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TRAPEZOIDAL MONTE CARLO INTEGRATION*

ELIAS MASRY† AND STAMATIS CAMBANIS‡

Abstract. The approximation of weighted integrals of random processes by the trapezoidal rule based on an ordered random sample is considered. For processes that are once mean-square continuously differentiable and for weight functions that are twice continuously differentiable, it is shown that the rate of convergence of the mean-square integral approximation error is precisely n^{-4}, and the asymptotic constant is also determined.

Key words. Monte Carlo integration of random processes, trapezoidal rule, rate of quadratic-mean convergence

AMS(MOS) subject classifications. 65D30, 60G12, 65U05, 62G05

1. Introduction, results, and discussion. The simplest Monte Carlo method for approximating the integral

$$I(f)=\int_0^1 f(t)\,dt \tag{1.1}$$

of a (square integrable) function f over a finite interval, uses n independent samples $U_1, \cdots, U_n$ from a uniform distribution over the unit interval and forms the average estimate

$$J_n(f)=\frac{1}{n}\sum_{i=1}^{n} f(U_i). \tag{1.2}$$

J_n is an unbiased estimator of I: $EJ_n(f)=I(f)$, and, in view of the strong law of large numbers, it is consistent, i.e., as the sample size n increases to infinity, $J_n(f)$ tends to $I(f)$ with probability one, i.e., for almost every realization of $\{U_n\}_{n=1}^{\infty}$. The variance or mean-square error of $J_n(f)$ is

$$E[I(f)-J_n(f)]^2=\frac{1}{n}\{I(f^2)-[I(f)]^2\} \tag{1.3}$$

and thus tends to zero at the rate of n^{-1}. As the constant $I(f^2)-I^2(f)$ is strictly positive for all functions f which are not almost everywhere constant, no improvement in the rate of convergence is to be expected from any smoothness assumptions on f.

Yakowitz, Krimmel, and Szidarovszky [9] proposed improving the convergence rate in (1.3) of the crude Monte Carlo method by using quadrature formulas instead of the simple averaging in (1.2). They specifically studied the trapezoidal rule, based on the ordered sample $t_{n,0} \triangleq 0 < t_{n,1} < t_{n,2} < \cdots < 1 \triangleq t_{n,n+1}$ obtained from the independent, uniformly distributed samples $U_1, U_2, \cdots, U_n$, which is given by

$$I_n(f)=\frac{1}{2}\sum_{i=0}^{n}[f(t_{n,i})+f(t_{n,i+1})](t_{n,i+1}-t_{n,i}) \tag{1.4a}$$

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and uses $n+2$ sample points, the n random samples, and the fixed-interval endpoints. Since the trapezoidal rule can be written in the form

$$I_n(f)=\frac{1}{2}\left\{t_{n,1}f(0)+\sum_{i=1}^{n}(t_{n,i+1}-t_{n,i-1})f(t_{n,i})+(1-t_{n,n})f(1)\right\} \tag{1.4b}$$

it can be considered as a weighted Monte Carlo rule with random weights. When f has continuous second derivative they proved that

$$E[I(f)-I_n(f)]^2 \leqq \frac{C(f)}{n^4} \tag{1.5}$$

for some constant $C(f)$ and all $n \geqq 1$, and they provided simulation evidence that the convergence rate of the mean-square error is in fact n^{-4}.

In this paper, we consider weighted integrals of random processes and establish the rate of convergence and asymptotic constant for the trapezoidal rule (1.4a) based on an ordered random sample. As a consequence, the rate of convergence and asymptotic constant for integrals of nonrandom functions are also determined.

Throughout $X=\{X(t,\omega), 0\leqq t\leqq 1\}$ is a second-order random process with mean zero, $EX(t)=0$, and covariance function $R(t,s)=E\{X(t)X(s)\}$, defined on some probability space. We assume, with no further notice, that

(i) The random process X and the random sample $\{U_i\}$ are mutually independent, and

(ii) $X(t,\omega)$ is jointly measurable in t and ω.

The first assumption simplifies the task of computing expected values and is quite natural as the randomness of the sampling mechanism is generally in no way dependent on or related to the randomness of the stochastic integrand. The second assumption enables us to integrate with respect to t for almost every fixed path ω and to obtain as a result of the integration a random variable, whose expectation may therefore be computed. This is a minimal regularity assumption on the process X; when X is mean-square continuous, as we will shortly assume, it always has a jointly measurable version. Following standard practice, we delete the probability variable ω and write $X(t)$ for $X(t,\omega)$ (just as we write $t_{n,i}$ and U_i for the random variables $t_{n,i}(\omega)$ and $U_i(\omega)$).

We first derive under general conditions an explicit expression for the mean-square error in the approximation of the random integral:

$$I(fX)=\int_0^1 f(t)X(t)\,dt$$

by the trapezoidal rule based on ordered random samples:

$$I_n(fX)=\frac{1}{2}\sum_{i=0}^{n}[f(t_{n,i})X(t_{n,i})+f(t_{n,i+1})X(t_{n,i+1})](t_{n,i+1}-t_{n,i}).$$

This expression can be used to evaluate finite sample size performance.

THEOREM 1. *If $\int_0^1 R(t,t)f^2(t)\,dt<\infty$, then for all $n\geqq 1$ we have*

$$E[I(fX)-I_n(fX)]^2$$

$$=\frac{R(0,0)f^2(0)+R(1,1)f^2(1)+R(0,1)f(0)f(1)}{2(n+1)(n+2)} \tag{1.6a}$$

$$(1.6b) \quad +\int_0^1 \{R(t,1)f(1)f(t)+R(0,1-t)f(0)f(1-t)\}$$
$$\cdot\left\{-\frac{(1-t)^{n+1}}{2(n+1)}+\frac{n^2+3n-2}{4(n+1)}t^{n+1}-\frac{n+1}{2}t^n+\frac{n}{4}t^{n-1}\right\}dt$$

$$(1.6c) \quad +\int_0^1 R(t,t)f^2(t)\left\{\frac{3}{2(n+1)}+\frac{n-1}{2(n+1)}[t^{n+1}+(1-t)^{n+1}]-\frac{1}{2}[t^n+(1-t)^n]\right\}dt$$

$$(1.6d) \quad +\iint_{0\leqq t<s<1} R(t,s)f(t)f(s)$$
$$\cdot\left\{\frac{3}{2}[t^n+(1-s)^n]-\frac{1}{2}(s-t)^n+\frac{1}{4}n(n-1)(1-s+t)^{n-2}\right.$$
$$\left.-\frac{1}{2}n^2(1-s+t)^{n-1}+\frac{1}{4}(n-2)(n+3)(1-s+t)^n\right\}dt\,ds.$$

Using the expression of the mean-square error in (1.6), we now show that when the function f has two continuous derivatives and the process X has one quadratic-mean derivative which is mean-square continuous, then the rate of convergence of the mean-square integral approximation error is precisely n^{-4}; and we also determine the asymptotic constant.

THEOREM 2. *If $f(t)$ has continuous second derivative on* $[0,1]$ *and $R(t,s)$ has continuous mixed partial derivatives $R^{i,j}(t,s)$ of order* $2, 0\leqq i+j\leqq 2$, *on the unit square* $[0,1]\times[0,1]$ *and of order* 3, $i+j=3$, *off its diagonal, then*

$$(1.7) \qquad E[I(fX)-I_n(fX)]^2=\frac{C(f,R)+o(1)}{(n+1)(n+2)(n+3)(n+4)}.$$

The asymptotic constant in (1.7) is given by

$$\begin{aligned}2C(f,R)=&\tfrac{1}{2}R(0,0)f'(0)^2+\tfrac{1}{2}R(1,1)f'(1)^2-R(0,1)f'(0)f'(1)\\ &+R^{1,0}(0,0)f(0)f'(0)+R^{1,0}(1,1)f(1)f'(1)-R^{1,0}(0,1)f(0)f'(1)\\ (1.8a)\quad &-R^{0,1}(0,1)f'(0)f(1)+3R^{2,0}(1,1)f^2(1)-3R^{2,0}(0,0)f^2(0)\\ &+2R^{1,1}(0,0)f^2(0)-R^{1,1}(1,1)f^2(1)-R^{1,1}(0,1)f(0)f(1)\\ &+3\int_0^1[R^{1,1}(t,t)-2R^{2,0}(t,t)]f(t)f'(t)\,dt-3\int_0^1 R^{3,0}(t-,t)f^2(t)\,dt.\end{aligned}$$

By putting in Theorem 2, $R(t,s)\equiv 1$, we obtain the precise rate conjectured in [9] for the integral approximation of nonrandom functions, plus of course the asymptotic constant.

COROLLARY. *If f has continuous second derivative on* $[0,1]$ *then*

$$(1.9a) \qquad E[I(f)-I_n(f)]^2=\frac{[f'(1)-f'(0)]^2+o(1)}{4(n+1)(n+2)(n+3)(n+4)}.$$

Thus,

$$(1.9b) \qquad \lim_{n\to\infty} n^4E[I(f)-I_n(f)]^2=\tfrac{1}{4}[f'(1)-f'(0)]^2.$$

It is interesting to note that the asymptotic constant in (1.9b) for the trapezoidal rule with random samples has the same functional form as the classical asymptotic constant for the trapezoidal rule with equally-spaced samples [7, Thm. 3.3], but is larger by a factor of 36.

Returning to the general case of Theorem 2, we note that the assumption of continuous mixed partial derivatives of R of order up to 2 on $[0, 1]^2$ is equivalent to the assumption that X has one mean-square continuous quadratic-mean derivative. The additional assumption of differentiability of order 3 off the diagonal is weak, and is always satisfied, e.g., when X is stationary, has rational spectral density, and exactly one quadratic-mean derivative.

The expression of $C(f, R)$ in (1.8a) is not symmetric and cannot be symmetrized under the conditions of Theorem 2. If $R(t, s)$ has continuous mixed partial derivatives of order 3 throughout the unit square (rather than merely off its diagonal) then the asymptotic constant in (1.8a) takes the following simple and symmetric form:

$$C_{sym}(f, R) = \tfrac{1}{4}\{M^{1,1}(0,0) + M^{1,1}(1,1) - 2M^{1,1}(0,1)\}, \tag{1.10}$$

where $M(t, s) = f(t)R(t, s)f(s)$. The expression within braces in (1.10) equals $E[(fX)'(1) - (fX)'(0)]^2$, where prime denotes quadratic-mean derivative. Thus under the slightly more stringent assumption of continuous mixed partial derivatives of order 3 on the unit square we obtain

$$E[I(fX) - I_n(fX)]^2 = \frac{E[(fX)'(1) - (fX)'(0)]^2 + o(1)}{4(n+1)(n+2)(n+3)(n+4)}. \tag{1.11}$$

This expression is the stochastic analogue of the nonrandom case in (1.9a) and it holds even though X is not assumed to have a second quadratic-mean derivative. It is clear from (1.11) that in general the asymptotic constant is positive and no faster rate of convergence can be achieved by requiring any further smoothness of f or of R. Under the conditions of Theorem 2, the asymptotic constant differs from its symmetric expression by the following:

$$\begin{aligned} C(f, R) = C_{sym}(f, R) + \frac{3}{2}\Big\{\int_0^1 [R^{1,1}(t,t) - 2R^{2,0}(t,t)]f(t)f'(t)\,dt \\ -\frac{1}{2}[R^{1,1}(t,t) - 2R^{2,0}(t,t)f^2(t)]_0^1 - \int_0^1 R^{3,0}(t-,t)f^2(t)\,dt\Big\}. \end{aligned} \tag{1.8b}$$

When the random process X is stationary, i.e., $R(t, s) = R(t - s)$, then the general form (1.8a) of the asymptotic constant simplifies to

$$\begin{aligned} 2C_{st}(f, R) &= \frac{1}{2}R(0)[f'(0)^2 + f'(1)^2] - R(1)f'(0)f'(1) + R'(1)[f(0)f'(1) - f'(0)f(1)] \\ &\quad -\frac{1}{2}R''(0)[f^2(0) + f^2(1)] + R''(1)f(0)f(1) - 3R'''(0-)\int_0^1 f^2 \\ &= \frac{1}{2}E[(fX)'(1) - (fX)'(0)]^2 - 3R'''(0-)\int_0^1 f^2. \end{aligned} \tag{1.12}$$

When instead of using in the trapezoidal rule (1.4a) ordered random samples, we use equidistant samples $t_i = i/(n+1)$, $i = 0, 1, \cdots, n+1$, then it has been shown in [2, App. B] that for stationary processes, under the assumptions of Theorem 2, we have

$$E[I(fX) - I_n(fX)]^2 = \frac{C_{ed}(f, R) + o(1)}{n^4}$$

where

$$C_{ed}(f, R)=\frac{1}{72}\left\{\frac{1}{2} R(0)[f'(0)^2+f'(1)^2]-R(1)f'(0)f'(1)+R'(1)[f(0)f'(1)-f'(0)f'(1)]\right.$$

$$\left.-\frac{1}{2} R''(0)[f^2(0)+f^2(1)]+R''(1)f(0)f(1)\right\}-\frac{1}{360} R'''(0-)\int_0^1 f^2$$

or

$$C_{ed}(f, R)=\frac{1}{144} E[(fX)'(1)-(fX)'(0)]^2+\frac{1}{360}[-R'''(0-)]\int_0^1 f^2<\frac{1}{36} C_{st}(f, R).$$

It follows that $(C_{st}/C_{ed})^{1/4}>(36)^{1/4}\simeq 2.45$ and thus, asymptotically, at least two-and-a-half times more random samples are required than equidistant samples for the same accuracy measured in mean-square error. It is of course quite natural that equidistant samples provide a superior approximation than ordered random samples with the same average distance.

The analysis carried out here suggests that kth-order quadrature rules based on ordered random samples should have mean-square error with rate of convergence $n^{-2(k+1)}$ when acting on nonrandom functions with continuous $(k+1)$st derivative, or random processes with (essentially) mean-square continuous kth quadratic-mean derivative, or on their products. (The trapezoidal rule considered here and in [9] is a first-order quadrature rule.)

It should be finally mentioned that, for integrals of nonrandom functions, Haber has developed a stratified Monte Carlo rule with rate n^{-3} [3]; a stratified and symmetrized Monte Carlo rule with rate n^{-5} [4]; and certain stratified stochastic quadrature formulas with rate n^{-1-2k} when approximating the integral of a nonrandom function f with continuous kth derivative [5]. For weighted integrals of random processes, a simple Monte Carlo rule with rate n^{-1} and a stratified Monte Carlo rule with rate n^{-3} have been developed by Schoenfelder [6] (see also [1]).

Example. We illustrate by an example the finite sample size performance of the trapezoidal rule with random sampling. We consider the stationary process $X(t)$ with correlation function

$$R(t)=(1+\beta|t|)\,e^{-\beta|t|}$$

where $\beta>0$. Note that $R(0)=1$ and that X has precisely one quadratic-mean derivative. For simplicity we take $f(t)\equiv 1$ so that the random integral to be estimated is $I(X)=\int_0^1 X(t)\,dt$ and its estimate is $I_n(X)$ of (1.4a). The variance σ^2 of $I(X)$ is given by

$$\sigma^2=E[I(X)]^2=\iint R=\frac{2}{\beta}\left(2-\frac{3}{\beta}+\left(1+\frac{3}{\beta}\right)e^{-\beta}\right).$$

Using Theorem 1, we find after some algebra that the mean-square error is given by

$$E[I(X)-I_n(X)]^2=-\frac{1}{2}\left\{\frac{(n+3)(n^3+4n^2+n+2)}{2\beta^2}\right.$$

$$\left.+\frac{n^4+4n^3+6n^2+9n+2}{\beta(n+1)}+\frac{n^4+4n^3+5n^2-4n-4}{2(n+1)(n+2)}\right\}$$

$$
\begin{aligned}
&+\frac{1}{2} e^{-\beta}\left\{\frac{\beta}{(n+1)(n+2)}+\frac{2n+5}{(n+1)(n+2)}+\frac{(n+3)^2}{\beta(n+1)}\right\} \\
&+\frac{1}{2} a(n,\beta)\left\{(n+2)-\frac{1}{\beta}(n+3)^2\right\} \\
&+\frac{1}{4} a(n-1,-\beta) n e^{-\beta}\left\{\beta(n+1)+(3n^2+6n+5)\right. \\
&\qquad +\frac{1}{\beta}(3n^3+12n^2+13n+10) \\
&\qquad \left.+\frac{1}{\beta^2}(n+3)(n^3+4n^2+n+2)\right\}
\end{aligned}
\tag{1.13}
$$

for $n \geqq 1$, where

$$a(n,\beta)=\int_0^1 e^{-\beta x} x^n \, dx.$$

For $n=0$, the mean-square error can be computed directly yielding

$$E[I(X)-I_0(X)]^2=\frac{1}{2}-\frac{6}{\beta^2}+e^{-\beta}\left[\frac{5}{2}+\frac{\beta}{2}+\frac{6}{\beta}+\frac{6}{\beta^2}\right].$$

The asymptotic constant $C_{st}(1, R)$ is given by

$$C_{st}(1,R)=\{1+6\beta+(\beta-1)\,e^{-\beta}\}\frac{\beta^2}{2}.$$

Let $m=2, 3, \cdots$ be the (true) sample size, $m=n+2$, with corresponding mean-square error mse $(m)=E[I(X)-I_{m-2}(X)]^2$. The fractional mean-square error is then given by mse $(m)/\sigma^2$.

In selecting appropriate values of β for numerical display of the finite sample size performance, the behavior of the fractional error mse $(0)/\sigma^2$ (based only on the endpoints $X(0)$ and $X(1)$) as a function of β was investigated. It is seen from Table 1 that for values $\beta \leqq 1$ the fractional mean-square error is too small so that $I_0(X)$ already provides a fairly accurate approximation of $I(X)$. We select therefore two

TABLE 1
The fractional error mse $(0)/\sigma^2$ *and the asymptotic constant* $C_{st}(1, R)$ *as functions of* β.

β	mse $(0)/\sigma^2$	$C_{st}(1, R)$
0.2	2.3624×10^{-4}	3.09×10^{-2}
0.4	1.686×10^{-3}	0.24
1	1.929×10^{-2}	3.5
2	9.862×10^{-2}	26.27
3	0.225	85.95
4	0.377	200.44
5	0.5376	387.84
6	0.698	666.22
7	0.854	1053.6
8	1.0058	1568.08
9	1.152	2227.54
10	1.295	3050.02

values $\beta=3$ and $\beta=5$ corresponding to moderate values of mse $(0)/\sigma^2$. The asymptotic constant $C_{st}(1, R)$ is monotonically increasing with β and is asymptotically equal to $3\beta^3$ for large β. Table 1 provides a few typical values.

In Fig. 1 the fractional mean-square error mse $(m)/\sigma^2$ is plotted as a function of the sample size $m=2,\cdots,26$, for $\beta=3$ and $\beta=5$. It is seen that for the smaller value of $\beta=3$, the fractional error is considerably smaller for each sample size m. This can be explained by the less rapid decay of $R(t)$ and hence the larger correlation, on the average, between consecutive samples so that $I_n(X)$ provides a better estimate of $I(X)$ in this case. The closeness of the fractional mean-square error to its asymptotic value,

$$\text{mse}\,(m)/\sigma^2 \sim \frac{C_{st}(1, R)/\sigma^2}{(m-1)m(m+1)(m+2)},$$

is displayed in Fig. 2 for parameter $\beta=3$. Note that the asymptotic value *overestimates* the true error for all sample sizes m in the plotted range. Naturally the discrepancy between the two values diminishes as m increases. To see more clearly the convergence of the scaled mean-square error $(m-1)m(m+1)(m+2)$ mse (m) to the asymptotic constant $C_{st}(1, R)$ as m increases, we display in Fig. 3 the values of these two quantities for $m=2,\cdots,26$ with the chosen values of β as parameter. Again for the smaller $\beta=3$ the discrepancy between these two quantities is smaller for each $m=2,\cdots,26$ than for $\beta=5$.

2. The mean-square error. In this section we derive the exact expression of the mean-square error given in Theorem 1.

Since the trapezoidal rule approximation I_n to the integral I is based on the ordered samples $t_{n,1}<t_{n,2}<\cdots<t_{n,n}$ obtained from n independent uniformly distributed samples on (0, 1), we need certain properties of the order statistics from the uniform distribution, which we summarize first. For brevity we will write t_k for $t_{n,k}$.

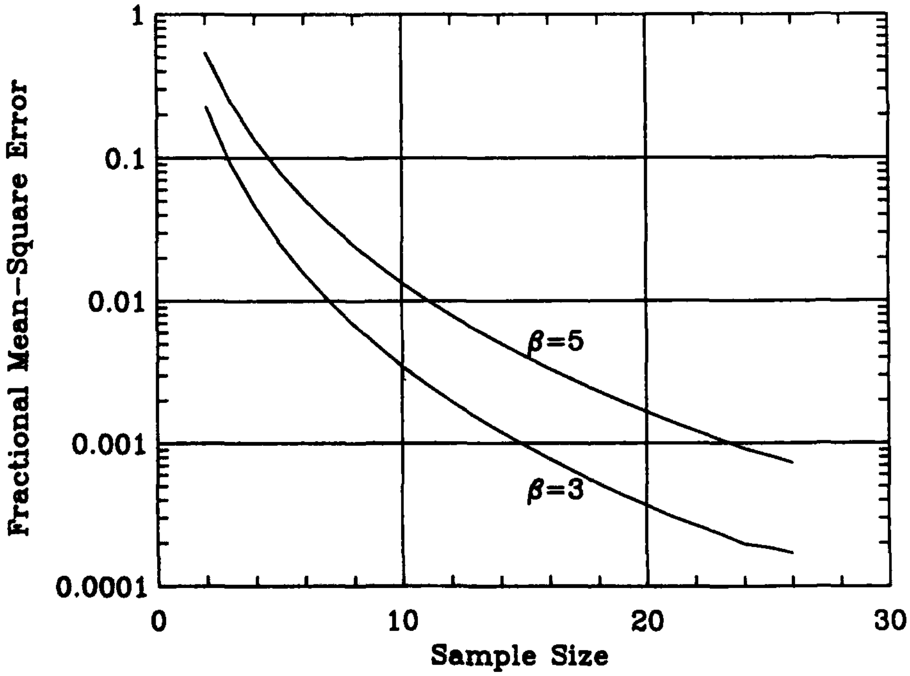

FIG. 1. *Fractional mean-square error mse $(m)/\sigma^2$ as a function of the sample size m.*

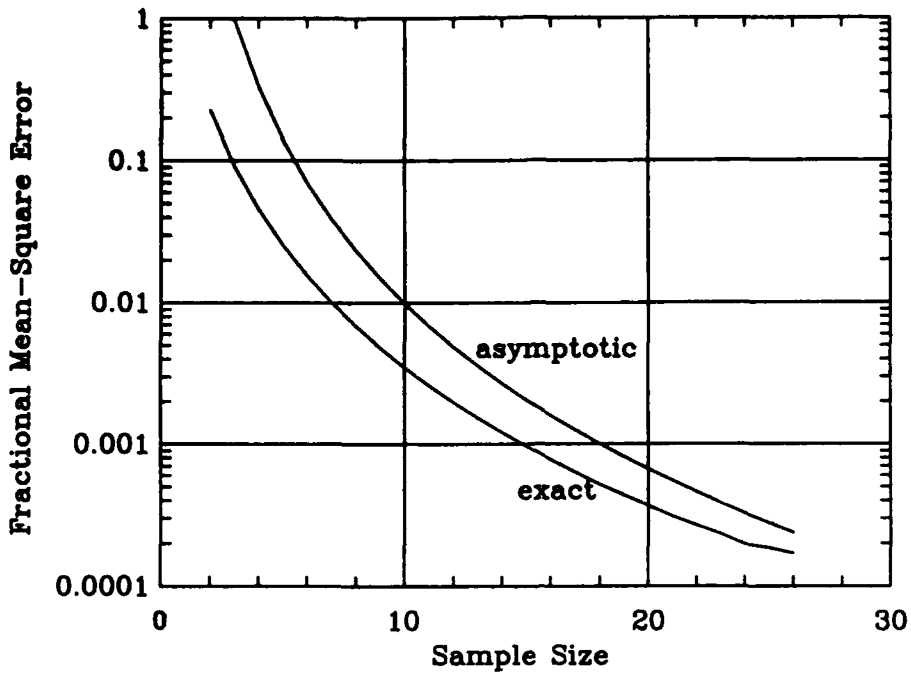

FIG. 2. *Exact and asymptotic fractional mean-square error as functions of the sample size* ($\beta = 3$).

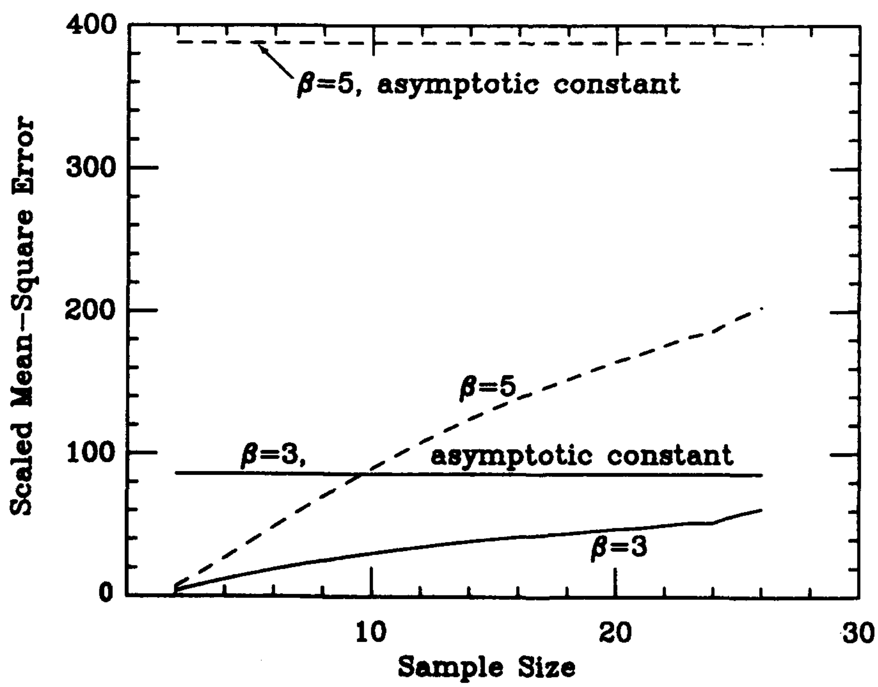

FIG. 3. *Scaled mean-square error* $(m-1)m(m+1)(m+2)\,\mathrm{mse}(m)$ *as a function of the sample size m.*

The joint distribution of order statistics is an ordered multivariate Dirichlet distribution; see Wilks [8, §§ 8.7.5 and 8.7.7]. Specifically, the ordered samples t_{k_1}, $t_{k_1+k_2}, \cdots, t_{k_1+k_2+\cdots+k_m}$ (where the k_i's are positive integers with $k_1+k_2+\cdots+k_m \leqq n$) have joint probability density function denoted by $p_{k_1,k_1+k_2,\cdots,k_1+\cdots+k_m}(x_1, x_2, \cdots, x_m)$ and given by

$$\frac{\Gamma(n+1)}{\Gamma(k_1)\cdots\Gamma(k_m)\Gamma(n+1-k_1-\cdots-k_m)} \cdot (x_1)^{k_1-1}(x_2-x_1)^{k_2-1}\cdots(x_m-x_{m-1})^{k_m-1}(1-x_m)^{n-k_1-\cdots-k_m}$$

for $0<x_1<x_2<\cdots<x_m<1$ and zero elsewhere. We will make explicit use of the following expressions.

The smallest- and largest-order statistics t_1 and t_n have densities

$$p_1(x)=n(1-x)^{n-1}, \qquad p_n(x)=nx^{n-1}, \qquad 0<x<1, \tag{2.1}$$

and joint density

$$p_{1,n}(x, y)=n(n-1)(y-x)^{n-2}, \qquad 0<x<y<1. \tag{2.2}$$

Two consecutive order statistics t_i and t_{i+1} $(1\leqq i\leqq n-1)$ have joint density

$$p_{i,i+1}(x, y)=n!\frac{x^{i-1}}{(i-1)!}\frac{(1-y)^{n-1-i}}{(n-1-i)!}, \qquad 0<x<y<1. \tag{2.3}$$

We will need the value of their sum

$$\sum_{i=1}^{n-1} p_{i,i+1}(x, y)=n!\sum_{k=0}^{n-2}\frac{x^k}{k!}\frac{(1-y)^{n-2-k}}{(n-2-k)!}=n(n-1)(1-y+x)^{n-2}. \tag{2.4}$$

Three consecutive order statistics t_i, t_{i+1}, t_{i+2} $(1\leqq i\leqq n-2)$ have joint density

$$p_{i,i+1,i+2}(x, y, z)=n!\frac{x^{i-1}}{(i-1)!}\frac{(1-z)^{n-2-i}}{(n-2-i)!}, \qquad 0<x<y<z<1,$$

and their sum will be used:

$$\sum_{i=1}^{n-2} p_{i,i+1,i+2}(x, y, z)=n!\sum_{k=0}^{n-2}\frac{x^k}{k!}\frac{(1-z)^{n-3-k}}{(n-3-k)!}=n(n-1)(n-2)(1-z+x)^{n-3}. \tag{2.5}$$

We will also use the following two trivariate densities

$$p_{1,i,i+1}(x, y, z)=n!\frac{(y-x)^{i-2}}{(i-2)!}\frac{(1-z)^{n-1-i}}{(n-1-i)!}, \qquad 0<x<y<z<1, \quad 2\leqq i\leqq n-1,$$

$$p_{i,i+1,n}(x, y, z)=n!\frac{x^{i-1}}{(i-1)!}\frac{(z-y)^{n-2-i}}{(n-2-i)!}, \qquad 0<x<y<z<1, \quad 1\leqq i\leqq n-2,$$

and their sums

$$\sum_{i=2}^{n-1} p_{1,i,i+1}(x, y, z)=n(n-1)(n-2)(1-z+y-x)^{n-3}, \tag{2.6}$$

$$\sum_{i=1}^{n-2} p_{i,i+1,n}(x, y, z)=n(n-1)(n-2)(z-y+x)^{n-3}. \tag{2.7}$$

Finally two pairs of consecutive order statistics t_i, t_{i+1}, t_j, t_{j+1} $(1<i+1<j<n)$ have joint density

$$p_{i,i+1,j,j+1}(x, y, z, w)=n!\frac{x^{i-1}}{(i-1)!}\frac{(z-y)^{j-i-2}}{(j-i-2)!}\frac{(1-w)^{n-1-j}}{(n-1-j)!}, \qquad 0<x<y<z<w<1.$$

We will need their double sum

$$
\begin{aligned}
\sum_{i=1}^{n-3}\sum_{j=i+2}^{n-1} p_{i,i+1,j,j+1}(x,y,z,w) &= n!\sum_{i=1}^{n-3}\frac{x^{i-1}}{(i-1)!}\sum_{k=0}^{n-3-i}\frac{(z-y)^k}{k!}\frac{(1-w)^{n-3-i-k}}{(n-3-i-k)!} \\
&= n!\sum_{i=1}^{n-3}\frac{x^{i-1}}{(i-1)!}\frac{(1-w+z-y)^{n-3-i}}{(n-3-i)!} \qquad (2.8)\\
&= n(n-1)(n-2)(n-3)(1-w+z-y+x)^{n-4}.
\end{aligned}
$$

Proof of Theorem 1. The expectation in $E[I(fX)-I_n(fX)]^2$ is with respect to both the random samples $\{t_i\}_{i=1}^n$ and the random process $\{X(t), 0\leqq t\leqq 1\}$ which are mutually independent. Performing first the expectation with respect to the random process $X(t)$ we find, with $M(t,s)=f(t)R(t,s)f(s)$,

$$
\begin{aligned}
E[I(fX)-I_n(fX)]^2 = E\Big\{&\int_0^1\int_0^1 M(t,s)\,dt\,ds \\
&-\sum_{i=0}^{n}\int_0^1 [M(t,t_i)+M(t,t_{i+1})]\,dt\cdot(t_{i+1}-t_i) \\
&+\frac{1}{4}\sum_{i=0}^{n}\sum_{j=0}^{n}[M(t_i,t_j)+M(t_i,t_{j+1})+M(t_{i+1},t_j) \qquad (2.9)\\
&+M(t_{i+1},t_{j+1})](t_{i+1}-t_i)(t_{j+1}-t_j)\Big\}.
\end{aligned}
$$

The interchange of expectation and integration is justified from

$$
\begin{aligned}
E\left\{\int_0^1 |f(t)X(t)|\,dt\right\}^2 &= \int_0^1\int_0^1 |f(t)f(s)|E\{|X(t)X(s)|\}\,dt\,ds \\
&\leqq \left\{\int_0^1 |f(t)|R^{1/2}(t,t)\,dt\right\}^2 < \infty
\end{aligned}
$$

since

$$E^2\{|X(t)X(s)|\}\leqq E\{X^2(t)\}E\{X^2(s)\}=R(t,t)R(s,s).$$

The more restrictive condition $\int_0^1 f^2(t)R(t,t)\,dt<\infty$ will be needed for the finiteness of the expected value with respect to the random samples and the further interchange of expectation and integration. As this is plain from the following expressions, it will not be discussed further.

To facilitate the computation of the expected value with respect to the random sampling points $t_1,\cdots,t_n$, we split the double summation in (2.9) into its diagonal part, the part corresponding to the two immediate parallels to the diagonal and the rest; which in view of the symmetry of $M(t,s)$ can be written as

$$\sum_{i=0}^{n}\sum_{j=0}^{n} = \sum_{i=j=0}^{n} + 2\sum_{i=0}^{n-1}(j=i+1) + 2\sum_{i=0}^{n-2}\sum_{j=i+2}^{n}.$$

Omitting for simplicity the terms in brackets, which are evident from (2.9), we write the mean-square error in the form

$$E[I(fX)-I_n(fX)]^2 = \int_0^1\int_0^1 M(t,s)\,dt\,ds \qquad (2.10)$$

$$(\triangleq A_1) \qquad -\sum_{i=0}^{n} E\int_0^1 [\cdots]\,dt\,\Delta t_i$$

$$(\triangleq A_2) \qquad +\frac{1}{4}\sum_{i=j=0}^{n} E[\cdots]\Delta t_i \Delta t_j$$

$$(\triangleq A_3) \qquad +\frac{1}{2}\sum_{i=0}^{n-1} E[\cdots]_{(j=i+1)}\Delta t_i \Delta t_{i+1}$$

$$(\triangleq A_4) \qquad +\frac{1}{2}\sum_{i=0}^{n-2}\sum_{j=i+2}^{n} E[\cdots]\Delta t_i \Delta t_j$$

with $\Delta t_i = t_{i+1} - t_i$. We now evaluate separately the cross term A_1, the diagonal term A_2, the second diagonal term A_3, and the off-diagonal term A_4, all clearly identified on the left margin in (2.10).

The cross term A_1. Since $t_0 = 0$ and $t_{n+1} = 1$, we isolate the first and last terms in the sum $\sum_{i=0}^{n}$, and write

$$\begin{aligned}
-A_1 &= E\int_0^1 [M(t, 0) + M(t, t_1)]\,dt\cdot t_1 \\
&\quad + \sum_{i=1}^{n-1} E\int_0^1 [M(t, t_i) + M(t, t_{i+1})]\,dt\cdot (t_{i+1} - t_i) \\
&\quad + E\int_0^1 [M(t, t_n) + M(t, 1)]\,dt\cdot (1 - t_n) \\
&= \int_0^1 dx \int_0^1 dt[M(t, 0) + M(t, x)]x p_1(x) \\
&\quad + \iint_{0<x<y<1} dx\,dy \int_0^1 dt[M(t, x) + M(t, y)](y - x)\left\{\sum_{i=1}^{n-1} p_{i,i+1}(x, y)\right\} \\
&\quad + \int_0^1 dy \int_0^1 dt[M(t, y) + M(t, 1)](1 - y)p_n(y).
\end{aligned}$$

We now use the expressions in (2.1) and (2.2) to write:

$$\begin{aligned}
-A_1 &= n\int_0^1 dt \int_0^1 dx\,[M(t, 0) + M(t, x)]x(1 - x)^{n-1} \\
&\quad + n(n-1)\int_0^1 dt \iint_{0<x<y<1} dx\,dy[M(t, x) + M(t, y)](y - x)(1 - y + x)^{n-2} \\
&\quad + n\int_0^1 dt \int_0^1 dy[M(t, y) + M(t, 1)](1 - y)y^{n-1}
\end{aligned}$$

and then we evaluate all inner integrals that can be computed to obtain

$$\begin{aligned}
-A_1 &= \frac{1}{n+1}\int_0^1 M(t, 0)\,dt + n\int_0^1\int_0^1 M(t, x)x(1 - x)^{n-1}\,dt\,dx \\
&\quad + n(n-1)\left\{\int_0^1\int_0^1 M(t, x)\left[\frac{1}{n(n-1)} - \frac{x^{n-1}}{n-1} + \frac{x^n}{n}\right]dt\,dx\right. \\
&\qquad \left. + \int_0^1\int_0^1 M(t, y)\left[\frac{1}{n(n-1)} - \frac{(1-y)^{n-1}}{n-1} + \frac{(1-y)^n}{n}\right]dt\,dy\right\} \\
&\quad + n\int_0^1\int_0^1 M(t, y)(1 - y)y^{n-1}\,dt\,dy + \frac{1}{n+1}\int_0^1 M(t, 1)\,dt.
\end{aligned}$$

Rearranging the terms we put A_1 in its final form:

$$
\begin{aligned}
A_1 &= -\frac{1}{n+1}\int_0^1 [M(x,1)+M(0,1-x)]\,dx \\
&\quad -\int_0^1\int_0^1 M(x,y)\{2-y^n-(1-y)^n\}\,dx\,dy.
\end{aligned} \tag{2.11}
$$

The diagonal term A_2. We proceed as for A_1. We first separate the terms with $t_0=0$ and $t_{n+1}=1$,

$$
\begin{aligned}
4A_2 &= E\{[M(0,0)+2M(0,t_1)+M(t_1,t_1)]t_1^2\} \\
&\quad +\sum_{i=1}^{n-1} E\{[M(t_i,t_i)+2M(t_i,t_{i+1})+M(t_{i+1},t_{i+1})](t_{i+1}-t_i)^2\} \\
&\quad + E\{[M(t_n,t_n)+2M(t_n,1)+M(1,1)](1-t_n)^2\}
\end{aligned}
$$

and then using (2.1) and (2.3) we obtain

$$
\begin{aligned}
4A_2 &= n\int_0^1 [M(0,0)+2M(0,x)+M(x,x)]x^2(1-x)^{n-1}\,dx \\
&\quad + n(n-1)\iint_{0<x<y<1} [M(x,x)+2M(x,y)+M(y,y)] \\
&\qquad \cdot (y-x)^2(1-y+x)^{n-2}\,dx\,dy \\
&\quad + n\int_0^1 [M(y,y)+2M(y,1)+M(1,1)](1-y)^2 y^{n-1}\,dy.
\end{aligned}
$$

We now evaluate all inner integrals that can be computed and regroup terms to reach the final expression

$$
\begin{aligned}
A_2 &= \frac{M(0,0)+M(1,1)}{2(n+1)(n+2)} + \frac{n}{2}\int_0^1 [M(x,1)+M(0,1-x)](1-x)^2 x^{n-1}\,dx \\
&\quad + \int_0^1 M(u,u)\left\{\frac{1}{n+1}+\frac{n}{2(n+1)}[u^{n+1}+(1-u)^{n+1}]-\frac{1}{2}[u^n+(1-u)^n]\right\}du \\
&\quad + \frac{1}{2}n(n-1)\iint_{0<x<y<1} M(x,y)(y-x)^2(1-y+x)^{n-2}\,dx\,dy.
\end{aligned} \tag{2.12}
$$

The second diagonal term A_3. We first split off the sum $\sum_{i=0}^{n-1}$ the first and last terms involving $t_0=0$ and $t_{n+1}=1$,

$$
\begin{aligned}
2A_3 &= E\{[M(0,t_1)+M(0,t_2)+M(t_1,t_1)+M(t_1,t_2)]t_1(t_2-t_1)\} \\
&\quad + \sum_{i=1}^{n-2} E\{[M(t_i,t_{i+1})+M(t_i,t_{i+2})+M(t_{i+1},t_{i+1})+M(t_{i+1},t_{i+2})] \\
&\qquad \cdot (t_{i+1}-t_i)(t_{i+2}-t_{i+1})\} \\
&\quad + E\{[M(t_{n-1},t_n)+M(t_{n-1},1)+M(t_n,t_n)+M(t_n,1)](t_n-t_{n-1})(1-t_n)\}.
\end{aligned}
$$

Next we use the values of the bivariate densities $p_{1,2}$ and $p_{n-1,n}$ from (2.3) and the sum of the consecutive bivariate densities in (2.5) and write

$$2A_3 = n(n-1)\iint_{0<x<y<1} [M(0,x)+M(0,y)+M(x,x)+M(x,y)] \cdot x(y-x)(1-y)^{n-2}\,dx\,dy$$
$$+\,n(n-1)(n-2)\iiint_{0<x<y<z<1} [M(x,y)+M(x,z)+M(y,y)+M(y,z)] \cdot (y-x)(z-y)(1-z+x)^{n-3}\,dx\,dy\,dz$$
$$+\,n(n-1)\iint_{0<x<y<1} [M(x,y)+M(x,1)+M(y,y)+M(y,1)] \cdot (y-x)(1-y)x^{n-2}\,dx\,dy.$$

We again evaluate all inner integrals which can be computed and regroup terms to derive the following expression, after considerable algebra,

$$A_3 = \frac{1}{2}n(n-1)\int_0^1 [M(x,1)+M(0,1-x)]x^{n-2}(1-x)\left[\frac{x^2}{n(n-1)}+\frac{1}{6}(1-x)^2\right]dx$$
$$+\frac{1}{2(n+1)}\int_0^1 M(u,u)[1-u^{n+1}-(1-u)^{n+1}]\,du$$
$$+\frac{1}{2}n(n-1)\iint_{0<x<y<1} M(x,y)(y-x)\left\{-\frac{1}{n-1}[x^{n-1}+(1-y)^{n-1}]\right.$$
$$\left.+\frac{1}{6}(n-2)(y-x)^2(1-y+x)^{n-3}+\frac{2}{n-1}(1-y+x)^{n-1}\right\}dx\,dy. \tag{2.13}$$

The off-diagonal term A_4. We first isolate the terms involving the points $t_0 = 0$ and $t_{n+1} = 1$ by splitting the double sum into

$$\sum_{i=0}^{n-2}\sum_{j=i+2}^{n} = (i=0, j=n) + \sum_{j=2}^{n-1}(i=0) + \sum_{i=1}^{n-2}(j=n) + \sum_{i=1}^{n-3}\sum_{j=i+2}^{n-1}.$$

We thus write A_4 as

$$2A_4 = E\{[M(0,t_n)+M(0,1)+M(t_1,t_n)+M(t_1,1)]t_1(1-t_n)\}$$
$$+\sum_{j=2}^{n-1} E\{[M(0,t_j)+M(0,t_{j+1})+M(t_1,t_j)+M(t_1,t_{j+1})]t_1(t_{j+1}-t_j)\}$$
$$+\sum_{i=1}^{n-2} E\{[M(t_i,t_n)+M(t_i,1)+M(t_{i+1},t_n)+M(t_{i+1},1)](t_{i+1}-t_i)(1-t_n)\}$$
$$+\sum_{i=1}^{n-3}\sum_{j=i+2}^{n-1} E\{[M(t_i,t_j)+M(t_i,t_{j+1})$$
$$+M(t_{i+1},t_j)+M(t_{i+1},t_{j+1})](t_{i+1}-t_i)(t_{j+1}-t_j)\}.$$

In calculating the expectation we use for the first term the value of $p_{1,n}$ from (2.2), for the second term the sum in (2.6), for the third term the sum in (2.7), and for the fourth term the sum in (2.8), and obtain

$$
\begin{aligned}
2A_4 = n(n-1) \iint_{0<x<y<1} & [M(0,y)+M(0,1)+M(x,y)+M(x,1)] \\
& \cdot x(1-y)(y-x)^{n-2}\,dx\,dy \\
+ n(n-1)(n-2) \iiint_{0<x<y<z<1} & [M(0,y)+M(0,z)+M(x,y)+M(x,z)] \\
& \cdot x(z-x)(1-z+y-x)^{n-3}\,dx\,dy\,dz \\
+ n(n-1)(n-2) \iiint_{0<x<y<z<1} & [M(x,z)+M(x,1)+M(y,z)+M(y,1)] \\
& \cdot (y-x)(1-z)(z-y+x)^{n-3}\,dx\,dy\,dz \\
+ n(n-1)(n-2)(n-3) \iiiint_{0<x<y<z<w<1} & [M(x,z)+M(x,w)+M(y,z) \\
& + M(y,w)](y-x)(w-z)(1-w+z-y+x)^{n-4}\,dx\,dy\,dz\,dw.
\end{aligned}
$$

Now we evaluate all inner integrals that can be computed and we regroup similar terms. After extensive but routine calculations we find

$$
\begin{aligned}
A_4 = \int_0^1 \int_0^1 & M(x,y)\,dx\,dy + \frac{M(0,1)}{2(n+1)(n+2)} + \frac{1}{2}\int_0^1 [M(x,1)+M(0,1-x)] \\
& \cdot \left\{ \frac{2}{n+1} - [x^n+(1-x)^n] + \frac{n}{n+1}[x^{n+1}+(1-x)^{n+1}] + x(1-x)^n \right. \\
& \left. + \frac{1}{2}n(n-1)(n-2)\left[-\frac{x^{n-2}}{3(n-2)} + \frac{x^{n-1}}{n-1} - \frac{x^n}{n} + \frac{x^{n+1}}{3(n+1)} \right] \right\} dx \\
& + \frac{1}{2}n(n-1) \iint_{0<x<y<1} M(x,y)A(x,y)\,dx\,dy
\end{aligned}
\tag{2.14a}
$$

where

$$
\begin{aligned}
A(x,y) = x(1-y)(y-x)^{n-2} &+ x\left\{ 2\frac{(1-x)^{n-1}}{n-1} - (1-x)[(y-x)^{n-2} \right. \\
& \left. + (1-y)^{n-2}] + \frac{n-2}{n-1}[(y-x)^{n-1}+(1-y)^{n-1}] \right\} \\
& + (1-y)\left\{ 2\frac{y^{n-1}}{n-1} - y[(y-x)^{n-2}+x^{n-2}] + \frac{n-2}{n-1}[(y-x)^{n-1}+x^{n-1}] \right\} \\
& + \frac{2}{n}[y^n+(1-x)^n+(1-y+x)^n] - \frac{1}{n}[x^n+(1-y)^n+(y-x)^n] \\
& - \frac{2}{n-1}[y^{n-1}+(1-x)^{n-1}+(1-y+x)^{n-1}] \\
& - \frac{n-2}{n-1}[x^{n-1}+(1-y)^{n-1}+(y-x)^{n-1}] \\
& + x^{n-2}y(1-y+x)+(1-y)^{n-2}(1-x)(1-y+x)+(y-x)^{n-2}(1-x)y \\
& + \frac{1}{2}(n-2)(n-3)\left\{ \frac{(1-y+x)^n}{3n} - \frac{(1-y+x)^{n-1}}{n-1} + \frac{(1-y+x)^{n-2}}{n-2} - \frac{(1-y+x)^{n-3}}{3(n-3)} \right\}.
\end{aligned}
\tag{2.14b}
$$

We now substitute in (2.10) the expressions for A_1 to A_4 we derived in (2.11) to (2.14), and after grouping similar terms and some algebra we arrive at the expression in (1.6). □

3. The rate of convergence. In this section we determine the rate of convergence to zero of the mean-square error given in Theorem 2.

We will use the following expression for the integral for a function F with L ($\geqq 1$) continuous derivatives:

$$\int_0^1 F(x)x^n\,dx = \sum_{l=1}^{L} \frac{(-1)^{l-1}}{n^{(l)}} F^{(l-1)}(1) + \frac{(-1)^L}{n^{(L)}} \int_0^1 F^{(L)}(x)x^{n+L}\,dx \tag{3.1}$$

where n is a positive integer and

$$n^{(l)} = (n+1)\cdots(n+l). \tag{3.2}$$

Expression (3.1) is obtained by repeated integration by parts or use of Taylor's expansion for $F(x)$ about the point 1.

We will also use the following version of the approximation of a function by a delta sequence (whose proof is standard).

LEMMA. *Let $F(x)$ and $\{K_n(x)\}_{n=1}^{\infty}$ be Borel functions defined on* $[0, 1]$. *If F is bounded and (left) continuous at* 1 *and if the kernels K_n satisfy the following conditions*:

(i) $\displaystyle\int_0^1 K_n(x)\,dx = 1$ *for all n,*

(ii) $\displaystyle\int_0^1 |K_n(x)|\,dx \leqq C < \infty$ *for all n,*

(iii) $\displaystyle\lim_{n\to\infty} \int_0^{1-\delta} |K_n(x)|\,dx = 0$ *for all* $\delta \in (0, 1)$,

then

$$\lim_{n\to\infty} \int_0^1 F(x)K_n(x)\,dx = F(1-). \tag{3.3}$$

In particular, when F is as in the lemma we have

$$\lim_{n\to\infty} (n+1) \int_0^1 F(x)x^n\,dx = F(1-), \tag{3.4a}$$

$$\lim_{n\to\infty} (n+1)(n+2) \int_0^1 F(x)x^n(1-x)\,dx = F(1-), \tag{3.4b}$$

$$\lim_{n\to\infty} \frac{1}{2}(n+1)(n+2)(n+3) \int_0^1 F(x)x^n(1-x)^2\,dx = F(1-). \tag{3.4c}$$

Proof of Theorem 2. We proceed by determining first more detailed expressions in inverse powers of n for each of the integral terms in (1.6), which are then combined to produce the rate of convergence of the mean-square error. We denote by B_1 the first integral (1.6b), by B_2 the second integral (1.6c) and by B_3 the third double integral (1.6d). For convenience we set $M(x, y) = f(x)R(x, y)f(y)$.

The sectional integral B_1. Putting $F(x)=M(x,1)+M(0,1-x)$ and using (3.1) with $L=2$ (and changing variables to $y=1-x$ for the final term) we obtain after some algebra

$$B_1=-\frac{F(0)}{2n^{(2)}}-\frac{F(1)}{n^{(2)}}-\frac{F'(0)}{2n^{(3)}}+\frac{F'(1)}{2n^{(3)}}-\frac{1}{2n^{(3)}}\int_0^1 F''(1-x)x^{n+3}\,dx$$
$$+\int_0^1 F''(x)\left\{\frac{(n^2+3n-2)x^{n+3}}{4(n+1)(n+2)(n+3)}-\frac{x^{n+2}}{2(n+2)}+\frac{x^{n+1}}{4(n+1)}\right\}dx.$$

It is easily checked that the polynomials $k_n(x)$ within braces in the last integral are positive on $(0,1)$, with

$$\int_0^1 k_n(x)\,dx=\frac{1}{2n^{(4)}},$$

and that the kernels $K_n(x)=2n^{(4)}k_n(x)$ satisfy the conditions of the lemma. It then follows from (3.3) and (3.4a) that

$$B_1=-\frac{F(0)+2F(1)}{2n^{(2)}}+\frac{F'(1)-F'(0)}{2n^{(3)}}+\frac{F''(1)-F''(0)}{2n^{(4)}}+o(n^{-4}),$$

and using the form of F to express the coefficients in terms of M we find

$$B_1=-\frac{1}{n^{(2)}}\{M(0,0)+M(1,1)+M(0,1)\}$$

$$+\frac{1}{2n^{(3)}}\{M^{1,0}(1,1)-M^{1,0}(0,0)-M^{1,0}(0,1)+M^{0,1}(0,1)\} \tag{3.5}$$

$$+\frac{1}{2n^{(4)}}\{M^{2,0}(0,0)+M^{2,0}(1,1)-M^{2,0}(0,1)-M^{0,2}(0,1)\}+o(n^{-4}).$$

The diagonal integral B_2. The integral B_2 may be written in the form

$$B_2=\frac{3}{2(n+1)}\int_0^1 M(u,u)\,du+\int_0^1\{M(u,u)+M(1-u,1-u)\}$$
$$\cdot\left\{\frac{n-1}{2(n+1)}u^{n+1}-\frac{1}{2}u^n\right\}du.$$

Since $F(u)=M(u,u)+M(1-u,1-u)$ has two continuous derivatives we obtain from (3.1) with $L=2$, applied to the second term,

$$B_2=\frac{3}{2(n+1)}\int_0^1 M(u,u)\,du-\frac{3}{2n^{(2)}}F(1)-\frac{2}{n^{(3)}}F'(1)$$
$$+\int_0^1 F''(u)\left\{\frac{n-1}{2n^{(3)}}u^{n+3}-\frac{u^{n+2}}{2n^{(2)}}\right\}du.$$

The polynomials $k_n(u)$ within braces in the last integral are negative on $(0,1)$ with

$$\int_0^1 k_n(u)\,du=-\frac{5}{2n^{(4)}},$$

and it is easily checked that the kernels $K_n = (2/5)n^{(4)}k_n$ satisfy the conditions of the lemma. It then follows from (3.3) that

$$B_2 = \frac{3}{2(n+1)} \int_0^1 M(u, u)\, du - \frac{3F(1)}{2n^{(2)}} - \frac{2F'(1)}{n^{(3)}} - \frac{5F''(1)}{2n^{(4)}} + o(n^{-4}),$$

and using the form of F we obtain

$$\begin{aligned} B_2 = {} & \frac{3}{2(n+1)} \int_0^1 M(u, u)\, du - \frac{3}{2n^{(2)}} \{M(0, 0) + M(1, 1)\} \\ & + \frac{4}{n^{(3)}} \{M^{1,0}(1, 1) - M^{1,0}(0, 0)\} \\ & - \frac{5}{n^{(4)}} \{M^{2,0}(0, 0) + M^{2,0}(1, 1) + M^{1,1}(0, 0) + M^{1,1}(1, 1)\} + o(n^{-4}). \end{aligned} \tag{3.6}$$

The double integral B_3. In the expression of B_3 in (1.6d), by changing variables appropriately in each of the six double integral terms and grouping separately the first three and the last three terms, we can write B_3 in the following form:

$$\begin{aligned} B_3 = {} & \int_0^1 \left\{ \frac{1}{2} \int_x^1 [3M(x, y) + 3M(1-x, 1-y) + M(y-x, y)]\, dy \right\} x^n\, dx \\ & + \int_0^1 \left\{ \int_0^x M(x-y, 1-y)\, dy \right\} \left\{ \frac{1}{4}(n-2)(n+3)x^n - \frac{1}{2} n^2 x^{n-1} + \frac{1}{4} n(n-1)x^{n-2} \right\} dx \\ \triangleq {} & \int_0^1 F(x)x^n\, dx + \int_0^1 G(x) g_n(x)\, dx \end{aligned} \tag{3.7}$$

with the obvious identification for F, G, and g_n. Since f is assumed to have only two continuous derivatives, so do F and G. We therefore use first (3.1) with $L=2$ and then identify those terms in F'', G'' which are not differentiable for separate treatment. We first obtain (after some algebra)

$$\begin{aligned} B_3 = {} & \frac{F(1)}{n+1} - \frac{F'(1)}{n^{(2)}} + \frac{1}{n^{(2)}} \int_0^1 F''(x)x^{n+2}\, dx \\ & - \frac{3G(1)}{2(n+1)} + \frac{G'(1)}{n^{(2)}} + \int_0^1 G''(x) \left\{ \frac{(n-2)(n+3)}{4n^{(2)}} x^{n+2} - \frac{n}{2(n+1)} x^{n+1} + \frac{1}{4} x^n \right\} dx. \end{aligned} \tag{3.8}$$

Evaluating $2F(1) - 3G(1)$ and $G'(1) - F'(1)$, and denoting by $h_n(x)$ the polynomial in braces in the last integral, we have

$$\begin{aligned} B_3 = {} & -\frac{3}{2(n+1)} \int_0^1 M(u, u)\, du + \frac{1}{n^{(2)}} \{2M(0, 0) + 2M(1, 1) + \tfrac{1}{2} M(0, 1)\} \\ & + \frac{1}{n^{(2)}} \int_0^1 F''(x)x^{n+2}\, dx + \int_0^1 G''(x) h_n(x)\, dx. \end{aligned} \tag{3.9}$$

From the definition of F and G in (3.7) we evaluate their second derivatives after which we separate those terms which are not differentiable. We thus write

$$F'' = F_1 + F_2, \qquad G'' = G_1 + G_2$$

where

$$F_1(x) = -\tfrac{9}{2}[M^{1,0}(x, x) - M^{1,0}(1-x, 1-x)] - \tfrac{1}{2}[M^{0,1}(0, x) - M^{1,0}(0, x)], \tag{3.10a}$$

(3.10b) $F_2(x)=\frac{1}{2}\int_x^1 \{3M^{2,0}(x,y)+3M^{2,0}(1-x,1-y)+M^{2,0}(y-x,y)\}\,dy,$

(3.11a) $G_1(x)=-M^{0,1}(0,1-x)+M^{1,0}(0,1-x),$

(3.11b) $G_2(x)=\int_0^x M^{2,0}(x-y,1-y)\,dy,$

and F_1, G_1 are differentiable, while F_2, G_2 are not. Proceeding as before, using (3.1) with $L=1$, we obtain

$$\begin{aligned} B_{3,1} &\triangleq \frac{1}{n^{(2)}}\int_0^1 F_1(x)x^{n+2}\,dx+\int_0^1 G_1(x)h_n(x)\,dx \\ &= \frac{F_1(1)}{n^{(3)}}-\frac{1}{n^{(3)}}\int_0^1 F_1'(x)x^{n+3}\,dx \\ &\quad + G(1)\times\{0\}-\int_0^1 G_1'(x)\left\{\frac{(n-2)(n+3)}{4n^{(3)}}x^{n+3}-\frac{n}{2n^{(2)}}x^{n+2}+\frac{1}{4(n+1)}x^{n+1}\right\}dx. \end{aligned} \tag{3.12}$$

It is easily checked that the polynomials $k_n(x)$ within braces in the last integral are positive on (0, 1) with

$$\int_0^1 k_n(x)\,dx=\frac{3}{2n^{(4)}}$$

and that the kernels $K_n(x)=(2/3)n^{(4)}k_n(x)$ satisfy the assumptions of the Lemma. Thus from (3.3) and (3.4a) we have

$$B_{3,1}=\frac{F_1(1)}{n^{(3)}}-\frac{F_1'(1)}{n^{(4)}}-\frac{3G_1'(1)}{2n^{(4)}}+o(n^{-4}).$$

Evaluating $F_1(1)$ and $2F_1'(1)+3G_1'(1)$ from (3.10a) and (3.11a) we finally find

$$\begin{aligned} B_{3,1} &= \frac{1}{2n^{(3)}}\{9[M^{1,0}(0,0)-M^{1,0}(1,1)]+M^{1,0}(0,1)-M^{0,1}(0,1)\} \\ &\quad +\frac{1}{2n^{(4)}}\{9M^{2,0}(1,1)+6M^{2,0}(0,0)+M^{0,2}(0,1) \\ &\qquad +9M^{1,1}(1,1)+12M^{1,1}(0,0)-M^{1,1}(0,1)\}+o(n^{-4}). \end{aligned} \tag{3.13}$$

To complete the evaluation of B_3 in (3.9) we need to evaluate

$$B_{3,2}\triangleq\frac{1}{n^{(2)}}\int_0^1 F_2(x)x^{n+2}\,dx+\int_0^1 G_2(x)h_n(x)\,dx.$$

While F_2 and G_2 are not differentiable, when $M(t,s)=f(t)R(t,s)f(s)$ is substituted in (3.10b), (3.11b), some of the terms in the resulting expressions are differentiable and we first deal with these. We thus decompose F_2 and G_2 into

$$F_2=F_{2,1}+F_{2,2},\qquad G_2=G_{2,1}+G_{2,2}$$

where

$$\begin{aligned} F_{2,1}(x) &= \frac{1}{2}\int_x^1 \{6f'(x)R(x,y)f(y)+6f'(1-x)R(1-x,1-y)f(1-y) \\ &\quad +2f'(y-x)R^{1,0}(y-x,y)f(y)+3f(x)R^{2,0}(x,y)f(y) \\ &\quad +3f(1-x)R^{2,0}(1-x,1-y)f(1-y) \\ &\quad +f(y-x)R^{2,0}(y-x,y)f(y)\}\,dy, \end{aligned} \tag{3.14a}$$

(3.14b) $$F_{2,2}(x)=\frac{1}{2}\int_x^1 \{3f''(x)R(x,y)f(y)+3f''(1-x)R(1-x,1-y)f(1-y)$$
$$+f''(y-x)R(y-x,y)f(y)\}\,dy,$$

(3.15a) $$G_{2,1}(x)=\int_0^x \{2f'(x-y)R^{1,0}(x-y,1-y)f(1-y)$$
$$+f(x-y)R^{2,0}(x-y,1-y)f(1-y)\}\,dy,$$

(3.15b) $$G_{2,2}(x)=\int_0^x f''(x-y)R(x-y,1-y)f(1-y)\,dy.$$

As $F_{2,1}$ and $G_{2,1}$ are differentiable, the term

$$B_{3,2,1}\triangleq\frac{1}{n^{(2)}}\int_0^1 F_{2,1}(x)x^{n+2}\,dx+\int_0^1 G_{2,1}(x)h_n(x)\,dx$$

can be evaluated by using (3.1) with $L=1$ to obtain

$$B_{3,2,1}=\frac{1}{n^{(3)}}F_{2,1}(1)-\frac{1}{n^{(3)}}\int_0^1 F'_{2,1}(x)x^{n+3}\,dx+G_{2,1}(1)\times\{0\}-\int_0^1 G'_{2,1}(x)k_n(x)\,dx$$

with the same polynomials $k_n(x)$ as in (3.12). It then follows as for $B_{3,1}$ that

$$B_{3,2,1}=\frac{F_{2,1}(1)}{n^{(3)}}-\frac{F'_{2,1}(1)}{n^{(4)}}-\frac{3G'_{2,1}(1-)}{2n^{(4)}}+o(n^{-4})$$

and evaluating $F_{2,1}(1)(=0)$ and $2F'_{2,1}(1)+3G'_{2,1}(1-)$ from (3.14a) and (3.15a) we obtain

(3.16)
$$B_{3,2,1}=\frac{1}{n^{(4)}}\Big\{6R^{1,0}(1,1)f(1)f'(1)+2R^{1,0}(0,1)f'(0)f(1)$$
$$+3R^{2,0}(1,1)f^2(1)+R^{2,0}(0,1)f(0)f(1)$$
$$-3\int_0^1 [2R^{1,0}(u,u)f(u)f''(u)+3R^{2,0}(u,u)f(u)f'(u)$$
$$+R^{3,0}(u-,u)f^2(u)]\,du\Big\}+o(n^{-4}).$$

To complete the evaluation of $B_{3,2}$ we finally need to evaluate

$$B_{3,2,2}\triangleq\frac{1}{n^{(2)}}\int_0^1 F_{2,2}(x)x^{n+2}\,dx+\int_0^1 G_{2,2}(x)h_n(x)\,dx$$

where $F_{2,2}$ and $G_{2,2}$ are given in (3.14b) and (3.15b) and h_n is the polynomial in braces in (3.8). Substituting $F_{2,2}$ and $G_{2,2}$ and isolating the nondifferentiable factor f'' we can write it in the form:

$$B_{3,2,2}=\frac{3}{2n^{(2)}}\int_0^1 f''(x)x^{n+2}\left\{\int_x^1 R(x,y)f(y)\,dy\right\}dx$$
$$+\frac{3}{2n^{(2)}}\int_0^1 f''(1-x)x^{n+2}\left\{\int_x^1 R(1-x,1-y)f(1-y)\,dy\right\}dx$$
$$+\frac{1}{2n^{(2)}}\int_0^1 f''(u)\left\{\int_u^1 R(u,v)f(v)(v-u)^{n+2}\,dv\right\}du$$
$$+\int_0^1 f''(u)\left\{\int_u^1 R(u,v)f(v)h_n(1+u-v)\,dv\right\}du.$$

We denote the four terms by T_i, $i=1, 2, 3, 4$, and the corresponding functions in braces by H_i. For the first two terms we use the Taylor expansion about 1,

$$H(x)=H(1)-H'(1)(1-x)+\tfrac{1}{2}H''(t_x)(1-x)^2$$

where the intermediate point t_x belongs to $(x, 1)$ and depends continuously on x. We thus find

$$\begin{aligned}T_1+T_2=\frac{3}{2n^{(2)}}\bigg\{&-H_1'(1)\int_0^1 f''(x)x^{n+2}(1-x)\,dx+\frac{1}{2}\int_0^1 f''(x)H_1''(t_{1,x})x^{n+2}(1-x)^2\,dx\\&-H_2'(1)\int_0^1 f''(1-x)x^{n+2}(1-x)\,dx\\&+\frac{1}{2}\int_0^1 f''(1-x)H_2''(t_{2,x})x^{n+2}(1-x)^2\,dx\bigg\}\end{aligned}$$

and from (3.4b), (3.4c) we obtain

$$T_1+T_2=\frac{3}{2n^{(4)}}\{-H_1'(1)f''(1)-H_2'(1)f''(0)\}+o(n^{-4})+O(n^{-5})$$

and finally

$$T_1+T_2=\frac{3}{2n^{(4)}}\{R(1,1)f(1)f''(1)+R(0,0)f(0)f''(0)\}+o(n^{-4}). \tag{3.17}$$

For the third term, T_3, we first integrate by parts the expression of H_3 to write it in the form

$$H_3(u)=\frac{1}{n+3}R(u,1)f(1)(1-u)^{n+3}-\frac{1}{n+3}\int_u^1 D_v[R(u,v)f(v)](v-u)^{n+3}\,dv$$

where D_v denotes partial derivative with respect to v, and substitute in T_3:

$$\begin{aligned}T_3=&\frac{f(1)}{2n^{(3)}}\int_0^1 f''(u)R(u,1)(1-u)^{n+3}\,du\\&-\frac{1}{2n^{(3)}}\iint_{0<u<v<1} f''(u)D_v[R(u,v)f(v)](v-u)^{n+3}\,dv.\end{aligned}$$

Since the integrand of the double integral is bounded and $\iint_{u<v}(v-u)^k\,du\,dv=[(k+1)(k+2)]^{-1}$, the double integral is $O(n^{-5})$ and thus by (3.4a) we obtain

$$T_3=\frac{1}{2n^{(4)}}f(1)f''(0)R(0,1)+o(n^{-4}). \tag{3.18}$$

For the fourth term T_4 we likewise first integrate by parts the inner integrand H_4. Noting that $h_n=k_n'$ where k_n is the polynomial within braces in the integral in (3.12), we obtain by integrating by parts

$$\begin{aligned}H_4(u)&=\int_u^1 R(u,1+u-z)f(1+u-z)k_n'(z)\,dz\\&=R(u,u)f(u)k_n(1)-R(u,1)f(1)k_n(u)\\&\quad-\int_u^1 D_z[R(u,1+u-z)f(1+u-z)]k_n(z)\,dz.\end{aligned}$$

Since $k_n(1)=0$, and as was pointed out following (3.12), $(2/3)n^{(4)}k_n$ satisfies the assumptions of the Lemma, we obtain by (3.3),

(3.19)
$$T_4 = -\int_0^1 \left\{ f''(z)R(z,1)f(1) + \int_0^z f''(u)D_z[R(u,1+u-z)f(1+u-z)]\,du \right\} k_n(z)\,dz$$
$$= -\frac{3}{2n^{(4)}}\left\{ f''(1)R(1,1)f(1) - \int_0^1 f''(u)[R^{0,1}(u,u)f(u) + R(u,u)f'(u)]\,du \right\} + o(n^{-4}).$$

Putting together the expressions in (3.17) to (3.19), we find

(3.20)
$$B_{3,2,2} = \frac{1}{2n^{(4)}}\left\{ 3R(0,0)f(0)f''(0) + R(0,1)f''(0)f(1) + 3\int_0^1 f''(u)[R^{0,1}(u,u)f(u) + R(u,u)f'(u)]\,du \right\} + o(n^{-4}).$$

Now from (3.9), (3.13), (3.16), and (3.20), we derive the final expression for B_3:

(3.21)
$$\begin{aligned} B_3 = {} & -\frac{3}{2(n+1)}\int_0^1 M(u,u)\,du + \frac{1}{n^{(2)}}\left\{ 2M(0,0) + 2M(1,1) + \frac{1}{2}M(0,1) \right\} \\ & + \frac{1}{2n^{(3)}}\{9[M^{1,0}(0,0) - M^{1,0}(1,1)] + M^{1,0}(0,1) - M^{0,1}(0,1)\} \\ & + \frac{1}{2n^{(4)}}\Big\{ 6M^{2,0}(0,0) + 9M^{2,0}(1,1) + M^{0,2}(0,1) + 12M^{1,1}(0,0) \\ & \quad + 9M^{1,1}(1,1) - M^{1,1}(0,1) + 3R(0,0)f(0)f''(0) + R(0,1)f''(0)f(1) \\ & \quad + 6R^{1,0}(1,1)f(1)f'(1) + 2R^{1,0}(0,1)f'(0)f(1) + 3R^{2,0}(1,1)f^2(1) \\ & \quad + R^{2,0}(0,1)f(0)f(1) + 3\int_0^1 [f''Rf' - f''R^{1,0}f - 3f'R^{2,0}f](u,u)\,du \\ & \quad - 3\int_0^1 R^{3,0}(u-,u)f^2(u)\,du \Big\} + o(n^{-4}). \end{aligned}$$

The rate of convergence. Substituting the expressions (3.5), (3.6), (3.21) of B_1, B_2, B_3 into the expression (1.6) of the mean-square error, we find that all lower-order terms cancel and we obtain (1.7), where the constant $C(f,R)$ is readily identified from the coefficients of $(n^{(4)})^{-1}$ in (3.5), (3.6), and (3.21). In addition to expressing M in terms of f and R in the coefficient of $(n^{(4)})^{-1}$ in (3.21), we also use the following expressions for some of the integrals involved, which follow by integration by parts:

$$\int_0^1 [f''Rf'](u,u)\,du = \frac{1}{2}\{[f'Rf'](1,1) - [f'Rf'](0,0)\} - \int_0^1 [f'R^{1,0}f'](u,u)\,du,$$

$$\int_0^1 [f''R^{1,0}f](u,u)\,du = [f'R^{1,0}f](1,1) - [f'R^{1,0}f](0,0) - \int_0^1 [f'(R^{2,0}f + R^{1,1}f + R^{1,0}f')](u,u)\,du.$$

The resulting expression of the asymptotic constant $C(f,R)$ is given in (1.8). This completes the proof of Theorem 2. □

The asymptotic constant. The expression of $C(f, R)$ given in (1.8a) cannot be symmetrized any further under the current assumptions (note the lack of symmetry in the constant terms involving $R^{1,1}$ and in the integrals). However, if $R(t, s)$ is further assumed to have continuous mixed partial derivatives of order 3 throughout the unit square (rather than off its diagonal—as has so far been assumed), then by integrating by parts we find (using the obvious shorthand)

$$\int_0^1 (R^{1,1}-2R^{2,0})ff' = \tfrac{1}{2}[(R^{1,1}-2R^{2,0})f^2]_0^1 + \int_0^1 R^{3,0}f^2 \tag{3.22}$$

and thus the integral terms in $C(f, R)$ can be evaluated. This produces the following symmetric expression:

$$\begin{aligned} 2C_{\text{sym}}(f, R) = {} & \tfrac{1}{2}R(0,0)f'(0)^2 + \tfrac{1}{2}R(1,1)f'(1)^2 - R(0,1)f'(0)f'(1) \\ & + R^{1,0}(0,0)f(0)f'(0) + R^{1,0}(1,1)f(1)f'(1) \\ & - R^{1,0}(0,1)f(0)f'(1) - R^{0,1}(0,1)f'(0)f(1) \\ & + \tfrac{1}{2}R^{1,1}(0,0)f^2(0) + \tfrac{1}{2}R^{1,1}(1,1)f^2(1) - R^{1,1}(0,1)f(0)f(1). \end{aligned} \tag{3.23}$$

It is easily checked that this can be written as in (1.10), so that under these slightly more stringent assumptions, which fall short of guaranteeing two quadratic-mean derivatives for X, we have (1.11). In view of (3.22) and (3.23), the general form of the asymptotic constant in (1.8a) can be written as in (1.8b).

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