Reflection and scattering of sound by correlated rough surfaces[a)]

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Pair correlations and the effects of dense packing are included to extend existing results on reflection and scattering by distributions of protuberances (bosses) on rigid or free base planes. The earlier energy conserving forms for the specularly reflected wave, the surface impedance, and the differential scattering cross section per unit area, are obtained in terms of a transform of the scattering amplitude of an isolated boss. Low-frequency approximations are developed with emphasis on the roles of the packing density and multipole coupling effects, and explicit results are given for semi-elliptic cylinders and hemi-ellipsoids. For lossless bosses, the reflection coefficient has a minimum, and the incoherent scattering a maximum, at the packing density corresponding to maximum fluctuations in the number of bosses per unit area. Multipole coupling effects may be misinterpreted in data inversion programs as changes in boss shape; if such effects are not included, then, e.g., hemispheres may be mistaken for hemiellipsoids broadened along the base plane and shortened along the normal.

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INTRODUCTION

Various physical aspects of the coherent reflection and incoherent scattering of sound by rough surfaces are exhibited by random distributions of protuberances on rigid or free base planes. For sparse uncorrelated distributions we obtained[1] energy-conserving approximations, general enough to account for the dependence on angle for near-grazing incidence and for pseudo-Brewster effects, and simple enough to analyze special shapes[1,2] and to reduce data in experimental contexts.[3]

The representations[1] of the reflection coefficient (R) and the differential scattering cross section (Q) in terms of an impedance (Z) proportional to the scattering amplitude (f) of one protuberance (boss) in isolation and to the number (ρ) of bosses per unit area, are restricted to sparse packings. At dense packings and low frequencies, i.e., for negligible incoherent scattering, we supplemented R with results for periodic distributions[4,5] for the case of one propagating mode (separation of bosses small compared to wavelength λ). To investigate domains of validity for the sparse and periodic cases for parallel cylindrical protuberances (analogs of the statistical mechanics one-dimensional cases of the sparse gas and crystal), we included the effects of pair correlations.[6] Using the Zernike–Prins[7] one-dimensional liquid state pair function $\rho p(x)$ for minimum separation b and average separation $\bar{b} = \rho^{-1}$ of centers, plus a rapidly converging residue series for $b \ll \bar{b}$ and $b \sim \bar{b}$, led to forms in terms of $\rho b = w$ for the full range $0 < w < 1$, from sparse gas on to the deterministic periodic limit. The present paper extends the earlier work on striated surfaces[6] to facilitate data inversion purposes, and provides analogous results for rough surfaces of arbitrary bounded bosses in terms of a pair function $\rho p(\mathbf{R})$ with $\mathbf{R}$ as the separation of centers of pairs. The development is based on recent work on correlated monolayers.[8]

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We obtain the original form[1] for $R(Z)$ in terms of a more complete impedance proportional to a transform of f that includes packing effects and correlations, plus a corresponding energy-conserving approximation for Q. We construct explicit low-frequency results (with semi-ellipses, and hemiellipsoids as illustrations) and emphasize the roles of packing density, and multipole coupling. It is shown, for example, that for lossless bosses, the specularly reflected energy has a minimum, and the incoherent scattering a maximum, at the packing density corresponding to maximum fluctuations in the number of bosses per unit area. Another application relates to the inversion of reflection data to determine boss shape: e.g., because of multipole coupling effects a hemisphere corresponds to an equivalent hemi-ellipsoid (broadened along the base plane and shortened along the normal), and such effects need to be included to the data reduction procedure.

For hemispheres with symmetry axis perpendicular to the plane, we take the packing density (w) either as the fraction of plane covered by the boss bases (of radius a), or more generally as the fractional area covered by cocentered larger acoustically transparent disks (of diameter $b > 2a$). The b disks correspond to impenetrable statistical mechanica particles[9–12] with b equal to the minimum separation of centers; equivalently, b is the radius of the exclusion region containing the center of only one base of radius a. The additional parameter b for the minimum separation, or the corresponding packing density $w = \rho\pi b^2/4$, enables us to model a broader class of surfaces. The size of w, from near zero on to $w_d \approx 0.84$ (an experimented value for the densest packing of identical circular disks) determines the correlation effects. More generally, the base of the boss and of the exclusion disk are not taken as circular; their shapes and orientation in the plane are specified by cocentered curves $a(\hat{\mathbf{R}})$ and $b(\hat{\mathbf{R}})$ that are neither similar nor similarly aligned. As before,[8] we assume that $\mathbf{b}(\hat{\mathbf{R}})$ fully determines $p(\hat{\mathbf{R}})$ and that both have the same reflection and inversion symmetries as an ellipse.

In the following we use expressions given in Refs. 1 and 8 and refer to earlier equations by (1:38), etc. For brevity, the recent work on monolayers[8] is cited for many detailed results that are required. We work with forms for the average multiple scattering amplitude F for one boss in terms of the isolated value f and in terms of transforms of f. (We could construct these forms by superposing results for the symmetrical monolayer, but we include some additional relations $F[f]$ to facilitate perturbation extensions to more general base planes.) For detailed applications, we consider series decompositions of F and f in terms of the same scattering coefficients A_n and a_n, and the distribution integrals $\mathscr{H}_n$ (continuum analogs of lattice sums) derived for the symmetrical monolayer.[8]

I. COHERENT REFLECTION

As before,[8] we use

$$\mathbf{r} = r\hat{\mathbf{r}}, \quad r^2 = z^2 + R^2, \quad \hat{\mathbf{r}} = \hat{\mathbf{z}}\cos\theta + \hat{\mathbf{R}}(\varphi)\sin\theta,$$
$$\hat{\mathbf{R}}(\varphi) = \hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi, \tag{1}$$

as well as the direction cosines $\hat{\mathbf{r}} = \hat{\mathbf{z}}\gamma + \hat{\mathbf{x}}\alpha + \hat{\mathbf{y}}\beta = \Sigma_i \hat{\mathbf{z}}_i \gamma_i$. Suppressing $e^{-i\omega t}$, we work with a pair of plane waves, imaged in the base plane $z = 0$,

$$\phi = e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{k} = k\hat{\mathbf{k}}, \quad k = 2\pi/\lambda,$$
$$\hat{\mathbf{k}} = \hat{\mathbf{r}}(\theta_0, \varphi_0) = \hat{\mathbf{r}}_0,$$
$$\phi' = e^{i\mathbf{k}'\cdot\mathbf{r}}, \quad \mathbf{k}' = k\hat{\mathbf{k}}',$$
$$\hat{\mathbf{k}}' = \hat{\mathbf{r}}(\pi - \theta_0, \varphi_0) = \hat{\mathbf{r}}_0' = \hat{\mathbf{r}}_{0'}. \tag{2}$$

We write $\hat{\mathbf{k}}' = \hat{\mathbf{k}} - 2\hat{\mathbf{k}}\cdot\hat{\mathbf{z}}\hat{\mathbf{z}} = \hat{\mathbf{k}} - 2\gamma_0\hat{\mathbf{z}}$ with $\gamma_0 = \cos\theta_0 = |\cos\theta_0|$. For cylinders with generators along $\hat{\mathbf{y}}$, we set $\varphi = \varphi_0, \beta = \gamma_3 = 0$, and $\hat{\mathbf{R}} = \hat{\mathbf{x}}$. In general we use three-dimensional terminology; if two forms of a factor arise, we list the cylindrical first.

For ϕ incident on an arbitrary obstacle at the phase origin, we write[1,8] the scattering amplitude as $g(\hat{\mathbf{r}}, \hat{\mathbf{k}})$. If the obstacle is symmetrical to reflection in $z = 0$, then $g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = g(\hat{\mathbf{r}}', \hat{\mathbf{k}}')$ with $\hat{\mathbf{r}}' = \hat{\mathbf{r}}(\pi - \theta, \varphi)$. The scattering amplitude for the corresponding boss on a rigid (+) or free (−) base plane excited by $\pm\phi'$ follows by superposition,

$$f_\pm(\hat{\mathbf{r}}, \hat{\mathbf{k}}') = g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) \pm g(\hat{\mathbf{r}}, \hat{\mathbf{k}}')$$
$$= g(\hat{\mathbf{r}}', \hat{\mathbf{k}}') \pm g(\hat{\mathbf{r}}', \hat{\mathbf{k}}) = f_\pm(\hat{\mathbf{r}}', \hat{\mathbf{k}}). \tag{3}$$

The boss amplitudes $f_\pm$ are obtained directly from known results for g, as discussed originally by Rayleigh[13] for fine semicircular cylinders. Equivalently, f_+ or f_- is twice the component of g symmetrical or antisymmetrical to reflection in $z = 0$. We normalize $g(\hat{\mathbf{r}}, \hat{\mathbf{k}})$ as before,[1,8] so that for lossless cases, $-\operatorname{Re} g(\hat{\mathbf{k}}, \hat{\mathbf{k}})$ equals the mean value of $|g(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2$ over all directions of observation r.

We restrict consideration to the half-space $z > 0$ (henceforth the back space). The choice $\pm\phi'$ for the incident wave for the corresponding rigid and free base planes allows for a single representation of the scattered and reflected fields, and for the general suppression of subscripts $\pm$. Thus the specular scattering theorem[14] for either case has the form

$$-\sigma_0 \operatorname{Re} f(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \sigma_a + \sigma_s, \quad \sigma_s = \sigma_0 \tfrac{1}{2}\mathscr{M}|f(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2;$$
$$\sigma_0 = \frac{4}{k}, \ \frac{4\pi}{k^2};$$
$$\mathscr{M} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta, \ \ \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \int_0^{\pi/2} d\theta \sin\theta, \tag{4}$$

where σ_a and σ_s are the absorption and scattering cross sections; $\mathscr{M}|f|^2$, the mean value over the back space of real directions ($0 \leqslant \theta \leqslant \pi/2$), is also (by symmetry) the mean overall directions. See Ref. 1 for additional theorems and discussion of $f_\pm$ for arbitrary bosses, and for explicit low- and high-frequency approximations for semicircular cylinders and hemispheres; results for elliptic semicylinders are given in Ref. 2, and for hemi-ellipsoids in Ref. 5 (p. 661, ff).

Similarly for ϕ incident on a configuration of obstacles ($s = 1,2,...N$) with centers at $\mathbf{R}_s$ on $z = 0$, we specify the response as in (8:7) in terms of the multiple scattering amplitudes G_s determined functionally by the isolated values g_s as in $G[g]$ of (8:8). For the corresponding boss problem we work with $F[f]$, such that in terms of G's for a symmetrical monolayer, $F_\pm(\hat{\mathbf{r}}, \hat{\mathbf{k}}') = G(\hat{\mathbf{r}}, \hat{\mathbf{k}}) \pm G(\hat{\mathbf{r}}, \hat{\mathbf{k}}') = G(\hat{\mathbf{r}}', \hat{\mathbf{k}}') \pm G(\hat{\mathbf{r}}', \hat{\mathbf{k}})$ equals twice the symmetry components of G. Thus, with $e^{i\mathbf{k}\cdot\mathbf{R}_s}F_s$ as the amplitude for the boss at $\mathbf{R}_s$, we write the solution (excess pressure) for the configuration of bosses as

$$\Psi_\pm = \pm\phi' + \phi + \mathscr{U}_\pm = \pm\phi' + \Psi_R^\pm,$$
$$\mathscr{U}(\mathbf{r}) = \sum_s e^{i\mathbf{k}\cdot\mathbf{R}_s}\int_c e^{i\mathbf{k}_c\cdot(\mathbf{r} - \mathbf{R}_s)}F_s(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}'),$$
$$\mathbf{k}_c = k\hat{\mathbf{r}}_c; \quad \hat{\mathbf{r}}_c = \hat{\mathbf{r}}(\theta_c, \varphi_c), \tag{6}$$

where $\int_c$ equals $(1/\pi)\int d\theta_c$ with contour as for $H_0^{(1)}$, or $(1/2\pi)\int d\Omega(\theta_c, \varphi_c)$ with contours as for $h_0^{(1)}$. Similarly,

$$F_t(\hat{\mathbf{r}}, \hat{\mathbf{k}}') = f_t(\hat{\mathbf{r}}, \hat{\mathbf{k}}')$$
$$+ \frac{1}{2}\Sigma_s' \int_c f_t(\hat{\mathbf{r}}', \hat{\mathbf{r}}_c)F_s(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}')e^{i(\mathbf{k}_c - \mathbf{k})\cdot\mathbf{R}_{ts}},$$
$$\mathbf{R}_{ts} = \mathbf{R}_t - \mathbf{R}_s, \tag{7}$$

where Σ_s' is the sum over $s \neq t$.

For a homogeneous ensemble of configurations of identical aligned bosses, we proceed as for (8:9) to obtain the corresponding ensemble average of (6),

$$\langle\mathscr{U}\rangle = \rho\int_c e^{i\mathbf{k}_c\cdot\mathbf{r}}\langle F_s(\hat{\mathbf{r}}_c, \hat{\mathbf{k}})\rangle_s \int d\,\mathbf{R}e^{i(\mathbf{k} - \mathbf{k}_c)\cdot\mathbf{R}}$$
$$= \phi\, 2CF(\hat{\mathbf{k}}', \hat{\mathbf{k}});$$
$$C = \rho/k\gamma_0, \quad \rho\pi/k^2\gamma_0, \tag{8}$$

where $F = \langle F_s\rangle_s$ is the ensemble averaged amplitude for any fixed boss. Thus

$$\langle\Psi\rangle = \pm\phi' + \phi + \langle\mathscr{U}_\pm\rangle = \pm\phi' + \langle\Psi_R^\pm\rangle, \tag{9}$$

with the coherent reflected wave as in (1:37),

$$\langle\Psi_R\rangle = \phi(1 + 2CF) = \phi\frac{1+Z}{1-Z}, \quad Z = \frac{CF}{1+CF},$$
$$F = F(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = F(\hat{\mathbf{k}}', \hat{\mathbf{k}}), \tag{10}$$

in terms of the nomalized impedance Z. The same form also arises for periodic surfaces with ρ^{-1} as the grating spacing for parallel cylinders[4] or as the area of unit cell for the rectangular lattice[5] and spacings small compared to λ (one propagating mode).

To obtain $F = \langle F_t \rangle_t$, we consider the ensemble average of (7) in terms of the pair-distribution function $\rho p(\mathbf{R}_{ts})$,

$$\langle F_t(\hat{\mathbf{r}}, \hat{\mathbf{k}}')\rangle_t = f(\hat{\mathbf{r}}, \hat{\mathbf{k}}') + \frac{1}{2}\rho \int d\,\mathbf{R}_s\, p(\mathbf{R}_{ts}) \times \int_c f(\hat{\mathbf{r}}', \hat{\mathbf{r}}_c)\langle F_s(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}')\rangle_{st} e^{i(\mathbf{k}_c - \mathbf{k})\cdot\mathbf{R}_{ts}}. \quad (11)$$

Proceeding as for (8:13), we approximate $\langle F_s\rangle_{st}$ by $\langle F_s\rangle_s$, and use briefer notation $F(\hat{\mathbf{r}}, \hat{\mathbf{k}}') = F_{r0}$, $F(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}') = F_{c0}$, etc., to obtain

$$F_{r0} = f_{r0} + \frac{1}{2}\rho \int d\,\mathbf{R}\, p(\mathbf{R}) \int_c f_{rc} F_{c0} e^{i(\mathbf{k}_c - \mathbf{k})\cdot\mathbf{R}}. \quad (12)$$

In terms of the Fourier transform of the pair correlation function

$$P[K] = \rho \int p(\mathbf{R}) e^{i\mathbf{K}\cdot\mathbf{R}} d\,\mathbf{R}, \quad \mathbf{K} = k(\hat{\mathbf{r}} - \hat{\mathbf{k}})\cdot(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) \quad (13)$$

and simple contours

$$\int_c = \frac{1}{\pi}\int_{-\pi/2 + i\infty}^{\pi/2 - i\infty} d\theta_c, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi_c \int_0^{\pi/2 - i\infty} d\theta_c \sin\theta_c,$$

we obtain the form (6:37) developed originally for striated surfaces

$$F_{r0} = f_{r0} + \mathbf{S} f_{rc} F_{c0},$$
$$\mathbf{S} = \frac{1}{2}\lim_{\epsilon \to 0}\int_c e^{-k\epsilon \gamma_c} P[K_c] = \frac{1}{2}\mathscr{M} P[K_c] + \mathbf{S}_e, \quad (14)$$

with $\mathscr{M}$ as in (4). [See (8:13)–(8:16) for discussion of the convergence factor and other aspects.] We showed that $\int_c$ operates on a function of the real variables φ_c and $\sin\theta_c$; because we assume that $p(\mathbf{R})$ has the same inversion and reflection symmetries as an ellipse, $P[K]$ is real, and it follows that $\mathscr{M}P$ and $\mathbf{S}_e$ are real and imaginary operators, respectively.

Introducing the Fourier transform of the total correlation function $p(\mathbf{R}) - 1$,

$$\mathscr{P}[K] = \rho \int [p(\mathbf{R}) - 1] e^{i\mathbf{K}\cdot\mathbf{R}} d\,\mathbf{R}, \quad (15)$$

we make the specular contribution explicit to obtain form (6:41),

$$F_{r0} = f_{r0}(1 + CF_{00}) + \mathbf{S}^{(0)} f_{rc} F_{c0},$$
$$\mathbf{S}^{(0)} = \tfrac{1}{2}\mathscr{M}\mathscr{P}[K_c] + \mathbf{S}_e. \quad (16)$$

If the effects of $\mathbf{S}^{(0)}$ are negligible, then F and the corresponding impedance reduce to the sparse-gas approximations[1] $F \approx F^0$ and $Z \approx Z^0$, with

$$F_{r0}^0 = f_{r0}(1 + CF_{00}^0) = f_{r0}/(1 - Cf),$$
$$Z^0 = Cf(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = Cf, \quad (17)$$

as discussed and applied earlier in detail.[1,2] For the general case (16), we reduce F by using the results of the analogous development of G for the monolayer.[8] As before, we first introduce the energy functions to provide motivation, but in view of the details that are available,[8] the present development is brief.

II. AVERAGE ENERGY FUNCTIONS

We write the average energy density as $\langle|\Psi|^2\rangle = |\langle\Psi\rangle|^2 + V$, with $|\langle\Psi\rangle|^2$ obtained from (9) as the coherent component. The variance $V = \langle|\mathscr{U} - \langle\mathscr{U}\rangle|^2\rangle$, the incoherent or fluctuation component, follows from (6) by proceeding as for (8:19). In particular, for $z \sim \infty$ we write V as an integral over all back space directions ($|\theta| < \pi/2$),

$$V \sim \rho c_1 \int \mathscr{W}[K]\,|F(\hat{\mathbf{r}}, \hat{\mathbf{k}}')|^2 \sec\theta \equiv \rho \int Q_{r0} \sec\theta;$$
$$c_1 = \frac{2}{\pi}k, \ \frac{1}{k^2};$$
$$\mathscr{W}[K] = 1 + \mathscr{P}[K] = 1 + \rho \int [p(\mathbf{R}) - 1] e^{i\mathbf{K}\cdot\mathbf{R}} d\,\mathbf{R}, \quad (18)$$

where $\mathscr{W}[K]$ is the structure factor, and ρQ_{r0} is the multiple scattered differential cross section per unit area.

For the corresponding average energy flux, we have

$$\mathbf{J}_\pm = \mathrm{Re}\langle\Psi_\pm^* \nabla\Psi_\pm / i\mathrm{k}\rangle = \mathbf{J}_\pm^c + \mathbf{I},$$
$$J_\pm^c = \hat{\mathbf{k}}' \pm \mathrm{Re}\,\phi'\langle\Psi_R^\pm\rangle^*(\hat{\mathbf{k}} + \hat{\mathbf{k}}') + |\langle\Psi_R^\pm\rangle|^2\hat{\mathbf{k}},$$
$$\mathbf{I} \sim \rho \int Q_{r0}\hat{\mathbf{r}} \sec\theta. \quad (19)$$

The first term of the coherent component $\mathbf{J}^c$ is the incident flux, the second arises from interference of the incident and reflected waves, and the third is the specularly reflected flux. The net returned flux is

$$\mathbf{J}_r = R\hat{\mathbf{k}} + \mathbf{I}, \quad R = |\langle\Psi_R\rangle|^2 = |1 + 2CF|^2 = \left|\frac{1+Z}{1-Z}\right|^2 \quad (20)$$

corresponding to coherent reflection plus incoherent scattering. The components of (19) normal to the base plane equal

$$\hat{\mathbf{z}}\cdot\mathbf{J}^c = (1 + R)\cos\theta_0, \quad \hat{\mathbf{z}}\cdot\mathbf{I} = \rho\sigma_0\tfrac{1}{2}\mathscr{M}\mathscr{W}|F_{r0}|^2 = \rho\mathscr{S}_s, \quad (21)$$

with $\rho\mathscr{S}_s$ as the average incoherent scattering cross section per unit area. The corresponding average absorption cross section per unit area is the average inward flux $-\hat{\mathbf{z}}\cdot\mathbf{J} = \rho\mathscr{S}_a$. Conservation of energy requires, $\hat{\mathbf{z}}\cdot\mathbf{J} = \hat{\mathbf{z}}\cdot(\mathbf{J}^c + \mathbf{I})$, from which[1]

$$1 = R + \rho\mathscr{S}\sec\theta_0, \quad \mathscr{S} = \mathscr{S}_a + \mathscr{S}_s, \quad (22)$$

i.e., the incident flux density equals the flux coherently reflected plus that absorbed and incoherently scattered by the area of distribution irradiated by unit area of the incident wave. From the development following (8:23), we may represent

$$\mathscr{S}_a = -\int \mathrm{Re}\langle\Psi^*\nabla\Psi/ik\rangle_t \cdot d\mathbf{A}$$

as an integral over the surface of the boss, or as one-half of the result of integrating over the boss plus its image.

Substituting $R = |1 + 2CF_{00}|^2$ in (22), and using $4C\cos\theta_0/\rho = \sigma_0$, we obtain

$$-\operatorname{Re} F_{00} = C|F_{00}|^2 + (\mathscr{S}_a + \mathscr{S}_s)/\sigma_0 = C|F_{00}|^2 + \tfrac{1}{2}\mathscr{M}\mathscr{W}[K]|F_{r0}|^2 + \mathscr{S}_a/\sigma_0, \tag{23}$$

which for lossless scatterers reduces to (6:65). See (8:24)ff for the detailed procedure for the more general problem of the nonsymmetrical monolayer. The essential feature of the procedure for present purposes is that the energy relation (23) corresponds to F of (16), and that the same procedure may be applied to a modified problem shorn of specular losses.

III. SCATTERING AMPLITUDE REDUCTIONS

The procedure (8:24)ff for obtaining the energy relations (23) applied to a modified scattering problem specified by

$$\mathscr{f}_{r0} = f_{r0} + \mathbf{S}^{(0)} f_{rc}\mathscr{f}_{c0}, \quad \mathbf{S}^{(0)} = \tfrac{1}{2}\mathscr{M}\mathscr{P}[K] + \mathbf{S}_e, \tag{24}$$

i.e., by the form (16) without the specular term, yields the corresponding form of (23). Thus for lossless scatterers,

$$-\operatorname{Re}\mathscr{f}_{00} = \tfrac{1}{2}\mathscr{M}\mathscr{W}[K]|\mathscr{f}_{r0}|^2. \tag{25}$$

In terms of $\mathscr{f}$ we reduce (16) and the corresponding impedance $Z = CF/(1+CF)$ to

$$F_{r0} = \mathscr{f}_{r0}(1 + CF_{00}) = \mathscr{f}_{r0}/(1 - C\mathscr{f}), \quad Z = C\mathscr{f} = C\mathscr{f}_{00} = C\mathscr{f}(\hat{\mathbf{k}}, \hat{\mathbf{k}}'), \tag{26}$$

i.e., the same forms as in (17) discussed earlier,[1] but with f replaced by the modified amplitude $\mathscr{f}$.

To analyze $\mathscr{f}$ and display the implicit physics we express f in terms of the amplitude f' for a radiationless boss,[4,5]

$$f_{10} = f'_{10} + \tfrac{1}{2}\mathscr{M} f'_{1r} f_{r0}, \tag{27}$$

such that for the lossless case, $f'_{01} + f'^{*}_{10} = 0$. The leading term of (27) includes phase and absorption effects but no radiative losses. For a lossless isolated boss,

$$-f_{01} - f^{*}_{10} = \mathscr{M} f_{r1} f^{*}_{r0}, \quad -\operatorname{Re} f_{00} = \tfrac{1}{2}\mathscr{M}|f_{r0}|^2, \tag{28}$$

with the special case for $\operatorname{Re} f_{00}$ corresponding to (4) for $\sigma_a = 0$.

Writing $\mathscr{P}[K] = \mathscr{W}[K] - 1$ in $\mathbf{S}^{(0)}$ of (24), we express $\mathscr{f}$ in terms of f' as

$$\mathscr{f}_{r0} = f'_{r0} + [\tfrac{1}{2}\mathscr{M}\mathscr{W} + \mathbf{S}_e] f'_{rc}\mathscr{f}_{c0}. \tag{29}$$

Suppressing $\mathbf{S}_e$ by means of the radiationless distribution amplitude,

$$f''_{r0} = f'_{r0} + \mathbf{S}_e f'_{rc} f''_{c0}, \tag{30}$$

such that for the lossless case, $\operatorname{Re} f''_{00} = 0$, we obtain

$$\mathscr{f}_{10} = f''_{10} + \tfrac{1}{2}\mathscr{M}\mathscr{W}[K] f''_{1r}\mathscr{f}_{r0} \tag{31}$$

as the analog of the isolated-boss form (27).

From $F_{00} = \mathscr{f}/(1 - C\mathscr{f}) = (\mathscr{f} - C|\mathscr{f}|^2)/|1 - C\mathscr{f}|^2$ of (26), we construct $-\operatorname{Re} F_{00} = (-\operatorname{Re}\mathscr{f} + C|\mathscr{f}|^2)/|1 - C\mathscr{f}|^2$. Using (25), we obtain

$$-\operatorname{Re} F_{00} = C|F_{00}|^2 + \tfrac{1}{2}\mathscr{M}\mathscr{W}[K]|F_{r0}|^2 \tag{32}$$

corresponding to (23) for lossless bosses. In $-\operatorname{Re} f_{00} = \tfrac{1}{2}\mathscr{M}|f_{r0}|^2$ of (28), the term $-\operatorname{Re} f_{00}$ represents the energy loss from the incident and image wave via interference with the radiated wave, and $\tfrac{1}{2}\mathscr{M}|f|^2$ shows it is balanced by scattering. In (32), the interference loss is balanced by specular reflection and fluctuation scattering.

The coherent power reflection coefficient equals

$$R = \left|\frac{1+Z}{1-Z}\right|^2 = 1 + \frac{4\operatorname{Re} Z}{|1-Z|^2} = 1 - \frac{\rho(\bar{\sigma}_a + \bar{\sigma}_s)\sec\theta_0}{|1-Z|^2}, \quad Z = C\mathscr{f}_{00}, \quad \bar{\sigma}_s = \sigma_0\tfrac{1}{2}\mathscr{M}\mathscr{W}|\mathscr{f}_{r0}|^2. \tag{33}$$

The corresponding differential scattering cross section per unit area is approximated by

$$\rho Q_{r0} = \rho\bar{q}_{r0}/|1 - Z|^2, \quad \bar{q}_{r0} = \mathscr{W}[K]c_1|\mathscr{f}_{r0}|^2; \quad c_1 = 2/\pi k,\ 1/k^2, \tag{34}$$

such that $\bar{\sigma}_s = \int\bar{q}_{r0}$, with

$$\sigma_s = c_1\int|f_{r0}|^2 = \int q_{r0}$$

for an isolated boss. The phase change (Θ) introduced by the distribution of bosses is given by

$$\tan\Theta = 2\operatorname{Im} Z/(1 - |Z|^2), \tag{35}$$

where Θ_+ and $\pi + \Theta_-$ are the total phase changes on reflection of $\pm\phi'$ from distributions on rigid (+) and free (−) base planes. These forms and $C = \rho/k\gamma_0$ or $\rho\pi/k^2\gamma_0$ are the same as before,[1] and the dependence of $\mathscr{f}_\pm$ on $\gamma_0 = \cos\theta_0$ and $\gamma = \cos\theta$ near grazing are determined by the corresponding behavior of $f_\pm$ discussed for (1:11). Thus $f_-(\hat{\mathbf{r}}, \hat{\mathbf{k}}')$ vanishes and f_+ does not, as either $\hat{\mathbf{r}}$ or $\hat{\mathbf{k}}'$ approaches grazing; consequently, $Z_+ \to \infty$ and $Z_- \to 0$, so that $\langle\Psi^{\pm}_R\rangle/(\pm\phi') \to -1$, and $Q \to 0$.

The operation over $\mathbf{S}_e$ in (30) is negligible for large kb for the present problem (which excludes complex θ_0, as well as multimode periodic cases), and also for small kb provided that boss width (a) is small compared to minimum separation b. (If a/b is not small, $\mathbf{S}_e$ may introduce k-independent factors arising from multipole coupling.[8]) Neglecting $\mathbf{S}_e$ corresponds to

$$\mathscr{f}_{10} \approx f'_{10} + \tfrac{1}{2}\mathscr{M}\mathscr{W} f'_{1r}\mathscr{f}_{r2}, \tag{36}$$

which differs from the isolated scattering amplitude f_{10} in that the radiation integral is modulated by the structure factor $\mathscr{W} = 1 + \mathscr{P}$. As discussed before,[8] the term 1 corresponds to radiation in isolation, $\mathscr{P}$ to correlations among pairs, and $\mathscr{W}$ represents fluctuation scattering; see (8:64) and (8:77).

For small $\operatorname{Im} f \gg \operatorname{Re} f$, or small $\mathscr{W}$ and small absorption, we iterate (37) and neglect absorption in the quadratic terms to obtain

$$\mathscr{f}_{00} \approx f'_{00} + \tfrac{1}{2}\mathscr{M}\mathscr{W} f'_{0r} f'_{r0} = f'_{00} - \tfrac{1}{2}\mathscr{M}\mathscr{W}|f'_{r0}|^2 \approx i\operatorname{Im} f_{00} - (\sigma_a + \bar{\sigma}_s)/\sigma_0,$$
$$\bar{\sigma}_s \approx \tfrac{1}{2}\mathscr{M}\mathscr{W}[K]|f'_{r0}|^2 \approx \rho c_1\int\mathscr{W}[K]q_{r0}. \tag{37}$$

To this approximation, $\mathscr{f}$ differs from f only by the presence of the structure factor $\mathscr{W}[K]$ in the radiation integral. Writing

$$Z = C\mathscr{f}_{00} = iZ_1 + Z_2, \quad Z_1 \approx C\operatorname{Im} f_{00} = \rho\sigma_0\operatorname{Im} f/4\gamma_0, \quad Z_2 \approx -\rho(\sigma_a + \bar{\sigma}_s)/4\gamma_0, \tag{38}$$

we simplify (33)–(35) for various practical purposes by

$$R \approx 1 - \frac{\rho(\sigma_a + \bar{\sigma}_s)\sec\theta_0}{|1-Z|^2},$$
$$\rho Q \approx \frac{\rho\mathscr{W}[K]q_{r0}}{|1-Z|^2}, \quad \tan\frac{\Theta}{2} \approx Z_1, \tag{39}$$

where we may approximate $|1-Z|^2$ by $1-2Z_2+Z_1^2 \approx 1+Z_1^2$. The factor $\sec^2\theta$ in Z_1^2 insures that $R_+ \to 1$ as $\theta_0 \to \pi/2$, and also relates to the pseudo-Brewster minimum in R_+. The present Q may be used to isolate $\mathscr{W}[K]$ versus $\hat{\mathbf{r}}$ from measurements essentially as in x-ray diffraction by liquids, and the resulting $\mathscr{W} - 1 = \mathscr{P}$ may then be Fourier transformed numerically to determine $p(\mathbf{R}) - 1$. For the direct problem we use $\mathscr{W}[K]$ as in (8:76) and (8:77), or numerical results obtained from other approximations[10] of p.

For cylindrical bosses, except for $w = \rho b \approx 1$, it follows from the Zernike–Prins $p(x)$, that[6–8]

$$\mathscr{W}(\mathscr{K}) = 1 + 2w\int_0^\infty (p-1)\cos\mathscr{K}u\,du = (1+\mathscr{D})^{-1},$$
$$\frac{\mathscr{D}}{2} = \frac{w\sin\mathscr{K}}{(1-w)\mathscr{K}} + \frac{w^2(1-\cos\mathscr{K})}{(1-w)^2\mathscr{K}^2},$$
$$\mathscr{K} = kb(\sin\theta - \sin\theta_0). \tag{40}$$

If kb is small, or if $\theta \approx \theta_0$, then

$$\mathscr{W}(\mathscr{K}) = \mathscr{W} + B\mathscr{K}^2 + \theta(\mathscr{K}^4), \quad \mathscr{W} = (1-w)^2,$$
$$B = \mathscr{W}w\tfrac{1}{3}(1-\tfrac{3}{4}w), \tag{41}$$

so that $\mathscr{W}(\mathscr{K})$ increases quadratically in $\mathscr{K}$ from its minimum $\mathscr{W} = \mathscr{W}(0)$. The first local maximum of $\mathscr{W}(\mathscr{K})$ of (40) with increasing $\mathscr{K}$ is the largest, and for large $\mathscr{K}(1-w)$,

$$\mathscr{W}(\mathscr{K}) \sim 1 - \frac{2w}{1-w}\frac{\sin\mathscr{K}}{\mathscr{K}} \tag{42}$$

approaches unity with oscillations decreasing as $\mathscr{K}^{-1}$.

For aligned bounded bosses and a circular exclusion curve of radius b, corresponding to hard statistical mechanics disks of diameter b, we have

$$\mathscr{W}(\mathscr{K}) = 1 + 8w\int_0^\infty (p-1)J_0(\mathscr{K}u)u\,du,$$
$$w = \rho\pi b^2/4, \tag{43}$$
$$(\mathscr{K}/kb)^2 = (\alpha-\alpha_0)^2 + (\beta-\beta_0)^2$$
$$= \sin^2\theta + \sin^2\theta_0 - 2\sin\theta\sin\theta_0\cos(\varphi-\varphi_0),$$

where $\mathscr{K}^2 = b^2\mathbf{K}\cdot(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}})\cdot\mathbf{K}$ with $\mathbf{K} = k(\hat{\mathbf{r}} - \hat{\mathbf{k}})$. In general, computations for $\mathscr{W}(\mathscr{K})$ can be based on numerical results for p obtained from integral equation approximations.[10] If kb is small, or if $\hat{\mathbf{r}} \approx \hat{\mathbf{k}}$, we may use the virial expansion of p to develop $\mathscr{W}(\mathscr{K})$ in powers of w and $\mathscr{K}^2$,

$$\mathscr{W}(\mathscr{K}) \approx 1 - 4w + 6.6159w^2 + \tfrac{1}{2}\mathscr{K}^2 w(1-5.9746w)$$
$$\approx \mathscr{W} + \mathscr{K}^2 B, \tag{44}$$

where the final form is the analog of (41). From the scaled particle[12] approximate equation of state, we showed

$$\mathscr{W} = \mathscr{W}(0) \approx (1-w)^3/(1+w) = 1 - 4w + 7w^2 + \cdots.$$

For large $(1-w)\mathscr{K}$, we integrate (43) by parts, and use $J_1(x) \sim (2/\pi x)^{1/2}\sin(x-\pi/4)$ and the scaled particle approximation for $p(b)$ to obtain

$$\mathscr{W}(\mathscr{K}) \sim 1 - \frac{4w(2-w)(2/\pi)^{1/2}}{(1-w)^2}\frac{\sin(\mathscr{K}-\pi/4)}{\mathscr{K}^{3/2}}, \tag{45}$$

which approaches unity more rapidly with increasing $\mathscr{K}$ than (42). For elliptically symmetric statistics specified by an exclusion ellipse with principal semidiameters b_2 and b_3 (corresponding to aligned hard elliptic disks with principal diameters b_2 and b_3), from geometrical consideration,[8] we work with (43)–(45) in terms of

$$\mathscr{K}^2/k^2 = b_2^2(\alpha-\alpha_0)^2 + b_3^2(\beta-\beta_0)^2, \quad w = \pi\rho b_2 b_3/4, \tag{46}$$

where $\mathscr{K}^2 = \mathbf{K}\cdot(b_2^2\hat{\mathbf{x}}\hat{\mathbf{x}} + b_3^2\hat{\mathbf{y}}\hat{\mathbf{y}})\cdot\mathbf{K}$. We apply existing results for circular disks $p(R) = p\{u\}$ given in terms of u as radial distance divided by disk diameter, by regarding u as the size parameter for a set of cocentered similar ellipses; $u = 1$ specifies the exclusion ellipse.

For small-spaced bosses (small kb), and small a/b, to lowest orders in k for the numerators and denominators in (39), we have

$$R \approx 1 - \frac{\rho(\sigma_a + \mathscr{W}\sigma_s)}{1+Z_1^2},$$
$$\rho Q \approx \frac{\rho\mathscr{W}q_{r0}}{1-Z_1^2}, \quad Z_1 = \frac{\rho\sigma_0\,\mathrm{Im}\,f_{00}}{4\cos\theta_0}, \tag{47}$$

where, except for pressure release bosses on a rigid base, Z_1 is of order k, and σ_s and q are of order k^3 for cylinders and k^4 for bounded bosses. To this approximation, the greater generality of the present distribution compared with the sparse gas case,[1] appears simply as a multiplicative packing factor: thus the earlier results can be carried over by inspection.

The packing factor

$$\mathscr{W} = 1 + \rho\int[p(\mathbf{R}) - 1]d\mathbf{R} \tag{48}$$

decreases the scattering effects as the packing density w increases. If there are n bosses in a central region A, such that $\rho = \langle n\rangle/A$, then

$$\mathscr{W} = [\langle n^2\rangle - \langle n\rangle^2]/\langle n\rangle = \langle(n - \langle n\rangle)^2\rangle/\langle n\rangle \tag{49}$$

is the variance (fluctuation) in the number of bosses in A. If there are no fluctuations, then Q and $\mathscr{S}_s = \mathscr{W}\sigma_s/(1+Z_1^2)$ vanish, and $R < 1$ corresponds solely to absorption. For hard statistical mechanics particles governed by an equation of state E, we used the theorem[9] $(\partial E/\partial\rho)^{-1} = \mathscr{W}$ and the scaled particle[12] approximations for E to obtain[11] $\mathscr{W}$ as simple rational functions of the packing density w:

$$\mathscr{W}_0 = (1-w), \quad \mathscr{W}_1 = (1-w)^2,$$
$$\mathscr{W}_2 = (1-w)^3/(1+w). \tag{50}$$

Here $\mathscr{W}_1$ is the function in (41), $\mathscr{W}_2$ is mentioned after (44), and $\mathscr{W}_0$ is the analog for a random lattice gas corresponding to uncorrelated space-occupying particles. The upper bound on $\mathscr{W}_0$ and $\mathscr{W}_1$ is unity; for $\mathscr{W}_2$, we used[11] $w \lesssim 0.84$, an experimental value for the densest random packing of identical circular disks on a plane, which approximates the mean of the square (0.785) and hexagonal (0.907) close packed values.

The fluctuation effects are determined by $\langle n\rangle\mathscr{W}$, or by the normalized variance, $S = w\mathscr{W}$. From (50), we see that $S(w)$ is small for w near its lower and upper bounds, and that

S has a maximum $S_\cap = S(w_\cap)$ corresponding to $w = w_\cap$ for which $\partial_w S = S' = 0$. As discussed before in a different[11] context,

$$S_{0\cap} = \tfrac{1}{4}, \quad w_{0\cap} = \tfrac{1}{4};$$
$$S_{1\cap} = \tfrac{4}{27} \approx 0.148, \quad w_{1\cap} = \tfrac{1}{3};$$
$$S_{2\cap} \approx 0.0856, \quad w_{2\cap} \approx 0.215. \tag{51}$$

To consider extrema of R and ρQ as functions of w, we write (47) as

$$R = 1 - \frac{w\Gamma + SB}{1 + w^2 A^2}, \quad \rho Q = \frac{SB'}{1 + w^2 A^2}, \tag{52}$$

and assume $\Gamma \ll B$.

For moderate θ_0 we drop A^2. Then, if there is no absorption, $R \approx 1 - SB$ has a minimum and $\rho Q = SB'$ has a maximum,

$$R_\cap \approx 1 - S_\cap B, \quad \rho Q_\cap \approx S_\cap B', \tag{53}$$

at the values $w_\cap$ of (51) corresponding to maximum fluctuations. For nonvanishing, small Γ/B, the extrema are shifted to $w_\wedge > w_\cap$. To first order in Γ, in terms of $S'' = \partial_w^2 S$,

$$R_\wedge \approx 1 - (W_\cap \Gamma + S_\cap B), \quad w_\wedge \approx w_\cap + \Gamma/B|S''_\cap|, \tag{54}$$

where $1/|S''_\cap| = \frac{1}{2}, \frac{1}{2}, 0.453$, for the sequence in (51). For small w, scattering losses dominate in R, but with increasing w absorption dominates. On the other hand if Γ/B is not small then there is in general no minimum of R with variation of w, and Γ dominates the loss effects for all w.

Similarly if we retain A^2 in (52) for $\Gamma = 0$, we obtain frequency dependent values w_ω for the extrema corresponding to vanishing $R'_\omega \approx R'_\cap + (w_\omega - w_\cap)R''_\cap$. To lowest orders,

$$R_\omega \approx 1 - S_\cap B(1 + w_\cap^2 A^2), \quad \rho Q_\omega \approx S_\cap B'/(1 + w_\cap^2 A^2),$$
$$w_\omega - w_\cap \approx -R'_\cap/R''_\cap \approx -S_\cap w_\cap 2A^2/|S''_\cap|. \tag{55}$$

Thus the extrema occur at smaller packings $w_\omega < w_\cap$, and the shift is proportional to A^2. Extrema with both Γ and A^2 retained correspond to w such that in terms of $R(w_\cap) = R_\cap$ and $Q(w_\cap) = Q_\cap$ and of (52), we have

$$R_\Delta \approx R_\cap, \quad Q_\Delta \approx Q_\cap,$$
$$w_\Delta \approx w_\cap - 2w_\cap S_\cap A^2/|S''_\cap| + \Gamma/B|S''_\cap| \tag{56}$$

when w_Δ may be greater or less than $w_\cap$.

IV. SCATTERING COEFFICIENT REDUCTIONS

In Ref. 8, we expanded the scattering amplitude g of an arbitrary obstacle and the multiple scattering amplitude G for a general monolayer as multipole series in terms of corresponding scattering coefficients a and A. From $G[g]$ of (8:16), we obtained an algebraic system $A[a]$ expressing A in terms of a and the distribution integrals $\mathscr{H}$ (continuum analogs of lattice sums determined by p). The same procedure applied to $F[f]$ of (14), or more directly to F of (5) in terms of G for the symmetrical monolayer, involves the same A's and $\mathscr{H}$'s.

Thus, from (5), for semicylinders

$$F_{r0}^{\pm} = G_{r0} \pm G_{r\cap} = \sum_{n=0}^{\infty} A_n [e^{in\theta} \pm e^{in(\pi - \theta)}], \tag{57}$$

with $A_n(\theta_0)$ as in (8:57)ff in terms of a_n and $\mathscr{H}_n$; see (8:59)–(8:70) for $\mathscr{H}_n$ in terms of $p(x)$, or in terms of $\mathscr{W}[K]$. Similarly for bounded bosses,

$$F_{r0}^{\pm} = \sum_{n=0}^{\infty} \sum_{n=-m}^{m} A_n^m [Y_n^m(\hat{\mathbf{r}}) + Y_n^m(\hat{\mathbf{r}}')]$$
$$= \sum_{nm} A_n^m Y_n^m(\hat{\mathbf{r}})[1 + (-1)^{n-m}], \tag{58}$$

with $A_n^m(\hat{\mathbf{r}}_0)$ as in (8:72)ff in terms of a_n^m and $\mathscr{H}_n^m$; see (8:74)–(8:89) for $\mathscr{H}_n^m$. All of the earlier results for the symmetrical monolayer[8] can be applied to the present problem. We illustrate the procedure by considering semi-ellipses and hemi-ellipsoids with principal diameters d_i small compared to λ. In the course of the development, we provide corresponding forms of f', f, f'', and $\not f$ to be used in (26)ff.

For ellipsoids, or for other obstacles having the same reflection and inversion symmetries, we retain only the monopole and dipole terms and write the scattering amplitude as

$$g_{r0} = a_0 + \sum_{i=1}^{n} a_i \gamma_i \gamma_i^0;$$
$$a_0 = a_0'/(1 - a_0'), \quad a_i - a_i'/(1 - a_i'/n), \tag{59}$$

where a_0' and a_i' are the corresponding coefficients of g' for the radiationless problem. Here $n = 2$ or 3 for cylinders or bounded obstacles, and γ_i and γ_i^0 are the direction cosines of $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}} = \hat{\mathbf{r}}_0$ with respect to $\hat{\mathbf{z}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}$ [see (8:90) for arbitrary alignment]. For a boss, from (3),

$$f_{r0}^{-} = 2a_1\gamma_1\gamma_1^0,$$
$$f_{r0}^{+} = 2(a_0 + a_2\gamma_2\gamma_2^0 + a_3\gamma_3\gamma_3^0), \tag{60}$$

with similar forms in terms of a' for the corresponding f'. For the distribution of bosses,

$$F_{r0}^{-} = 2A_1\gamma_1,$$
$$F_{r0}^{+} = 2(A_0 + A_2\gamma_2 + A_3\gamma_3), \tag{61}$$

with

$$A_1 = a_1(\gamma_1^0 + \mathscr{H}_{11}A_1), \quad A_i = a_i\Big(\gamma_i^0 + \sum_{j \neq 1} \mathscr{H}_{ij}A_j\Big),$$
$$\mathscr{H}_{ij} = \mathscr{H}_{ji} = 2\mathbf{S}\gamma_i\gamma_j, \quad \gamma_0 = 1, \tag{62}$$

where i and j equals 0, 2, 3 in the form for A_i, and the $\mathscr{H}_{ij}$ are given in terms of $\mathscr{H}_n$ and $\mathscr{H}_n^m$ in (8:93) and (8:94).

For A_1, we have directly

$$A_1 = a_1\gamma_1^0/(1 - a_1\mathscr{H}_{11}) \equiv \mathscr{A}_1\gamma_1^0,$$
$$F_{r0}^{-} = 2\mathscr{A}_1\gamma_1\gamma_1^0 = 2\mathscr{A}_1 \cos\theta \cos\theta_0. \tag{63}$$

The three coupled equations for A_0, A_2, and A_3 are reduced after (8:95). In particular, if $\varphi_0 = 0$, so that $\gamma_3^0 = 0$ and $\gamma_2^0 = \sin\theta_0$, then

$$A_0 = \mathscr{A}_0(1 + \mathscr{H}_{02}A_2), \quad A_2 = \mathscr{A}_2(\gamma_2^0 + \mathscr{H}_{20}A_0), \quad \mathscr{A}_i = a_i/(1 - a_i\mathscr{H}_{ii}),$$
$$F_{r0}^{+} = 2\,\frac{\mathscr{A}_0 + \mathscr{A}_2\gamma_2\gamma_2^0 + \mathscr{A}_0\mathscr{A}_2\mathscr{H}_{02}(\gamma_2 + \gamma_2^0)}{1 - \mathscr{A}_0\mathscr{A}_2(\mathscr{H}_{02})^2} = 2\,\frac{a_0 + a_2\gamma_2\gamma_2^0 - a_0a_2[\mathscr{H}_{22} + \mathscr{H}_{00}\gamma_2\gamma_2^0 - \mathscr{H}_{20}(\gamma_2 + \gamma_2^0)]}{1 - a_0\mathscr{H}_{00} - a_2\mathscr{H}_{22} + a_0a_2[\mathscr{H}_{00}\mathscr{H}_{22} - (\mathscr{H}_{02})^2]}. \tag{64}$$

This result holds for all $\hat{\mathbf{k}}$ for spheroids with symmetry axis along z, and for the two-dimensional problem ($\gamma_2 = \sin\theta$) of elliptic cylinders.

More generally, we reduce the coefficients to determine the modified amplitude $\mathscr{f}$ to use in (33)ff. To isolate the specular contributions, we decompose the distribution integrals as in (8:105),

$$\mathscr{H}_{ij} = 2C\gamma_i^0\gamma_j^0 - m_i\delta_{ij} + \mathscr{H}'_{ij},$$
$$\mathscr{H}'_{ij} = \mathscr{J}'_{ij} + i\mathscr{V}_{ij}, \tag{65}$$

where $m_0 = 1$, and $m_i = 1/n$ with $n = 2$ or 3 if $i \neq 0$. Substituting into the initial form of A_1 in (62), and using $F_{00}^- = 2A_1\gamma_1^0$ of (61), we obtain

$$A_1 = a_1\{\gamma_1^0 + [2C(\gamma_1^0)^2 + \mathscr{H}'_{11} - 1/n]A_1\}$$
$$= \alpha_1\gamma_1^0(1 + CF_{00}^-),$$
$$\alpha_i = a_1/[1 - a_1(\mathscr{H}'_{11} - 1/n)]. \tag{66}$$

Thus, from $F_{r0} = \mathscr{f}_{r0}(1 + CF_{00})$ of (26), and $F_{r0} = 2A_1\gamma_1$ of (61), we have

$$\mathscr{f}_{r0}^- = 2\alpha_1\gamma_1\gamma_1^0,$$
$$\alpha_1 = a'_1/(1 - a'_1\mathscr{H}'_{11}) = a''_1/(1 - a''_1\mathscr{J}'_{11}), \tag{67}$$

where $a'_1 = a_1/(1 + a_1/n)$ and $a''_1 = a'_1/(1 - ia'_1\mathscr{V}_{11})$ are the corresponding coefficients of f' and f''.

For F^+, we introduce

$$\tilde{\mathbf{a}} = a_2\hat{\mathbf{x}}\hat{\mathbf{x}} + a_3\hat{\mathbf{y}}\hat{\mathbf{y}}, \quad \mathbf{A} = a_2\hat{\mathbf{x}} + a_3\hat{\mathbf{y}},$$
$$\tilde{\mathscr{H}} = \mathscr{H}_{02}\hat{\mathbf{x}} + \mathscr{H}_{03}\hat{\mathbf{y}},$$
$$\tilde{\tilde{\mathscr{H}}} = \mathscr{H}_{22}\hat{\mathbf{x}}\hat{\mathbf{x}} + \mathscr{H}_{23}(\hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}}) + \mathscr{H}_{33}\hat{\mathbf{y}}\hat{\mathbf{y}}, \tag{68}$$

or the corresponding matrices, and rewrite the system in (62) as

$$A_0 = a_0(1 + \mathscr{H}_{00}A_0 + \mathscr{H}\cdot\mathbf{A}),$$
$$\mathbf{A} = \tilde{\mathbf{a}}\cdot(\hat{\mathbf{k}} + \mathscr{H}A_0 + \tilde{\mathscr{H}}\cdot\mathbf{A}). \tag{69}$$

Using (65), and compacting $F_{00}^+ = F$ as in (61), we write $\gamma_2\hat{\mathbf{x}} + \gamma_3\hat{\mathbf{y}}$ as $\hat{\mathbf{k}}'$ (or $\hat{\mathbf{k}}$) to obtain

$$A_0 = \mathscr{A}'_0(1 + CF + \mathscr{H}'\cdot\mathbf{A}),$$
$$\mathbf{A} = \tilde{\mathscr{A}}\cdot[\hat{\mathbf{k}}'(1 + CF) + \mathscr{H}A_0],$$
$$A'_0 = a'_0/(1 - a'_0\mathscr{H}'_{00}) = a''_0/(1 - a''_0\mathscr{J}'_{00}),$$
$$\tilde{\mathbf{A}} = (\tilde{\mathbf{I}} - \tilde{\mathbf{a}}'\cdot\tilde{\mathscr{H}}')^{-1}\cdot\tilde{\mathbf{a}}' = (I - \tilde{\mathbf{a}}''\cdot\tilde{\mathscr{J}}')^{-1}\cdot\tilde{\mathbf{a}}''. \tag{70}$$

Here $a'_0 = a_0/(1 + a_0)$ and $\tilde{\mathbf{a}}' = (\tilde{\mathbf{I}} + \tilde{\mathbf{a}}/n)^{-1}\cdot\tilde{\mathbf{a}} = a'_2\hat{\mathbf{x}}\hat{\mathbf{x}} + a'_3\hat{\mathbf{y}}\hat{\mathbf{y}}$ are the coefficients of f', and $a''_0 = a'_0/(1 - ia'_0\mathscr{V}_{00})$ and $\tilde{\mathbf{a}}'' = (I - i\tilde{\mathbf{a}}'\cdot\tilde{\mathscr{V}})^{-1}\cdot\tilde{\mathbf{a}}'$ are the coefficients of f''. Eliminating the cross terms,

$$A_0 = (\alpha_0 + \alpha\cdot\hat{\mathbf{k}}')(1 + CF),$$
$$\mathbf{A} = (\alpha + \tilde{\alpha}\cdot\hat{\mathbf{k}}')(1 + CF),$$
$$\alpha_0 = (1 - \mathscr{A}'_0\mathscr{H}'\cdot\tilde{\mathscr{A}}\cdot\mathscr{H}')^{-1}\mathscr{A}'_0 = \tilde{\mathscr{A}}\cdot(\tilde{\mathbf{I}} + \alpha_0\mathscr{H}\mathscr{H}\cdot\tilde{\mathscr{A}}),$$
$$\alpha = \mathscr{H}'\cdot\tilde{\mathscr{A}}\alpha_0 = \tilde{\mathscr{A}}\cdot\mathscr{H}'\alpha_0. \tag{71}$$

From (26) and (71),

$$\mathscr{f}_{r0}^+ = 2[\alpha_0 + \alpha\cdot(\hat{\mathbf{r}} + \hat{\mathbf{k}}') + \hat{\mathbf{r}}\cdot\tilde{\alpha}\cdot\hat{\mathbf{k}}'], \tag{72}$$

where only the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components of $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}'$ arise. We may also obtain (72) by using $f_{r1} = 2(a_0 + \hat{\mathbf{r}}\cdot\tilde{\mathbf{a}}\cdot\hat{\mathbf{r}}_1)$ and $\mathscr{f}_{r0} = 2(\mathfrak{u}_0 + \hat{\mathbf{r}}\cdot\mathbf{u})$ in (24) to obtain $\mathfrak{u} = \alpha_0 + \alpha\cdot\hat{\mathbf{k}}'$ and $\mathbf{u} = \alpha + \tilde{\alpha}\cdot\hat{\mathbf{k}}'$. See analogous discussion of $\mathscr{g}$ following (8:109). Specializing to $\varphi_0 = 0$, reduces (72) to the form (64) in terms of $\mathscr{A}'_0$, $\mathscr{A}'_2 = a'_2/(1 - a'_2\mathscr{H}'_{22})$, and $\mathscr{H}'_{02}$.

We may apply the above to Rayleigh's results[15] for elliptic cylinders and ellipsoids of area or volume $\mathscr{V}$, specified by two relative parameters C and B (compressibility and inverse mass density in the simplest cases, but complex in general). We have

$$a'_0 \approx i\Gamma\mathscr{C}, \quad \mathscr{C} = C - 1;$$
$$a'_i \approx -i\Gamma\mathscr{B}/(1 + \mathscr{B}q_i) = -i\Gamma\mathscr{B}'_i, \quad \mathscr{B} = B - 1,$$
$$\Gamma = k^2\mathscr{V}/4, \quad k^3\mathscr{V}/4\pi; \quad \operatorname{Im} C \geqslant 0, \quad \operatorname{Im} B \leqslant 0. \tag{73}$$

These results are correct to $\mathscr{O}(k^n)$, and the next terms are $\mathscr{O}(k^{n+2})$. Here q_i is the standard depolarization integral[16] written in terms of the principal diameters d_i,

$$q_i = d_1d_2d_3\int_0^\infty \{2(d_i^2 + x)[(d_1^2 + x)(d_2^2 + x)$$
$$\times(d_3^2 + x)]^{1/2}\}^{-1}dx, \quad \sum_{i=1}^n q_i = 1. \tag{74}$$

For cylinders, $d_3 \to \infty$ and $q_3 = 0$,

$$q_1 = 1 - q_2 = \frac{d_2}{d_1 + d_2} = \frac{1}{1 + r}, \quad r = \frac{d_1}{d_2}, \tag{75}$$

with $q_1 = q_2 = \frac{1}{2}$ for the circle. For spheroids with d_1 as the symmetry axis, $q_1 + 2q_2 = 1$. If $r = d_1/d_2 < 1$, the oblate case, then

$$q_1 = \frac{1}{1 - r^2}\left(1 - \frac{r\cos^{-1}r}{(1 - r^2)^{1/2}}\right),$$
$$\frac{1}{3} < q_1 < 1, \quad 0 < q_2 < \frac{1}{3}, \tag{76}$$

such that for near disks, $q_1 \approx 1 - \pi r/2 + 2r^2$. If $r > 1$, the prolate case,

$$q_1 = \frac{1}{r^2 - 1}\left(\frac{r\cosh^{-1}r}{(r^2 - 1)^{1/2}} - 1\right),$$
$$0 < q_1 < \frac{1}{3}, \quad \frac{1}{3} < q_2 < \frac{1}{2}, \tag{77}$$

such that for near needles, $q_1 \approx (\ln 2r - 1)/r^2$. If d_2 is the symmetry axis, then $q_2 + 2q_1 = 1$, and we interchange d_1 and d_2.

If $r = 1$, then $q_i = \frac{1}{3}$ for the sphere. For near spheres, $r \approx 1$, and (76) and (77) give $q_1 \approx \frac{1}{3} - 4(r - 1)/15$. More generally, for a near sphere, $d_i = d(1 + \delta_i)$ with $\delta_i \approx 0$. From (74) to first order in δ, we have $q_1 = \frac{1}{3} + (-4\delta_1 + 2\delta_2 + 2\delta_3)$; similarly for q_2 and q_3, so that $\Sigma q_i = 1$. For a volume preserving deformation ($d_1d_2d_3 = d^3$), we require $\Sigma\delta_i = 0$, and consequently $q_1 = \frac{1}{3} - 2\delta_i/5$. These points are stressed to facilitate geometrical interpretation of the multipole coupling effects we consider subsequently.

The form in (73) for a'_0 holds for arbitrary shapes; if $B \approx 1$, then $a'_i \approx -i\Gamma\mathscr{B}$ is similarly shape independent. For a rigid obstacle, we let C and B equal zero in (73) to obtain

$$a'_0 \approx -i\Gamma, \quad a'_i \approx i\Gamma/(1 - q_i). \tag{78}$$

For pressure release obstacles (e.g., gas bubbles for underwater sound)

$$a'_{02} \approx -i\pi/2\ell_0, \quad a'_{03} \approx -ik\ell, \quad a'_i \approx -i\Gamma/q_i,$$
$$\ell_0 = \ln[8/kc(d_1+d_2)], \quad c \approx 1.781;$$
$$\ell = \left(\int_0^\infty [(d_1^2+x)(d_2^2+x)(d_3^2+x)]^{-1/2}\,dx\right)^{-1}. \qquad (79)$$

The dipole terms a'_i, obtained by letting $B \to \infty$ in (73), are negligible in g but dominate in f^-. The values of ℓ(the electrostatic capacity) for the spheroid equal

$$\ell 2/d_2 = (1-r^2)^{1/2}/\cos^{-1} r \ \text{for}\ r<1,$$
$$\ell 2/d_2 = (r^2-1)^{1/2}/\cosh^{-1} r \ \text{for}\ r>1. \qquad (80)$$

These reduce to $\ell 2/d = 1$ for the sphere, to $\ell_2/d_2 \approx (2/\pi)(1+2r/\pi)$ for near disks, and to $\ell 2/d_1 \approx 1/\ln 2r$ for near needles.

More generally than in (73), we use a dyadic parameter $\tilde{\mathbf{B}}$ and incorporate the shape integrals q_i in a dyadic $\tilde{\mathbf{q}}$. For present purposes, we assume that the principal axis of all dyadics are aligned with the Cartesian axes of the exclusion ellipse and its normal, and replace $\mathscr{B}$ by $\mathscr{B}_i = B_i - 1$. See Ref. 8 for the general case.

For small spaced scatterers, we use

$$\mathscr{J}'_{ij} \approx \mathscr{W} m_i \delta_{ij}, \quad \mathscr{N}_{ij} \approx \mathscr{N}_{ii}\delta_{ij}, \qquad (81)$$

where $\mathscr{W}$ is the k-independent packing factor. The $\mathscr{N}_{ii}$ for $k \approx 0$, reduce to $\mathscr{N}_{00} \approx \mathfrak{N}_0$, $\mathfrak{N}_0^0/kb_2$, and $\mathscr{N}_{ii} \approx \mathfrak{N}_{ii}(kb_2)^3$ for $i \neq 0$; except for $\mathfrak{N}_0$ which depends on $\ln k$, the $\mathfrak{N}$'s are k independent.

For such cases (for $i = 0, 1, 2, 3$),

$$a_i \approx \frac{a''_i}{1 - a''_i \mathscr{W} m_i}, \quad a''_i \approx \frac{a'_i}{1 - ia'_i \mathfrak{n}_{ii}} \equiv \frac{a'_i}{\mathscr{D}_i}. \qquad (82)$$

Except for the monopole terms for pressure release scatterers (which we consider separately) all coefficients are of order k^n. The dominant terms in (81) for cylinders, from (8:70) and (8:93) are given by $(kb)^2 \mathscr{N}_{ii} \to \mathfrak{N}_{ii}$,

$$\begin{Bmatrix}\mathfrak{N}_{11}\\ \mathfrak{N}_{22}\end{Bmatrix} = \pm \frac{1}{2}\mathfrak{N}_2 = \mp \frac{2}{\pi} I_2, \quad I_2 = 2w\int_1^\infty \frac{p}{u^2}\,du, \qquad (83)$$

in terms of the Zernike–Prins p. If w is near zero, then $I_2 \approx 2w(1 + w\,0.307)$; if w is near unity, $I_2 \approx \zeta(2) = \pi^2/6 \approx 1.645$.

For bounded obstacles and elliptically symmetric statistics determined by $t = b_3/b_2$ (the ratio of the principal diameters of the exclusion ellipse), from (8:87) and (8:94) in terms of $(kb_2)^2 \mathscr{N}_{ii} \to \mathfrak{N}_{ii}$,

$$\mathfrak{N}_{11} = \tfrac{2}{3}\mathfrak{N}_2^0 = -I_2 M_2^0, \quad I_2 = 8w\int_1^\infty \frac{p}{u^2}\,du,$$
$$\begin{Bmatrix}\mathfrak{N}_{22}\\ \mathfrak{N}_{33}\end{Bmatrix} = -\tfrac{1}{3}\mathfrak{N}_2^0 \pm \tfrac{1}{8}\mathfrak{N}_2^2 = \tfrac{1}{2}I_2 M_2^0 \pm \tfrac{3}{2}I_2 M_2^2$$
$$= \tfrac{1}{2}I_2(M_2^0 \pm 3M_2^2),$$
$$M_2^0 = (2/\pi t)E(v),$$
$$M_2^2 = (2/\pi 3tv)[(2-v)E(v) - 2(1-v)K(v)],$$
$$v = 1 - t^2, \qquad (84)$$

where

$$E(v) = \int_0^{\pi/2} (1 - v\sin^2\tau)^{1/2}\,d\tau$$

and

$$K(v) = \int_0^{\pi/2} (1 - v\sin^2\tau)^{-1/2}\,d\tau$$

are the complete elliptic integrals, and p is the two-dimensional radial distribution function.[10] If w is small, $I_2 \approx 8w(1 + 0.4157w)$. See discussion of (8:89) for illustrations for $t = 2$ and 4, and for comparisons with analogs for the rectangular periodic lattice.[5]

For all such cases, $\mathscr{D}_0 = 1 + O(k^2) \approx 1$, and the monopoles satisfy

$$a''_0 \approx a'_0 \approx i\Gamma\mathscr{C}. \qquad (85)$$

The dipoles $a''_i = a'_i/\mathscr{D}_i$ differ from a'_i by the k-independent factors

$$\mathscr{D}_i = 1 - ia'_i\mathscr{N}_{ii} \approx 1 - \Gamma\mathfrak{B}'_i\mathfrak{N}_{ii}/(kb_2)^n = 1 + \mathfrak{B}'_i\mathscr{E}_i. \qquad (86)$$

For cylindrical dipoles, from (73) and (83),

$$\epsilon_1 = \epsilon, \quad \epsilon_2 = -\epsilon; \quad \epsilon = \mathscr{V} I_2/2\pi b^2, \qquad (87)$$

where $\mathscr{V} = \pi d_1 d_2/4$. For bounded dipoles, from (73) and (84)

$$\epsilon_1 = \epsilon, \quad \epsilon_2 = -\tfrac{1}{2}(\epsilon + \epsilon'), \quad \epsilon_3 = -\tfrac{1}{2}(\epsilon - \epsilon');$$
$$\epsilon = \mathscr{V} I_2 M_2^0/4\pi b_2^3, \quad \epsilon' = 3\mathscr{V} I_2 M_2^2/4\pi b_2^3, \qquad (88)$$

where $\mathscr{V} = \pi d_1 d_2 d_3/6$. Both sets satisfy $\Sigma\epsilon_i = 0$, so that if we introduce new depolarization factors

$$Q_i = q_i + \epsilon_i, \quad \sum Q_i = 1, \quad \sum \epsilon_i = 0, \qquad (89)$$

we reduce a''_i to the same form as a'_i,

$$a''_i = \frac{a'_i}{\mathscr{D}_i} = \frac{-i\Gamma\mathfrak{B}'_i}{1+\mathfrak{B}'_i\epsilon_i} = \frac{-i\Gamma\mathscr{B}_i}{1+\mathscr{B}_i(q_i+\epsilon_i)}$$
$$= \frac{i\Gamma\mathscr{B}_i}{1+\mathscr{B}_i Q_i} = -i\Gamma\mathfrak{B}_i. \qquad (90)$$

The relations (85) and (90) specify an ellipsoid having the same volume and physical parameters as the original but a different shape. From the discussion following (77), the packing effects incorporated in Q_i determine the shape. The equivalent ellipsoid is flatter along the array normal ($i = 1$) and broader in the base plane, the elongation being greater along the smaller diameter of the exclusion ellipse (along b_2 for $b_2 < b_3$).

For elliptic cylinders with shapes specified by $r = d_1/d_2 = q_2/q_1$, the equivalent cylinders correspond to $r_e = d_1^e/d_2^e = Q_2/Q_1$ such that

$$\frac{r_e}{r} = \frac{d_1^e d_2}{d_2^e d_1} = \left[\frac{d_1^e}{d_1}\right]^2 = \left[\frac{d_2}{d_2^e}\right]^2 = \frac{1-\epsilon/q_2}{1+\epsilon/q_1}$$
$$\approx 1 - \frac{\epsilon}{q_1 q_2} = 1 - \frac{(d_1+d_2)^2}{8b^2} I_2. \qquad (91)$$

We substitute (85) and (90) in a_i of (82), and write $\mathscr{C} = \mathscr{C}_{(1)} + i\mathscr{C}_{(2)}$ and $\mathfrak{B}_i = \mathfrak{B}_{i(1)} - i\mathfrak{B}_{i(2)}$ in terms of real components with $\mathscr{C}_2$ and $\mathfrak{B}_2^i$ positive and small compared to $\mathscr{C}_1$ and $\mathfrak{B}_1^i$. Keeping only the leading loss terms, we have

$$a_0 = a_0''/(1-a_0''\mathscr{W}) \approx i\Gamma\mathscr{C}_{(1)} - \Gamma[\mathscr{C}_{(2)} + \Gamma\mathscr{W}\mathscr{C}_{(1)}^2],$$
$$a_i = a_i''/(1-a_i''\mathscr{W}/n) \approx -i\Gamma\mathfrak{B}_{i(1)} - \Gamma(\mathfrak{B}_{i(2)} + \Gamma\mathscr{W}\mathfrak{B}_{i(1)}^2)$$
$$\approx \frac{-i\Gamma\mathfrak{B}_{i(1)}}{1+\mathscr{B}_{i(1)}Q_i} - \frac{\Gamma}{(1+\mathscr{B}_{i(1)}a_i)^2}\left(\mathscr{B}_{i(2)} + \frac{\Gamma\mathscr{W}\mathscr{B}_{i(1)}^2}{n}\right). \quad (92)$$

Using these in (67) and (68) in the forms

$$f_{r0}^+ = 2(a_0 + a_2\gamma_2\gamma_2^0 + a_3\gamma_3\gamma_3^0), \quad f_{r0}^- = 2a_1\gamma_1\gamma_1^0, \quad (93)$$

we substitute into $F = f/(1-Z)$ with $Z = Cf$.

Writing $\gamma_i^0 = \gamma_i$ for brevity, the imaginary part of Z is given by

$$\operatorname{Im} Z_{\pm} = \frac{\rho k \mathscr{Y}}{2\gamma_1}\mathfrak{F}_{(1)}^{\pm};$$
$$\mathfrak{F}_{(1)}^{+} = \mathscr{C}_{(1)} - \mathfrak{B}_{2(1)}\gamma_2^2 - \mathfrak{B}_{3(1)}\gamma_3^2, \quad \mathfrak{F}_1^{-} = -\mathfrak{B}_{1(1)}\gamma_1^2. \quad (94)$$

The corresponding real part involves absorption and scattering terms

$$\operatorname{Re} Z_{\pm} = \frac{-\rho(\bar\sigma_a + \bar\sigma_s)}{4\gamma_1} = -\frac{\rho k \mathscr{Y}}{2\gamma_1}\mathfrak{F}_{(a)}^{\pm} - \frac{\rho\mathscr{W}\mathscr{Y}^2}{8\gamma_1}\left(k^3, \frac{k^4}{\pi}\right)\mathfrak{F}_{(s)}^{\pm}, \quad (95)$$
$$\mathfrak{F}_{(a)}^{+} = \mathscr{C}_{(2)} + \mathfrak{B}_{2(2)}\gamma_2^2 + \mathfrak{B}_{3(2)}\gamma_3^2, \quad \mathfrak{F}_{(a)}^{-} = \mathfrak{B}_{1(2)}\gamma_1^2,$$
$$\mathfrak{F}_{(s)}^{+} = \mathscr{C}_{(1)}^2 + \frac{\mathfrak{B}_{2(1)}^2\gamma_2^2}{n} + \frac{\mathfrak{B}_{3(1)}^2\gamma_3^2}{n}, \quad \mathfrak{F}_{(s)}^{-} = \frac{\mathfrak{B}_{1(1)}^2\gamma_1^2}{n}.$$

In $\bar\sigma_s$, for semicylinders we use k^3 and $n=2$, and for bounded bosses, k^4/π and $n=3$.

For rigid bosses on rigid or free base planes, by letting $\mathscr{C} \to -1$ and $\mathscr{B}_i \to -1$ in (92)–(95),

$$\mathfrak{F}_{(1)}^{+} = -1 + \frac{\gamma_2^2}{(1-Q_2)} + \frac{\gamma_3^2}{(1-Q_3)}, \quad \mathfrak{F}_{(1)}^{-} = \frac{\gamma_1^2}{(1-Q_1)}, \quad (96)$$
$$\mathfrak{F}_{(s)}^{+} = 1 + \frac{\gamma_2^2}{(1-Q_2)^2 n} + \frac{\gamma_3^2}{(1-Q_3)^2 n}, \quad \mathfrak{F}_{(s)}^{-} = \frac{\gamma_1^2}{(1-Q_1)^2 n}.$$

The corresponding f_+ and Z_+ specify reflection for the homogeneous case of a fully rigid rough surface; f_- is of interest for underwater sound incident on the submerged part of a rigid obstacle on the sea surface, and Z_- specifies reflection for the corresponding distribution.

The present development also suffices for the homogeneous pressure release surface, free bosses on a free base,

$$\mathfrak{F}_{(1)}^{-} = -\gamma_1^2/Q_1, \quad \mathfrak{F}_{(s)}^{-} = \gamma_1^2/Q_1^2 n. \quad (97)$$

For various practical purposes, we would use $\mathfrak{F}^+$ of (96) and $\mathfrak{F}^-$ of (97) for the reflection of sound from the sea surface, $\mathfrak{F}^+$ for airborne sound, and $\mathfrak{F}^-$ for underwater sound.

The one remaining case, that of free bosses on a rigid base plane is exceptional in that the monopole as in (79) is not of order k^n. Here $f_- \approx a_0$ is appropriate for underwater sound incident on a gaseous boss adhering to a rigid bottom. The effects of dense packing on the imaginary coefficient a_0' as determined by $\mathscr{N}_{00}$, may be appreciable. We have

$$a_0'' \approx \frac{a_0'}{1 - a_0' i \mathscr{N}_{00}} = \frac{a_0'}{\mathscr{D}_0}, \quad \mathscr{N}_{00} \approx \mathfrak{N}_0, \ \frac{\mathfrak{N}_0^0}{kb_2}. \quad (98)$$

For cylinders $\mathscr{N}_{00} \approx \mathfrak{N}_0$,

$$\mathfrak{N}_0 = -\frac{2}{\pi}I^0, \quad I^0 = I_0 \ln\frac{2}{ckbu} - I_\ell,$$
$$I^0 = 2w\int_0^\infty (p-1)\ln\frac{2}{ckbu}\,du = I_0' - I_\ell,$$
$$I_0' = 2w\ln\frac{2}{ckb}\int_0^\infty (p-1)du = -2w\left(1 - \frac{w}{2}\right)\ln\frac{2}{ckb},$$
$$I_\ell = 2w\int_0^\infty (p-1)\ln u\,du. \quad (99)$$

If w is small, $I_\ell \approx -2w[1 + w(2\ln 2 - 1.25)]$; if $w \sim 1$, then $I_\ell \sim \ln(2\pi)$, and $I^0 \sim -\ln(4\pi/ckb)$. For bounded obstacles, $\mathscr{N}_{00}/kb_2 \to \mathfrak{N}_0^0$,

$$\mathfrak{N}_0^0 = -I_0 M_0^0, \quad I_0 = 8w\int_0^\infty (p-1)du,$$
$$M_0^0 = (2/\pi t)K(\nu), \quad \nu = 1 - t^{-2}. \quad (100)$$

If w is small, then $I_0 \approx -8w(1 - 0.6413w)$. For the rectangular lattice, we replace $I_0 M_0^0$ by $L_0^0 \approx -2\ln(4\pi/ct) \approx -3.906 + 2\ln t$.

We have

$$\mathscr{D}_{02} \approx 1 + I^0/\ell_0 = 1 - |I^0|/\ell_0,$$
$$\mathscr{D}_{03} \approx 1 + (\ell/b_2)I_0 M_0^0 = 1 - (\ell/b_2)|I_0|M_0^0, \quad (101)$$

where only $\mathscr{D}_{03}$ is k independent. The corresponding impedance is

$$Z_+ = C f^+ = C\,2a_0$$
$$= C\,2a_0''/(1 - a_0''\mathscr{W}) = C\,2a_0'/(\mathscr{D} - a_0'\mathscr{W})$$
$$= C\,2a_0'/(1 - ia_0'\mathfrak{N} - a_0'\mathscr{W}), \quad (102)$$

with a_0' as in (79). For cylindrical bosses, $a_0' = -i\pi/2\ell_0$,

$$Z_+ = -i2\rho/k\gamma_1[2(\ell_0 + I^0)/\pi + i\mathscr{W}], \quad (103)$$

where for $w \sim 1$, $\ell_0 + I^0 \sim \ln[2b/\pi(d_1 + d_2)]$ becomes independent of k, and $\mathscr{W} = (l - w)^2 \sim 0$. For bounded bosses, $a_0' = -ik\ell$,

$$Z_+ = \frac{-i2\pi\rho\ell}{k\gamma_1}\left(\frac{1}{1 + \ell I_0 M_0^0/b_2 + ik\ell\mathscr{W}}\right)$$
$$= \frac{-i2\pi\rho b_2}{k\gamma_1}\left(\frac{1}{b_2/\ell + I_0 M_0^0 + ikb_2\mathscr{W}}\right), \quad (104)$$

so that for both (103) and (104), Im Z is essentially of order $1/k$.

We generalize the monopole development to include the first resonance for large finite $\mathscr{C} = C - 1 \approx C$, by using

$$a_0' \approx \frac{i\Gamma\mathscr{C}}{(1 - \Gamma\mathscr{C}\mathscr{L})}; \quad \mathscr{L} = \frac{2\ell_0}{\pi}, \ \frac{1}{k\ell}. \quad (105)$$

The denominator is of the form $1 + \mathscr{C}\mathscr{O}(k^2)$; if $\mathscr{C}$ is moderate in size, we neglect the $\mathscr{O}(k^2)$ term to obtain (73), and if $\mathscr{C} \to \infty$ we obtain (79). The resonant frequencies, given by $\Gamma\mathscr{C}\mathscr{L} = 1$, equal

$$k_{0\wedge}^2 = 2\pi/\mathscr{V}\mathscr{C}\ell_0, \quad 4\pi\ell/\mathscr{V}\mathscr{C} \quad (106)$$

and the corresponding isolated monopole coefficient $a_0 = a_0'/(1 - a_0')$ reduce to $a_{0\wedge} = -1$.

For the scatterer in the array, because of multipole coupling, the resonances occur at different frequencies, and the resonant magnitudes are higher. From

$$a_0 \approx a_0'/(1 - a_0' i \mathscr{I}_{00} - a_0' \mathscr{W}) = a_0''/(1 - a_0'' \mathscr{W}) \tag{107}$$

and

$$a_0'' = \frac{a_0'}{1 - a_0' i \mathscr{I}_{00}} = \frac{i\Gamma\mathscr{C}}{1 - \Gamma\mathscr{C}(\mathscr{L} - \mathscr{I}_{00})} = \frac{\ell\Gamma\mathscr{C}}{1 - k^2/k_\wedge^2}, \tag{108}$$

we have

$$k_\wedge^2 = \frac{k_{0\wedge}^2}{1 - x} > k_{0\wedge}^2, \quad x_2 = \frac{\pi \mathfrak{R}_0}{2\ell_0} = \frac{|I^0|}{\ell_0},$$

$$x_3 = \frac{\mathfrak{N}_0^0 \ell}{b} = \frac{|I_0| M_0^0 \ell}{b}, \tag{109}$$

and $a_{0\wedge} = -1/\mathscr{W}'$, $|a_{0\wedge}| > |a_{0\wedge}| = 1$. The reflection coefficient $R = |(1+Z)/(1-Z)|^2$, $Z = C/\ell \approx 2Ca_0$, at $k = k_\wedge$ reduces to

$$R_\wedge = 1 - \frac{8C\mathscr{W}'}{|\mathscr{W}' + 2C|^2} = 1 - \rho \mathscr{S}_s \sec\theta_0. \tag{110}$$

With increasing w in (50), $\mathscr{W}'$ becomes small (zero for the periodic cases), and $R_\wedge \approx 1$.

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[1]V. Twersky, "On Scattering and Reflection of Sound by Rough Surfaces," J. Acoust. Soc. Am. **29**, 209–225 (1957); **22**, 539–546 (1950).

[2]J. E. Burke and V. Twersky, "On Scattering and Reflection by Elliptically Striated Surfaces," J. Acoust. Soc. Am. **40**, 883–895 (1966).

[3]See, for example, O. I. Diachok, "Effect of Sea-Ice Ridges on Sound Propagation in the Ocean," J. Acoust. Soc. Am. **59**, 1110–1120 (1976).

[4]V. Twersky, "On the Scattering of Waves by an Infinite Grating," IRE Trans. Ant. Propag. **AP-4**, 330–345 (1956); "On Scattering of Waves by the Infinite Grating of Circular Cylinders," **AP-10**, 737–765 (1962); J. E. Burke and V. Twersky, "On Scattering of Waves by the Infinite Grating of Elliptic Cylinders," IEEE Trans. **AP-14**, 465–480 (1960).

[5]V. Twersky, "Multiple Scattering of Waves by the Doubly Periodic Planar Array of Obstacles," "Lattice Sums and Scattering Coefficients for the Rectangular Planar Array," "Low Frequency Coupling in the Planar Rectangular Lattice," J. Math. Phys. **16**, 633–666 (1975).

[6]V. Twersky, "Scattering by Quasi-Periodic and Quasi-Random Distributions," IRE Trans. **AP-7**, S307–S319 (1959); "Multiple Scattering of Waves by Planar Random Distributions of Cylinders and Bosses," Inst. Math. Sci., NYU Rep. EM58, 1953, NYC.

[7]F. Zernike and J. A. Prins, "Die Beuging von Röntgenstrahlen in Flussigkeiten als Effekt der Molekülanordnung," Z. Phys. **41**, 184–194 (1927).

[8]V. Twersky, "Multiple Scattering of Sound by Correlated Monolayers," J. Acoust. Soc. Am. **73**, 68–84 (1983).

[9]H. L. Frisch and J. L. Lebowitz, *The Equilibrium Theory of Classical Fluids* (Banjamin, New York, 1964). R. J. Baxter, "Distribution Functions," in *Physical Chemistry*, Vol. VIII A, edited by H. Eyring, D. Henderson, and W. Jost (Academic, New York, 1971), Chap. 4, pp. 267–334.

[10]Y. Uehara, T. Ree, and F. H. Ree, "Radial Distribution Function for Hard Disks from the BGY2 Theory," J. Chem. Phys. **70**, 1876–1883 (1979). J. Woodhead-Galloway and P. A. Machin "X-ray Scattering from a Gas of Uniform Hard-Disks using the Percus–Yevick Approximation," Mol. Phys. **32**, 41–48 (1976). F. Lado, "Equation of State for the Hard-Disk Fluid from Approximate Integral Equations," J. Chem. Phys. **49**, 3092–3096 (1968).

[11]V. Twersky, "Acoustic Bulk Parameters in Distributions of Pair-Correlated Scatterers," J. Acoust. Soc. Am. **64**, 1710–1719 (1978); "Transparency of Pair-Correlated Random Distributions of Small Scatterers with Applications to the Cornea," J. Opt. Soc. Am. **65**, 524–530 (1975); "Propagation in Pair-Correlated Distributions of Small-Spaced Lossy Scatterers," J. Opt. Soc. Am. **69**, 1567–1572 (1979).

[12]E. Helfand, H. L. Frisch, and J. L. Lebowitz, "The Theory of the Two- and One-Dimensional Rigid Sphere Fluids," J. Chem. Phys. **34**, 1037–1042 (1961).

[13]Lord Rayleigh, "On the Light Dispersed from Fine Lines Ruled upon Reflecting Surfaces or Transmitted by very Narrow Slits," Phil. Mag. **14**, 350–359 (1907).

[14]V. Twersky, "Certain Transmission and Reflection Theorems," J. Appl. Phys. **25**, 589–862 (1954).

[15]Lord Rayleigh, "On the Incidence of Aerial and Electric Waves upon Small Obstacles," Phil. Mag. **44**, 28–52 (1897).

[16]J. A. Osborn, "Demagnetizing Factors of the General Ellipsoid," Phys. Rev. **67**, 351–357 (1945). E. C. Stoner, "The Demagnetizing Factors for Ellipsoids," Phil. Mag. **36**, 803–821 (1945).