad (89)$$

where  $a_i$  is given by Eq. 4 and  $D_c$  depends only on quantities, which are available from the canonical eigenvalue equation. We will refer to  $D_c$  as the canonical normalization coefficient. The advantages of this particular breakdown will soon be apparent.

For purposes of analysis, it is convenient to express  $D_c$  as

$$D_c = D_s + D_b + \sum_{i=1}^{I-1} D_{ni} \quad (90)$$

The quantities  $D_s$  and  $D_b$  represent the contributions at the surface and bottom boundaries. The quantity  $D_{ni}$  represents the contribution to  $D_b$  at interface  $i$ . This contribution is summed over all interfaces, which total  $I-1$  for a bounded profile with  $I$  layers.

From Eq. 11, it follows that

$$a_{i+1}^{-1} = -\rho_i a_i^{-1} \quad (91)$$

Hence

$$a_i^{-1} = D_{pi} a_1^{-1} \quad (92)$$

where

$$D_{pi} = \prod_{j=1}^{i-1} (-\rho_j) ; D_{p1} = 1 \quad (93)$$

Equation 92 follows from the repeated use of Eq. 91 and expresses  $a_i^{-1}$  as  $a_1^{-1}$  times the  $D_{pi}$  of Eq. 93, which is a product function of the negative of  $\rho_j$  for all  $j$  less than  $i$ . From Eq. 87, we see that  $a_i^{-1}$  is a factor associated with the layer  $i$  contribution to  $D_n$ . Equation 92 allows us to express  $D_n$  in the form of Eq. 89.

We now evaluate the contribution to  $D_c$  at the surface boundary. In the case of a free surface,  $F_1 = 0$  and the lower limit of Eq. 87 leads to

$$D_s = -\pi^{-2} \quad (94)$$

where  $\pi^{-1}$  is the Wronskian for Airy's differential equation. In the case of a rigid surface,  $F_1' = 0$ , and the lower limit of Eq. 87 can be simplified to

$$D_s = -\xi_{11}\pi^{-2}. \quad (95)$$

We next proceed to evaluate the contribution at interface  $i+1$ . Equation 87 is by layers, not interfaces. At interface  $i+1$ , we have the upper limit contribution of Eq. 87 for layer  $i$  and the lower limit contribution of Eq. 87 for layer  $i+1$ . We make use of Eq. 77 and 78 to express the layer  $i+1$  contribution in terms of the layer  $i$  contribution. The application of Eq. 87 then leads to

$$a_i^{-1} D_{ni} = (a_i^{-1}\xi_{i0} - a_{i+1}^{-1}\xi_{i+1,1})F_i^2(-\xi_{i0}) + (a_i^{-1} - a_{i+1}^{-1}\rho_i^2)F_i'^2(-\xi_{i0}). \quad (96)$$

Equation 10 and 11 allow a further simplification to

$$D_{ni} = D_{pi} (1 + \rho_i^3) [\xi_{i0}F_i^2(-\xi_{i0}) + F_i'^2(-\xi_{i0})]. \quad (97)$$

In the case of an unbounded medium, the contributions to Eq. 87 at infinity are taken as zero. Thus, there is nothing more to add and  $D_b$  of Eq. 90 is zero. In the case of a medium with lower bound, we must evaluate the contribution at this boundary. In the case of a free boundary at the bottom of layer  $I$ , we obtain

$$D_b = D_{pI} F_I'^2(\xi_{I0}). \quad (98)$$

In the case of a rigid bottom, we obtain

$$D_b = D_{pI} \xi_{I0} F_I^2(-\xi_{I0}). \quad (99)$$

All components in Eq. 94, 95, and 97 to 99 are expressed in terms of quantities that are available from the canonical eigenvalues and canonical profile parameters. The particular profile parameters  $\gamma_1$  and  $C_1$  and the frequency enter into  $D_n$  through the simple multiplicative factor  $a_i^{-1}$  in Eq. 89. The bulk of the normalization evaluation,  $D_c$ , can be evaluated for the canonical profile class.

## SECTION 2. OUTLINE OF EVALUATION BY THE CANONICAL APPROACH

This section outlines the general steps necessary for the evaluation of normal-mode fields by the canonical approach. It also serves as a summary of the method.

We start with a bounded sound-speed profile class of  $I$  layers. The class is characterized by  $(I-1)$  values of  $\rho_i$  and  $S_i$  and by a given pair of boundary conditions. The  $I \times I$  eigenvalue matrix of the customary approach is formed by using the boundary and interface conditions. The arguments of the Airy functions in layer  $i$  are given generically by  $-\zeta_{i1}$  at the upper interface and  $-\zeta_{i0}$  at the lower interface.

The next step is to use Eq. 32 and 33 to replace these arguments by linear functions of  $x$  and  $y$ . The coefficients of these linear functions are functions  $\rho_i$  and  $S_i$  and are generated by recursion with Eq. 18. Equations 34 and 39 give these coefficients for the first three layers. At the end of this process, we have the canonical eigenvalue equation, which we may describe generically as

$$G(x, y, \rho_i, S_i) = 0. \quad (100)$$

The solution of this equation gives the canonical eigenvalues,  $y$  as a function of  $x$ . These eigenvalues apply to the canonical class of the sound-speed profile defined by the set of  $\rho_i$  and  $S_i$  and the given boundary conditions.

The next stage in the process is to generate the eigenvalue curves,  $y$  as a function of  $x$ , for the given set of  $\rho_i$  and  $S_i$ . This is accomplished by iteration using Newton's method where  $\partial G/\partial x$  or  $\partial G/\partial y$  are the required derivatives. These partial derivatives are also used to obtain  $dy/dx$  so that the canonical group velocity factor,  $L$  of Eq. 67, can be evaluated for later use. The canonical frequency function,  $(y-x)^{3/2}$ , and the canonical normalization coefficient,  $D_c$  of Eq. 90, are also evaluated for later use.

All the quantities generated thus far apply to the canonical profile class. To proceed further, we must select the parameters  $C_1$ ,  $C_2$ , and  $\gamma_1$  of the first layer of the particular profile we wish to evaluate. These parameters plus the set of  $\rho_i$  and  $S_i$  completely define the sound-speed profile.

The next step is to use Eq. 14, the canonical frequency function,  $C_1$ ,  $C_2$ , and  $\gamma_1$  to determine the frequency,  $f$ , as a function of  $x$ . The next step is to use Eq. 17,  $x$ ,  $f$ ,  $C_1$ , and  $\gamma_1$  to determine the phase velocity,  $C_p$ , as a function of  $f$ . The next step is to use Eq. 69,  $L$ ,  $C_p$ ,  $C_1$ , and  $C_2$  to determine the group velocity,  $C_g$ , as a function of  $f$ . The final step is to use Eq. 89,  $D_c$ ,  $f$ ,  $C_1$ , and  $\gamma_1$  to determine the eigenfunction normalization coefficient,  $D_n$ , as a function of frequency. The latter step does not depend on the phase or group velocity and could be carried out, if so desired, before  $C_p$  and  $C_g$  are evaluated.

The depth functions of Eq. 84 may now be evaluated as a function of depth to give the standing waves. The range dependence is given by  $H_0^2(\omega r/C_p)$ , where  $H_0^2$  is a Hankel function of the second kind. The acoustic field is then evaluated by multiplying the depth functions by the range dependence and summing over the modes.

Reference 2 applies the various stages of this approach to the class of one-layer bounded profiles. Three pairs of boundary conditions are treated—a free surface and rigid bottom, a free surface and free bottom, and a rigid surface and rigid bottom. Numerical results are presented to illustrate the desired outputs and other key elements of the evaluation.

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