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19. Abstract

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In this paper we study topological properties of multi-Hilbertian spaces and their duals, hoping that this will serve as an introduction to a study of probability problems on these spaces. We tried to clearly distinguish properties that are consequences of nuclearity from those that hold on non-nuclear spaces.

In section 5, we propose a non-standard completion theorem, removing the condition of "compatibility" of norms, a condition that seems to be overlooked in most probability papers in this area. Also, we give a detailed account on open, bounded and compact sets. Elaborated proofs are left for appendices, and, as a result, appendices occupy a considerable space. This is mostly due to results related to seminorms that we wanted to make rigorous.

The results that are given in this paper are selected with a purpose to serve as a basis for probability investigation; the topology alone was not the aim. As a continuation of this work, we plan to investigate $\sigma-$ algebras and probability measures, and weak convergence of measures in a general context of multi-Hilbertian spaces. Also, we plan to investigate examples of interest in applications.

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MULTI-HILBERTIAN SPACES AND THEIR DUALS

by

Milan J. Merkle

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Multi-Hilbertian spaces and their duals

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and
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0 Introduction

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The results that are given in this paper are selected with a purpose to serve as a basis for probability investigation; the topology alone was not the aim. As a continuation of this work, we plan to investigate $\sigma-$ algebras and probability measures, and weak convergence of measures in a general context of multi-Hilbertian spaces. Also, we plan to investigate examples of interest in applications.

1 Seminorms

Let E be a real vector space. Unless stated otherwise, it will be assumed that all vector spaces are defined over the field of real numbers.

1. Definition. A real valued function p defined on E is called a *seminorm* if for all $x, x_1, x_2 \in E$ and all $a \in R$:
(i) $p(x) \geq 0$
(ii) $p(ax) = |a|p(x)$
(iii) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$
(iv) $p(x) > 0$ for some $x \in E$.

2. Definition. A seminorm on E is called a *Hilbertian seminorm* if, for all $x_1, x_2 \in E$:

$$p^2(x_1 + x_2) + p^2(x_1 - x_2) = 2(p^2(x_1) + p^2(x_2))$$

For a Hilbertian seminorm p , define

$$p(x_1, x_2) = \frac{1}{4}(p^2(x_1 + x_2) - p^2(x_1 - x_2))$$

3. Definition. A seminorm p is called *separable* if there is a countable set $D \subset E$ such that

$$(\forall x \in E)(\forall \varepsilon > 0)(\exists d \in D)p(x - d) \leq \varepsilon,$$

i.e., if the set D is $p-$ *dense* in E.

4. Remark. It can be shown that $p(\cdot,\cdot)$ satisfies axioms of an inner product, except that $p(x,x)$ can be 0 even if $x \neq 0$. It is important to remark that Cauchy-Schwartz inequality $p(x_1, x_2) \leq p(x_1)p(x_2)$ holds for seminorms. Using this inequality it can be shown that $p(x_n - x) \to 0$ and $p(y_n - y) \to 0$ together imply $p(x_n, y_n) \to p(x, y)$. More details on seminorms are given in Appendix A.

2 Operators on Hilbert spaces

1. Definition. Let H_1, H_2 be separable Hilbert spaces with norms $\|.\|_1$ and $\|.\|_2$, inner products $\langle\cdot\rangle_1$ and $\langle\cdot\rangle_2$ and orthonormal bases $\{e_n\}$ and $\{h_n\}$ respectively. Let A be a linear map $H_1 \to H_2$. We define the following classes of operators:

A is a *compact operator* if

$$Ax = \sum t_n \langle x, e_n \rangle h_n, \tag{1}$$

where $t_n \downarrow 0$.

A is a *Hilbert-Schmidt operator*(HS) if (1) holds with $\sum t_n^2 < \infty$.

A is a *nuclear operator* if (1) holds with $\sum t_n < \infty$. ($t_n > 0$).

2. Remark. For a compact operator A the image $A(S)$ of the unit sphere of H_1 is a relatively compact set in H_2.

3. Definition. Let A be an operator $A : H \to H$. Then

A is a *trace class operator* if $\sum |\langle Ae_n, e_n\rangle| < \infty$, for any ON basis $\{e_n\}$ in H.

A^* is the *adjoint mapping of* A if $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in H$.

A is a *self-adjoint operator* if $A^* = A$.

A is *positive* if A is self-adjoint and $\sum a_i \bar{a}_j \langle x_i, x_j\rangle \geq 0$, for any set of complex numbers $\{a_i\}$.

4. Theorem. Let A,B be operators $H \to H$. Then:

(*i*) If A and B are HS operators, then AB is a nuclear operator.

(*ii*) If A is a compact positive operator, then A is nuclear if and only if it is a trace class operator.

(*iii*) If A is a nuclear operator, then A^* is also nuclear. The same holds true for HS property.

(*iv*) If A is a positive nuclear operator, then there exists a positive HS operator $A^{1/2}$, such that $(A^{1/2})^2 = A$.

(*v*) A positive compact operator A can be represented as

$$Ax = \sum t_n \langle x, e_n \rangle e_n,$$

where $\{e_n\}$ are eigenvectors and t_n are eigenvalues of A.

Let A be an operator $H_1 \to H_2$.

(*vi*) A is HS if and only if $\sum_n \|Ae_n\|_2^2 < \infty$ and it is nuclear if and only if $\sum_n |\langle Ae_n, e_n\rangle_2| < \infty$, where $\{e_n\}$ is an orthonormal base in H_2. These properties hold for one orthonormal base if and only if they hold for every orthonormal base in H_2.

(vii) A is a compact operator if and only if the image of every bounded set in H_1 is a compact set in H_2.

Proof: See Gelfand and Vilenkin, ch.1.

5. Lemma. *If p is a Hilbertian seminorm on a Hilbert space H, continuous with respect to the norm $\|\cdot\|$ on H, then $p(x) = \|Ax\|$, for some continuous linear operator A.*

Proof: Assume first that p is a norm. Define the inner product $p(x,y)$, as in the Definition 2. For any fixed $x \in H$, $p(x,\cdot)$ is a continuous linear functional on H; therefore, by Riesz representation theorem, $p(x,y) = \langle x_0, y\rangle$. Define an operator φ, by $\varphi(x) = x_0$ ($x \in H$). By properties of an inner product $p(\cdot,\cdot)$, φ is a positive linear operator; let $A = \varphi^{1/2}$. Then $p(x,y) = \langle Ax, Ay\rangle$ and the assertion follows.

3 Definition of Multi-Hilbertian spaces

The family of all *separable* Hilbertian seminorms will be denoted by HSN

1. Lemma. If $p \in HSN$ then $cp \in HSN (c > 0)$.

2. Lemma. If $p_1, \ldots, p_n \in HSN$, then $p \in HSN$, where p is defined by

$$p(x) = (p_1^2(x) + \ldots p_n^2(x))^{1/2} \stackrel{\text{def}}{=} \bigvee_{i=1}^{n} p_i(x) \tag{1}$$

Proof: Only the triangle inequality needs to be proved. Using Minkowski inequality, we have:

$$\begin{aligned} p(x+y) &= (\sum p_i^2(x+y))^{1/2} \leq (\sum (p_i^2(x) + p_i^2(y))^{1/2} \\ &\leq (\sum p_i^2(x))^{1/2} + (\sum p_i^2(y))^{1/2} \\ &= p(x) + p(y). \end{aligned} \tag{2}$$

3. Definition. If $p, q \in HSN$, we write $p \prec q$ if $p(x) < cq(x)$ for some $c \in R_+$ and all $x \in E$.

4. Let $E_p = (E, p)$, i.e. E with the topology defined by a seminorm p. Let $\ker(p)=\{x \in E | p(x) = 0\}$ Then $E_p/\ker(p)$ removes the defficiency of seminorms and becomes a normed space with an inner product $p(\cdot, \cdot)$. It may not be a Hilbert space, as it need not be complete. Let $\bar{E}_p$ be the completion of $E/\ker(p)$ with respect to p ; it can be identified with the union of E and all limit points of Cauchy sequences in E. $\bar{E}_p$ is a Hilbert space.

If $p \prec q$ then the identity mapping

$$i_{p,q} : E_q \to E_p \tag{3}$$

is continuous, by $q(x) < \varepsilon \Rightarrow p(i(x)) = p(x) < \varepsilon / c$. Let us extend it to

$$i_{p,q} : E_q/\ker(q) \to E_p/\ker(p)$$

by

$$i_{p,q}(x + \ker(q)) = x + \ker(p) \tag{4}$$

Let us show that the mapping defined by (4) is well defined, i.e.

$$y_1, y_2 \in x + \ker(q) \Rightarrow y_1, y_2 \in x + \ker(p) \tag{5}$$

Indeed, $y_1, y_2 \in x + \ker(q) \iff q(y_1 - y_2) = 0$ and $q(y_1 - x) = 0 \Rightarrow p(y_1 - y_2) = 0$ and $q(y_1 - x) = 0 \iff y_1, y_2 \in x + \ker(p)$, where in the last step we used the fact that $p \prec q$. Note that the mapping (4) is *not invertible*, by $p(x - y) = 0 \not\Rightarrow q(x - y) = 0$.

Let us now go through completion. Suppose $x \in \bar{E}_q$. Then there is a sequence $x_n \in E_q/\ker(q), q(x_n - x) \to 0$. So, $\{x_n\}$ is a q-Cauchy sequence. thus

$$q(x_m - x_n) \to 0 \qquad (m, n \to \infty)$$

Therefore:

$$p(i_{p,q}(x_m) - i_{p,q}(x_n)) \to 0$$

and $i_{p,q}(x_n)$ is a p-Cauchy sequence. So, there is an $y \in \bar{E}_p$, such that p-lim $i_{p,q}(x_n) = y$. Define $i_{p,q}(x) = y$. Now the mapping $i_{p,q}$ is extended to

$$i_{p,q} : \bar{E}_q \to \bar{E}_p \tag{6}$$

This is clearly a linear continuous mapping.

For $x \in \bar{E}_q$, let us define $p(x) = p(i_{p,q}x)$. On $\bar{E}_q, p$ is only a seminorm,whereas q is a norm on the same space, as we have seen.

5. Definition. Let p, q be Hilbertian semi-norms.We say that $p <_{HS} q$ if $p \prec q$ and $i_{p,q} : \bar{E}_q \rightarrow \bar{E}_p$ is a *HS* operator.

6. Lemma. Suppose that $\{e_n\}$ is an orthonormal base in $\bar{E}_q$. Then $p <_{HS} q$ if and only if $p \prec q$ and

$$\sum_n p^2(i_{p,q}(e_n)) < \infty \tag{7}$$

Proof: By 2.4(vi).

7.Lemma. Suppose that $p <_{HS} q$. Then the quantity defined by (7) does not depend on the choice of an orthonormal base $\{e_n\}$ and depends only on p and q.

Proof: See Appendix A.

8. Definition. Let $\Pi = \{p_j\}_{j \in J}$ be a family of HSN (separable Hilbertian semi-norms) on E such that

$$(\forall x, y \in E: \ x \neq y)\,(\exists p \in \Pi)\, p(x - y) \neq 0 \tag{8}$$

Define the topology τ by a basis of neighborhoods of 0:

$$\{x \in E \mid p_1(x) < \varepsilon_1, \ldots, p_n(x) < \varepsilon_n\} \qquad (n \in N, p_1, \ldots, p_n \ \in \Pi)$$

Topology τ is called a *multi-Hilbertian topology*(determined by Π).

The space E with the topology τ determined by Π is called a *multi-Hilbertian space*. In this situation, we will use notation (E, τ, Π) or (E, τ).

To avoid trivial cases, we will assume that Π contains *at least* countably many topologically non-equivalent seminorms.

9. Lemma. If τ is a multi-Hilbertian topology determined by a family Π, then it is also determined by a family Π' defined by:

$$\Pi' = \{q \mid q = \bigvee_{i=1}^{n} p_i, n \in N\}$$

Proof: We have to show that neighborhood bases of 0 are equal . Let U be an element of neigborhood basis of 0 generated by Π :

$$U = \{x \in E \mid p_1(x) < \varepsilon_1, \ldots, p_n(x) < \varepsilon_n\}$$

Then $U \subset V$,where

$$V = \{x \in E \mid p_1^2(x) + \ldots + p_n^2(x) < n\varepsilon^2\} \quad (\varepsilon = \max\{\varepsilon_1^2, \ldots, \varepsilon_n^2\})$$

i.e.,

$$V = \{x \in E \mid \bigvee_{i=1}^{n} p_i(x) < \varepsilon\sqrt{n}\}$$

Suppose that V is a neighborhood of 0 in Π':

$$V = \{x \in E \mid \bigvee_{i=1}^{n_1} p_{1i}(x) < \varepsilon_1, \ldots, \bigvee_{i=1}^{n_k} p_{ki}(x) < \varepsilon_k\}.$$

Then $V \subset U$, where

$$U = \{x \in E \mid p_1(x) < \varepsilon_1, \ldots, p_k(x) < \varepsilon_k\}.$$

Therefore, both Π and Π' generate the same topology τ.

10. Remark. In the view of Lemma 9, we may assume (when we need it) that τ is determined by a directed system of seminorms. If Π is a countable set, then then we may assume :

$$\Pi = \{p_1, p_2, \ldots\} \qquad (p_1 < p_2 < \ldots)$$

In any multi-Hilbertian space we may also assume (without loss of generality) the following:

If $p_1, \ldots, p_n \in \Pi$, then there is a $p_0 \in \Pi$ such that $\bigvee_{i=1}^{n} p_i \prec p_0$.

The latter asssumption is a techical requirement that will be useful later. By Lemma 9, it does not affect the topological structure of τ.

11. Remark. Notation $p = \bigvee_{i=1}^{n} p_i$ is due to the fact that the p-topology on E is coarser (weaker) than any topology which is finer (stronger) than any of p_i . Let us prove this fact:

Suppose that ψ is a topology finer than any of topologies determined by $p_i (i = 1, \ldots, n)$. So, for every $i = 1, \ldots, n$: If a set A is p_i-open, then A is ψ-open. Suppose now that A is a p-open set. For every $x \in A$ there is an $\varepsilon > 0$ such that

$$\{y \mid p(x-y) < \varepsilon\} \subset A.$$

Therefore,

$$W = \bigcap_{i=1}^{n} \{y \mid p_i(x-y) < \varepsilon/\sqrt{n}\} \subset A.$$

W is an intersection of sets in p_i topologies, so $W \subset \psi$, thus A is open in ψ, which shows that p- topology is weaker than ψ. If $p_1 < p_2 < \ldots p_n$, seminorms $p = \bigvee_{i=1}^{n}$ and p_n generate the same topology and so $E_p = E_{p_n}$.

12. Definition. We say that a set $B \subset E$ is *bounded* if for every $p \in \Pi$, the set of real numbers $\{p(x)|x \in B\}$ is bounded. Thus, B is (τ-) bounded if and only if it is bounded for every $p \in \Pi$ and if and only if for every neighborhood U of 0 there is a $\lambda \in R$ such that $\lambda B \subset U$.

13. Theorem. *(Properties of* (E, τ).)

(i) (E, τ) is a locally convex linear topological space , and E_p is a locally convex linear topological space for every $p \in \Pi$.

(ii) $x_\alpha \to x$ if and only if $p(x_\alpha - x) \to 0$ for all $p \in \Pi$

(iii) $\bigcap_{p \in \Pi} \ker(p) = 0$.

(iv) (E, τ) is a Hausdorff space.

(v) If a set A is open in at least one $p \in \Pi$ then it is open in τ. (I.e., τ is stronger than any p-topology.)

(vi) If a set B is compact in τ-topology then it is compact in every p-topology (but not necessarily closed in p-topology, because P is only a seminorm.

Proof: (i) follows from the convexity of the set $\{x|p(x) < \varepsilon\}$. The proof of (ii) is easy. (iii) can be proved using the condition in Definition 8: for every $x \in E, x \neq 0$ there is a $p \in \Pi$ such that $p(x) \neq 0$; thus (iii). (iv) follows from (iii) and the fact that Ker(p) is a τ-closed set for every p ; this shows the closedness of $\{0\}$ and thus the Hausdorff property. To prove (v) it suffices to remark that, by Definition 8, any set $\{x|p(x) < \varepsilon\}$ is an element of neighborhood basis of τ topology. (vi) follows directly from (v).

4 Dual spaces

1.Definition. Let (E, τ) be a multi-Hilbertian space. We say that a linear functional F defined on E is τ-continuous if

$$x_\alpha \xrightarrow{\tau} x \Rightarrow F(x_\alpha) \rightarrow F(x)$$

(α is a net). The set of all continuous linear functionals on (E, τ) is denoted by E' or E'_τ. If $F \in E'_\tau$, then F evaluated at $x \in E$ is denoted by $\langle F, x \rangle$.

2.Lemma. $\langle F, x \rangle$ separates points in both E and E'.

Proof: If $F_1 \neq F_2$, then there is an $x \in E$ such that $\langle F_1, x \rangle \neq \langle F_2, x \rangle$. Conversely, suppose $x_1 \neq x_2$ and $\langle F, x_1 \rangle = \langle F, x_2 \rangle$ for all $F \in E'$. Then by a Hahn-Banach theorem (appendix D.5) we conclude that $x_1 - x_2 = 0$, which is false.

3.Definition. The *strong topology* on E' is determined by seminorms

$$\|F\|_B = \sup_{x \in B} |\langle F, x \rangle|,$$

where B is a bounded set in E.

We say that a subset A of E' is *bounded* (or *strongly bounded*) if it is bounded in all seminorms $\|F\|_B$.

4. Definition. The *weak-* (or w-*) topology* on E' is determined by seminorms

$$\|F\|_x = |\langle F, x \rangle| \quad (x \in E)$$

We say that a subset A of E' is *weakly bounded* if for every $x \in E$, $\sup_{F \in A} |\langle F, x \rangle| < \infty$.

5. Definition. The *weak topology* on E is determined by seminorms

$$\|x\|_F = |\langle F, x \rangle|$$

We say that a subset B of E is *weakly bounded* if for every $F \in E'$, $\sup_{x \in B} |\langle F, x \rangle| < \infty$.

6. Remark. A set B is weakly bounded if and only if it is strongly bounded (appendix D.8).

7. Definition. Let E'_p denote the topological dual of E with respect to the seminorm p; i.e., E'_p contains all linear functionals on E satisfying

$$p(x_\alpha - x) \to 0 \Rightarrow F(x_\alpha - x) \to 0$$

8. Lemma. E'_p is a Hilbert space with the *norm*

$$\widetilde{p}(F) = \sup_{p(x)\leq 1} |\langle F, x\rangle|$$

Proof: In appendix C it is proved that E'_p is isomorphic and isometric to the dual space of $\bar{E}_p$, which is a Hilbert space.

9. Lemma. Let (E, τ, Π) be a multi-Hilbertian space . Then

$$E' = \bigcup_{p\in\Pi} E'_p$$

Proof: (a) Let F $\in E'_p$ for some fixed p. Suppose that $x_\alpha \to x$ in τ. Then $p(x_\alpha - x) \to 0$, and thus, $\langle F, x_\alpha\rangle \to \langle F, x\rangle$; so $F \in E'$.

(b) In this part of the proof, we assume that Π is directed (in the sense of Remark 3.10). Let $F \in E'_\tau$, i.e., F is continuous with respect to τ. Then by continuity at 0, we have:

$$(\forall\varepsilon > 0)(\exists V_\varepsilon)(\forall x \in V_\varepsilon) \mid \langle F, x\rangle| < \varepsilon \ ,$$

where $V_\varepsilon = \{x \mid p_{i_1}(x) < \varepsilon_1, \ldots, p_{i_n}(x) < \varepsilon_n\}$. If $p \in \Pi$, such that $p \prec \bigvee_{i=1}^n p_{i_n}$, then:

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\exists p \in \Pi)(\forall x \in E)(p(x) < \delta \Rightarrow |\langle F, x\rangle| < \varepsilon) \qquad (1)$$

Now, take $\varepsilon = 1$ and let p_1, δ_1 be as in (1).Then we have:

$$(\forall x \in E)p_1(x) < \delta \Rightarrow |\langle F, x\rangle| < 1 \qquad (2)$$

Then (2) implies:

$$\begin{aligned}(\forall n \in N)(\forall x \in E)\ p_1(x) < \delta/n \ &\Rightarrow\ p_1(nx) < \delta \Rightarrow |\langle F, nx\rangle| < \varepsilon \\ &\Rightarrow\ |\langle F, x\rangle| < \varepsilon/n.\end{aligned}$$

Therefore, F is continuous with respect to p_1, i.e , $F \in E'_{p_1}$.

The following two lemmas are easy to prove.

10. Lemma. If $p \prec q$ then $\widetilde{q} \prec \widetilde{p}$ and $E'_p \subset E'_q$.

11. Lemma. If $p \prec q$ then every $\widetilde{q}$-open set is also a $\widetilde{p}$-open set; so the $\widetilde{q}$-topology induced on E'_p is coarser than $\widetilde{p}$-topology.

12. Remark. On each E'_p there are four different topologies that can be considered. These are:

a) Strong topology of E'_τ
b) Weak-* topology of E'_τ
c) $\widetilde{p}$-topology
d) All $\widetilde{q}$-topologies with $p \prec q$.

In Lemma 9, a relation between c) and d) is given. For simplicity, let us assume that $p < q$. Then, if K_p, K_q are unit balls in p and q respectively, we have for every $x \in E$:

$$\lambda\{x\} \subset B \subset K_q \subset K_p$$

for some $\lambda \in R$ and some bounded set $B \subset E$. Therefore,

weak-* topology is weaker than
strong topology is weaker than
$\widetilde{q}$-topology is weaker than
$\widetilde{p}$-topology.

If the topology is not explicitly specified, we will always assume that E'_p is equipped with $\widetilde{p}$- topology and E'_τ with strong topology.

13. Lemma. All four topologies mentioned in **10.** are Hausdorff topologies.

Proof: Since by lemma 2,the weak-* topology is Hausdorff, it follows that all other (stronger) topologies are Hausdorff.

14. Lemma. If U is an open set in the strong topology of E'_τ then V= $U \cap E'_p$ is an open set in the $\widetilde{p}$-topology of E'_p.

Proof: Let U be an open set in the strong topology of E'_τ. If $U = \emptyset$, then it is also open in E'_p. If not, let F be any element of U. By $F \in U$ and U being open, there is a strong open neighborhood of F contained in U:

$$O_F = \{G \mid \sup_{x \in B} |\langle F - G, x\rangle| < \varepsilon\} \subset U,$$

where B is a bounded set. Let

$$\sup_{x \in B} p(x) = \eta$$

Then

$$B_\eta = \{x | p(x) \leq \eta\} \supset B,$$

and therefore,

$$U_F = \{G | \sup_{x \in B_\eta} |\langle F - G, x\rangle| < \varepsilon\} \subset O_F \subset U$$

This shows that V is open in $\tilde{p}$-topology, because, if $F \in E'_p$, then

$$U_F = \{G | \tilde{p}\,(F - G) \leq \varepsilon/\eta\} \subset E'_p.$$

However, E'_p is *not open* in E'_τ as the next lemma shows.

15. Lemma. Let U be any non-empty subset of E'_p. Then it is not open in E'_τ.

Proof: If $U \subset E'_p$ and U open in E'_τ ,then a translation of U, call it U_0 , which contains 0 is also open in E'_τ. From appendix D.1, it follows that U_0 contains an absorbing set, i.e.

$$(\forall F \in E'_\tau)(\exists \lambda \neq 0)\lambda F \in U_0$$

which is possible if and only if $E'_p = E'_\tau$ and this case is excluded. (See definition 3.8.)

16. Lemma. Every closed $\tilde{p}$ $-$ball in E'_p is closed in $w - *$ and in the strong topology of E'_τ.

Proof: Let

$$B = \{F \in E'_p \mid \tilde{p}\,(F) \leq r\}$$

Suppose $F_\alpha \in B$, $F_\alpha \to F$ in the $w - *$ topology. So, for every $x \in E$, $\langle F_\alpha, x\rangle \to \langle F, x\rangle$. Therefore, for every fixed x with $p(x) \leq 1$ there is a sequence $\langle F_n, x\rangle \to \langle F, x\rangle$. By $|\langle F_n, x\rangle| \leq r$ we have $|\langle F, x\rangle| \leq r$, so $F \in B$.

17. Lemma. A set $A \in E'_p$ is an open set in the $w-*$ topology of E'_p if and only if $A = U \cap E'_p$ for some $w-*$ open set in E'_τ.

Proof: Suppose $A \in E'_p$ is an open set in the $w-*$ topology of E'_p. For any $F \in A$, there is a set

$$U_F = \{G \in E'_\tau \mid |\langle F-G, x_1\rangle| < \varepsilon_1, \ldots |\langle F-G, x_n\rangle| < \varepsilon_n\},$$

such that $U_F \subset A$. Therefore,

$$A = \bigcup_{F\in A} (U_F \bigcap E'_p) = (\bigcup_{F\in A} U_F) \bigcap E'_p.$$

Let $U = \cup_{F\in A} U_F$. Then U is an open set in the $w-*$ topology of E'_τ and $A = U \cap E'_p$.

Conversely, let U be an open set in the $w-*$ topology of E'_τ. Then it is easy to see that $U \cap E'_p$ is w-* open in E'_p.

18. Lemma. (i) Any bounded set in E'_p is bounded in E'_τ.

(ii) Any compact set in E'_p is compact in E'_τ.

Proof: (i) Let $A \subset E'_p$ be bounded. Then

$$(\exists N > 0)(\forall F \in A)\, \tilde{p}(F) < N$$

Let B be any bounded set in E. Then $B \subset \{p(x) < M\}$, for some positive M ; so we have:

$$\begin{aligned}
(\forall F \in A) \sup_{p(x)\le 1} |\langle F, x\rangle| < N \quad &\Rightarrow \quad \sup_{p(Mx)\le M} |\langle F, Mx\rangle| \le MN \\
&\Rightarrow \quad \sup_{p(x)\le M} |\langle F, x\rangle \le MN \\
&\Rightarrow \quad \sup_{x\in B} |\langle F, x\rangle| \le MN \\
&\Rightarrow \quad \|F\|_B \le MN,
\end{aligned}$$

so A is bounded in E'_τ.

(ii) Suppose $K \subset E'_p$ is compact in $\tilde{p}$ - topology. Then let U_α be any collection of strongly τ-open sets that covers K. By lemma 14, $U_\alpha \cap E'_p$ is a collection of $\tilde{p}$-open sets that covers K ; by $\tilde{p}$ - compactness, there is a finite subcover $U_1 \cap E'_p, \ldots, U_n \cap E'_p$; thus $U_1, \ldots, U_n$ is a finite cover of K.

19. Lemma. Let $B_r = \{F | \tilde{p}\,(F) \le r\}$, for some $r > 0$ and $p \in \Pi$. Then B_r is a w-* compact set and the w-* topology on it is metrizable.

Proof: By Banach-Alaoglu theorem (Appendix D).

5 Countably Hilbert spaces. Baire category argument

1. Definition. Multi-hilbertian space E with the τ - topology determined by countably many seminorms $p_1, \ldots, p_n, \ldots$ is called a *countably Hilbert space* .

In the spirit of Remark 3.10., we will assume that

$$p_1 < p_2 < \ldots < p_n < p_{n+1} \cdots \tag{1}$$

In a countably Hilbert space, define

$$d(x_1, x_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x_1 - x_2)}{1 + p_n(x_1 - x_2)} \tag{2}$$

In the situation described above, we will use the notation (E, τ, Π, d) , sometimes abbreviated to (E, τ, d) , (E, Π, d) and so on.

2. Remark. It is easy to see that d satisfies all axioms of distance. Besides, $d(x_1, x_2) = 0$ if and only if $p_n(x_1 - x_2) = 0$ for all n, which (by Definition 3.8) implies $x_1 = x_2$. So, d is a metric and (E, d) is a metric space. Note that $d(x_1, x_2) < 1$ for all $x_1, x_2 \in E$.

3. Remark. In E we define τ-topology as in Definition 3.8. Note that in countably Hilbert spaces, τ-topology has a countable neighborhood basis, i.e, satisfies the first axiom of countability. Thus, all questions of convergence can be viewed via sequences. The same is true for the topology defined by d. Our first task is to prove the following

4. Lemma. The topology defined by the metric d coincides with the τ - topology defined in 3.8.

Proof: By Remark 3., it suffices to show

$$x_n \xrightarrow{d} x \iff x_n \xrightarrow{\tau} x$$

By Theorem 3.13(ii), it amounts to showing

$$x_n \xrightarrow{d} x \iff (\forall k) p_k(x_n - x) \to 0,$$

which follows immediately from (2).

Note that this result implies that every d-ball contains a p_k- ball for some k.

5. Lemma. A countably Hilbert space (E, τ) is separable in τ - topology.

Proof: By separability of seminorms.

6. Remark. By lemma 4.10 we have $E'_{p_1} \subset E'_{p_2} \subset \ldots$, and by lemma 4.9, $E'_\tau = \cup_i E'_{p_i}$. In the following two lemmas we will give two separability results on E'_τ.

7.Lemma. The dual E'_τ of a countably Hilbert space is separable in the strong topology (and therefore is separable in the w-* topology too).

Proof: All spaces E'_{p_n} are p_n - separable. So, there are dense sets

$$\begin{array}{ll} d_{11}, d_{12}, \ldots & \text{in } E'_{p_1} \\ d_{21}, d_{22}, \ldots & \text{in } E'_{p_2} \\ \ldots & \end{array}$$

Let D be the union of all d_{ij} above. Let $F \in E'_\tau$. Fix a neighborhood of F :

$$V_F = \{G \mid \|F - G\| < \varepsilon\},$$

where B is a bounded set in E. Since $F \in E'_\tau$, we have that $F \in E'_{p_n}$ for some p_n. So, for some $\eta > 0$, $B \subset \{x | p_n(x) < \eta\} = B_{p_n}$, and

$$V_F \supset \{G \mid \sup_{x \in B_{p_n}} |(F - G, x)|\} = \{G \mid \widetilde{p}_n(F - G) < \varepsilon'\}$$

But the latter set contains elements from the dense set D, and this ends the proof.

8. Lemma. Let (E, τ, Π) be a countably Hilbert space. Let B be any set in E. For an $r \in R$, let

$$A = \{F \in E'_\tau \mid \sup_{x \in B} |\langle F, x\rangle| \leq r\}$$

$$A_p = \{F \in E'_p \mid \sup_{x \in B} |\langle F, x\rangle| \leq r\}.$$

Then there is a countable set $\{y_k\} \in B$ such that

$$A = \bigcap_{k=1}^{\infty} \{F \in E'_\tau \mid |\langle F, y_k\rangle| \leq r\}$$

$$A = \bigcap_{k=1}^{\infty} \{F \in E'_p \mid |\langle F, y_k\rangle| \leq r\}$$

Proof: By separability, there is a countable dense set $D = \{x^n_{ij}\} \subset B$, such that

$$(\forall j \in N)(\forall n \in N)(\forall x \in B)(\exists i \in N)\ p_j(x - x^n_{ij}) \leq \frac{1}{n} \tag{3}$$

Let

$$\begin{aligned} A^0 &= \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{\infty} \{F \in E'_\tau \mid |\langle F, x^n_{ij}\rangle| \leq r\} \\ &= \bigcap_{k=1}^{\infty} \{F \in E'_\tau \mid |\langle F, y_k\rangle| \leq r, \} \end{aligned} \tag{4}$$

where $\{y_k\}$ obtained from $\{x^n_{ij}\}$ by renumeration. Then clearly $A \subset A^0$. To show the converse, suppose $F \in A^0$. Then there is a $q \in \Pi$, such that $F \in E'_q$. Let x be an arbitrary point in B. Then by (3), there is a sequence $x_n \in D$, such that $q(x_n - x) \to 0$, by continuity we have $\langle F, x_n\rangle \to \langle F, x\rangle$, and this implies $F \in A$. For A_p we may use the same D as in (3), or a simpler choice of $D_p = \{x^n_i\} \in B$ such that

$$(\forall n \in N)(\forall x \in B)(\exists i \in N)\ p(x - x^n_i) \leq \frac{1}{n}. \tag{5}$$

9. Remark. In 3.4. we have defined $\bar{E}_p$ to be the completion of $E/\ker(p)$ with respect to p. But, also E can be completed with respect to p, by taking

$$\bar{E}_p^s = \{ \{x_n\} \mid x_n \text{ is a } p-\text{Cauchy sequence in } E\}$$

with a seminorm p on $\bar{E}_p^s$ defined by

$$p(\{x_n\}) = \lim_{n\to\infty} p(x_n).$$

Note that we are not taking equivalence classes with respect to p, so we obtain a seminormed space. We will call $\bar{E}_p^s$ a *seminormed completion* of E.

As seminorms are increasing, all p_j-Cauchy sequences are p_i-Cauchy for $i < j$, and, consequently, $\bar{E}_{p_i}^s \supset \bar{E}_{p_j}^s$. On the biggest space $\bar{E}_{p_1}^s$ define

$$d(\{x_n\},\{y_n\}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \overline{\lim_{n\to\infty}} \frac{p_i(x_n - y_n)}{1 + p_i(x_n - y_n)}. \tag{6}$$

Then on each $\bar{E}_p^s$ introduce an equivalence relation by

$$\{x_n\} \sim \{y_n\} \iff d(\{x_n\},\{y_n\}) = 0$$

The collection of equivalence classes will be denoted by $\bar{E}_p^s/\ker(d)$. It is easy to see that $\bar{E}_{p_i}^s/\ker(d) \supset \bar{E}_{p_j}^s/\ker(d)$ for $i < j$, and if we define

$$\bar{E}^s = \bigcap_{i=1}^{\infty} \bar{E}_{p_i}^s/\ker(d) = (\bigcap_{i=1}^{\infty} \bar{E}_{p_i}^s)/\ker(d) \tag{7}$$

then d becomes a metric on $\bar{E}^s$.

More details and a discussion are presented in appendix B.

10. Theorem. (*Completion of a countably Hilbert space.*) Let (E, Π, d) be a countably Hilbert space. Then there is an isometric and isomorphic, 1-1 mapping π of E onto a dense linear subspace of a complete countably Hilbert space $(\bar{E}^s, \Pi, d)$, where $\bar{E}^s$ is defined by (7).

(E, d) is a complete metric space if and only if

$$\pi(E) = \bar{E}^s \tag{8}$$

If (8) holds, we will write $E = \bar{E}^s$.

Proof: a) We will first show that d, defined by (6) is a metric on $\bar{E}^s$, and that $(\bar{E}^s, d)$ is a complete metric space.

Elements of $\bar{E}^s$ are (equivalence classes of) sequences that are p_i-Cauchy for all i. For any such sequence x_n, $\lim_n p_i(x_n)$ exists and is finite. So, (6) becomes

$$d(\{x_n\},\{y_n\}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\lim_n p_i(x_n - y_n)}{1 + \lim_n p_i(x_n - y_n)}. \tag{9}$$

If $d(\{x_n\},\{y_n\}) = 0$, then sequences $\{x_n\}$ and $\{y_n\}$ belong to a same equivalence class in $\bar{E}^s$. Therefore, d is a metric on $\bar{E}^s$. Suppose that $\bar{x}_k = \{x_n\}_k$ is a $d-$Cauchy sequence in $\bar{E}^s$(more precisely, $\{x_n\}_k$ is a Cauchy sequence of equivalence classes in $\bar{E}^s$). Then for each fixed k, $\bar{x}_k$ is an equivalence class of $\bar{E}^s$ that contains the following Cauchy sequence of E:

$$x_{k,1}, x_{k,2}, \ldots, x_{k,n} \cdots$$

and we have

$$d(\bar{x}_j - \bar{x}_k) = \lim_{n\to\infty} d(x_{j,n} - x_{k,n}) \to 0 \quad \text{as } j,k \to \infty. \tag{10}$$

For each fixed k, choose n_k such that

$$d(x_{k,m} - x_{k,n_k}) < \frac{1}{k} \quad \text{if } m \geq n_k, \tag{11}$$

which is possible because, for each fixed k, the sequence $\{x_{k,n}\}_n$ is Cauchy in E. Define $\bar{x}$ to be the equivalence class in $\bar{E}^s$ that contains the following sequence :

$$x_{1,n_1}, x_{2,n_2}, \ldots, x_{k,n_k} \cdots \tag{12}$$

Then $\bar{x}$ is indeed in $\bar{E}^s$ because, for $m \geq \max(n_j, n_k)$:

$$d(x_{j,n_j} - x_{k,n_k}) \leq d(x_{j,n_j} - x_{j,m}) + d(x_{j,m} - x_{k,m}) + d(x_{k,m} - x_{k,n_k}).$$

Letting $m \to \infty$ and then $j,k \to \infty$ and using (10) and (11) we get

$$d(x_{j,n_j} - x_{k,n_k}) \to 0 \quad \text{as } j,k \to \infty,$$

and so, (12) is a Cauchy sequence in E.

For any fixed k we have

$$d(\bar{x}_k - \bar{x}) = \lim_{n\to\infty} d(x_{k,n} - x_{k,n_k}) \leq \frac{1}{k}.$$

Therefore, $d(\bar{x}_k - \bar{x}) \to 0$, which shows that $\bar{E}^s$ is a complete space.

For every $p \in \Pi$ and $\{x_n\} \in \bar{E}^s$, we may define

$$p(\{x_n\}) = \lim_{n\to\infty} p(x_n).$$

So, $d(\{x_n\},\{y_n\}) \to 0$ if and only if $p(\{x_n\} - \{y_n\}) \to 0$ for every $p \in \Pi$. Note that p need not be a norm on $\bar{E}^s$ even if it is a norm on E.

b) Let us define a mapping $\pi:\ E \to \bar{E}^s$ by

$$\pi(x) = \{x, x, \ldots, x, \ldots\} + \ker(d) \tag{13}$$

Then $\pi(E)$ is a linear subspace of $\bar{E}^s$. π is an isomorphism by

$$\pi(\alpha x + \beta y) = \alpha\pi(x) + \beta\pi(y).$$

From (9) it follows

$$d(\pi(x), \pi(y)) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1 + p_i(x-y)} = d(x,y), \tag{14}$$

so π is an isometry.

From (13) and (14) it follows that π is an 1-1 mapping.

Finally, let us show that $\pi(E)$ is dense in $\bar{E}^s$. Let $\{x_n\} + \ker(d) \in \bar{E}^s$. Then $\{x_n\}$ is a d-Cauchy sequence in E, so $\pi(x_n)$ is a d-Cauchy sequence in $\pi(E)$, and, as $n \to \infty$ we have:

$$d(\pi(x_n) - \{x_n\}) \to 0.$$

c) If (8) holds, then E is complete, because, as shown in a), $\bar{E}^s$ is complete. Conversely, let (E, d) be a complete space. Let π be defined by (13). We need to show that $\bar{E}^s \subset \pi(E)$. To this end, let $y = \{x_n\} + \ker(d) \in \bar{E}^s$. Then, as shown in b), $\pi(x_n) \to y$. By completeness of E, there is a $z \in E$ such that $x_n \to z$. By uniqueness of a limit, we have that $y = \pi(z) \in \pi(E)$, which had to be proved.

11. Lemma. Let (E, τ, Π) be a complete countably Hilbert space , *or* only a countably Hilbert space , but τ determined by norms. Then $A \subset E$ is a compact set in E if and only if it is compact in every p-topology.

Proof: "Only if" part follows from Theorem 3.13(vi). So, suppose that A is compact in every p-topology . Let us first consider the case of a complete countably Hilbert space. For a given sequence $\{x_n\}$, there is a p_1 - convergent subsequence $x_{1,n}$ such that

$$p_1(x_{1,n} - y_1) \to 0,$$

for some $y_1 \in E$. Now take $x_{1,n}$ as a sequence , to obtain a p_2-convergent subsequence

$$p_2(x_{2,n} - y_2) \to 0.$$

In this way , we obtain subsequences $x_{k,n}$ and a sequence y_k , such that

$$p_k(x_{k,n} - y_k) \to 0 \text{ as } n \to \infty \tag{15}$$

and $\{x_{k,n}\}$ is a subsequence of $\{x_{k-1,n}\}$, for $k = 1, 2, \ldots$. So, the sequence $\{x_{n,n}\}$ is a Cauchy sequence for every fixed p_k, i.e,

$$p_k(x_{n,n} - x_{m,m}) \to 0 \text{ as} n, m \to \infty$$

By completeness, $x_{n,n}$ is a convergent sequence in (E, τ), and by compactness in p_k, it has a limit in A.

If p_k are norms,then from (15) and Corollary 17 of appendix B, we conclude that $y_1 = y_2 = \ldots = y$; so $p_k(x_{n,n} - y) \to 0$ and this ends the proof.

12. Remark. In the proof of the next lemma, we will use some results from appendix B. As the proof relies on Baire category theorem, the next lemma is often referred to as a *Baire category argument.*

13. Definition. A positive function $V : E \to R_+$ is called a *G-function* if the set $G = \{x \in E | u(x) \leq n\}$ is closed for every $n = 1, 2, \ldots$.

14. Lemma. Let (E, τ, Π, d) be a complete countably Hilbert space. Let B be an open $d-$ball in E :

$$B = \{x \in E \mid d(u, x) \leq a\} \quad \text{for some } u \in E, a \in R.$$

For an arbitrary index set I , let $\{V_i\}_{i \in E}$ be a family of functions $E \to R_+$ such that

(i)	V_i is a G – function for every $i \in I$,
(ii)	$V_i(x+y) \leq V_i(x) + V_i(y)$ for all $x, y \in E$,
(iii)	$V_i(ax) = \lvert a \rvert V_i(x)$ for all $x \in E, a \in R$,
(iv)	For every $x \in B$, the set $\{V_i(x) \mid i \in I\}$ is bounded.

Then for each $i \in I$, V_i is a continuous function on E , and there is a $p \in \Pi$ and a constant $\theta > 0$ such that for every $i \in I$ and $x \in E$:

$$V_i(x) \leq \theta p(x)$$

$$|V_i(x) - V_i(y)| \leq \theta p(x-y)$$

Proof: By (i) , (iv) and Corollary 21 of Appendix B, there are: $p \in \Pi, z \in E, r > 0, M > 0$, such that

$$p(z-x) \leq r \Rightarrow V_i(x) \leq M. \tag{16}$$

Therefore, for every $x \in E, x \neq 0$, we have:

$$V_i(-z + \frac{xr}{p(x)}) \leq M.$$

In particular , (16) implies $|V_i(z)| \leq M$. Then by (ii) we conclude that

$$V_i(\frac{xr}{p(x)}) = V_i(\frac{xr}{p(x)} - z + z) \leq 2M,$$

and by (iii) we have

$$V_i(x) \leq p(x)\frac{2M}{r} = \theta p(x), \tag{17}$$

for every $x \in E$ and $\theta = 2M/r$.

To show continuity, note that by (ii) and (17) :

$$|V_i(x) - V_i(y)| \leq V_i(x-y) \leq \theta p(x-y).$$

15. Remark. Consider the following conditions:

(ii')	$(\forall M > 0)(\exists M_1 > 0)(\forall x, y \in E)\ V_i(x) \leq M \wedge V_i(y) \leq M \Rightarrow V_i(x+y) \leq M_1$
(iii')	$(\exists k \in R_+)(\forall x \in E)(\forall a \in R) V_i(ax) = \lvert a \rvert^k V_i(x)$.

If (iii) is replaced by (iii') then the conclusion of the theorem 12 is

$$|V_i(x) - V_i(y)| \leq \frac{2Mp^k(x)}{r^k} \to 0 \text{ as } p(x-y) \to 0,$$

so V_i is a continuous function.

If, in addition, (ii) is replaced by (ii'), we can still conclude the continuity at 0:

$$V_i(x) \leq \frac{M_1 p^k(x)}{r^k} \to 0 \text{ as } p(x) \to 0.$$

16. Lemma. Let (E, τ, Π, d) be a complete countably Hilbert space, and let $A \in E'_\tau$ be a weakly bounded set, i.e, for any fixed $x \in E$:

$$\sup_{F \in A} |\langle F, x \rangle| < \infty. \tag{18}$$

Then there is a $p \in \Pi$ such that $A \subset E'_p$ and the set A is strongly bounded in A, i.e, for any bounded set $B \in E$:

$$\sup_{F \in A} \|F\|_B < \infty.$$

Proof: By continuity of F, a mapping $x \to |\langle F, x \rangle|$ is a G-function, and assumptions of lemma 14 are fulfilled, so $|\langle F, x \rangle| \leq \theta p(x)$, for some $\theta > 0, p \in \Pi$ and all $F \in A$; therefore $A \subset E'_p$ with $\widetilde{p}\,(F) \leq \theta$ for $F \in A$. Strong boundness follows then from lemma 4.14(i).

17. Corollary. A set $A \subset E'_\tau$, where E'_τ is the dual of a complete countably Hilbert space (E, τ, Π) is strongly bounded if and only if $A \in E'_p$ for some $p \in \Pi$ and the set $\{\widetilde{p}\,(F) \mid F \in A\}$ is bounded.

Proof: Straightforward from lemma 16 and its proof.

18. Lemma. Let E'_τ be the dual of a complete countably Hilbert space (E, τ, Π, d). Suppose that F_n is a sequence in E'_τ such that for every $x \in E$ the sequence $\langle F_n, x \rangle$ of real numbers converges to some real number. Then

(i) There is an $F \in E'_\tau$ such that $F_n \to F$ in the w-* topology of E'_τ.

(ii) There is a $p \in \Pi$, such that $F \in E'_p$ and $F_n \in E'_p$ for all $n \geq 1$.

Proof: Let $f(x) = \lim_{n \to \infty} \langle F_n, x \rangle$. Then by assumption, f is well defined and finite for all $x \in E$. Since E is a complete metric space, it follows from Banach-Steinhaus theorem (appendix D.3) that $f(x) = \langle F, x \rangle$ for some $F \in E'_\tau$. Let $A = \{F, F_1, F_2, \ldots\}$. Then, A is weakly bounded set, so by lemma 16 and corollary 17, there is a $p \in \Pi$ such that $F_n, F \in E'_p$.

19. Remark. From lemma 18, we can conclude that if $F_n \to F$ in the strong topology of E'_τ, then $F_n, F \in E'_p$, for some $p \in \Pi$. However, this does not imply $\tilde{p}\,(F_n - F) \to 0$. For this conclusion, we will need an additional assumption that will be made in the next section.

20. Remark. For a countably Hilbert space, define its *second dual* E''_τ as a set of all linear functionals φ defined on E'_τ that are continuous with respect to the strong topology on E'_τ.

If $E = E''_\tau$, (in the sense of usual identification by means of a bicontinuous bijection), the space E is called ***reflexive***. It is always true that $E \subset E''_\tau$.

21. Theorem. A complete countably Hilbert space (E, τ, Π, d) is reflexive.

Proof: For a $\varphi \in E''_\tau$ and a bounded set $A \subset E'_\tau$, define

$$\|\varphi\|_A = \sup_{F \in A} |\langle \varphi, F \rangle|. \tag{19}$$

For an $x \in E$ define $\varphi \in E''_\tau$ by

$$\langle \varphi, F \rangle = \langle F, x \rangle. \tag{20}$$

Then

$$\|\varphi\|_A = \sup_{F \in A} |\langle F, x \rangle|. \tag{21}$$

Let us show that the mapping $\pi : E \to E''_\tau$, $\pi(x) = \varphi$, with φ defined as in (21) is continuous with respect to topology on E''_τ determined by norms (19). By corollary 17, $A \subset E'_p$ for some $p \in \Pi$ and $\tilde{p}\,(F) \le M$ for some $M > 0$. Using the duality in Hilbert spaces, we have (appendix D.10):

$$\sup_{F \in A} |\langle F, x \rangle| \le \sup_{\tilde{p}(F) \le M} |\langle F, x \rangle| = \frac{1}{M} p(x), \tag{22}$$

and the continuity of φ follows from (21) and (22).

We shall now show that π has an inverse which is also continuous. For every $p \in \Pi$, $\bar{E}_p$ is a Hilbert space and E'_p its dual. By duality in Hilbert spaces, every $\varphi \in E''_p$ is determined uniquely by an $x \in \bar{E}_p$. So, it follows that for every $\varphi \in E''_p$ there is a (not unique) $x \in \bar{E}^s_p / \ker(d)$, such that $\langle \varphi, F \rangle = \langle F, x \rangle$ for all $F \in E'_p$. Therefore, all functionals on E'_{p_1} are completely determined by elements of $\bar{E}^s_{p_1} / \ker(d)$. All functionals on E'_{p_2} are completely determined by elements of $\bar{E}^s_{p_2} / \ker(d)$,etc. Since $E'_{p_{i+1}} \supset E'_{p_i}$, we see that all functionals on $E'_\tau = \cup E'_{p_i}$ are completely determined by elements of $E = \cap \bar{E}^s_{p_i} / \ker(d)$.

So, π is a 1-1 map with $\pi(E) = E''_\tau$. Let us show now that π^{-1} is a continuous mapping. Observe that, for

$$A = \{F \in E'_\tau \mid \tilde{p}\,(F) \leq M\} \tag{23}$$

we have

$$\sup_{F \in A} |\langle F, x\rangle| = p(x). \tag{24}$$

So, if $\varphi_\alpha \to 0$ in the topology determined by (19), then, in particular, $\|\varphi_\alpha\|_A \to 0$, with A as specified in (23), and by (21) and (24) we have $\pi^{-1}(\varphi_\alpha) \to 0$ in $\tau-$topology.

The above proof shows that there is a bijection and isomorphism $\pi : E_\tau \to E''_\tau$, such that E_τ and E''_τ have the same topological structure. Therefore, we may identify E''_τ withE_τ.

22. Remark. On a countably Hilbert space, as mentioned in remark 3, the strong topology satisfies the first axiom of countability. Moreover, since it is a metric space, compact sets can be characterised as those in which every sequence has a convergent subsequence.

However, the weak topology on E does not satisfy the first countability axiom, that is, does not have a countable neighborhood basis. Also, neither the strong nor the w-* topology on E'_τ can be defined with a countable neighborhood basis.

Therefore, sequences cannot be used as a convenient replacement for nets.

6 $I(\tau)$ topology and nuclear spaces

1.Definition. Let E be a multi-Hilbertian space with a topology τ determined by a family Π of Hilbertian seminorms. Let P_I be the family of all seminorms that are $<_{HS}$ to some seminorm in Π ,i.e:

$$P_I = \{q \in HSN | (\exists p \in \Pi)\; q <_{HS} p\}$$

The $I(\tau)$ topology is the topology determined by P_I (in the sense of 3.8).

In the next lemma we will use the results stated in section 2.

2.Lemma. Let $(H, \|\cdot\|)$ be a Hilbert space. Let τ, τ_H, τ_N, S be topologies determined by $\|\cdot\|, P_H, P_N, P_S$ respectively , where

$$\begin{aligned} P_H &= \{p | p(x) = \|Ax\| \text{ , } A \text{ is a Hilbert} - \text{Schmidt}\} \\ P_N &= \{p | p(x) = \langle Ax, x\rangle^{1/2} \text{ , } A \text{ is positive nuclear}\} \\ P_S &= \{p | p(x) = \langle Ax, x\rangle^{1/2} \text{ , } A \text{ is positive , compact , trace class}\} \end{aligned}$$

Then $\tau_H = \tau_N = S = I(\tau) \subset \tau$.

Proof: $I(\tau) = \tau_H$: Let $\{e_i\}$ be an orthonormal base in H. Let $p \in P_H$. Then $\sum p^2(e_i) = \sum \|Ae_i\|^2 < \infty$. Besides, $p(x) \leq \|ax\| \cdot \|x\|$; so $p <_{HS} \|\cdot\|$ and $p \in P_I$. Conversely , if $p \in P_I$ then $p(x) = \|Ax\|$ for some continuous linear operator A ; by $\sum p^2(e_i) < \infty$ we have that A is a Hilbert- Schmidt operator.

$\tau_H = \tau_N$: Let A be a Hilbert-Schmidt operator. Let $p(x) = \|AX\|$. Then $p^2(x) = \langle Ax, AX\| = \langle A^*Ax, x\rangle$. A^*A is a positive nuclear operator as a product of two Hilbert-Scmidt operators. Conversely , let A be a positive (thus a self-adjoint) nuclear operator. Then $A^{1/2}$ is a positive Hilbert-Schmidt operator and $p(x) = \langle Ax, x\rangle^{1/2} = \langle A^{1/2}x, A^{1/2}x\rangle = \|A^{1/2}x\|$.

$\tau_N = S$ follows from Theorem 2.4.

$\tau_I \subset \tau$ follows from $p(x) < c\|x\|$ for some $c > 0$.

3.Remark. Lemma 2 gives different ways of obtaining the $I(\tau)$ topology in a Hilbert space;this is applicable in a multi-Hilbertian space in an obvious manner,because P_I in a multi-Hilbertian space is the union of corresponding families of seminorms in Hilbert spaces.

The last statement of Lemma 2 implies that $I(\tau)$ is a Hausdorff topology. Note that P_I need not be a subset of Π.

Throughout this section, we will make use of the fact (proved in appendix A) that for every $q \in \Pi$ there is a $q-$orthonormal base $\{e_i\}$ in E (rather than in $\bar{E}_p$). This enables us to drop $i_{p,q}$ in lemma 3.6. So, $p <_{HS} q$ if and only if $\sum p^2(e_i) < \infty$ for any $q-$orthonormal base $\{e_i\}$ in E.

In the next lemma we will give a construction of seminorms that determine $I(\tau)$ and this will show that the P_I in multi-Hilbertian spaces is not an empty set.

4.Lemma. For every $q \in \Pi$ there is a $p \in HSN$ so that $p <_{HS} q$.

Proof: Let $\{e_n\}$ be an orthonormal base in $\bar{E}_q$. Let $c_i, i = 1, 2, \ldots$ be a sequence of real numbers such that $\sum c_i^2 < \infty$. Define p by

$$p^2(x) = \sum_{i=1}^{\infty} c_i^2 q^2(x, e_i)$$

Then $p^2(e_i) = c_i^2$ and therefore $\sum p^2(e_i)$ converges. Also , $p(x) \le Mq(x)$, where $M = \sup_i c_i$. Thus , $p <_{HS} q$.

5. Definition. A countably Hilbert space (E, τ, Π, d) is a *nuclear space* if $I(\tau) = \tau$, i.e, if

$$(\forall p \in \Pi)(\exists q \in \Pi)(p < q \wedge \sum_{i=1}^{\infty} p^2(e_i) < \infty),$$

for a q-orthonormal base $\{e_i\}$.

6. Theorem. Let (E, τ, Π) be a complete nuclear space. A set $A \subset E$ is compact if and only if it is bounded and closed.

Proof: It suffices to prove "if" part only. Let A be a bounded and closed set in E. Let $p \in \Pi$ be fixed, and let $p <_{HS} q$. Then there are $p-$ and $q-$orthonormal bases in E, denote them by $\{e_i\}$ and $\{f_i\}$ respectively. By assumption, $q(x) \le M$ for all $x \in A$; so using theorem 3 of appendix A, we have:

$$\begin{aligned}
\sum_{i=1}^{\infty} p^2(x, e_i) &= \sum_i p^2(\sum_j q(x, f_j) f_j, e_i) \\
&= \sum_i (\sum_j q(x, f_j) p(f_j, e_i))^2 \\
&\le \sum_i \sum_j q^2(x, f_j) \sum_j p^2(f_j, e_i) \\
&\le M \sum_i \sum_j p^2(f_j, e_i) \\
&= M \sum_j \sum_i p^2(f_j, e_i) = M \sum_j p^2(f_j) < \infty
\end{aligned}$$

This shows that the series $\sum_i p^2(x, e_i)$ converges uniformly for $x \in A$. By corollary 7 of appendix A, A is relatively compact with respect to $p-$ convergence, for every $p \in \Pi$. Then,using completeness and $\tau-$closedness of A, it can be proved that it is $\tau-$compact, similarly to the proof of lemma 5.11.

7. Remark. From the proof of lemma 6, we can conclude that nuclear spaces have the following property: For any $p \in \Pi$ there is a $q \in \Pi$ such that any q-bounded set is a $p-$relatively compact set.

It is not only in nuclear spaces that every bounded and closed set is compact. A complete countably Hilbert space with this structure of compact sets will be called *perfect space*. In fact, every complete countably Hilbert space with aforementioned property is perfect.In the next lemma, we give another sufficient condition for a space to be perfect.

8. Lemma. Let (E, τ, Π) be a complete countably Hilbert space. Suppose that for every bounded sequence $x_n \in E$, and for each $p \in \Pi$ there is a $p-$Cauchy subsequence $x_{n'}$. Then E is a perfect space.

Proof: Let $p_1 < p_2 < \ldots$ be seminorms in Π. Let x_n be a sequence in a bounded and closed set A. By assumption, there are infinitely many subsequences $\{x_{i,j}\}_j$ such that $\{x_{i,j}\}_j$ is a subsequence of $\{x_{i-1,j}\}_j$ and $\{x_{i,j}\}_j$ is a p_i-Cauchy sequence. Then $\{x_{n,n}\}_n$ is a Cauchy sequence in all $p \in \Pi$; by completeness it converges to some $x \in E$, and by closedness of A, $x \in A$.

9. Lemma. Let (E, τ, Π, d) be a perfect space. Then

(i) Strong and weak sequential convergence coincide on E.

(ii) Strong and w-* sequential convergence coincide on E'_τ.

Proof: (i) Let $x_n \to 0$ in the weak topology on E. Then $\langle F, x_n \rangle \to 0$ for all $F \in E'_\tau$. Therefore, the set $\{x_n\}$ is weakly bounded, and so is strongly bounded (appendix D.8). So, the sequence $\{x_n\}$ belongs to a compact set .

Suppose that, for some $p \in \Pi$, $p(x_n)$ does not converge to zero. Then, there is a subsequence, say $x_{n'}$, such that $p(x_{n'}) \geq \varepsilon$ for some $\varepsilon > 0$. By compactness, there is a convergent subsequence $x_{n''}$ of $x_{n'}$, by weak convergence, $p(x_{n''}) \to 0$; but $p(x_{n''}) \geq \varepsilon$, which is a contradiction.

(ii) Suppose that $F_n \to 0$ in the w-* topology on E'. Then the sequence $\{F_n\}$ is weakly bounded; by lemma 5.16, it is also strongly bounded, and so, for every $n \geq 1$ and every bounded closed set $B \subset E$:

$$\sup_{x \in B} |\langle F_n, x \rangle| \leq M_B \tag{1}$$

Suppose now that F_n does not converge to 0 in the strong topology. Then there is a closed bounded (therefore compact) set $B \subset E$ and a sequence $\{x_{n'}\} \in B$, such that for some $\varepsilon > 0$:

$$|\langle F_{n'}, x_{n'} \rangle| \geq \varepsilon \tag{2}$$

By compactness, we may assume that $x_{n'}$ is a convergent sequence, $x_{n'} \to x_0$ in the strong topology of E. Then, as in the proof of theorem 5.21, we have:

$$\sup_{F \in A} |\langle F, x_{n'} - x_0 \rangle| \leq cp(x_{n'} - x_0)$$

for any bounded set $A \subset E'$, some $c > 0$ and some $p \in \Pi$. If we take $A = \{F_{n'}\}$, we have $\langle F_{n'}, x_{n'} - x_0 \rangle \to 0$, and by assumption, $\langle F_{n'}, x_0 \rangle \to 0$, so $\langle F_{n'}, x_{n'} \rangle \to 0$, which is a contradiction to (2).

10. Lemma. Let (E, τ, Π) be a complete nuclear space. Let $\{F_n\}$ be a sequence in E'_τ. Then $F_n \to F$ in the strong or w-* topology if and only if there is a $p \in \Pi$ such that $F, F_1, F_2, \ldots \in E'_p$ and $\widetilde{p}\,(F_n - F) \to 0$.

Proof: By lemma 9 and remark 5.18, if $F_n \to F$ then all F_n and F belong to some space $E'_r, r \in \Pi$. By remark 7, there is a $p \in \Pi$, $r <_{HS} p$, such that the $p-$unit ball B_p is compact in $r-$topology. We will now prove that $\widetilde{p}\,(F_n - F) \to 0$. If not, then there is a subsequence $F_{n'}$ and a sequence $x_{n'}$ such that $x_{n'} \in B_p$ and

$$|\langle F_{n'} - F, x_{n'} \rangle| \geq \varepsilon, \tag{3}$$

for some $\varepsilon > 0$. By compactness, we may assume that $r(x_{n'} - x_0) \to 0$ for some $x_0 \in B_p$. Then we have:

$$\begin{aligned} |\langle F_{n'} - F, x_{n'} \rangle| &\leq |\langle F_{n'} - F, x_{n'} - x_0 \rangle| + |\langle F_{n'} - F, x_0 \rangle| \\ &\leq \widetilde{r}\,(F_{n'} - F) r(x_{n'} - x_0) + |\langle F_{n'} - F, x_0 \rangle| \\ &\to 0, \end{aligned}$$

which is a contradiction to (3).

Appendix

A Orthogonalization in (E, τ)

In this section we will consider certain questions related to the existance of an orthonormal base with respect to a fixed seminorm. We will also derive a necessary and sufficient condition for a linear functional to be continuous with respect to a given seminorm. Although some proofs might appear to be unecessary elaborated , it is important to understand specific techniques one has to employ while dealing with seminorms.

We will use the same notation as in the main body of the text. Let us first consider a space E_p , topologized by one seminorm p. Denote by π the cannonical mapping $E_p \to E_p/\ker(p)$, defined by $\pi(x) = x + \ker(p) = x^*$, where x^* stands for the equivalence class relative to x.

1. Lemma. If D is a dense set in E_p , then $D^* = \{\pi(d), d \in D\}$ is dense in $E_p/\ker(p)$ and also in $\bar{E}_p$.

Proof: Let $x^* \in E_p/\ker(p)$ and let $\varepsilon > 0$. Choose $x \in \pi^{-1}(x^*)$. Since D is dense in E_p , there is a $d \in D$ so that $p(x - d) < \varepsilon$. Let $\pi(d) = d^*$. Then $p(x^* - d^*) = p(x - d) < \varepsilon$; so D^* is dense in $E_p/\ker(p)$. To show densness in $\bar{E}_p$, note that if x_n^* is a (Cauchy) sequence that converges to x^*, and if we choose d_n such that $p(x_n^* - d_n^*) \leq 1/n$ then we have $d_n^* \to x^*$.

2. Remark. By separability of p and Lemma 1, there is a countable dense set $D^* = \{d_1^*, d_2^*, \ldots\}$ in $E_p/\ker(p)$. Let x_1^* be the first non-zero element in the sequence D^*, let x_2^* be the first d_i^* which is not in the closed subspace spanned by x_1^* , etc ..., let x_{n+1}^* be the first d_i^* which is not in the closed subspace spaned by $\{x_1^*, x_2^*, \ldots, x_n^*\}$ (the idea is to remove dependent d_i^*'s from the sequence). Therefore,

$$\operatorname{span}\{d_1^*, d_2^*, \ldots, d_n^*, \ldots\} = \operatorname{span}\{x_1^*, x_2^*, \ldots, x_n^*, \ldots\}$$

As D^* is dense in $E_p/\ker(p)$ we have (a bar denotes the closure):

$$\bar{E}_p = \overline{\text{span}}\{d_1^*, d_2^*, \ldots, d_n^*, \ldots\} = \overline{\text{span}}\{x_1^*, x_2^*, \ldots, x_n^*, \ldots\}$$

Now let us apply the Gramm-Schmidt orthonormalization, starting from $x_1^*, x_2^*, \ldots$ to obtain

$$e_j^* = \sum_{k=1}^{j} b_{jk} x_k^* \qquad j = 1, 2, \ldots \tag{1}$$

From the above construction it follows that d_1^* is either 0 or x_1^*, d_2^* is either 0 or x_1^* or x_2^*, ..., in general, $d_n^* \in \{0, x_1^*, x_2^*, \ldots x_n^*\}$, which by means of inverting equation (1) gives

$$d_j^* = \sum_{k=1}^{j} a_{jk} e_k^* \tag{2}$$

Let us now show that $\{e_j^*\}$ is a complete orthonormal base in $\bar{E}_p$. Indeed, *if there is an* $y \in \bar{E}_p$ *such that* $p(y, e_j^*) = 0$ for every j, then by (2) we also have $p(y, d_j^*) = 0$ for every j, which by denseness of d_j^* gives $p(y, f) = 0$ for every $f \in \bar{E}_p$; in particular $p(y, y) = 0$, and so $y = 0$.

From above considerations it follows that $\bar{E}_p$ is a Hilbert space with a complete orthonormal base $\{e_1^*, e_2^*, \ldots\}$. Then for every $f \in \bar{E}_p$ we have

$$f = \sum p(f, e_i^*) e_i^*$$

$$p^2(f) = \sum p^2(f, e_i^*)$$

Now, getting down to $E_p/\ker(p)$ and then to E_p we can state the following:

3. Theorem. There is a countable set $\{e_1, e_2, \ldots\} \subset E$ (*Orthonormal base for* E) and a p-dense set $\{d_1, d_2, \ldots\} \subset E$ such that

$$p(e_i, e_j) = \delta_{ij} \qquad i, j = 1, 2, \ldots$$

$$d_j = \sum_{k=1}^{j} a_{jk} e_k + r_j,$$

where r_j are elements in ker(p). For every $x \in E$:

$$x = \sum_{i=1}^{\infty} p(x, e_i) e_i \ + \ r(x),$$

with the convergent series in the following sense:

$$\lim_n p(x - \sum_{i=1}^{n} p(x, e_i)\, e_i) \ = \ 0$$

and $r(x)$ is an element in ker(p), depending on x. The following equality holds for every $x \in E$:

$$p^2(x) = \sum_{i=1}^{\infty} p^2(x, e_i)$$

4.Theorem. Let p and q be Hilbertian seminorms on E such that $p \prec q$.Let $\{e_i\}$ be any q-orthonormal base in E (in the sense of theorem 3). Then

$$\sum_{i=1}^{\infty} p^2(e_i) \text{ does not depend on a choice of } \{e_i\} \tag{3}$$

$$\sum_{i=1}^{\infty} p^2(e_i) = \sup_{\alpha} \sum_{i=1}^{n} p^2(e_{\alpha_i}) \tag{4}$$

Proof: Let $\{e_n\}$ and $\{f_n\}$ be q-orthonormal bases in the sense of theorem 3. It suffices to prove (3) under the assumption $\sum_{i=1}^{\infty} p^2(e_i) < \infty$. Then by theorem 3 we have:

$$\lim_N q(f_i - \sum_{j=1}^{N} a_{ij} e_j) = 0 \qquad (a_{ij} = q(f_i, e_j)) \tag{5}$$

$$\lim_N q(e_j - \sum_{i=1}^{N} b_{ji} f_i) = 0 \qquad (b_{ji} = q(e_j, f_i)) \tag{6}$$

and, therefore

$$a_{ij} = b_{ji} \tag{7}$$

Let us recall that $q(a - a_n) \to 0$ and $q(b - b_n) \to 0$ implies $q(a, b) - q(a_n, b_n) \to 0$ (see Remark 1.4), and by $p \prec q$ and the triangle inequality , this implies $p(a, b) - p(a_n, b_n) \to 0$. Therefore, using (5), (6), (7) and orthonormality , we have:

$$\begin{aligned} q(e_j, e_k) &= \lim_N q(\sum_{i=1}^N b_{ji} f_i \, , \, \sum_{i=1}^N b_{ji} f_i) \\ &= \lim_N \sum_{i=1}^N b_{ji} b_{ki} \, q(f_i, f_i) \\ &= \lim_N \sum_{i=1}^N b_{ji} b_{ki} \\ &= \lim_N \sum_{i=1}^N a_{ij} a_{ik} = 0 \qquad \text{if } j \neq k, \end{aligned} \tag{8}$$

and

$$\lim_N \sum_{i=1}^N a_{ij}^2 = 1 \, . \tag{9}$$

Further,

$$\begin{aligned} p^2(f_i) &= p(f_i, f_i) = \lim_N p(\sum_{j=1}^N a_{ij} e_j \, , \, \sum_{k=1}^N a_{ik} e_k) \\ &= \lim_N \sum_{j=1}^N a_{ij}^2 p^2(e_j) \, + \, \lim_N \sum_{j \neq k} a_{ij} a_{ik} p(e_j, e_k) \end{aligned}$$

After summation,we obtain:

$$\begin{aligned} \sum_{i=1}^M p^2(f_i) &= \lim_N \sum_{i=1}^M \sum_{j=1}^N a_{ij}^2 p^2(e_j) + \lim_N \sum_{i=1}^M \sum_{j \neq k} a_{ij} a_{ik} p(e_j, e_k) \\ &= \lim_N \sum_{j=1}^N (\sum_{i=1}^M a_{ij}^2) p^2(e_j) + \lim_N \sum_{j \neq k} (\sum_{i=1}^M a_{ij} a_{ik}) p(e_j e_k) \end{aligned}$$

Letting $m \to \infty$, using (9), (8) and the dominated convergence theorem we obtain

$$\sum_{i=1}^{\infty} p^2(f_i) = \sum_{i=1}^{\infty} p^2(e_i),$$

which had to be proved. Let us remark that the dominated convergence theorem could be used here, because of the following inequalities:

$$|\sum_{i=1}^M a_{ij} a_{ik}| \leq (\sum_{i=1}^M |a_{ij}|^2)^{1/2} (\sum_{i=1}^M |a_{ik}|^2)^{1/2} \leq 1$$

$$|\sum_{j\neq k} p(e_i,e_j)| \leq (\sum_j p^2(e_j))^{1/2}(\sum_k p^2(e_k))^{1/2} < \infty$$

Let us show (4). From the construction of orthonormal base it follows that for any finite orthonormal system $\{e_1, e_2, \ldots, e_n\}$ there is an orthonormal base $\{f_1, f_2, \ldots\}$ (in the sense of theorem 3) that contains the given system. Therefore,

$$\sum_{i=1}^{n} p^2(e_i) \leq \sum_{i=1}^{\infty} p^2(f_i) = R,$$

and the quantity R does not depend on a choice of $\{f_i\}$, by (3).Therefore,

$$\sup_{\alpha} \sum_{i=1}^{n} p^2(e_{\alpha_i}) \leq R$$

On the other hand, the reverse inequality holds by the definition of R, and (4) is proved.

5. Remark. Suppose that, under assumptions of theorem 4,

$$\sum_{i=1}^{\infty} p^2(e_i) < \infty.$$

Then define

$$\mathrm{HS}(p,q) = \sum_{i=1}^{\infty} p^2(e_i),$$

$$\mathrm{C}(p,q) = \sup_{q(x)\leq 1} \{p(x)\}.$$

The following holds:

$$\mathrm{C}(p,q) \leq \mathrm{HS}(p,q),$$

$$\mathrm{C}(p,r) \leq \mathrm{C}(p,q)\mathrm{C}(q,r),$$

$$\mathrm{HS}(p,r) \leq \mathrm{C}(p,q)\mathrm{HS}(q,r),$$

$$\mathrm{HS}(p,r) \leq \mathrm{HS}(p,q)\mathrm{C}(q,r).$$

This shows that the relation $<_{HS}$ is transitive and also

$$p <_{HS} q \prec r \Rightarrow p <_{HS} r$$

and similar variations. For a proof of these statements see Itô(1984).

6. Lemma. Let A be a bounded subset of a separable Hilbert space H, with a norm $\|\cdot\|$, the corresponding inner product $\langle\cdot,\cdot\rangle$ and the orthonormal base $\{e_i\}$, such that

$$(\forall\varepsilon > 0)(\exists N)(\forall x \in A) \sum_{i=N}^{\infty}\langle x, e_i\rangle^2 \leq \varepsilon \tag{10}$$

(i.e, $\sum\langle x, e_i\rangle^2$ converges uniformly on A). Then A is a relatively compact set.

Proof: Let ε_k be a sequence of positive numbers such that $\sum\varepsilon_k < \infty$ and let N_k be the corresponding sequence of natural numbers according to (10). Let us define seminorms $p_k(x)$ by

$$p_k^2(x) = \sum_{i=1}^{N_k-1}\langle x, e_i\rangle^2.$$

Let x_n be a sequence in A. By a finite-dimensional result , there is a sequence $x_{1,n}$ such that $p_1(x_{1,n} - x_1) \to 0$. Taking this subsequence as a sequence , and repeating this procedure , we get

$$p_k(x_{k,n} - x_k) \to 0 \quad \text{as } n \to 0\ ,\ k = 1, 2, \ldots$$

and $x_{k,n}$ is a subsequence of $x_{k-1,n}$. By $p_k > p_{k-1}$ we have $p_{k-1}(x_k - x_{k-1}) = 0$. Further,

$$\|x_k - x_{k-1}\|^2 \leq p_{k-1}^2(x_k - x_{k-1}) + \varepsilon_{k-1} = \varepsilon_{k-1},$$

and by $\sum\varepsilon_k < \infty$ we conclude that x_k is a Cauchy sequence and thus , for some x_∞ , $\|x_k - x_\infty\| \to 0$ as $k \to \infty$. Then we have

$$\begin{aligned}\|x_{n,n} - x_\infty\|^2 &\leq 2(\|x_{n,n} - x_k\|^2 + \|x_k - x_\infty\|^2)\\ &\leq 2(p_k^2(x_{n,n} - x_k) + \varepsilon_k + \|x_k - x_\infty\|^2),\end{aligned}$$

so, letting $n \to \infty$ and then $k \to \infty$, we conclude that $\|x_{n,n} - x_\infty\| \to 0$.

7. Corollary. If A is a subset of E, bounded in p-seminorm, and if (10) holds with $\langle\cdot,\cdot\rangle$ replaced with $p(\cdot,\cdot)$, then A is a relatively compact set in E with respect to p-topology.

Proof: Let π be a natural mapping from E to $\bar{E}_p$, $\pi(x) = x + \ker(p)$. Then $\pi(A)$ is a bounded set in a separable Hilbert space $\bar{E}_p$; by previous lemma, it is relatively compact in $\bar{E}_p$, therefore, A is relatively compact in E.

B Seminorm completion

We will first consider here some questions related to the theorem 5.10, then give a usefull convergence lemma 15, and finally (theorem 21) prove a Baire category result. All statements in this part, that involve only one seminorm, hold in a general multi-Hilbertian space (not necessarily a countably Hilbert).

1. Let (E, τ, Π, d) be a countably Hilbert space. We define a *seminorm completion* $\bar{E}_p^s$ as follows:

Let $\bar{E}_p^s$ be the set of all $p-$ Cauchy sequences $\{x_n\}$ on E , with

$$\{x_n\} + \{y_n\} = \{x_n + y_n\}$$

$$a\{x_n\} = \{ax_n\} \quad (a \in R)$$

and a seminorm

$$p(\{x_n\}) = \lim_{n \to \infty} p(x_n)$$

The limes above exists, because $p(x_n)$ is a Cauchy sequence in R. We will show that $\bar{E}_p^s$ is a complete space with the norm as introduced above.

Let $\bar{x}_k = \{x_{k,n}\}_k$ be a Cauchy sequence in $\bar{E}_p^s$. That means

$$\lim_{n \to \infty} p(x_{j,n} - x_{k,n}) \to 0 \quad \text{as } j, k \to \infty.$$

For each fixed k, choose n_k such that

$$p(x_{k,m} - x_{k,n_k}) < \frac{1}{k} \quad \text{if } m \geq n_k.$$

Let $\bar{x} = \{x_{1,n_1}, \ldots, x_{k,n_k} \ldots\}$. Then $\bar{x} \in \bar{E}_p^s$, because for $m \geq \max(n_j, n_k)$:

$$\begin{aligned} p(x_{j,n_j} - x_{k,n_k}) &\leq p(x_{j,n_j} - x_{j,m}) + p(x_{j,m} - x_{k,m}) + p(x_{k,m} - x_{k,n_k}) \\ &\to 0 \quad \text{as } j, k \to \infty, \end{aligned}$$

and, for a fixed k :

$$p(\bar{x}_k - \bar{x}) = \lim_{n \to \infty} p(x_{k,n} - x_{k,n_k}) \leq \frac{1}{k}.$$

So, $p(\bar{x}_k - \bar{x}) \to 0$ as $k \to \infty$, and the completeness is proved.

Let us define a map $\pi : E \to \bar{E}_p^s$ by

$$\pi(x) = \{x, x, \ldots, x, \ldots\} \tag{1}$$

It is clear that $\pi(\alpha x+\beta y) = \alpha\pi(x)+\beta\pi(y)$, for $\alpha, \beta \in R$, and $p(\pi(x)) = p(x)$, so π is an isomorphism and an isometry. Let us show that $\pi(E)$ is dense in $\bar{E}_p^s$.

An arbitrary element $\{x_k\}$ in $\bar{E}_p^s$ is a limit of a sequence $\{x\}_n$, with

$$\{x\}_n = \{x_n, x_n, \ldots, x_n, \ldots\}.$$

Actually,

$$p(\{x\}_n - \{x_k\}) = \lim_{k\to\infty} p(x_n - x_k)$$

and

$$\lim_{n\to\infty} p(\{x\}_n - \{x_k\}) = \lim_n \lim_k p(x_n - x_k) = 0,$$

because $\{x_k\}$ is a Cauchy sequence. Therefore, we have proved

2. Theorem. The mapping π as defined by (1) is a 1-1 isometrical and isomorphical mapping of E_p onto a dense linear subspace of the space $\bar{E}_p^s$ whose elements are all $p-$ Cauchy sequences in E.

3. Remark. An obvious danger of an uncautious use of $\bar{E}_p^s$ is in the fact that $\bar{E}_p^s$ is only a seminormed space, even if p is a norm.

Let us define a semi-metric d on $\bar{E}_p^s$ by

$$d(\{x_n\}, \{y_n\}) = \sum_{i=1}^{\infty} \overline{\lim_{n\to\infty}} \frac{p_i(x_n - y_n)}{1 + p_i(x_n - y_n)}.$$

Then $d(\{x_n\}, \{y_n\}) = 0$ if and only if for all $p_i \in \Pi$:

$$\lim_{n\to\infty} p_i(x_n - y_n) = 0.$$

So, an equivalence relation can be defined in $\bar{E}_p^s$ by

$$\{x_n\} \sim \{y_n\} \iff d(\{x_n\}, \{y_n\}) = 0.$$

The set of all such equivalence classes, equipped by the topology of $\bar{E}_p^s$ will be denoted by $\bar{E}_p^s/\ker(d)$. A class corresponding to each $x \in E$ consists of a single element, because d is a metric on E. Therefore,

4. Lemma. There is a 1-1, isometrical and isomorphical mapping of E_p onto a dense linear subspace of $\bar{E}_p^s/\ker(d)$.

5. A usual way of completion of E with respect to p is similar to the previous procedure, except that one first takes $E/\ker(p)$ and then defines an equivalence relation in the set of all $p-$ Cauchy sequences of $E/\ker(p)$, by

$$\{x_n + \ker(p)\} \sim \{y_n + \ker(p)\} \iff \lim_{n\to\infty} p(x_n - y_n) = 0.$$

New elements obtained in this way should be actually denoted by $\{x_n + \ker(p)\} + \ker(p)$, but it is customary to use the notation $\{x_n + \ker(p)\}$. Denote the set of all such equivalence classes by $\bar{E}_p$. Proceeding in an identical way as in the case of $\bar{E}_p^s$, one can prove the following

6. Lemma. There is a 1-1, isometrical and isomorphical mapping of $E/\ker(p)$ to a dense subspace of a space $\bar{E}_p$, whose elements are equivalence classes of all $p-$ Cauchy sequences in $E/\ker(p)$.

7. Let us define on $\bar{E}_p^s$:

$$\ker(p) = \{\ \{x_n\} \in \bar{E}_p^s\ |p(\{x_n\}) = 0\}.$$

Let $\bar{E}_p^s/\ker(p)$ be the set of all equivalence classes with respect to $\ker(p)$. Observe the following:

Two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ belong to a same class if and only if $\lim_n p(x_n - y_n) = 0$.

Two elements $x, y \in E$, regarded as Cauchy sequences (by mapping (1)) belong to a same class in $\bar{E}_p^s/\ker(p)$ if and only if $p(x - y) = 0$.

Let us define $\pi_s : \bar{E}_p^s/\ker(p) \to \bar{E}_p$ by:

$$\pi_s(\{x_n\} + \ker(p)) = \{x_n + \ker(p)\}. \tag{2}$$

By the observations above, we have:

8. Lemma. The mapping π_s, defined by (2) is an isomorphic, isometric, 1-1 mapping of $\bar{E}_p^s/\ker(p)$ onto $\bar{E}_p$, so we may write

$$\bar{E}_p^s/\ker(p) = \bar{E}_p.$$

9. Since $p(\{x_n\} - \{y_n\}) = 0$ if $d(\{x_n\}, \{y_n\}) = 0$, we have that $(\bar{E}_p^s / \ker(d)) / \ker(p) = \bar{E}_p^s / \ker(p)$, and therefore, $(\bar{E}_p^s / \ker(d)) / \ker(p) = \bar{E}_p$.

10. Let us now consider two seminorms p, q on E, such that $p < q$. Since every $q-$ Cauchy sequence is also a $p-$ Cauchy sequence, we have that $\bar{E}_q^s \subset \bar{E}_p^s$ and also $\bar{E}_q^s / \ker(d) \subset \bar{E}_p^s / \ker(d)$. So, the identity injection

$$i_{p,q} \; : \; \bar{E}_q^s \to \bar{E}_p^s$$

is a 1-1 continuous mapping.

The "1-1" property fails to hold in the case of the map

$$i_{p,q} \; : \; \bar{E}_q \to \bar{E}_p,$$

as we pointed out in 3.4. Even if p, q are norms, $i_{p,q}$ may not be 1-1 because two or more classes in $\bar{E}_q$ may be mapped in a same class in $\bar{E}_p$; i.e, two Cauchy sequences may converge to the same limit in the $p-$seminorm, but to different limits in the $q-$seminorm. If this occurs, we cannot write $\bar{E}_q \subset \bar{E}_p$, which is important if we want to define $\cap_i \bar{E}_{p_i}$ as in the theorem 5.10 (there is no a common space in which all $\bar{E}_{p_i}$ can be embedded). To avoid such a difficulty, it is suggested (Gelfand and Shilov, section 2.2.) that a compatibility of p and q should be required. We will now show that our construction by means of seminorm completion yields the same result as the classical one in a special case of compatible norms.

11. Definition. If $p, q (p < q)$ are norms on E, we say that they are *compatible* if any $q-$Cauchy sequence $\{x_n\}$ that converges to 0 in $p-$norm, also converges to 0 in $q-$norm.

12. Theorem. Let (E, τ, Π, d) be a countably Hilbert space, with $\Pi = \{p_i\}_{i=1}^{\infty}$, where for every $i < j$, p_i and p_j are compatible norms. Then

$$\bigcap_{i=1}^{\infty} \bar{E}_{p_i}^s / \ker(d) = \bigcap_{i=1}^{\infty} \bar{E}_{p_i} \qquad (3)$$

Proof: Elements of the set on the left hand side of (3) are equivalence classes of Cauchy sequences in E, defined in such a way that two Cauchy sequences x_n,y_n belong to a same class if and only if $d(x_n - y_n) \to 0$. By $\bar{E}_{p_1}^s \subset \bar{E}_{p_2}^s \subset \ldots$, sequences in the intersection are Cauchy in all p_i, and so they are $d-$Cauchy sequences.

Elements of the set on the right hand side of (3) are also equivalence classes with respect to $\ker(p_1)$, i.e, two d−Cauchy sequences belong to the same class if and only if $p_1(x_n - y_n) \to 0$.

The compatibility of norms is, indeed, equivalent to $\ker(d) = \ker(p_1)$ in the space of all Cauchy sequences; that is , if x_k is a d−Cauchy sequence and $p(x_k) \to 0$ then $d(x_k) \to 0$.

Therefore, (3) is proved.

13. Corollary. Under assumptions of theorem 12, E is a complete metric space if and only if

$$E = \bigcap_i \bar{E}_{p_i}.$$

Proof: Follows from (3) and theorem 5.10.

14. Remark. Let us give a simple model to understand seminorms: Let $E = R_3$ and for $u \in E$, $u = (x, y, z)$ let $p_1(u) = |x|$, $p_2(u) = |y|$, $p_3(u) = |z|$. Then topology τ on E (Euclidean topology) can be thought of as being determined by p_1, p_2, p_3 , because

$$u_n \xrightarrow{\tau} u \iff p_i(u_n - u) \to 0 \text{ for } i = 1, 2, 3.$$

We have that $\ker(p_1)$ is yOz plane, and similarly for p_2 and p_3.

Let $u_n = (1 + \frac{1}{n}, 2 + \frac{1}{n}, 3 + \frac{1}{n})$. Let $v_1 = (1,0,0)$, $v_2 = (0,2,0)$, $v_3 = (0,0,3)$. Then

$$u_n \xrightarrow{p_1} v_1$$

$$u_n \xrightarrow{p_1} v_2$$

$$u_n \xrightarrow{p_1} v_3$$

All three limit points are different , but we can choose $v = (1,2,3)$, such that $u_n \to v$ in all three seminorms. A natural question one can ask is does it work this way in general. A partial answer is given by the following lemma.

15. Lemma. Let E be a topological vector space with seminorms p, q defined on E. Suppose that $p(x_n - x_p) \to 0$ and $q(x_n - x_q) \to 0$, as $n \to \infty$, where $x_n, x_p, x_q \in E$. Then there exists an x_0 in the completion $\bar{E}$ of E with respect to $p \vee q$ such that $p(x_p - x_0) = q(x_q - x_0) = 0$.

Proof: From $p(x_n - x_p) \to 0$ and $q(x_n - x_0) \to 0$ we conclude that x_n is both $p-$ and $q-$ Cauchy sequence. Therefore, x_n is $p \vee q-$ Cauchy sequence , and there is $x_0 \in \bar{E}$ such that $(p \vee q)(x_n - x_0) \to 0$, which implies $p(x_n - x_0) \to 0$ and $q(x_n - x_0) \to 0$. Thus, $p(x_p - x_0) = q(x_q - x_0) = 0$, which had to be proved.

16. Remark. The assertion of lemma 15 holds regardless of the way of completion (seminorm or norm completion).

17. Corollary. If p and q are *norms* then $p(x_n - x_p) \to 0$ and $q(x_n - x_q) \to 0$ imply $x_p = x_q$.

Proof: By lemma 15, these assumptions imply $p(x_p - x_0) = q(x_q - x_0) = 0$, and this implies $x_p = x_0 = x_q$.

18. Lemma.. Let (E, τ, Π, d) be a complete countably Hilbert space with seminorms $p_1 < p_2 < \ldots$. Suppose that $p_i(x_n - y_i) \to 0$ as $n \to \infty$, for all i. Then there is an $y_0 \in E$ such that $p_i(y_i - y_0) = 0$ for all i.

Proof: For a fixed i and $m > i$ we have:

$$p_i(x_n - y_m) \leq p_m(x_n - y_m) \to 0 \quad \text{as } n \to \infty.$$

Using the assumption $p_i(x_n - y_i) \to 0$, we have $p_i(y_{m_1} - y_{m_2}) = 0$ if $m_1, m_2 \geq i$.Thus, y_m is a Cauchy sequence ; by completeness, there is $y_0 = \lim_m y_m$. Then

$$p_i(y_i - y_0) \leq p_i(y_i - y_m) + p_i(y_m - y_0) \to 0 \quad \text{as } m \to \infty$$

19. Remark. In 5.13 we defined a G-function as a map $V : E \to R_+$, such that for every $n \in N$, the set $G = \{x \in E | V(x) \leq n\}$ is closed. A typical example of a G-function is a positive function V that satisfy

$$x_n \to x \Rightarrow \overline{\lim} V(x_n) \geq V(x) \tag{4}$$

A function that satisfy (4) with $\underline{\lim}$ is called a *lower semicontinuous function*.The next theorem and also lemma 5.14 are usually stated in terms of lower semicontinuous functions and taking the whole space E instead of a ball B as below.

20. Theorem. Let (E, d) be a complete metric space and let I be an arbitrary index set. Let $\{U_i\}_{i \in I}$ be a familly of G-functions. Let $B \subset E$

be an open d–ball ($B = \{x \in E \mid d(u,x) < a\}$, for some $u \in E$, $a \in R$). Suppose that for each $x \in B$, the set $\{U_i(x) \mid i \in E\}$ is bounded. Then there is a ball

$$B_r(z) = \{x \in E | d(z,x) \leq r\}$$

and a positive constant M such that

$$(\forall x \in B_r(z))(\forall i \in I)\; U_i(x) \leq M.$$

Proof: Let

$$G_{in} = \{x \in E | U_i(x) \leq n\}.$$

Since U_i are G-functions, G_{in} is a closed set for every i and n. Let

$$F_n = \bigcap_{i \in I} G_{in} \qquad n = 1, 2, \ldots.$$

Then F_n is a closed set, and by assumption, $\cup_n F_n \supset B$,so $\cap_n F_n^c \subset B^c$. Since B is an open ball, $\cap_n F_n^c$ is not dense in E; therefore, by Baire theorem (appendix D.11), there is a set F_M^c which is not dense in E, so there is a ball in E that belongs to F_M which had to be proved.

21. Corollary Let (E, τ, Π, d) be a complete countably Hilbert space. Then under assumptions of Theorem 20, there is a seminorm $p \in \Pi$, a ball $B = \{x \in E | p(x - z) \leq r\}$ and a constant $M > 0$ such that $U_i(x) \leq M$ for all $x \in B$ and all $i \in I$.

Proof: By Lemma 5.4, every d-ball contains a p-ball, and the assertion follows directly from Theorem 20.

C Linear functionals

In the next part of the text we will consider linear functionals on a multi-Hilbertian space (E, τ, Π). Although E'_p is a Hilbert space, and every $F \in E'_\tau$ belongs to a E'_p, we cannot *a priori* make full use of Hilbert space theory due to the following facts:

(i) F is defined only on E.

(ii) E need not be complete with respect to p,

(iii) p is, in general, a seminorm.

Fortunately enough, this is not a serious nuisance. The only real difference that above facts make is contained in the requirement (8) of corollary 2 below.

Let us remark that F is a linear functional with respect to a seminorm p, if and only if (appendix D.2) :

$$\tilde{p}(F) \stackrel{def}{=} \sup_{p(x)\leq 1} |\langle F,x\rangle| < \infty \tag{1}$$

1. Theorem. Let F be a linear functional (not necessarily continuous) on E. Then for every $p \in \Pi$:

$$\tilde{p}^2(F) = \sup_{p(x)\leq 1} |\langle F,x\rangle|^2 = \sup_{\alpha} \sum_{i=1}^{n} \langle F, e_{\alpha_i}\rangle^2, \tag{2}$$

where sup goes over all finite p-orthonormal sets $\alpha = \{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_n}\} \in E$. If F is continuous with respect to p then

$$\tilde{p}^2(F) = \sum_{i=1}^{\infty} \langle F, e_i\rangle^2 < \infty, \tag{3}$$

where $\{e_i\}$ is a p-orthonormal base as in theorem A.3.

Proof: It suffices to give a proof for $F \neq 0$. Let us first show that for every $x \in E, p(x) \leq 1$, there is an element $x_0 \in E - \ker(p)$ such that $\langle F,x\rangle^2 \leq \langle F,x_0\rangle^2$ and $p(x_0) = 1$.

If $x \in E - \ker(p)$, take $x_0 = x/p(x)$. If $x \in \ker(p)$ and $\langle F,x\rangle = 0$, then for x_0 take any element in $E - \ker(p)$ with $p(x_0) = 1$. If $x \in \ker(p)$ and $\langle F,x\rangle = M > 0$, then let $y \in E - \ker(p)$ such that $\langle F,y\rangle > 0$ and $p(y) = 1$. (To show that such an y exists, it is enough to show that there is a $z \in E - \ker(p)$ with $\langle F,z\rangle \neq 0$. If such z does not exists, then there must be an $w \in \ker(p)$ such that $\langle F,w\rangle \neq 0$. Then $w + z \notin \ker(p)$ and $\langle F, x+y\rangle \neq 0$, which is a contradiction.) Now take $x_0 = x + y$ to obtain $p(x_0) = p(y) = 1$ and $\langle F,x_0\rangle = \langle F,x\rangle + \langle F,y\rangle > M$ by construction. In the case $\langle F,x\rangle < 0$ we replace x with $-x$ and apply the above proof. Therefore, we have just proved

$$(\forall x : p(x) \leq 1)(\exists x_0 \in E - \ker(p),\, p(x_0) = 1)(\langle F,x\rangle^2 \leq \langle F,x_0\rangle^2)$$

But every x_0 with $p(x_0) = 1$ can be a member of some orthonormal set; therefore, for every x such that $p(x) \leq 1$ we have

$$\langle F,x\rangle^2 \leq \sup_{\alpha} \sum_{i=1}^{n} \langle F, e_{\alpha_i}\rangle^2$$

and

$$\tilde{p}^2(F) = \sup_{p(x)\leq 1} \langle F, x\rangle^2 \leq \sup_{\alpha} \sum_{i=1}^{n} \langle F, e_{\alpha_i}\rangle^2 \tag{4}$$

To show the converse inequality, let $\alpha = (e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_n})$ be a fixed orthonormal system.Define

$$x_\alpha = \frac{\sum_{k=1}^{n} \langle F, e_{\alpha_k}\rangle e_{\alpha_k}}{(\sum_{k=1}^{n} \langle F, e_{\alpha_k}\rangle^2)^{1/2}}$$

Then $p(x_\alpha) = 1$ and

$$\langle F, x_\alpha\rangle^2 = \sum_{k=1}^{n} \langle F, e_{\alpha_k}\rangle^2.$$

This shows that for every α

$$\tilde{p}^2(F) \geq \sum_{k=1}^{n} \langle F, e_{\alpha_k}\rangle^2$$

and, therefore,

$$\tilde{p}^2(F) \geq \sup_{\alpha} \sum_{k=1}^{n} \langle F, e_{\alpha_k}\rangle^2 \tag{5}$$

Inequalities (4) and (5) together give the desired equality (3).

For the second part, note that if F is continuous with respect to p, then $\tilde{p}(F) < \infty$ and

$$\begin{aligned} |\langle F, x\rangle| &= |\langle F, \frac{x}{p(x)}\rangle| \, p(x) \leq \sup_{p(x)\leq 1} |\langle F, x\rangle| \, p(x) \\ &= \tilde{p}(F) p(x) \end{aligned}$$

So, we have for $x = \sum_{i=1}^{\infty} p(x, e_i)\, e_i + r$:

$$|\langle F, x\rangle - \langle F, \sum_{i=1}^{n} p(x, e_i)\, e_i\rangle| \leq \tilde{p}(F)\, p(x - \sum_{i=1}^{n} p(x, e_i)\, e_i),$$

and by theorem A.3,the righthand side above tends to 0 as $n \to \infty$. Hence we have:

$$|\langle F, x\rangle| = \lim_{n\to\infty} |\langle F, \sum_{i=1}^{n} p(x, e_i)\, e_i\rangle|$$

$$
\begin{aligned}
&= \lim_{n\to\infty} |\sum_{i=1}^{n} p(x,e_i)\langle F,e_i\rangle| \\
&\leq (\sum_{i=1}^{n} p^2(x,e_i))^{1/2}(\sum_{i=1}^{n}\langle F,e_i\rangle^2)^{1/2} \\
&= (\sum_{i=1}^{\infty} p^2(x,e_i))^{1/2}(\sum_{i=1}^{\infty}\langle F,e_i\rangle^2)^{1/2},
\end{aligned}
$$

so we conclude that $\tilde{p}^2(F) \leq \sum_{i=1}^{\infty}\langle F,e_i\rangle^2$.

Now to show the converse inequality , let

$$x_n = \frac{\sum_{i=1}^{n}\langle F,e_i\rangle e_i}{(\sum_{i=1}^{n}\langle F,e_i\rangle^2)^{1/2}}.$$

Then $p(x_n) = 1$ and $\langle F,x_n\rangle = \sum_{i=1}^{n}\langle F,e_i\rangle^2$,so letting $n\to\infty$ we obtain

$$\tilde{p}^2(F) \geq \sum_{i=1}^{\infty}\langle F,e_i\rangle^2.$$

2. Corollary Let F be a linear functional defined on E. Then $F \in E_p'$ if and only if

$$\sup_{\alpha}\sum_{i=1}^{n}\langle F,e_{\alpha_i}\rangle^2 < \infty, \tag{6}$$

and if and only if the following two conditions are *both* satisfied:

$$\sum_{i=1}^{\infty}\langle F,e_i\rangle^2 < \infty, \tag{7}$$

where $\{e_i\}$ is a p-orthonormal base as in theorem A.3, and

$$(\forall x \in E)\ (p(x) = 0 \Rightarrow \langle F,x\rangle = 0). \tag{8}$$

Proof: The condition (6) follows directly from the previous theorem and (1), as well as the necessity of (7) and (8). So, let us assume that (7) and (8) hold. Then let x_n be a p-null sequence in E, i.e. $x_n \to 0$ in p-seminorm. By Theorem 3 we have

$$x_n = \sum a_{in}e_i + r_n \qquad (p(r_n) = 0),$$

and using (7) and (8), we have:

$$\begin{aligned}\langle F, x_n\rangle^2 &= (\sum a_{in}\langle F, e_i\rangle)^2 \\ &\leq (\sum a_{in}^2)(\sum\langle F, e_i\rangle^2) \\ &= p^2(x_n)(\sum\langle F, e_i\rangle^2) \to 0,\end{aligned}$$

which ends the proof.

3. Remark. Let us note that (6) is not a consequence of (7) without (8): Suppose that (7) holds, and there is a $x_0 \in \ker(p)$, $\langle F, x_0\rangle = K \neq 0$. Define another $p-$orthonormal base by $f_i = e_i + x_0$. Then $\sum\langle F, f_i\rangle^2 = \infty$, and (6) is clearly violated. As an actual example, take the space of all real valued bounded functions with a seminorm $p(f) = |f(0)|$ and a linear functional F defined by $\langle F, f\rangle = f(1)$.

4. Theorem. Let (E, τ, Π) be a multi-Hilbertian space, Suppose that a metric d is defined by means of countably many seminorms in Π, and let $p \in \Pi$ be one of these seminorms. Then

$$E_p' \cong (\bar{E}_p^s)' \cong (\bar{E}_p^s / \ker(d))' \cong (\bar{E}_p)', \tag{9}$$

where $X \cong Y$ means that there is an isometrical and isomorphic mapping between X and Y, which is 1-1 and onto.

Proof: We shall firstly prove

$$(E_p / \ker(p))' \cong E_p' \tag{10}$$

and

$$(\overline{E_p / \ker(p)})' \cong (E_p / \ker(p))', \tag{11}$$

so these two relations will lead to $E_p' \cong (\bar{E}_p)'$.To prove (10) , let $\pi : E_p' - (E_p / \ker(p))'$ be defined by $\pi(F) = F_\pi$, $\langle F_\pi, x + \ker(p)\rangle = \langle F, x\rangle$. This mapping is well defined because , by the continuity of F it follows that $p(x-y) = 0 \Rightarrow \langle F, x\rangle = \langle F, y\rangle$. Further, it is obviously 1-1 map and onto. By $\pi(\alpha F + \beta G) = \alpha\pi(F) + \beta\pi(G)$, it is an isomorphic map. Finally , it is an isometry by

$$\sup_{p(x)\leq 1} |\langle F, x\rangle| = \sup_{p(x+\ker(p))\leq 1} |\langle F_\pi, x + \ker(p)\rangle|.$$

(11) is a consequence of a more general relation: If X is a dense subspace of a semi-normed vector space Y, then $X' \cong Y'$. Namely, by Hahn-Banach theorem (see appendix D.4), every linear functional F defined on X can be extended to Y; by denseness of X in Y, this extension is unique and preserves the norm of F.

The rest of (9) can be proved in a similar way, using theorem B.2 and lemmas B.4 and B.8.

D References

In addition to listing bibliographical items, we state the results that we used in this paper.

1. In a topological vector space X, every neighborhood U of zero has the following property:

$$(\forall x \in X)(\exists \lambda \in \mathrm{R})\,(\lambda x \in U)$$

[Trèves, theorem 3.1]

2. Let E be a topological vector space with a seminorm p. Let F be a linear functional on E. Then F is continuous if and only if

$$\sup_{p(x)\leq 1} |\langle F, x\rangle < \infty$$

[Yosida, I.6.1]

Proofs of following 6 statements can be found in Rudin (theorems 2.8, 3.3, 3.4, 3.15, 3.16, 3.18).

3. (*Banach-Steinhaus*) Let (E, d) be a complete metric space and let F_n be a sequence of linear continuous functionals defined on E, such that

$$f(x) = \lim_{n\to\infty} \langle F_n, x\rangle$$

exists for every $x \in E$. Then f is a continuous linear functional on E.

4. (*Hahn-Banach*) Suppose M is a subspace of a vector space E, and p is a seminorm on E. Let f be a lineear functional on M such that

$$|\langle f, x\rangle| \leq cp(x) \qquad (x \in M).$$

Then f can be extended to a linear functional F on E that satisfies

$$\langle F, x\rangle = \langle f, x\rangle \qquad (x \in M)$$

$$|\langle F, x\rangle| \leq cp(x) \qquad (x \in E)$$

5. (*Hahn-Banach*) Let A and B be disjoint nonempty convex sets in a Hausdorff topological vector space E. If A is open, then there is a linear continuous functional F, defined on E, such that

$$\langle F, x\rangle < a \leq \langle F, y\rangle,$$

for every $x \in A, y \in B$ and some $a \in R$.

6. (*Banach-Alaoglu*) Let V be a neighborhood of 0 in a topological vector space E and let

$$K = \{F \in E' \mid |\langle F, x\rangle| \leq 1 \text{ for every } x \in V\}.$$

Then K is a w-* compact set.

7. Let E be a separable topological vector space, and let $K \in E'$ be a w-* compact set. Then K is metrizable in the w-* topology.

8. Let E be a locally convex topological vector space. We say that the set $B \in E$ is *(strongly) bounded* if, for any neighborhood U of 0 there is a $\lambda \in R$ such that $\lambda B \subset U$. We say that B is *weakly bounded* if for any $F \in E'$, the set $\{\langle F, x\rangle \mid x \in B\}$ is bounded.

Any set $B \subset E$ is (strongly) bounded if and only if it is weakly bounded.

The following two results are taken from Yosida (III.6).

9. (*Riesz*) Let H be a Hilbert space with an inner product $p(\cdot,\cdot)$ and the norm $p(\cdot)$, and let F be a continuous linear functional on H. Then there is a uniquelly determined $y \in H$ such that for every $x \in H$,

$$\langle F, x\rangle = p(y, x)$$

10. Under the assumptions of previous lemma, if for a continuous linear functional F

$$\tilde{p}\,(F) = \sup_{p(x)\leq 1} |\langle F, x\rangle|,$$

then for every $x \in H$:

$$p(x) = \sup_{\tilde{p}(F)\leq 1} |\langle F, x\rangle|$$

11. (*Baire category theorem*) If E is a complete metric space, then the intersection of every countable collection of dense open subsets of E is dense in E.

[Rudin, theorem 2.2]

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