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Filters With Small Non-Linearities

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ABSTRACT

The Kalman filter provides a finite dimensional solution when the signal and observation processes are linear and have Gaussian noise. In this paper the effect of a small non-linearity in the signal is discussed by considering stochastic flows for the signal and a Girsanov transformation for the observation. The result can be expressed in terms of Gaussian densities.

1. THE LINEAR FILTER

In this section we first describe the linear Kalman filter. For simplicity real valued signal and observation processes will be considered; the vector case can be discussed with more complicated notation and calculations. ω_t, B_t, $t \geq 0$, are two independent Brownian motions defined on a probability space (Ω, F, P) which has a complete, right continuous filtration $\{F_t\}$ to which ω and B are adapted. a_t, $t \geq 0$, is a locally integrable, measurable function, and h_t, $t \geq 0$, is a function with a locally integrable derivative.

Suppose the SIGNAL is described by the linear equation

$$x_t = x_s + \int_s^t a_u x_u du + \omega_t. \qquad (1.1)$$

Write the solution of (1.1) as $\xi_{s,t}(x_s)$. Supose $\Phi(s,t)$ is the solution of

$$\frac{d\Phi(s,t)}{dt} = a_t \Phi(s,t)dt, \qquad t \geq s, \qquad (1.2)$$
$$\Phi(s,s) = 1.$$

Clearly, $\Phi(s,t) = \exp\left(\int_s^t a_u du\right)$ and

$$\xi_{s,t}(x_s) = \Phi(s,t)\Big\{x_s + \int_s^t \Phi(s,u)^{-1} d\omega_u\Big\}. \qquad (1.3)$$

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The OBSERVATION process is taken to be of the form

$$y_t = \int_0^t h_s \xi_{0,s}(x_0) ds + B_t. \qquad (1.4)$$

As usual, we shall suppose x_0 is a Gaussian F_0 measurable random variable independent of ω_t, B_t, $t > 0$.

Write $\{Y_t\}$, $t \geq 0$, for the right continuous complete filtration generated by the observations and

$$\hat{x}_t(x_s) = E[x_t \mid x_s,\ Y_t] \qquad \text{for} \quad t \geq s.$$

Then it is known that $\hat{x}_t(x_s)$ is a Gaussian random variable for $t > s$ and

$$\hat{x}_t(x_s) = x_s + \int_s^t a_u \hat{x}_u(x_s) du + \int_s^t P_{s,u} h_u (dy_u - h_u \hat{x}_u(x_s))du \qquad (1.5)$$

where

$$P_{s,t} = E[x_t^2 \mid x_s,\ Y_t] - (E[x_t \mid x_s,\ Y_t])^2$$

satisfies the deterministic equation

$$\frac{dP_{s,t}}{dt} = -h_t^2 P_{s,t}^2 + 2a_t P_{s,t} + 1, \qquad (1.6)$$
$$P_{s,s} = 0.$$

Consequently, $\hat{x}_t(x_s)$ is Gaussian with conditional mean $\hat{x}_t(x_s)$ and variance $P_{s,t}$.

Writing $\hat{x}_t = E[x_t \mid Y_t]$ we see $\hat{x}_t$ is Gaussian with mean and variance P_t given by

$$\hat{x}_t = E[x_0] + \int_0^t a_s \hat{x}_s ds + \int_0^t P_s h_s (dy_s - h_s \hat{x}_s ds) \qquad (1.7)$$

$$\frac{dP_t}{dt} = -h_t^2 P_t^2 + 2a_t P_t + 1, \qquad (1.8)$$
$$P_0 = E[x_0^2] - (E[x_0])^2.$$

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The equations (1.5) and (1.6), or (1.7) and (1.8) are forms of the Kalmar filter. The inovation processes

$$\beta_t(x_s) = y_t - \int_s^t h_u \hat{x}_u(x_s)du, \qquad t \geq s,$$

$$\beta_t = y_t - \int_0^t h_u \hat{x}_u du, \qquad t \geq 0,$$

are $\{Y_t\}$ Brownian motions. They generate the same filtration as $\{y_t\}$.

The Gaussian measure on R with mean m and variance P will be denoted by $\mu(m, P, dx)$. If g is a Borel measurable function on R we shall write

$$\Gamma(g, m, P) = \int_R g(x)\mu(m, P, dx).$$

If Z_t is an integrable process, $t \geq 0$, $\Pi_t(Z)$ will denote the $\{Y_t\}$-predictable projection of Z, so $\Pi_t(Z) = E[Z_t \mid Y_t]$ a.s. For a function $g(t, x)$ such that

$$|g(t,x)| \leq K(1 + |x|^m)$$

for some $K > 0$, $m > 0$, we shall write

$$\Pi_t(g) = \Pi_t(g(t, x_t)).$$

From [2] we quote the following results:

LEMMA 1.1. *a) Suppose* $0 \leq s \leq t$. *The conditional law of* x_s *given* Y_t *is*

$$\mu(m_s^t, P_s^t, dx)$$

where

$$m_s^t = \hat{x}_t + \frac{P_s}{\gamma_s}\int_s^t \gamma_u h_u d\beta_u \tag{1.9}$$

$$P_s^t = P_s - \Big(\frac{P_s}{\gamma_s}\Big)^2 \int_s^t \gamma_u^2 h_u^2 du \tag{1.10}$$

and γ *is the solution of*

$$\gamma_t = 1 + \int_0^t (a_s - P_s h_s^2)\gamma_s ds \tag{1.11}$$

so

$$\gamma_t = \exp\int_0^t (a_s - P_s h_s^2)ds.$$

b) Suppose $g(t, x)$ *and* $g_x(t, x)$ *are Borel functions satisfying growth conditions as above. Then*

$$\Pi_t\Big(\int_0^t g(s, x_s)ds\Big) = \int_0^t \Pi_s(g)ds + \int_0^t \Pi_s\Big(\int_0^s g_x(u, x_u)\frac{P_u}{\gamma_u}\,du\Big)\gamma_s h_s d\beta_s. \tag{1.12}$$

From (1.3) we see the map

$$x \to \xi_{s,t}(x)$$

is a diffeomorphism of R and

$$\frac{\partial \xi_{s,t}(x)}{\partial x} = \Phi(s,t).$$

From (1.5) we can write

$$\hat{x}_t(x_s) = \Phi(s,t)\Big[x_s + \int_s^t \Phi(s,u)^{-1} P_{s,u} h_u d\beta_u(x_s)\Big] \tag{1.13}$$

and

$$\frac{\partial \hat{x}_t(x_s)}{\partial x_s} = \gamma_{s,t}$$

where

$$\gamma_{s,t} = 1 + \int_s^t (a_u - P_{s,u}h_u^2)\gamma_{s,u} du \tag{1.14}$$

so

$$\gamma_{s,t} = \exp\int_s^t (a_u - P_{s,u}h_u^2)du. \tag{1.15}$$

2. NONLINEAR SIGNAL EQUATIONS

For linear signal and observations the Kalman filter provides a finite dimensional solution to the filtering problem. Consider a measurable function $f(t,x)$ on $[0,\infty] \times R$ which is twice differentiable in x and which satisfies the growth condition

$$|f(t,x)| + |f_x(t,x)| \leq K(1 + |x|). \tag{2.1}$$

Let $\varepsilon > 0$ be a small positive number. Consider a signal process given by the non-linear equation

$$\bar{x}_t = x_0 + \int_0^t (a_s \bar{x}_s + \varepsilon f(s, \bar{x}_s))ds + \omega_t. \tag{2.2}$$

Consider the process z defined by

$$z_t = x_0 + \int_0^t \Phi(0,s)^{-1}\varepsilon f(s, \xi_{0,s}(z_s))ds \tag{2.3}$$

where $\xi_{0,s}(\cdot)$ is the diffeomorphism defined by (1.1).

LEMMA 2.1. *The process* $\xi_{0,t}(z_t)$ *is the solution of (2.2).*

PROOF. Substituting (2.3) in (1.3) we have

$$\xi_{0,t}(z_t) = \Phi(0,t)\Big[x_0 + \int_0^t \Phi(0,s)^{-1}\varepsilon f(s, \xi_{0,s}(z_s))ds + \int_0^t \Phi(0,s)^{-1} d\omega_s\Big]. \tag{2.4}$$

Differentiating (2.4) in t the result follows.

REMARKS 2.2. Because f satisfies the linear growth con dition (2.1) $\bar{x}_t = \xi_{0,t}(z_t)$ has finite moments of all orders.

If Z_t is a process we shall write $Z_t = O(\epsilon^k)$ if

$$\left(E\left(\sup_{s\leq t}|Z_t|^p\right)\right)^{1/p} = O(\epsilon^k)$$

for every $p \geq 1$.

NOTATION 2.3. Write

$$\Delta_{0,t} = \Phi(0,t)\int_0^t \Phi(0,s)^{-1} f(s,x_s)ds.$$

Using the mean value theorem we can quickly deduce

PROPOSITION 2.4. $\bar{x}_t - x_t = D_{0,t} = \epsilon\Delta_{0,t} + O(\epsilon^2)$.

REMARKS 2.5. To discuss the effect of the non-linear signal $\bar{x}_t = \xi_{0,t}(z_t)$ on the observations consider the measure $\overline{P}$ defined by

$$\left.\frac{d\overline{P}}{dP}\right|_{F_t} = \Lambda_t^\epsilon$$

where

$$\Lambda_t^\epsilon = \exp\left(\int_0^t h_s D_{0,s} dB_s - \frac{1}{2}\int_0^t h_s^2 D_{0,s}^2 ds\right).$$

Then under $\overline{P}$

$$\overline{B}_t = B_t - \int_0^t h_s D_{0,s} ds$$

is a Brownian motion, i.e.,

$$y_t = \int_0^t h_s \xi_{0,s}(z_s)ds + \overline{B}_t. \tag{2.5}$$

Therefore, under $\overline{P}$ the signal process is $\bar{x}$ and this now influences the observations as in (2.5). The non-linear filtering expression we wish to consider is

$$\overline{E}[\xi_{0,t}(z_t) \mid Y_t].$$

By Baye's theorem this is

$$E[\Lambda_t^\epsilon \xi_{0,t}(z_t) \mid Y_t]\cdot(E[\Lambda_t^\epsilon \mid Y_t])^{-1}.$$

LEMMA 2.6. $\Lambda_t^\epsilon = 1 + \epsilon\int_0^t h_s\Delta_{0,s}dB_s + O(\epsilon^2)$.

PROOF. $\Lambda_t^\epsilon = 1 + \int_0^t \Lambda_s^\epsilon h_s D_{0,s} dB_s$ and the result follows by substituting for Λ_s^ϵ on the right and using Proposition 2.4.

From Proposition 3.3 of Picard [2] we have

LEMMA 2.7. $\Pi_t(\Lambda)^{-1} = 1 - \epsilon\Pi_t\left[\int_0^t h_s\Delta_{0,s}dB_s\right] + O(\epsilon^2)$.

The main result is the following theorem:

THEOREM 2.8. *Writing* $\bar{x}_t = \xi_{0,t}(z_t)$, $x_t = \xi_{0,t}(x_0)$

$$\begin{aligned}\overline{E}[\bar{x}_t \mid Y_t] &= E[x_t \mid Y_t] + \epsilon E\left[x_t\int_0^t h_s\Delta_{0,s}dB_s \mid Y_t\right]\\ &\quad + \epsilon E[\Delta_{0,t} \mid Y_t] - \epsilon E[x_t \mid Y_t]E\left[\int_0^t h_s\Delta_{0,s}dB_s \mid Y_t\right]\\ &\quad + O(\epsilon^2).\end{aligned} \tag{2.6}$$

PROOF.

$$\begin{aligned}\overline{E}[\bar{x}_t \mid Y_t] &= E[\Lambda_t^\epsilon \bar{x}_t \mid Y_t]\cdot E[\Lambda_t^\epsilon \mid Y_t]^{-1}\\ &= E[\Lambda_t^\epsilon(x_t + \epsilon\Delta_{0,t}) \mid Y_t]\\ &\quad\times E\left[\left(1 - \epsilon\int_0^t h_s\Delta_{0,s}dB_s\right) \mid Y_t\right] + O(\epsilon^2)\\ &= E\left[\left(1 + \epsilon\int_0^t h_s\Delta_{0,s}dB_s\right)(x_t + \epsilon\Delta_{0,t}) \mid Y_t\right]\\ &\quad\times\left[1 - \epsilon E\left[\int_0^t h_s\Delta_{0,s}dB_s \mid Y_t\right]\right] + O(\epsilon^2)\end{aligned}$$

by Proposition 2.3 and Lemma 2.7.

REMARKS 2.9. These expectations are all expressible in terms of Gaussian measures because they are all expectations of functions of the original linear process x_t under the original measure P. For example, $E[x_t \mid Y_t] = \hat{x}_t$ is given by the Kalmar filter. The remaining terms in (2.6) can be expressed in a recursive way; proofs can be found in [1]. For example, we have

LEMMA 2.10.

$$\begin{aligned}E[\Delta_{0,t} \mid Y_t] &= \Phi(0,t)\left[\int_0^t \Phi(0,s)^{-1}\Pi_s(f(s,x_s))ds\right.\\ &\quad\left.+\int_0^t \Pi_s\left\{\int_0^s f_x(u,x_u)P_u\gamma_u^{-1}du\right\}\Phi(0,s)^{-1}h_s\gamma_s d\beta_s\right].\end{aligned}$$

3. CONCLUSION

As in the paper of Picard [2] the first two terms in an expansion of the conditional mean in powers of ϵ have been determined. These coefficients have been expressed explicitly in terms of Gaussian measures by using stochastic flows.

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[2] J. Picard, "A filtering problem with a small nonlinear term." Stochastics, 18, pp. 313–341, 1986.