Wavelength-dependent bulk parameters for coherent sound in correlated distributions of small-spaced scatterers

Victor Twersky
Mathematics Department, University of Illinois, Chicago, Illinois 60680

(Received 25 May 1984; accepted for publication 8 August 1984)

Earlier results for coherent propagation of sound in correlated random distributions of two-parameter particles of radius a (with minimum separation $b \geq 2a$ small compared to wavelength $\lambda = 2\pi/k$) are generalized to obtain the refractive and absorptive terms and the corresponding bulk parameters to order $(ka)^2$. The present development includes higher order terms of the earlier multiple scattering by monopoles and dipoles, as well as scattering and multipole-coupling effects through quadrupole terms. The correlation aspects are determined by the statistical mechanics radial distribution function $f(R)$ for impenetrable particles of diameter b. The new terms for slab scatterers and spheres involve the integral of fR (first moment), or of $Rf \ln R$ for cylinders. The new packing factor is evaluated exactly for slabs as a simple algebraic function of the volume fraction w, and it is shown that the bulk index of refraction reduces to that of one particle in the limit $w = 1$. Similar results are obtained for spheres in terms of the Percus–Yevick approximation and the unrealizable limit $w = 1$.

PACS numbers: 43.20.Fn, 43.20.Bi

INTRODUCTION

We apply general results for coherent propagation in pair-correlated random distributions[1] of particles with minimum separation (b) of centers small compared to wavelength ($\lambda = 2\pi/k$) to obtain additional terms in k of the bulk parameters (C,B) and index of refraction ($\eta^2 = C/B$) considered before.[2] Using γ for either C, B, or η^2, and the form $\gamma = \gamma_r + i\gamma_a + i\gamma_s$, for small kb, we obtained[2] results for the refractive (γ_r) and absorptive (γ_a) terms that were explicitly independent of k, and corresponding results for the scattering (γ_s) loss term to lowest order in k. The explicit approximations for γ_r and γ_a for spheres, cylinders, and slabs ($m = 3, 2, 1$, respectively) depended only on the particles' radius or half-width (a), their acoustic parameters, and on their average number (ρ) per unit volume: They exhibited the statistical aspect of the problem only in the volume fraction $w = \rho v$ with $v = v(a)$ as the volume of one particle. The corresponding scattering terms γ_s were additionally dependent on $(ka)^m$ and on the low-frequency limit of the structure factor $\mathscr{W}(W)$ with $W = w(b/2a)^m$ as the volume fraction of impenetrable statistical particles with diameter $b > 2a$, i.e., in general, each particle was visualized as having an acoustically transparent coating of thickness $(b/2) - a$. The present paper applies the general theory[1] to derive the leading λ-dependent terms of γ_r and γ_a; these depend explicitly on $(ka)^2$ and a/b for all cases, and on appropriate correlation integrals $\mathscr{N}(W)$. We parallel a recent development for the simpler one-parameter optical case.[3]

The correlation aspects of the distribution we consider are determined by the statistical-mechanics radial distribution function[4] $f(R)$ for impenetrable particles, and are exhibited explicitly as simple integrals over all R of the total correlation function $F = f - 1$. The integrals for spherical and slab particles are of the form $\int FR^n\, dR$ (moments of F), but cylindrical particles also involve $\int RF(\ln R)dR$. These can all be evaluated numerically from existing statistical mechanics forms or approximations[4–15] for f. We obtained explicit closed form approximations before[2,16] for the integrals that arise in the $\mathscr{W}$ set, and also used the required $\mathscr{N}$ integral for spheres in a related development[17] for large kb; for slabs, we obtain both the $\mathscr{W}$ and $\mathscr{N}$ integrals from our earlier Laplace transformation[13] of the exact Zernike–Prins result[12] for f. For cylinders, we may use the virial expansion for f to consider some of the properties of $\mathscr{N}$.

In the following, for brevity, we use, for example (1:113) to indicate Eq. (113) of Ref. 1, as well as essentially the same notation as before.[1–3] We generalize the earlier[2] multiple scattering monopole and dipole approximations for C, B, and η^2 by including scattering and multipole coupling to higher orders in k, as well as quadrupoles (for spheres and cylinders) to lowest order. For slabs and $b = 2a$ (minimum separation of slab centers equal to slab thickness), the explicit approximations for the bulk values (γ) reduce to the single particle values (γ') if $w \rightarrow 1$, as required from physical considerations: $\gamma \rightarrow \gamma'$ because the particles occupy all space. The limit $w \rightarrow 1$ is not realizable for identical spheres, and we take $w \lesssim w_d \approx 0.63$, with w_d as the densest random packing introduced earlier[16] to define the amorphous solid. However, our explicit approximations for γ for spheres also reduce to γ' as w increases to 1; we regard this as consistent with the approximations involved in scaled particle[5] and Percus–Yevick[4,6] statistical mechanics theory, and with the closure approximation used in the multiple scattering theory.[1] For cylinders, we take $w \lesssim w_d \approx 0.84$ as before.[18,2] Were an analogous closed form available for the $\mathscr{N}$ integral for this case, we would expect the corresponding approximations for γ to show the same behavior for the nonrealizable limit $w \rightarrow 1$.

The present application of the general theory[1] to larger kb than before[2] plus the recent applications[17] to large kb provide simple forms which explicitly display the functional dependence on all key parameters for many practical applications. Thus, in these ranges of kb, elaborate machine com-

 0001-4966/85/010029-09$00.80

putations are no longer required, and the results help delineate the fundamental physical processes.

I. GENERAL CONSIDERATIONS

For a slab-region distribution and a normally incident wave $\phi e^{-i\omega t}$ (the excess pressure) we write

$$\phi = e^{ikz}, \quad k = 2\pi/\lambda = 2\pi\eta_e/\lambda_e, \qquad (1)$$

with η_e as the index of the embedding medium. The corresponding bulk coherent propagation coefficient is given by

$$K = k\eta_b/\eta_e = k\eta, \quad \eta^2 = C/B, \qquad (2)$$

where in the simplest cases, C is the relative compressibility and B^{-1} is the relative mass density. The values are to be expressed in terms of ρ and F for pair correlated particles specified by their isolated scattering amplitudes $g(\hat{\mathbf{r}},\hat{\mathbf{z}})$. The normalization for g is such that for lossless particles

$$-\operatorname{Re} g(\hat{\mathbf{z}},\hat{\mathbf{z}}) = -\operatorname{Re} g = \mathscr{M}|g(\hat{\mathbf{r}},\hat{\mathbf{z}})|^2, \qquad (3)$$

with $\mathscr{M}$ as the mean over all directions of observation $\hat{\mathbf{r}}$. The corresponding known[19] scattering coefficients a_n are normalized by the form

$$g = \sum a_n; \quad a_n = a_n(C',B';x), \quad x \equiv ka, \qquad (4)$$

where

$$C' = \frac{C_p}{C_e}, \quad B' = \frac{B_p}{B_e}; \quad C' - 1 \equiv \mathscr{C}, \quad B' - 1 \equiv \mathscr{B}, \qquad (5)$$

with C' and B' as the particle's relative acoustic parameters, such that $\operatorname{Im} C' > 0$ and $\operatorname{Im} B' < 0$ to account for energy losses; $\mathscr{C}$ and $\mathscr{B}$ are the parametric contrasts. In addition to the dependence on the relative parameters and on x (the normalized radius or half-width), the coefficients depend on the dimensionality (m) of the problem.

We obtain results for the bulk relative values $\gamma = \{C,B,\eta^2\}$ in the form

$$\gamma = \gamma_1 + \gamma_c + i\gamma_s = \Gamma_1 + x^2\Gamma_c + ix^m\Gamma_s, \qquad (6)$$

where Γ_i (with $i = 1,c,s$) depends on m. The forms for Γ_1 and Γ_s, corresponding to multiple scattering by monopoles and dipoles to lowest order in x for the real and imaginary parts of a_0 and a_1, were discussed before[2] in detail. Now we obtain Γ_c.

From Rayleigh's results for spherical dipoles,[20] the first approximation for sparse uncorrelated distributions corresponds to

$$\eta_R - 1 = -\frac{cg}{2}, \quad c = \frac{i2\rho}{k}, \frac{i4\rho}{k^2}, \frac{i4\pi\rho}{k^3}, \qquad (7)$$

where the order for c (and subsequent sets) corresponds to $m = 1, 2, 3$. In Ref. 18 (1975), for lossless particles with small parametric contrast, we multiplied $\operatorname{Im} \eta_R$ by the statistical mechanics packing factor $\mathscr{W}$ to obtain the appropriate η_s for the correlated case. Appropriate forms for C_1 and B_1 for spheres and slabs were given by Maxwell,[21] and for cylinders and spheres by Rayleigh.[22] We obtained γ_1 and γ_s from (2:59), which followed from either the full multiple scattering procedure or an equivalent slab procedure (interface approximation).[1,2] For γ_c the two procedures lead to different results, as illustrated for circular cylinders by (1:101)ff. Now we apply the multiple scattering procedure to obtain Γ_c explicitly for an unbounded distribution. See discussion after (1:102) and (23:37), as well as after (23:50) which considers the impedance; discrepancies may arise for measurements on plane-bounded distributions because of transition region boundary layers.

Rewriting (2:30) in the original form (1:89)ff, we use

$$-\frac{(\eta^2 - 1)}{c} = \sum P_n \approx P_0 + P_1 + P_2, \qquad (8)$$

where P_2 does not arise for slabs. The coefficients P_n are determined by the linear system of algebraic equations (matrix equation),

$$P_n = a_n\, \eta^{2n}\Big(1 + \sum \mathscr{H}_{n\nu}\, \eta^{-n-\nu} P_\nu\Big), \qquad (9)$$

with $\mathscr{H}_{n\nu}$ given in terms of the correlation integrals $\mathscr{H}_n$ by (2:21)ff. All cases we consider are covered by

$$P_0 = a_0\Big(1 + P_0\mathscr{H}_0 + \frac{P_1\mathscr{H}_1}{\eta} + \frac{P_2\mathscr{H}_2}{\eta^2}\Big),$$
$$P_1 = a_1\eta^2\Big(1 + \frac{P_0\mathscr{H}_1}{\eta} + \frac{P_1\mathscr{H}_{11}}{\eta^2} + \frac{P_2\mathscr{H}_{12}}{\eta^3}\Big), \qquad (10)$$
$$P_2 = a_2\eta^4\Big(1 + \frac{P_0\mathscr{H}_2}{\eta^2} + \frac{P_1\mathscr{H}_{12}}{\eta^3} + \frac{P_2\mathscr{H}_{22}}{\eta^4}\Big).$$

Introducing

$$R = \sum P_\nu \Big/ \sum P_n\left(\frac{d_n}{\eta^n}\right), \qquad (11)$$

with $d_n = d_n^B - d_n^C$ as in (1:95)–(1:100), we write the corresponding bulk parameters as

$$\frac{C-1}{c} = R\sum \frac{P_n d_n^C}{\eta^n} \approx -P_0 - \frac{x^2}{m(m+2)}\Big(P_0\,\eta'^2\,\frac{\mathscr{B}}{\mathscr{C}}(1 - c\,P_0) - m\,\frac{\mathscr{C}}{\mathscr{B}}\,P_1\Big), \qquad (12)$$

$$\frac{\eta^2(B-1)}{c} = R\sum\frac{P_n d_n^B}{\eta^n} \approx P_1 - \frac{x^2}{m(m+2)}\Big(P_0\,\eta'^2\,\frac{\mathscr{B}}{\mathscr{C}}(\eta^2 + cP_1) - m\,\frac{\mathscr{C}}{\mathscr{B}}\,P_1\Big) + P_2, \quad \eta'^2 = C'/B'. \qquad (13)$$

The sets of d's are expressed in terms of Bessel functions of arguments ka and Ka in the forms (1:96)–(1:100); we use the forms in terms of j_n for spheres, in terms of J_n for cylinders, and in terms $\mathscr{J}_0 = \cos$ and $\mathscr{J}_1 = \sin$ for slabs. See cited equations for full details, and (1:101)–(1:107) for illustrations based on cylinders and for comparisons with the interface approximations. As shown in (1:99) and (1:100), the d^C and d^B sets are proportional to $\mathscr{C}$ and $\mathscr{B}$, respectively; thus $C = 1$ if $C' = 1$, and $B = 1$ if $B' = 1$, as required by the theorems (2:13) and (2:14) discussed earlier[23] in detail.

Using (11), we see that the equality $\eta^2(B-1) - (C-1) = 1 - \eta^2$ is satisfied by the rigorous series forms in (13), (12), and (8). To the accuracy indicated for (6), we may replace $\eta^2 + cP_1$ by $1 - cP_0$ within the large parentheses of the approximations; thus, the approximations in (13), (12), and (8) satisfy the equality to the required accuracy.

The isolated scattering coefficients[19] for the slab, $a_n = a_n'/(1-a_n')$ with a_n' imaginary for lossless scatterers, are given by

$$a_0' = ix(\mathscr{C} - x^2 T_0)/(1 - x^2 D_0) + O(x^5),$$
$$T_0 = [\mathscr{C}(1+\eta'^2) + 2\mathscr{B}\eta'^2]/6,$$
$$D_0 = (\eta'^2 - 1 - 2\mathscr{C})/2; \tag{14}$$
$$a_1' = -ix(\mathscr{B} - x^2 T_1)/[1 + \mathscr{B} - x^2 D_1],$$
$$T_1 = [\mathscr{B}(1+\eta'^2) + 2\mathscr{C}]/6, \quad D_1 = (\mathscr{B} + \mathscr{C})/2. \tag{15}$$

For the cylinder, we use $a_0 = a_0'/(1-a_0')$ and $a_n = a_n'/(1-a_n'/2)$ in terms of

$$a_0' = i(\pi x^2/4)(\mathscr{C} - x^2 T_0)/(1 - x^2 D_0) + O(x^6),$$
$$T_0 = [\mathscr{C}(1+\eta'^2) + \mathscr{B}\eta'^2]/8,$$
$$D_0 = (\eta'^2 - 1 + 2\mathscr{C}L)/4 \equiv D_0' + 2\mathscr{C}L/4, \tag{16}$$

where $L = \ln(2/xc')$ with $c' = 1.781...$;

$$a_1' = -i(\pi x^2/4)(\mathscr{B} - x^2 T_1)/[1 + \mathscr{B}/2 - x^2 D_1] + O(x^6),$$
$$T_1 = [\mathscr{B}(1+\eta'^2) + 2\mathscr{C}]/8,$$
$$D_1 = [3\mathscr{C} - 2\mathscr{B}(1+2L) + \eta'^2 - 1)]/16$$
$$\equiv D_1' - 4\mathscr{B}L/16; \tag{17}$$
$$a_2' = -i\pi(x/2)^4\mathscr{B}/(2+\mathscr{B}) + O(x^6). \tag{18}$$

For the sphere, $a_n = a_n'/[1 - a_n'/(2n+1)]$,

$$a_0' = i(x^3/3)(\mathscr{C} - x^2 T_0)/(1 - x^2 D_0) + O(x^7),$$
$$T_0 = [\mathscr{C}(1+\eta'^2) + 2\mathscr{B}\eta'^2/3]/10,$$
$$D_0 = (\eta'^2 - 1 + 2\mathscr{C})/6; \tag{19}$$
$$a_1' = -i(x^3/3)(\mathscr{B} - x^2 T_1)/(1 + \mathscr{B}/3 - x^2 D_1) + O(x^7),$$
$$T_1 = [\mathscr{B}(1+\eta'^2) + 2\mathscr{C}]/10,$$
$$D_1 = [3\mathscr{C} - 5\mathscr{B} + 2(\eta'^2 - 1)]/30; \tag{20}$$
$$a_2' = -ix^5 2\mathscr{B}/9(5 + 2\mathscr{B}) + O(x^7). \tag{21}$$

Although alternative decompositions may be constructed to the same accuracy, the present sets of T's and D's help delineate the dependence on $\mathscr{C} = C' - 1$ and $\mathscr{B} = B' - 1$; comparison of a given set for $m = 1, 2, 3$ indicates the role of dimensionality. If $\mathscr{B} = 0$, then the monopole a_0 dominates; if $\mathscr{C} = 0$, then the dipole a_1 dominates. The ratios of parametric contrasts in (12) and (13) do not introduce singularities; the explicit approximations displayed in the following show that the ratios insure fulfillment of theorems (2:13) and (2:14), so that the development is consistent to the required accuracy. For $m = 2$ and 3, cylinders and spheres, we also consider η^2 for rigid particles ($C' \to 0$, $B' \to 0$) as the limit for $\mathscr{C} \to -1$ and $\mathscr{B} \to -1$.

Using (8)–(21) to lowest orders in x of the real and imaginary parts of all components, we reconstruct (2:63), (2:67), and (2:70) in the present forms γ_1 and γ_s corresponding to (6) and use the results to simplify the subsequent forms for $\gamma_c = x^2\Gamma_c$. We have

$$C_1 = 1 + w\mathscr{C};$$
$$B_1 = 1 + \frac{w\mathscr{B}}{\mathscr{D}}, \quad \mathscr{D} = 1 + \frac{\mathscr{B}(1-w)}{m}, \tag{22}$$

and

$$\eta_1^2 = C_1/B_1 = (1 + w\mathscr{C})/(1 + w\mathscr{B}/\mathscr{D}), \tag{23}$$

where $\mathscr{C} = C' - 1$ and $\mathscr{B} = B' - 1$ are complex if absorption is present. In distinction to the notation in Ref. 2, the present set γ_1 includes absorption losses, such that $\text{Im}\, C_1 > 0$, $\text{Im}\, B_1 < 0$, and $\text{Im}\, \eta_1^2 = 2\,\text{Im}\,\eta_1\,\text{Re}\,\eta_1 > 0$. The corresponding scattering terms are

$$C_s = x^m d_m \mathscr{C}^2 w \mathscr{W}_m, \quad B_s = -x^m \frac{d_m \mathscr{B}^2}{m\mathscr{D}^2} w \mathscr{W}_m;$$
$$d_m = \{1, \pi/4, \tfrac{1}{3}\}, \tag{24}$$

with packing factors as in (2:33),

$$\mathscr{W}_m = \frac{(1-W)^{m+1}}{[1 + (m-1)W]^{m-1}}. \tag{25}$$

For small absorption, i.e., for $|\text{Re}\,\gamma'| \gg |\text{Im}\,\gamma'|$, we use $\mathscr{C}^2 \approx |\mathscr{C}|^2$ and $(\mathscr{B}/\mathscr{D})^2 \approx |\mathscr{B}/\mathscr{D}|^2$.

From (2:70)

$$\frac{C_1 + iC_s}{B_1 + iB_s} \approx \eta_1^2 \left[1 + i\left(\frac{C_s}{C_1} - \frac{B_s}{B_1}\right)\right] \equiv \eta_1^2 + i\eta_s^2;$$
$$\eta_s^2 = \frac{C_s - \eta_1^2 B_s}{B_1} = \frac{x^m d_m w \mathscr{W}}{B_1}\left(\mathscr{C}^2 + \frac{\mathscr{B}^2}{m\mathscr{D}^2}\eta_1^2\right), \tag{26}$$

where $\text{Re}\,(\eta_s^2/2\eta_1)$ corresponds to attenuation via incoherent scattering losses. Similarly for the present generalization based on (6), from analogous decomposition of $\eta^2 = C/B$, we obtain the additional relation

$$\eta_c^2 = \eta_1^2\left(\frac{C_c}{C_1} - \frac{B_c}{B_1}\right) = \frac{C_c - \eta_1^2 B_c}{B_1}, \tag{27}$$

which provides a check for the independent derivation of the new terms γ_c.

II. DISTRIBUTION OF SLABS

For slabs, we use $c = i2\rho/k = iw/x$, $w = \rho 2a$, and retain only P_0 and P_1 in (8)–(13) with $\mathscr{H}_{11} = c + \mathscr{H}_0$, and $\mathscr{H}_0$ and $\mathscr{H}_1$ as in (1:177) or (2:23). The algebraic system for P_0 and P_1 is valid for all $ka = x$, but we consider only the forms (14) and (15) for a_n and the corresponding leading terms of the correlation integrals

$$\mathscr{H}_0 \approx 2\rho \int_0^\infty F\,dR + ik\,2\rho \int_0^\infty FR\,dR \equiv \mathscr{W} - 1 - ixN,$$
$$\mathscr{H}_1 \approx ix\eta N, \tag{28}$$

where the next terms are $O(x^2)$. We use the rigorous functions[3]

$$\mathscr{W} = (1-W)^2, \quad N = \left(\frac{b}{a}\right)W\left(1 - \frac{4W}{3} + \frac{W^2}{2}\right),$$
$$W = \rho b, \tag{29}$$

with $\mathscr{W} = \mathscr{W}_1$ of (25). For $b = 2a$ corresponding to $W = w$ (minimum separation of particle centers equal to particle width),

$$\mathscr{W} = (1-w)^2, \quad N = 2w(1 - 4w/3 + w^2/2). \tag{29'}$$

For full packing $w \to 1$ (the limit of a uniform slab), $\mathscr{W} \to 0$ and $N \to 1/3$. Note that the present N differs in sign from that used earlier[3]; all packing functions are now defined as nonnegative for $w \to 1$.

Solving (10) to the required accuracy in terms of the T's and D's of (14), (15), and $\mathscr{D} = 1 + \mathscr{B}(1-w)$, we obtain

$$\begin{aligned} cP_0 &= -w(\mathscr{C} - x^2 U_0 + ix\mathscr{C}^2\mathscr{W}); \\ U_0 &= T_0 - \mathscr{C} D_0 - \mathscr{C}^2 N - \mathscr{C}\mathscr{B} N\eta^2/\mathscr{D}, \\ T_0 - \mathscr{C} D_0 &= [\mathscr{C}(2 - \eta'^2 + 3\mathscr{C}) + \mathscr{B}\eta'^2]/3. \end{aligned} \tag{30}$$

$$\begin{aligned} CP_1 &= \eta^2 w(\mathscr{B} - x^2 U_1 - ix\mathscr{B}^2\mathscr{W}/\mathscr{D})/\mathscr{D}; \\ U_1 &= [T_1(1+\mathscr{B}) - \mathscr{B} D_1 + \mathscr{B}^2 N]/\mathscr{D} + \mathscr{C}\mathscr{B} N, \\ T_1(1+\mathscr{B}) - \mathscr{B} D_1 &= [\mathscr{B}(1-\mathscr{B}) + \mathscr{C}]/3. \end{aligned} \tag{31}$$

Substituting into (8) gives a linear equation for η^2 whose solution plus (30) and (31) also determine C and B of (12) and (13). To lowest orders in x, we obtain the sets $\gamma_1 = \Gamma_1$ and $\gamma_s = x\Gamma_s$ as in (22)–(26) for $m = 1$. Thus we require only the k^2 correction to γ_1, i.e., $\gamma_c = x^2\Gamma_c$ for $m = 1$.

We have

$$\begin{aligned} \eta_c^2 &= \frac{x^2 w}{B_1}\left(-U_0 + \frac{U_1\eta_1^2}{\mathscr{D}}\right) = \frac{x^2 w}{B_1}\left(A_0 + \frac{A_1}{\mathscr{D}}\eta_1^2\right), \\ 3A_0 &\equiv \mathscr{C}[\eta'^2 - 2 - 3\mathscr{C}(1-N)] - \mathscr{B}\eta'^2, \\ 3A_1 &\equiv [\mathscr{B}(1 - \mathscr{B} + 3\mathscr{B}N) + \mathscr{C}]/\mathscr{D} + 6\mathscr{C}\mathscr{B}N, \end{aligned} \tag{32}$$

$$\begin{aligned} C_c &= x^2 w\left(-U_0 + \frac{\mathscr{B}\eta'^2 C_1}{3} + \frac{\mathscr{C}\eta_1^2}{3\mathscr{D}}\right) \\ &= \frac{x^2 w\mathscr{C}}{3}\Big(\eta'^2 - 2 - 3\mathscr{C}(1-N) \\ &\quad + \mathscr{B}\eta'^2 w + \eta_1^2\frac{(1 + 3\mathscr{B}N)}{\mathscr{D}}\Big), \end{aligned} \tag{33}$$

$$\begin{aligned} B_c &= x^2 w\left(-\frac{U_1}{\mathscr{D}} + \frac{\mathscr{B}\eta'^2 B_1}{3} + \frac{\mathscr{C}}{3\mathscr{D}}\right) \\ &= \frac{x^2 w\mathscr{B}}{3}\Big(\frac{\mathscr{B} - 1 - 3\mathscr{B}N + \mathscr{C}(1-w)}{\mathscr{D}^2} \\ &\quad - \frac{3\mathscr{C}N}{\mathscr{D}} + \eta'^2 B_1\Big), \end{aligned} \tag{34}$$

such that $\eta_c^2 B_1 = C_c - \eta_1^2 B_c$ as required by (27).

If $W = w \to 1$ (the limit of a uniform slab), then $N \to 1/3$ (as well as $\mathscr{D} \to 1$ and $\gamma_1 \to \gamma'$), and the set γ_c vanishes. Since $\mathscr{W} \to 0$ for $w \to 1$, the set γ_s also vanishes, and the results for $\gamma = \{\eta^2, C, B\}$ reduce to the values $\gamma' = \{\eta'^2, C', B'\}$ for a single particle as required by elementary physical considerations.

From (23) and (26), plus (32) with η_1^2 replaced by η^2 we reconstruct the linear equation for η^2 that follows directly from (8) in the form

$$\begin{aligned} \eta^2 = 1 &+ w(\mathscr{C} + x^2 A_0 + ix\mathscr{C}^2\mathscr{W}) \\ &- (w/\mathscr{D})[\mathscr{B} - x^2 A_1 - ix(\mathscr{B}^2/\mathscr{D})\mathscr{W}]\eta^2. \end{aligned} \tag{35}$$

The first approximation is $\eta^2 = \eta_1^2$ as in (23), and iteration gives $\eta_2^2 = \eta_1^2 + \eta_c^2 + i\eta_s^2$. We proceed similarly in the following sections, where (8) gives a quadratic equation for η^2; in order to minimize elementary algebraic manipulations, the γ_c terms are listed as in (32)–(34), and followed by the equation for η^2 as in (35).

The results simplify for the special cases where either one of the relative parameters equals unity. Thus if $B' = 1$, then $\mathscr{B} = 0$, $B_c = B_s = 0$, and B reduces to unity as required by the general theorem.[23] For this case

$$\eta_c^2 = C_c = -x^2 w\,\mathscr{C}^2(2 - w - 3N)/3, \tag{36}$$

with corresponding

$$\eta_1^2 = C_1 = 1 + w\,\mathscr{C}, \quad \eta_s^2 = C_s = xw\,\mathscr{W}\,\mathscr{C}^2. \tag{37}$$

The resulting sum $C = C_1 + C_c + iC_s$ is the same form as (3:21) for ϵ in terms of ϵ' and $\delta = \epsilon' - 1$.

Similarly, if $C' = 1$, then $\mathscr{C} = 0$, and C reduces to unity. We have

$$B_c = -\eta_c^2 B_1^2 = x^2 w\,\mathscr{B}^2(2 - w - 3N)/3\mathscr{D}^2, \tag{38}$$

with corresponding

$$\begin{aligned} B_1 &= \frac{1}{\eta_1^2} = 1 + \frac{\mathscr{B}\,w}{\mathscr{D}}, \\ B_s &= -\eta_s^2 B_1^2 = -\frac{xw\,\mathscr{W}\,\mathscr{B}^2}{\mathscr{D}^2}. \end{aligned} \tag{39}$$

For the present one-dimensional problem, we may reduce the forms for η_i^2 in (38) and (39) to the same forms as in (36) and (37) by introducing $\epsilon' = 1/B'$ and identifying $\epsilon' - 1$ with $\mathscr{C}$. The general acoustic two-parameter results (32)–(34) also correspond to the general two-parameter (ϵ,μ) electromagnetic cases for $\boldsymbol{\phi} = \hat{\mathbf{x}}\phi$; if $\boldsymbol{\phi} = \mathbf{E}$, then $C' = \epsilon'$ and $B' = 1/\mu'$; if $\boldsymbol{\phi} = \mathbf{H}$, then $C' = \mu'$ and $B' = 1/\epsilon'$.

For $b = 2a$, we define

$$\begin{aligned} \mathscr{P} &= w(2 - w - 3N)/2 = w(2 - 7w + 8w^2 - 3w^3)/2 \\ &= w(2 - 3w)(1 - w)^2/2, \end{aligned} \tag{40}$$

where $(1-w)^2 = \mathscr{W}$ decreases monotone from 1 to 0 as w increases from 0 to 1. As discussed before, $\mathscr{S} = w\,\mathscr{W} = w(1-w)^2$, which vanishes at $w = 0$ and 1, has a maximum at $w = 1/3$ corresponding to maximal loss through incoherent scattering (maximal fluctuation effects).[2,18] The function $\mathscr{P}/w$ decreases from unity at $w = 0$ to 0 at $w = 2/3$, reaches its minimum value at $w = 7/9$ and then increases to 0 at $w = 1$. The function $\mathscr{P}$ which vanishes at $w = 0$, 2/3, and 1, has a maximum at $2w = 1 - (1/3)^{1/2}$ and a minimum at $2w = 1 + (1/3)^{1/2}$. The sign of the correction terms γ_c changes at $w = 2/3$.

III. DISTRIBUTION OF CYLINDERS

For cylinders, we use $c = i4\rho/k^2 = i4w/\pi x^2$, $w = \rho\pi a^2$ in (8)–(13) with

$$\begin{aligned} 2\mathscr{H}_{11} &= c + \mathscr{H}_0 + \mathscr{H}_2, \quad 2\mathscr{H}_{12} = c\eta + \mathscr{H}_1 + \mathscr{H}_3, \\ 2\mathscr{H}_{22} &= c(\eta^2 + 1) + \mathscr{H}_0 + \mathscr{H}_4, \end{aligned}$$

in terms of the correlation integrals $\mathscr{H}_n$ in (1:71) or (2:22). We have

$$\begin{aligned} \mathscr{H}_0 &\approx 2\pi\rho\int FR\,dR + i4\rho\int F\ln\left(\frac{c'kR}{2}\right)R\,dR \\ &\equiv \mathscr{W} - 1 + i\mathscr{N}, \\ \mathscr{H}_n &\approx -i\left(\frac{\eta^n 2\pi\rho}{n\pi}\right)\int FR\,dR = -\frac{i\eta^n(\mathscr{W} - 1)}{n\pi}, \end{aligned} \tag{41}$$

where the next terms are $O(x^2)$. We work with

$$\mathscr{W}=1-8W\overline{F}_1;\quad W=\frac{\pi\rho b^2}{4},\quad \overline{F}_1=-\int_0^\infty F(u)u\,du, \tag{42}$$

and

$$-M=L-\frac{\pi\mathscr{N}}{2}=\ln\left(\frac{b}{a}\right)+\mathscr{W}\ln\left(\frac{2}{c'kb}\right)+8W\overline{F}_l;$$
$$\overline{F}_l=-\int_0^\infty F(\ln u)u\,du. \tag{43}$$

Note that the present M differs in sign from that used earlier.[3]

We may evaluate $\overline{F}_1$ and $\overline{F}_l$ numerically by using tabulated values of F, or the original integral equation approximations in the computing routine.[9]

To first order in W, we use the virial expansion[14,3]: $F=-1$ for $u<1$,

$$F=\frac{8W}{\pi}\left\{\cos^{-1}\frac{u}{2}-\frac{u}{2}\left[1-\left(\frac{u}{2}\right)^2\right]^{1/2}\right\},\quad 1<u<2, \tag{44}$$

and $F=0$ for $u>2$. This provides results for $\mathscr{W}$ and M correct to $O(W^2)$, e.g.,

$$\mathscr{W}=1-4W+\sqrt{3}\,12W^2/\pi\simeq 1-4W+6.6159W^2. \tag{45}$$

The closed form approximation derived earlier,[18,2] i.e., $\mathscr{W}=\mathscr{W}_2$ of (25),

$$\mathscr{W}=(1-W)^3/(1+W), \tag{46}$$

gives $\mathscr{W}=1-4W+7W^2+\dots$. For the unrealizable value $W=1$, the closed form $\mathscr{W}$ vanishes; as shown in the following, a comparable approximation of M would reduce to 3/4 for $b=2a$ and $W=1$ in order for γ to equal γ'. The corresponding moments are then $\overline{F}_1=1/8$ and $\overline{F}_l=-(\ln 2)/8-3/32$.

Solving (10) to the required accuracy in terms of a_n and the T's and D's of (16)–(18), and $\mathscr{D}=1+\mathscr{B}(1-w)/2$, we obtain

$$cP_0=-w(\mathscr{C}-x^2U_0+i\pi x^2\mathscr{C}^2\mathscr{W}/4);$$
$$U_0=T_0-\mathscr{C}D_0'+\mathscr{C}^2M/2-\mathscr{C}\mathscr{B}(1-\mathscr{W})\eta^2/4\mathscr{D},$$
$$T_0-\mathscr{C}D_0'=[\mathscr{C}(3-\eta'^2)+\mathscr{B}\eta'^2/8]. \tag{47}$$

$$cP_1=\eta^2w(\mathscr{B}-x^2U_1-i\pi x^2\mathscr{B}^2\mathscr{W}/8\mathscr{D})/\mathscr{D};$$
$$U_1=\left(\frac{T_1(2+\mathscr{B})}{2}-\mathscr{B}D_1'-\frac{\mathscr{B}^2M}{4}\right)\frac{1}{\mathscr{D}}+\frac{\mathscr{C}\mathscr{B}(1-\mathscr{W})}{4}-\frac{\mathscr{B}^2\eta^2}{16}\left(\frac{1-\mathscr{W}}{\mathscr{D}}+w\frac{1+B_1}{1+B'}\right),$$
$$\tfrac{1}{2}T_1(2+\mathscr{B})-\mathscr{B}D_1'=\tfrac{1}{16}[\mathscr{B}(3\mathscr{B}+4)+4\mathscr{C}]. \tag{48}$$

$$cP_2=\eta^4wx^2U_2,$$
$$U_2=\frac{\mathscr{B}(1+B_1)}{8(1+B')}=\frac{\mathscr{B}}{4(2+\mathscr{B})}\left(1+\frac{\mathscr{B}w}{2\mathscr{D}}\right)=\frac{\mathscr{B}}{8\mathscr{D}}. \tag{49}$$

The forms for U_2 illustrate other versions that also arise for U_1. Substituting into (8) gives a quadratic equation for η^2 whose solution plus (47)–(49) also determine C and B of (12) and (13). To lowest orders in x, we obtain the sets γ_1 and γ_s as in (22)–(27) for $m=2$, and as before we require only the set $\gamma_c=x^2\Gamma_c$.

We have

$$\eta_c^2=\frac{x^2w}{B_1}\left(-U_0+\frac{U_1\eta_1^2}{\mathscr{D}}-U_2\eta_1^4\right)=\frac{x^2w}{B_1}\left(A_0+\frac{A_1}{\mathscr{D}}\eta_1^2+A_2\eta_1^4\right);$$
$$8A_0=\mathscr{C}(\eta'^2-3-4\mathscr{C}M)-\mathscr{B}\eta'^2,$$
$$16A_1=[\mathscr{B}(4+3\mathscr{B}-4\mathscr{B}M)+4\mathscr{C}]/\mathscr{D}+8\mathscr{B}\mathscr{C}(1-\mathscr{W}),$$
$$16A_2=-\mathscr{B}^2(1-\mathscr{W})/\mathscr{D}^2-\mathscr{B}[(1+B_1)^2/(1+B')], \tag{50}$$

$$C_c=x^2w\left(-U_0+\frac{\mathscr{B}\eta'^2C_1}{8}+\frac{\mathscr{C}\eta_1^2}{4\mathscr{D}}\right)=\frac{x^2w\mathscr{C}}{8}\left(\eta'^2-3-4\mathscr{C}M+\mathscr{B}\eta'^2w+\frac{2\eta_1^2}{\mathscr{D}}[\mathscr{B}(1-\mathscr{W})+1]\right), \tag{51}$$

$$B_c=x^2w\left(-\frac{U_1}{\mathscr{D}}+\frac{\mathscr{B}\eta'^2B_1}{8}+\frac{\mathscr{C}}{4\mathscr{D}}+\eta_1^2U_2\right)$$
$$=\frac{x^2w\mathscr{B}}{16}\left(\frac{\mathscr{B}[-3+4M+\eta_1^2(1-\mathscr{W})]-4+2\mathscr{C}(1-w)}{\mathscr{D}^2}-\frac{4\mathscr{C}(1-\mathscr{W})}{\mathscr{D}}+2\eta'^2B_1+\eta_1^2\frac{(B_1+1)^2}{(B'+1)}\right). \tag{52}$$

The set (50)–(52) satisfies $\eta_c^2B_1=C_c-\eta_1^2B_c$ as required by (27). For the unrealizable limit $W=w\rightarrow 1$, we have $\mathscr{W}=\mathscr{W}_2\rightarrow 0$ and $\gamma_s=0$, $\gamma_1=\gamma'$; if we use $M\rightarrow 3/4$, then $\gamma_c\rightarrow 0$, and $\gamma\rightarrow\gamma'$.

The original equation for η^2 obtained directly from (8) by proceeding as for (35) is a quadratic in η^2,

$$\eta^2=1+w\left(\mathscr{C}+x^2A_0+\frac{i\pi x^2}{4}\mathscr{C}^2\mathscr{W}\right)-\frac{w}{\mathscr{D}}\left(\mathscr{B}-x^2A_1-\frac{i\pi x^2}{8}\frac{\mathscr{B}^2}{\mathscr{D}}\mathscr{W}\right)\eta^2+wx^2A_2\eta^4. \tag{53}$$

The first approximation gives η_1^2, and we obtain $\eta_2^2 = \eta_1^2 + \eta_c^2 + i\eta_s^2$ by iteration.

For the one-parameter case corresponding to $B' = 1$, we have

$$\eta_c^2 = C_c = x^2 w\,\mathscr{C}^2(1 + 2w - 4M)/8, \tag{54}$$

$$\eta_1^2 = C_1 = 1 + w\,\mathscr{C}, \quad \eta_s^2 = C_s = \pi x^2 w\,\mathscr{W}\,\mathscr{C}^2/4. \tag{55}$$

The resulting sum C is the same form as that of ϵ in terms of ϵ' given in (3:26) for the optical case of the electric field parallel ($\mathbf{E}_\parallel$) to the axes of purely dielectric cylinders.

Similarly if $C' = 1$, then

$$B_c = -\eta_c^2 B_1^2 = -\frac{x^2 w\,\mathscr{B}}{16}\left(\frac{3B' + 1 - \mathscr{B}[4M + \eta_1^2(1 - \mathscr{W})]}{\mathscr{D}^2} - \frac{2B_1}{B'} - \eta_1^2\frac{(1 + B_1)^2}{(1 + B')}\right), \tag{56}$$

$$B_1 = \frac{1}{\eta_1^2} = 1 + \frac{\mathscr{B}\,w}{\mathscr{D}},$$

$$B_s = -\eta_s^2 B_1^2 = -\frac{\pi x^2 w\,\mathscr{W}\mathscr{B}^2}{8\,\mathscr{D}^2}. \tag{57}$$

If we introduce $\epsilon' = 1/B'$ and $\delta = \epsilon' - 1$, then the corresponding sum $\eta^2 = \eta_1^2 + \eta_c^2 + i\eta_s^2 = \epsilon$ is the same as in (3:33) for $\mathbf{E}_\perp$. The general acoustic two-parameter results (50)–(52) correspond to the two-parameter (ϵ,μ) general electromagnetic cases; for $\mathbf{E}_\parallel$, we take $C' = \epsilon'$ and $B' = 1/\mu'$; for $\mathbf{E}_\perp$, we take $C' = \mu'$ and $B' = 1/\epsilon'$.

For comparison with (54) and (55) to $O(\mathscr{B}^2)$, we have

$$B_c \approx -\eta_c^2 \approx -x^2 w\,\mathscr{B}^2(1 + 2w - 4M + \mathscr{W})/16, \tag{58}$$

$$B_1 \approx 1 + w\,\mathscr{B} - w(1 - w)\mathscr{B}^2/2,$$

$$\eta_1^2 \approx 1 - w\,\mathscr{B} + w(1 + w)\mathscr{B}^2/2, \tag{59}$$

$$B_s \approx -\eta_s^2 \approx -\pi x^2 w\,\mathscr{W}\,\mathscr{B}^2/8. \tag{60}$$

The difference in the statistical factors in η_c^2 of (54) and (58) is the packing term $\mathscr{W}$, but both factors vanish for $w = W \to 1$ and $\mathscr{W} \to 0$, $M \to 3/4$.

IV. DISTRIBUTION OF SPHERES

For spheres, we use $c = i4\pi\rho/k^3 = i3w/x^3$, $w = \rho 4\pi a^3/3$ in (8)–(13) with

$$3\,\mathscr{H}_{11} = c + \mathscr{H}_0 + 2\,\mathscr{H}_2, \quad 5\,\mathscr{H}_{12} = 2c\eta + 2\,\mathscr{H}_1 + 3\,\mathscr{H}_3,$$

$$35\,\mathscr{H}_{22} = c(7 + 17\eta^2) + 7\,\mathscr{H}_0 + 10\,\mathscr{H}_2 + 18\,\mathscr{H}_4,$$

in terms of the correlation integrals $\mathscr{H}_n$ of (1:148) or (2:21). We have

$$\mathscr{H}_0 \approx \mathscr{W} - 1 + iN/x, \quad \mathscr{H}_n \approx i\eta^n N/x(2n + 1), \tag{61}$$

where

$$\mathscr{W} = 1 + 4\pi\rho\int_0^\infty FR^2\,dR = 1 - 24W\overline{F}_2;$$

$$W = \frac{4\pi\rho}{3}\left(\frac{b}{2}\right)^3, \quad \overline{F}_2 = -\int_0^\infty F(u)u^2\,du, \tag{62}$$

and

$$N = -4\pi\rho a\int_0^\infty FR\,dR = 24\left(\frac{a}{b}\right)W\overline{F}_1, \tag{63}$$

with the first moment as in (42). The next terms are $O(x)$.

By differentiating the scaled particle[5] approximation for the equation of state, we obtained[16] $\mathscr{W} = \mathscr{W}_3$ of (25), i.e.,

$$\mathscr{W} = (1 - W)^4/(1 + 2W)^2, \tag{64}$$

which also follows from the Percus–Yevick approximation.[6] The corresponding first moment[7] $\overline{F}_1$ of the Wertheim–Thiele solution of the PY integral equation[6] gives

$$N = \frac{2a}{b}\,\frac{6W}{1 + 2W}\left(1 - \frac{W}{5} + \frac{W^2}{10}\right). \tag{65}$$

For $b = 2a$ and the unrealizable value $w = 1$, it follows from the closed forms (64) and (65) that $\mathscr{W} = 0$ and $N = 9/5$; then as shown in the following, $\gamma = \gamma'$. See (2:27)ff for virial expansions in powers of W. The physically realizable range corresponds to $W < W_d \approx 0.63$, and although the range of validity of PY theory is more restricted, the limiting behavior of the closed forms provides a useful check. The simplicity of (64) and (65) obviates detailed computations, and their relations to the rigorous one-dimensional results (29) emphasize the essential physics.

Solving (10) to the required accuracy in terms of the a_n and T's and D's of (19)–(21), and $\mathscr{D} = 1 + \mathscr{B}(1 - w)/3$, we obtain

$$cP_0 = -w(\mathscr{C} - x^2U_0 + ix^3\mathscr{C}^2\,\mathscr{W}/3);$$

$$U_0 = T_0 - \mathscr{C}\,D_0 + \frac{\mathscr{C}^2 N}{3} - \frac{\mathscr{C}\,\mathscr{B}\,N\eta^2}{9\,\mathscr{D}},$$

$$T_0 - \mathscr{C}\,D_0 = \tfrac{1}{15}[\mathscr{C}(4 - \eta'^2 - 5\,\mathscr{C}) + \mathscr{B}\,\eta'^2], \tag{66}$$

$$cP_1 = \eta^2 w[\mathscr{B} - x^2U_1 - ix^3\mathscr{B}^2\,\mathscr{W}/9\,\mathscr{D}]/\mathscr{D};$$

$$U_1 = \left(\frac{T_1(3 + \mathscr{B})}{3} - \mathscr{B}\,D_1 - \frac{\mathscr{B}^2N}{9}\right)\frac{1}{\mathscr{D}} + \frac{\mathscr{B}\,\mathscr{C}\,N}{9} - \frac{\eta^2\mathscr{B}^2 2}{15}\left[\frac{N}{3\,\mathscr{D}} + \frac{2w}{5}\left(\frac{3 + 2B_1}{3 + 2B'}\right)\right],$$

$$T_1[(3 + \mathscr{B})/3] - \mathscr{B}\,D_1 = \tfrac{1}{5}[\mathscr{B}(\mathscr{B} + 1) + \mathscr{C}], \tag{67}$$

$$cP_2 = \eta^4 w x^2 U_2, \quad U_2 = \mathscr{B}\,\frac{2}{15}\left(\frac{3 + 2B_1}{3 + 2B'}\right). \tag{68}$$

We proceed as before for (47)–(49). We have

$$\eta_c^2 = \frac{x^2w}{B_1}\left(-U_0 + \frac{U_1\eta_1^2}{\mathscr{D}} - U_2\eta_1^4\right) = \frac{x^2w}{B_1}\left(A_0 + \frac{A_1}{\mathscr{D}}\eta_1^2 + A_2\eta_1^4\right),$$

$$15A_0 = \mathscr{C}[\eta'^2 - 4 - 5\,\mathscr{C}(N - 1)] - \mathscr{B}\,\eta'^2,$$

$$45A_1 = \{9[\mathscr{B}(\mathscr{B} + 1) + \mathscr{C}] - 5\,\mathscr{B}^2N\}/\mathscr{D} + 10\mathscr{C}\,\mathscr{B}\,N,$$

$$15A_2 = -\frac{2N\,\mathscr{B}^2}{3\,\mathscr{D}^2} - \mathscr{B}\,\frac{2}{5}\,\frac{(3 + 2B_1)^2}{(3 + 2B')}, \tag{69}$$

$$C_c = x^2w\left(-U_0 + \frac{\mathscr{B}\,\eta'^2}{15}C_1 + \frac{\mathscr{C}\,\eta_1^2}{5\,\mathscr{D}}\right) = (x^2w\,\mathscr{C}/15)[\eta'^2 - 4 - 5\,\mathscr{C}(N - 1) + \mathscr{B}\,\eta'^2 w + (\eta_1^2/3\,\mathscr{D})(5\,\mathscr{B}\,N + 9)], \tag{70}$$

$$B_c = x^2 w\left(-\frac{U_1}{\mathscr{D}} + \frac{\mathscr{B}\,\eta'^2 B_1}{15} + \frac{\mathscr{C}}{5\,\mathscr{D}} + \eta_1^2 U_2 \right)$$

$$= \frac{x^2 w\,\mathscr{B}}{15}\left(\frac{-3(\mathscr{B}+1) + \mathscr{C}(1-w) + \mathscr{B}\,N(5+2\eta_1^2)/3}{\mathscr{D}^2} - \frac{5\,\mathscr{C}\,N}{3\,\mathscr{D}} + \eta'^2 B_1 + \eta_1^2\,\frac{2}{5}\,\frac{(3+2B_1)^2}{(3+2B')} \right). \quad (71)$$

The set (69)–(71) satisfies $\eta_c^2 B_1 = C_c - \eta_1^2 B_c$. For the unrealizable limit $W = w \to 1$, we have $\mathscr{W} = \mathscr{W}_3 = 0$ and $N = 9/5$, so that $\gamma \to \gamma'$.

The original equation for η^2 obtained from (8) by proceeding as for (35) is a quadratic in η^2,

$$\eta^2 = 1 + w[\mathscr{C} + x^2 A_0 + (ix^3/3)\mathscr{C}^2\mathscr{W}] - \frac{w}{\mathscr{D}}\left(\mathscr{B} - x^2 A_1 - \frac{ix^3\mathscr{B}^2}{9\,\mathscr{D}}\,\mathscr{W} \right)\eta^2 + wx^2 A_2 \eta^4 \quad (72)$$

corresponding to the analog of (53) for $m = 3$.

For the one-parameter case $B' = 1$, we have

$$\eta_c^2 = C_c = x^2 w\,\mathscr{C}^2(6+3w-5N)/15, \quad (73)$$

$$\eta_1^2 = C_1 = 1 + w\,\mathscr{C}, \quad \eta_s^2 = C_s = x^2 w\,\mathscr{W}\,\mathscr{C}^2/3. \quad (74)$$

Similarly if $C' = 1$, then

$$B_c = -\eta_c^2 B_1^2 = \frac{x^2 w\,\mathscr{B}}{15}\left(\frac{-3B' + \mathscr{B}\,N(5+2\eta_1^2)/3}{\mathscr{D}^2} + \frac{B_1}{B'} + \eta_1^2\,\frac{2}{5}\,\frac{(3+2B_1)^2}{(3+2B')} \right), \quad (75)$$

$$B_1 = \frac{1}{\eta_1^2} = 1 + \frac{\mathscr{B}\,w}{\mathscr{D}},$$

$$B_s = -\eta_s^2 B_1^2 = \frac{-x^3 w\,\mathscr{W}\,\mathscr{B}^2}{9\,\mathscr{D}^2}, \quad (76)$$

where $\mathscr{D} = 1 + \mathscr{B}(1-w)/3$.

If we introduce $1/B' = V'$ and $1/B_1 = V_1$, then the resulting form of η_1^2 equals

$$\eta_1^2 = V_1 = 1 + \frac{\mathscr{V}\,w}{\mathscr{D}_v}; \quad \mathscr{V} = V' - 1,$$

$$\mathscr{D}_v = 1 + \frac{\mathscr{V}(1-w)2}{3}. \quad (77)$$

For $m = \{1,2,3\}$ we have $\mathscr{D}_v = 1 + \mathscr{V}(1-w)(1-m^{-1})$ as compared to $\mathscr{D} = 1 + \mathscr{B}(1-w)m^{-1}$. We also have $B_1\,\mathscr{D} = \mathscr{D}_v/V'$, $\mathscr{B}/B_1\,\mathscr{D} = -\mathscr{V}/\mathscr{D}_v$, so that

$$\eta_s^2 = -B_s/B_1^2 = x^3 w\,\mathscr{W}\,\mathscr{V}^2/9\,\mathscr{D}_v^2, \quad (78)$$

$$\eta_c^2 = -\frac{B_c}{B_1^2} = \frac{x^2 w\,\mathscr{V}}{15}\left(\frac{-3 + \mathscr{V}\,N(5+2\,\eta_1^2)/3}{\mathscr{D}_v^2} + V_1 + \eta_1^2\,\frac{2}{5}\,\frac{(3V_1+2)^2}{3V'+2} \right). \quad (79)$$

For the simplest acoustic problems, $V' = 1/B'$ is a particle's relative mass density. The electromagnetic two-parameter and one-parameter[3] problems for spheres give different forms than (69)–(79).

For comparison with (73) and (74) to $O(\mathscr{B}^2)$, we have from (75) and (76),

$$B_c \approx -\eta_c^2 \approx -x^2 w\,\mathscr{B}^2(6+3w-5N)7/(15)^2, \quad (80)$$

$$B_1 \approx 1 + w\,\mathscr{B} - w(1-w)\mathscr{B}^2/2,$$

$$\eta_1^2 \approx 1 - w\,\mathscr{B} + w(1+2w)\mathscr{B}^2/3, \quad (81)$$

$$B_s \approx -\eta_s^2 \approx -x^3 w\,\mathscr{W}\,\mathscr{B}^2/9. \quad (82)$$

Thus, to $O(\mathscr{B}^2)$ the statistical packing function in B_c is the same as in C_c of (73) and (3:44).

For $b = 2a$, we define

$$\mathscr{P} = \frac{w(6+3w-5N)}{6} = \frac{w(2-5w+4w^2-w^3)}{2(1+2w)} = \frac{w(2-w)(1-w)^2}{2(1+2w)} \quad (83)$$

as the analog of (40). The relation of $\mathscr{P}$ to $\mathscr{W}_3$ of (25) is indicated by $(1-w)^2/(1+2w) = \mathscr{W}_3^{1/2}$. The functions $\mathscr{W}$ and $\mathscr{P}/w$ decrease monotone from 1 to 0 as w increases from 0 to 1. The product[2,18] $\mathscr{S} = w\,\mathscr{W}$ has a maximum at $w \approx 0.13$, and $\mathscr{P}$ has a maximum at $w \approx 0.22$.

V. COLLECTIVE FORMS

The results (35), (53), and (72) correspond to (8), i.e., to $\eta^2 = 1 - c\,\Sigma P_m$ expressed as

$$\eta^2 = 1 + w(\mathscr{C} + x^2 A_0 + ix^m d_m\,\mathscr{C}^2\,\mathscr{W}) - \frac{\eta^2 w}{\mathscr{D}}\left(\mathscr{B} - x^2 A_1 - ix^m\,d_m\,\frac{\mathscr{B}^2\mathscr{W}}{m\,\mathscr{D}} \right) + \eta^4 wx^2 A_2, \quad (84)$$

with $A_2 = 0$ for $m = 1$. The solution of the matrix equation (10) for the P_n isolated corresponding $x^2 U_n(\eta^2)/c$ terms; the regrouping of the sum of U's in (8) in order to make the dependence on η^2 explicit, introduced the coefficients A_n. The explicit versions are special cases of

$$\eta^2 \approx \eta_1^2 + \frac{wx^2}{B_1}\left(A_0 + \frac{A_1\,\eta_1^2}{\mathscr{D}} + A_2\,\eta_1^4 \right) + \frac{ix^m\,d_m w\,\mathscr{W}}{B_1}\left(\mathscr{C}^2 + \frac{\mathscr{B}^2}{m\,\mathscr{D}^2}\,\eta_1^2 \right), \quad (85)$$

with coefficients defined in (32), (50), and (69).

Similarly (33), (51), and (70) are covered by

$$C \approx C_1 + x^2 w\left(-U_0 + \frac{\eta'^2\,\mathscr{B}\,C_1}{m(m+2)} + \frac{\mathscr{C}\,\eta_1^2}{(m+2)\mathscr{D}} \right) + ix^m\,d_m\,\mathscr{C}^2 w\,\mathscr{W}, \quad (86)$$

and (34), (52), and (71) by

$$B \approx B_1 + x^2 w\left(-\frac{U_1}{\mathscr{D}} + \frac{\mathscr{B}\,B_1\eta'^2}{m(m+2)} + \frac{\mathscr{C}}{(m+2)\mathscr{D}} + \eta_1^2\,U_2 \right) - ix^m\,\frac{d_m}{m}\,\frac{\mathscr{B}^2}{\mathscr{D}^2}\,w\,\mathscr{W}, \quad (87)$$

with the U_n defined in (30), (31), (47)–(49), and (66)–(68), for $m = 1,2,3$.

VI. RIGID PARTICLES

We derive the analogous results for η^2 for distributions of rigid spheres and cylinders as the limits of the two-parameter functions for $C' \to 0$ and $B' \to 0$. We consider spheres first, because explicit closed forms in W are available for the required packing functions $\mathscr{W}$ and N. Then we consider cylinders and use the development to infer an additional property of the implicit function $M(W)$.

For rigid spheres we substitute $\mathscr{C} = -1$, $\mathscr{B} = -1$, and $\mathscr{D} = 1 + \mathscr{B}(1-w)/3 = (2+w)/3$ in (69) to obtain

$$A_0 = \frac{9-5N}{15}, \quad A_1 = -\frac{27-5N(1+2w)}{45(2+w)},$$
$$A_2 = \frac{2[(10-w)^2 - 45N]}{[15(2+w)]^2}. \tag{88}$$

Although these vanish for $W = w = 1$ (for which case $N = 9/5$), we show that for $W = w$ each coefficient contains $1-w$ as a factor; this enables us to use the same iteration procedure in (72) as for the two-parameter case even for the unrealizable limit of $w \to 1$.

Substituting (88) into (72) for $\mathscr{C} = \mathscr{B} = -1$, we have

$$\eta^2 = 1 - w + \frac{3w}{2+w}\eta^2 + x^2 w\left(A_0 + \frac{3A_1}{2+w}\eta^2 + A_2\eta^4\right) + \frac{iwx^3\mathscr{W}}{3}\left(1 + \frac{3\eta^2}{(2+w)^2}\right). \tag{89}$$

Solving by iteration yields the leading term of the real part

$$\eta_1^2 = (2+w)/2 = 1 + w/2, \tag{90}$$

as well as the leading term of the imaginary part

$$\eta_s^2 = \frac{x^3 w\,\mathscr{W}(7+2w)}{12(1-w)}; \quad \mathscr{W} = \frac{(1-W)^4}{(1+2W)^2}, \quad W \geqslant w, \tag{91}$$

and the x^2 correction to the real part

$$\eta_c^2 = x^2\frac{w(2+w)}{2(1-w)}\left(A_0 + A_1\frac{3}{2} + A_2\frac{(2+w)^2}{4}\right). \tag{92}$$

The leading terms follow directly from (23) and (26), or from the closed form (2:74); the factor 1/3 in (2:75) should be replaced by 1/12. The physically realizable domain corresponds to $W < W_d \approx 0.63$, and $w \leqslant W$. For $W = w$, (91) has a maximum at $w \approx 0.16$.

It is clear for (91), that even for $W = w \to 1$, the iteration procedure is valid: $1-w$ is a factor of $\mathscr{W}$, and $\mathscr{W}/(1-w)$ vanishes as $O(1-w)^3$. Similarly for (92), for $W = w$ in N of (65) we may factor $1-w$ from the coefficients:

$$A_n = (1-w)\mathscr{A}_n;$$
$$\mathscr{A}_0 = \frac{3-w+w^2}{5(1+2w)}, \quad \mathscr{A}_1 = -\frac{9-w+w^2}{15(2+w)},$$
$$\mathscr{A}_2 = \frac{2(20+2w+5w^2)}{45(2+w)^2(1+2w)}. \tag{93}$$

Substituting into (92) we obtain

$$\eta_c^2 = \frac{x^2 w(2+w)}{180}\left(\frac{74-16w+23w^2}{1+2w} - \frac{9(9-w+w^2)}{2+w}\right) = \frac{x^2 w(1-w)}{180(1+2w)}(67-44w-5w^2), \tag{94}$$

which has a maximum at $w \approx 0.28$ and vanishes in the unrealizable limit $w \to 1$. We use these limiting results as guides for the analogous case of cylinders.

For cylinders, we substitute $\mathscr{C} = \mathscr{B} = -1$ and $\mathscr{D} = 1 + \mathscr{B}(1-w)/2 = (1+w)/2$ in (50) to obtain

$$A_0 = \frac{3-4M}{8}, \quad A_1 = \frac{(3-4M)-8}{8(1+w)} + \frac{(1-\mathscr{W})}{2},$$
$$A_2 = \frac{\mathscr{W}}{4(1+w)^2}, \tag{95}$$

which vanish for $W = w = 1$, if we use $\mathscr{W} = \mathscr{W}_2 = 0$ and $M = 3/4$. Substituting into (53) yields

$$\eta^2 = 1 - w + \frac{2w\eta^2}{1+w} + x^2 w\left(A_0 + \frac{2A_1\eta^2}{1+w} + A_2\eta^4\right) + \frac{i\pi x^2 w}{4}\left(1 + \frac{2\eta^2}{(1+w)^2}\right), \tag{96}$$

and iteration gives

$$\eta_1^2 = 1 + w, \tag{97}$$

$$\eta_s^2 = \frac{\pi x^2 w\,\mathscr{W}(3+w)}{4(1-w)}, \quad \mathscr{W} = \frac{(1-W)^3}{1+W}, \tag{98}$$

as obtained directly from (23) and (26), or from the closed form (2:74); see (2:76). In addition, the correction to η_1^2 is given by

$$\eta_c^2 = x^2 w\,[(1+w)/(1-w)]\,[A_0 + 2A_1 + A_2(1+w)^2]. \tag{99}$$

The physically realizable domain corresponds to $W < W_d \approx 0.84$, and $w \leqslant W$.

For (98) and $W = w$, we factor $1-w$ from $\mathscr{W}(w)$, and the result for η_s^2 vanishes for the unrealizable $w = 1$. For (99) however, although the individual A_n vanish for the limiting values $\mathscr{W} = 0$ and $M = 3/4$, this does not insure that η_c^2 vanishes. To eliminate a nonvanishing residue, we require that for large w,

$$3 - 4M = \frac{(1-w)8 + O(1-w)^n}{3+w}, \tag{100}$$

with $n > 1$. This inference, based on the behavior of η_c^2 for spheres in terms of the known closed forms for N and $\mathscr{W}_3$, may facilitate development of a corresponding closed form for M to use with $\mathscr{W}_2$.

ACKNOWLEDGMENTS

This work was supported in part by grants from the National Science Foundation and the U. S. Office of Naval Research.

[1] V. Twersky, Coherent scalar field in pair-correlated random distributions of aligned scatterers," J. Math. Phys. **18**, 2468–2486 (1977).
[2] V. Twersky, "Acoustic bulk parameters in distributions of pair-correlated scatterers," J. Acoust. Soc. Am. **64**, 1710–1719 (1978).
[3] V. Twersky, "Wavelength-dependent refractive and absorptive terms for propagation in small-spaced correlated distributions," J. Opt. Soc. Am. **73**, 1562–1567 (1983).
[4] H. L. Frisch and J. L. Lebowitz, *The Equilibrium Theory of Fluids* (Benjamin, New York, 1964); R. L. Baxter, "Distribution functions," in *Physical Chemistry*, edited by H. Eyring, D. Henderson and W. Jost (Academic, New York, 1971), Vol. VIII A, Chap. 4, pp. 267–334.
[5] H. Reiss, H. L. Frisch, and J. L. Lebowitz, "Statistical mechanics of rigid spheres," J. Chem. Phys. **31**, 369–380 (1959); E. Helfand, H. L. Frisch, and J. L. Lebowitz, "The theory of the two- and one-dimensional rigid sphere fluids," J. Chem. Phys. **34**, 1037–1042 (1961).
[6] M. S. Wertheim, "Exact solution of the Percus–Yevick integral equation for hard spheres," Phys. Rev. Lett. **10**, 321–323 (1963); E. Thiele, "Equa-

tion of state for hard spheres," J. Chem. Phys. **39**, 474–479 (1963).

[7]H. D. Jones, "Method for finding the equation of state of liquid metals," J. Chem. Phys. **55**, 2640–2642 (1971).

[8]G. J. Throop and R. J. Bearman, "Numerical solution of the Percus–Yevick equation for the hard-sphere potential," J. Chem. Phys. **42**, 2408–2411 (1965); F. Mandel, R. J. Bearman, and M. Y. Bearman, "Numerical solution of the Percus–Yevick equation for the Lennard–Jones (6–12) and hard sphere potentials," J. Chem. Phys. **52**, 3315–3323 (1970); D. Levesque, J. J. Weis, and J. P. Hansen, "Simulation of classical fluids," in *Monte Carlo Methods in Statistical Physics*, edited by K. Binder (Springer-Verlag, New York, 1979), pp. 121–144.

[9]Y. Uehara, T. Ree, and F. H. Ree, "Radial distribution function for hard disks from the BGY2 theory," J. Chem. Phys. **70**, 1876–1883 (1979); J. Woodhead-Galloway and P. A. Machin, X-ray scattering from a gas of uniform hard-disks using the Percus–Yevick approximation," Mol. Phys. **32**, 41–48 (1976); F. Lado, "Equation of state for the hard-disk fluid from approximate integral equations," J. Chem. Phys. **49**, 3092–3096 (1968).

[10]G. Placzek, B. R. A. Nijober, and L. Van Hove, "Effects of short wavelength interference on neutron scattering by dense systems of heavy nuclei," Phys. Rev. **82**, 392–403 (1951). B. R. A. Nijboer and L. Van Hove, "Radial distribution function of a gas of hard spheres and the superposition approximation," Phys. Rev. **85**, 777–783 (1952).

[11]B. Larsen, J. C. Rasaiah, and G. Stell, "Thermodynamic perturbation theory for multipolar and ionic liquids," Mol. Phys. **33**, 987–1027 (1977). G. Stell and K. C. Wu, "Pade approximant for the internal energy of a system of charged particles," J. Chem. Phys. **63**, 491–498 (1975).

[12]F. Zernike and J. A. Prins, "Die Beuging von Rontgenstrahlen in Flussigkeiten als Effekt der Molekulanordnung," Z. Phys. **41**, 184–194 (1927).

[13]V. Twersky, "Scattering by quasi-periodic and quasi-random distribution," IRE Trans. **AP-7**, S307-S319 (1959); "Multiple scattering of waves by planar random distributions of cylinders and bosses," Inst. Math. Sci. NYU Rep. EM58 (1953).

[14]V. Twersky, "Multiple scattering of sound by correlated monolayers," J. Acoust. Soc. Am. **73**, 68–84 (1983).

[15]L. Tonks, "The complete equation of state of one, two, and three-dimensional gases of hard elastic spheres," Phys. Rev. **50**, 955–963 (1936).

[16]S. W. Hawley, T. H. Kays, and V. Twersky, "Comparison of distribution functions from scattering data on different sets of spheres," IEEE Trans. Antennas Propag. **AP-15**, 118–135 (1967).

[17]V. Twersky, "Propagation in correlated distributions of large-spaced scatterers," J. Opt. Soc. Am. **73**, 313–320 (1983).

[18]V. Twersky, "Transparency of pair-correlated, random distributions of small scatterers with applications to the cornea," J. Opt. Soc. Am. **65**, 524–530 (1975); "Propagation in pair-correlated distributions of small-spaced lossy scatterers," J. Opt. Soc. Am. **69**, 1567–1572 (1979).

[19]Lord Rayleigh, *The Theory of Sound* (Cambridge, 1878; Dover, New York, 1945); the original derivations for the slab, cylinder, and sphere are analyzed in Secs. 271, 343, and 335, respectively. We use the forms for the coefficients given in (43)ff of V. Twersky, "Acoustic bulk parameters of random volume distributions of small scatterers," J. Acoust. Soc. Am. **36**, 1314–1329 (1964).

[20]Lord Rayleigh, "On the transmission of light through the atmosphere containing small particles in suspension, and on the origin of the color of the sky," Philos. Mag. **47**, 375–383 (1899).

[21]J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Cambridge, U. P., Cambridge, 1873; Dover, New York, 1954); spheres are considered in Sec. 314, and slabs in Sec. 321.

[22]Lord Rayleigh, "On the influence of obstacles arranged in rectangular order upon the properties of a medium," Philos. Mag. **34**, 481–501 (1892).

[23]V. Twersky, "Propagation parameters in random distributions of scatterers," J. Anal. Math. **30**, 498–511 (1976).