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In this report, we give a brief summary of the work done by the PIs and their students on the design and analysis of robust feedback systems. Under this contract we have conducted research on a number of problems in system and control theory. In particular, we have obtained significant new results on H_∞ control theory, robust stability analysis of feedback systems, robust stabilization of uncertain systems, approximation and control of infinite dimensional systems, etc. In this report, we present a brief summary of the main results obtained under the AFOSR contract.

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Final Report for the AFOSR Contract AFOSR- 88-0020 Generalized Interpolation Theory for the Design and Analysis of Robust Feedback Systems

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Abstract

In this report, we give a brief summary of the work done by the PIs and their students on the design and analysis of robust feedback systems. Under this contract, we have conducted research on a number of problems in system and control theory. In particular, we have obtained significant new results on H_∞ control theory, robust stability analysis of feedback systems, robust stabilization of uncertain systems, approximation and control of infinite dimensional systems, etc. In this report, we will present a brief summary of the main results obtained under the AFOSR contract.

1 Introduction

Our (PIs' and their students') research work in systems and control has been supported for the last 3 years by the AFOSR. We have made significant contributions to H_∞ control theory, robust control - analysis and synthesis, control of infinite-dimensional systems, control of time-varying systems, etc. Our work on these problems naturally divides up into 2 parts: work done Professor Khargonekar and his group and the work done by Professor Tannenbaum and his group. While they have addressed related problems, their work and approaches are complement each other, and it is easiest to describe the research accomplishments under the AFOSR contract into two separate parts.

We have published most of the work under the contract in various journals and conferences as can be seen in the attached list of publications. For reasons of space and time, we will only present brief descriptions of the results and give appropriate references where the results may be found. We have strictly restricted the discussion to our own work and references to our own papers since a complete discussion of related references will make this report too long and unwieldy. The interested reader is referred to our published papers where extensive discussions and lists of related references may be found.

2 Work done by Professor Khargonekar and his group

2.1 State-Space Approach to H_∞ Control Theory

Motivated by robustness considerations, Zames introduced the problem of H_∞ optimal control. The essential idea was to design a controller to optimize performance for the worst exogenous input. Thus, while in Kalman filtering and the LQG problem, the power spectrum of the exogenous input (noise) is assumed known (usually white noise), in the H_∞ control problem the power spectrum is assumed unknown and the controller is designed for the *worst case.*

Early research in H_∞ control theory was conducted using frequency domain methods. The key tools were the Youla-Jabr-Bongiorno-Kucera parametrization of all stabilizing controllers, inner-outer factorizations of transfer functions, Nevanlinna-Pick interpolation theory, Nehari distance theorem, the commutant lifting theorem, etc. This frequency-domain operator theoretic approach is currently a very active area of research and a number of different avenues are being explored. The work done by Professor Tannenbaum and his group is along these lines and a summary of results obtained along this approach may be found elsewhere in this report.

A major new development in H_∞ control theory in the last two years has been the introduction of state-space methods. This has led to a rather transparent solution to the standard problem of H_∞ control theory. This solution is remarkably similar to the classical LQG solution with appropriate differences that reflect the differences in the H_∞ and the LQG performance criteria. As a result it is possible to give a simple interpretation to the so-called H_∞ *central* controller. Analogous to the LQG theory, the solutions are given in terms of solutions to algebraic and differential matrix Riccati equations.

Consider the finite-dimensional linear time-invariant system Σ

$$\begin{aligned} \frac{dx}{dt} &= Fx + G_1 w + G_2 u, \\ z &= H_1 x + J_{11} w + J_{12} u, \\ y &= H_2 x + J_{21} w + J_{22} u \end{aligned} \tag{1}$$

Here x, w, u, z, y denote respectively the state, the excgenous input, the control input, the regulated output, and the measured output. Kalman made pioneering contributions to the problem of designing a controller for optimizing the variance of z when w is a stochastic process. This classical solution, commonly known as the *LQG theory,* has had a very significant influence on linear multivariable control theory over the last 30 years.

The *standard problem* of H_∞ control theory is: Given the linear system Σ and a positive number γ, find a causal (dynamic) controller K such that the closed loop system is well-posed, internally stable, the closed loop input-output operator

$$T_{zw} : L_2 \longrightarrow L_2 : w \mapsto z$$

is bounded, and the induced norm

$$\|T_{zw}\| := sup\{\frac{\|z\|_2}{\|w\|_2} : \|w\|_2 \neq 0\} < \gamma.$$

A controller K is called *admissible* iff it is causal, the closed loop system is well-posed and internally stable. If K is also an FDLTI system

$$\begin{aligned} \frac{d\xi}{dt} &= A\xi + By, \\ u &= C\xi + Dy \end{aligned} \tag{2}$$

then the closed loop system is well-posed iff $(I - J_{22}D)$ is invertible and the closed loop system is internally stable iff the closed loop system matrix

$$\begin{bmatrix} F + G_2(I - DJ_{22})^{-1}DJ_{22} & G_2(I - DJ_{22})^{-1}C \\ B(I - J_{22}D)^{-1}H_2 & A + BJ_{22}(I - DJ_{22})^{-1}C \end{bmatrix}$$

has no eigenvalues in the closed right half complex plane. Moreover, in this case, if we let $T_{zw}(s)$ denote the closed loop transfer function matrix, then

$$\|T_{zw}\| = \|T_{zw}\|_\infty := sup\{\bar{\sigma}(T_{zw}(s)) : Re(s) \geq 0\}.$$

It is a well known fact that under suitable assumptions, the *LQG* controller is also the unique solution to the problem of minimizing the quadratic norm of the closed loop transfer function matrix T_{zw} over all internally stabilizing controllers. Thus, the key difference between the *LQG* and the H_∞ control problems is in the choice of norm on transfer functions.

This is intimately related to the underlying assumptions on the exogenous signals as mentioned in the introduction. Also, it turns out that the H_∞ norm arises naturally in many robust control problems.

In order to present the most important concepts clearly, we will make certain simplifying assumptions. Most of these assumptions can be easily removed.

A.1 $J_{11} = J_{22} = 0$.

A.2 $J_{12}' \begin{bmatrix} H_1 & J_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

A.3 $J_{21} \begin{bmatrix} G_1' & J_{21}' \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

A.4 (F, G_1, H_1) is stabilizable and detectable.

A.5 (F, G_2, H_2) is stabilizable and detectable.

A.6 The controller K is an FDLTI system.

Assumption A.1 implies that there is no direct transmission from the exogenous input to regulated output and from the control input to the measured output. The latter assumption ensures that any proper rational matrix $K(s)$ leads to a well-posed feedback system. Assumption A.2 is quite common in the LQG literature and amounts to assuming that there is no *cross term* in the formula for $\|z\|^2$ and the penalty on the control input u is normalized, i.e.,

$$\|z\|^2 = x' H_1' H_1 x + u'u.$$

Assumption A.3 is the dual of assumption A.2 and is analogous to the standard assumption in the Kalman filtering problem that the process noise and the measurement noise are uncorrelated and that the measurement noise is nonsingular and normalized . Assumption A.4 is a technical assumption and guarantees that the corresponding controllers are admissible. Assumption A.5 is necessary and sufficient to guarantee the existence of an internally stabilizing controller for the system Σ.

The assumption A.6 also causes no loss of generality as shown by Khargonekar and Poolla [1989]. They showed that the infimum of the norm of T_{zw} over all causal internally stabilizing nonlinear controllers is no less than that over all FDLTI internally stabilizing controllers.

2.2 The State-Feedback H_∞ Control Problem

We will first consider the state-feedback H_∞ problem. In other words, in the system Σ, let us assume that

$$H_2 = I \qquad \textit{and} \qquad J_{21} = 0. \tag{3}$$

(It should be noted that the state-feedback problem does not satisfy the assumption A.3 above.) Under the assumption $J_{12} = 0$, Petersen considered the problem of finding a real matrix L such that $F + G_2L$ is asymptotically stable and

$$\|H_1(sI - F - G_2L)^{-1}G_1\|_\infty < \gamma.$$

He obtained a necessary and sufficient condition for the existence of L in terms of an algebraic Riccati equation with an indefinite quadratic term. This was quite an interesting result and led to a number of subsequent developments.

In view of the fact that frequency-domain H_∞ control theory results usually led to (apparently) high dimensional controllers, Khargonekar, Petersen, and Rotea [1988] investigated whether *static gains* were actually H_∞ optimal. This problem was motivated by the one of Kalman's key insights that control is a function of the state. Thus, we expected the following result to be true:

Theorem 2.1 *Consider the system Σ and suppose (3) holds. Then*
$\gamma^* = inf\ \{\ \|T_{zw}\|_\infty : K$ *is an internally stabilizing dynamic state-feedback controller* $\}$
$= inf\ \{\ \|T_{zw}\|_\infty : K$ *is an internally stabilizing static state-feedback gain.* $\}$

Theorem (2.1) was proved by Khargonekar, Petersen, and Rotea [1988] under the condition that $J_{12} = 0$. It was generalized further by Khargonekar, Petersen, and Zhou [1987], Zhou and Khargonekar [1988], Doyle, Glover, Khargonekar, and Francis [1989].

The following theorem shows how one can obtain state-feedback control laws by solving certain algebraic Riccati equations. It is taken from Doyle, Glover, Khargonekar, and Francis [1989]. Results analogous to Theorem (2.2) were also obtained in our earlier papers Khargonekar, Petersen, and Zhou [1987], Zhou and Khargonekar [1988].

Theorem 2.2 *Consider the system Σ. Suppose assumptions A.1, A.2, and (3) hold. Then there exists an admissible controller K such that*

$$\|T_{zw}\|_\infty < \gamma \tag{4}$$

if and only if there exists a (unique) symmetric matrix P such that

$$F'P + PF + P(\frac{G_1G_1'}{\gamma^2} - G_2G_2')P + H_1'H_1 = 0, \tag{5}$$

$F + (\frac{G_1G_1'}{\gamma^2} - G_2G_2')P$ *is asymptotically stable, and $P \geq 0$. In this case, the control law*

$$u = -G_2'Px \tag{6}$$

internally stabilizes Σ and (4) holds.

There are a number of interesting features to this result. Note that the control law is obtained by solving an algebraic Riccati equation which is analogous to the classical results of Kalman on the linear-quadratic optimal control problem. The algebraic Riccati equation (5) is similar to the corresponding equation in the linear-quadratic optimal control problem except that the quadratic term in (5) is indefinite. Indeed, (5) is identical to the ARE that arises in linear quadratic optimal *game* problems. This is intuitively appealing since in the H_∞ control problem the inputs w and u act as opposing players: the exogenous input w tries to maximize (the norm of) of z while u is designed to minimize it. This connection between linear-quadratic games and H_∞ control theory has been discussed and explored in a number of recent papers. See, e.g., Khargonekar, Petersen, and Zhou [1987], Limebeer, Anderson, Khargonekar, and Green [1989], and the references therein.

The paper by Limebeer, Anderson, Khargonekar, and Green [1989] contains analogues of Theorem (2.2) for linear time-varying systems.

Note that as γ goes to ∞, the controller (6) approaches the LQR solution. In other words, as the H_∞ norm constraint on the closed loop transfer function is relaxed, the control law given by (6) approaches the corresponding LQR control law.

The results of Khargonekar, Petersen, and Zhou [1987], Zhou and Khargonekar [1988] also apply to the general singular case (J_{21} is not necessarily full rank). In these papers the singular case is taken care of by introducing certain small perturbations to make the problem nonsingular. Since we are dealing with a strict inequality in (4), the existence of an appropriate small perturbation is not difficult to establish.

Theorem (2.2) naturally leads to the H_∞ analog of the *inverse problem of optimal control* formulated by Kalman. This inverse problem of H_∞ control theory was considered by Fujii and Khargonekar [1988] who showed that state-feedback H_∞ controllers inherit the nice robustness properties of the LQR controllers and in a certain sense are even more robust.

2.3 The H_∞ Filtering Problem

In the LQG theory, state-feedback linear-quadratic optimal control and Kalman filtering problems turn out to have dual structures. The situation is somewhat similar in the H_∞ case. Filtering problems were first considered by Doyle, Glover, Khargonekar, and Francis [1989] under the assumptions that (i) the process is stable, and (ii) the initial state in known. Recently, Nagpal and Khargonekar [1989] have developed a fairly complete H_∞ estimation theory. Here we will give a brief summary of some of their results.

Consider the FDLTI system

$$\begin{aligned} \frac{dx}{dt} &= Fx + G_1 w, \\ z &= H_1 x, \\ y &= H_2 x + J_{21} w. \end{aligned} \tag{7}$$

In this section, we will assume that (F, G_1) is reachable, (F, H_2) is detectable, and A.3 holds.

The problem is to estimate the output z using the measurements y. Nagpal and Khargonekar [1989] have considered both the filtering (estimator is required to be causal) and the smoothing (estimator is allowed to be noncausal) problems. So, let $\mathcal{F}$ be a causal estimator and let

$$\hat{z} = \mathcal{F}(y).$$

The estimator $\mathcal{F}$ is called *unbiased* if

$$y(\tau) = 0 \quad \forall \tau \leq t \Rightarrow \hat{z}(\tau) = 0 \quad \forall \tau \leq t.$$

Let us now define an H_∞ i.e., a worst case, performance measure. There are two cases to consider: (i) initial state is known (and, without loss of generality =0), and (ii) the initial state is unknown. In the latter case, it is assumed that the best a priori estimate of $x(0)$ is zero. In other words, the estimator initial state is taken to be zero. Let $0 \leq T \leq \infty$. Define

$$J_1(\mathcal{F}, T) := sup\{\frac{\|z - \hat{z}\|_2}{\|w\|_2} : w \in L_2[0,T], 0 \neq \|w\|_2, x(0) = 0\}, \tag{8}$$

and

$$J_2(\mathcal{F}, R, T) := sup\{\frac{\|z - \hat{z}\|_2}{[\|w\|_2^2 + x_0' R x_0]^{1/2}} : w \in L_2[0,T], x(0) = x_0, \|w\|_2^2 + x_0' R x_0 \neq 0\} \tag{9}$$

In the definition of J_2, R is taken to be positive definite. The key difference between J_1 and J_2 is that J_2 mesures the worst case performance over all possible w and $x(0)$. The expression $[\|w\|_2^2 + x_0' R x_0]^{1/2}$ is a total measure of energy in the exogenuos variables $w, x(0)$.

The following results are taken from Nagpal and Khargonekar [1989].

Theorem 2.3 *Consider the FDLTI system (7). Let $\gamma > 0$ be given and let $T = \infty$. Then there exists an unbiased linear filter such that*

$$J_2(\mathcal{F}, R, T) < \gamma \tag{10}$$

if and only if there exists a bounded symmetric matrix function $Q(t), t \in [0, \infty)$ such that

$$\dot{Q}(t) = FQ(t) + Q(t)F' - Q(t)H_2'H_2Q(t) + \frac{Q(t)H_1'H_1Q(t)}{\gamma^2} + G_1G_1', \quad Q(0) = R^{-1}, \tag{11}$$

and the system

$$\dot{e}(t) = [F - Q(t)(H_2'H_2 - \frac{H_1'H_1}{\gamma^2})]e(t)$$

is exponentially stable.

Moreover, if these conditions hold then one filter that satisfies the performance bound (10) is given by

$$\dot{\hat{x}}(t) = F\hat{x}(t) + Q(t)H_2'[y(t) - H_2\hat{x}(t)], \qquad \hat{x}(0) = 0, \tag{12}$$

$$\hat{z}(t) = H_1\hat{x}(t). \tag{13}$$

Theorem 2.4 *Consider the FDLTI system (7). Let $x(0) = 0$. Let $\gamma > 0$ be given and let $T = \infty$. Then there exists an unbiased linear filter such that*

$$J_1(\mathcal{F}, T) < \gamma \tag{14}$$

if and only if there exists a real symmetric matrix Q such that

$$FQ + QF' - QH_2'H_2Q + \frac{QH_1'H_1Q}{\gamma^2} + G_1G_1' = 0, \tag{15}$$

the matrix $(F - Q(H_2'H_2 - \frac{H_1'H_1}{\gamma^2}))$ has no eigenvalues in the closed right half plane, and $Q \geq 0$.

Moreover, if these conditions hold then with $Q(t)$ replaced by Q, the filter given by (12) and (13) satisfies the performance bound (14).

These results are the H_∞ analog of the well known Kalman filtering results. The key differences are in the Riccati differential equation (11) and the algebraic Riccati equation (15) which are very similar to the covariance equations for the Kalman filtering problem with the exception of the term $QH_1'H_1Q/\gamma^2$. Thus, in H_∞ filtering, the states to be estimated influence the filter itself unlike in Kalman filtering where the optimal estimate of any state-functional is obtained from the optimal state-estimator. Note that as $\gamma \to \infty$, the H_∞ filters approach the standard Kalman filters.

There is another qualitative interpretation of the term $QH_1'H_1Q/\gamma^2$. It can be regarded as additional *process noise*. As a result, the H^∞ filter has robustness (in the sense of satisfying (10) or (14)) to variation in the spectra of the exogenous signals at the expense of performance if the exogenous input w is actually a zero mean Gaussian *white* stochastic process.

The paper by Nagpal and Khargonekar [1989] contains similar results on H_∞ filtering and smoothing for infinite as well as finite horizon cases for linear time-invariant as well as linear time-varying systems.

2.4 The Output-Feedback H_∞ Control Problem

Let us now consider the output feedback problem. A key result from the linear-quadratic-Gaussian control theory is that in the output feedback case, the optimal controller is the Kalman filter for the optimal state-feedback law. The above mentioned results on the state-feedback H_∞ control problem indicate that something similar may be true in the output feedback H_∞ problem. The following result is taken from Doyle, Glover, Khargonekar, and Francis [1989].

Theorem 2.5 *Consider the linear system Σ. Suppose assumptions A.1-A.6 hold. Then there exists an admissible output feedback controller K such that with the control law $u = Ky$, the closed loop transfer function satisfies*

$$\|T_{zw}\|_\infty < \gamma \tag{16}$$

if and only if the following conditions hold:

1. *there exists a (unique) real symmetric matrix P such that*

$$F'P + PF + P(\frac{G_1G_1'}{\gamma^2} - G_2G_2')P + H_1'H_1 = 0,$$

$F + (\frac{G_1G_1'}{\gamma^2} - G_2G_2')P$ *is asymptotically stable, and* $P \geq 0$*;*

2. *there exists a (unique) real symmetric matrix Q such that*

$$FQ + QF' + Q(\frac{H_1'H_1}{\gamma^2} - H_2'H_2)Q + G_1G_1' = 0, \tag{17}$$

$F + Q(\frac{H_1'H_1}{\gamma^2} - H_2'H_2)$ *is asymptotically stable, and* $Q \geq 0$*; and*

3. $\rho(\frac{PQ}{\gamma^2}) < 1$.

If these conditions hold then one controller that satisfies the inequality (16) is given by

$$\begin{aligned} \frac{d\hat{x}}{dt} &= [F + \frac{G_1G_1'}{\gamma^2}P]\hat{x} + ZQH_2'(y - H_2\hat{x}) + G_2u \\ u &= -G_2'P\hat{x}, \end{aligned} \tag{18}$$

where

$$Z = (I - \frac{QP}{\gamma^2})^{-1}.$$

(Here $\rho(.)$ denotes the spectral radius.)

The notation in the above theorem is meant to be suggestive. Note that the above controller (18) has many similarities to (and some differences from) the classical LQG controller. It is called the H_∞ central controller and has many interesting interpretations and properties. As γ goes to ∞, this controller approaches the LQG controller.

It is shown in Doyle, Glover, Khargonekar, and Francis [1989] that the above controller is an H_∞ estimator for the state-feedback control law (6) in the presence of the disturbance $w = \frac{G_1'P}{\gamma^2}x$. More specifically, let

$$r := w - \frac{G_1'P}{\gamma^2}x.$$

Then the equations for system Σ become

$$\begin{aligned} \frac{dx}{dt} &= (F + \frac{G_1G_1'P}{\gamma^2})x + G_1r + G_2u, \\ y &= H_2x + J_{21}r. \end{aligned} \tag{19}$$

Then if we consider the problem of estimating the state-feedback law (6)

$$u = -G_2' P x$$

and apply Theorem (2.4) to the above system, then the resulting filter is precisely the output feedback controller in Theorem (2.5).

The above result is also closely connected to the separation principle in the *risk sensitive control* problem. See Doyle, Glover, Khargonekar, and Francis [1989] for further details along these lines.

As mentioned previously, in Limebeer, Anderson, Khargonekar, and Green [1989], we have extended these results to linear time-varying.

2.5 Robust Control: Synthesis Problems

The primary reason for the use of feedback in control systems is the fact that almost all control systems operate in uncertain environments. There are two types of uncertainties that affect control systems: plant uncertainty and signal uncertainty. Plant uncertainty arises due to errors in modeling, changes in parameters, inexact and incomplete data, modeling approximations while signal uncertainty arises due to exogenous signals such as disturbances, sensor noise, etc. Control theory attempts to provide systematic techniques and methods to solve control problems where uncertainty is a dominant issue. Indeed, from this point of view, robust linear control and adaptive control methodologies are different solutions to this problem. Now one can regard an adaptive controller as a nonlinear time-varying controller. Therefore, a fundamental question of interest is:

Given a feedback control problem for a linear multivariable system, what advantages (if any) do nonlinear time-varying (NLTV) controllers offer over linear time-invariant (LTI) controllers?

Our work with a number of colleagues T. Georgiou, A. Pascoal, K. Poolla, R. Ravi, M. Rotea, and A. Tannenbaum on various aspects of this problem has led us to formulate a qualitative principle which captures known results obtained so far.

THE PLANT UNCERTAINTY PRINCIPLE (Khargonekar-Poolla [1989])

In robust multivariable control problems, nonlinear time-varying controllers yield (significant) adavantages over linear time-invariant controllers if and only if there is (significant) parametric or structured uncertainty.

The strong and simultaneous stabilization problems have been considered in many papers in the literature. It is well known that in general, not every LTI plant can be stabilized by a stable controller. Also, not every pair of LTI plants can be simultaneously stabilized by an LTI controller. In contrast to the above situation, Khargonekar, Pascoal and Ravi [1988] have proved the following result:

Theorem 2.6 *Let $F := P_1, P_2, ..., P_n$ be a finite collection of finite-dimensional linear time-varying (LTV) plants such that each $P_i; i = 1, .., n$ is internally stabilizable. Then, F can* *be simultaneously (internally, and in the L_p-sense) stabilized by a stable finite- dimensional LTV controller.*

As a corollary, it was also shown that any finite-dimensional LTI plant can be internally stabilized by a stable, periodic controller. Further, every finite collection of finite-dimensional LTI plants can be simultaneously stabilized by a stable periodic controller.

The papers by Khargonekar [1989] and Khargonekar and Poolla [1989] contain a complete discussion of known results on the Plant Uncertainty Principle and a number of related references.

In a parallel direction, we also investigated robust control synthesis using Liapunov function methods - 'quadratic stabilization' and robust control of uncertain systems. In Khargonekar, Petersen, and Zhou [1987, 1990] we showed that there is a very deep connection between H_∞ control quadratic stability and stabilization and H_∞ control theory. Actually, this connection goes back to some early work of Popov, but it had apparently been overlooked in the more recent robust control literature. In Khargonekar, Petersen, and Zhou [1987, 1990], we gave a complete and constructive solution of the problem of quadratic stabilization by LTI control by output feedback - an unsolved problem then in the uncertain systems literature. Related technical results were obtained in Khargonekar and Zhou [1988]. An interesting result supporting the plant uncertainty principle was recently obtained by Khargonekar and Rotea [1988b], Rotea and Khargonekar [1989] where we showed that for systems with norm bounded uncertainty, quadratic stabilizability via NLTV controllers implies quadratic stabilizability via LTI controllers in the state-feedback case. Related results were obtained in Rotea and Khargonekar [1988] where we showed that if the plant states are available for feedback then quadratic stabilizability by linear dynamic state-feedback implies quadratic stabilizability by nondynamic state-feedback.

In Khargonekar and Gu [1989] we have given a new state-space solution to a robust stabilization problem for systems with combined parametric and dynamic uncertainty under a minimum phase assumption. This is an important problem area where very few results are currently available.

It is well known that control design invloves tradeoffs among competing objectives. In Khargonekar and Rotea [1989], we have recently given a complete solution to the multiple objective optimal control problem (in the sense of Pareto optimality) in the H_2 norm case. Our results include a complete characterization and construction of all Pareto optimal controllers. These are the most general results available on the problem to-date. The computations invlove only solving algebraic Riccati equations and are easily implementable using stnadard matrix algebra packages such as MATLAB. An interesting conclusion from this work is that the (state) dimension of the Pareto optimal controllers often exceed the (state) dimension of the controlled plant. This is in sharp contrast to the existing results in the LQG and H_∞ control problems, where the controller dimension can always be chosen to not exceed that of the controlled plant. This is a result on the controllers complexity and raises a host of issues for future research.

Motivated by problems in adaptive control, a complete solution to a certain robust synthesis problem for positive real functions was given in Anderson, Dasgupta, Khargonekar, Kraus, and Mansour [1989]. Roughly speaking, given a polytope of polynomials, it is desired to find a fixed polynomial (or a stable rational function) such that the ratio of the fixed polynomial and any polynomial from the given polytope is a positive rational function. We gave a necessary and sufficient condition for a solution to exist and also a construction for a solution if it exists. This result brought out an interesting connection to the growing body of literature on Kharitonov mathods for robust stability analysis.

We have obtained some key results on the construction of coprime factorizations for linear time-varying systems. These are presented in Khargonekar and Rotea [1988]. It was shown that a given a linear time-varying system admits a coprime factorization if and only if it can be stabilized by dynamic output feedback. In a related direction, a generalization of the Youla parametrization for a class of multirate digital control systems was obtained by Ravi, Khargonekar, Minto, and Nett [1989]. This result may open up anew research direction on the design of multirate control systems - a very important problem from a practical point of view.

2.6 Robust stability analysis

In Barmish and Khargonekar [1988, 1990] we formulated and solved a robust stability analysis problem for SISO systems containing uncertain real parameters and unstructured uncertainty. This paper still remains one of a handful of papers dealing with robustness analysis for simultaneous parametric and unstructured uncertainty. The result was an algorithm to check the stability of a SISO feedback system containing uncertain real parameters and multiplicative unmodeled dynamic uncertainty.

In a very recent paper Barmish, Khargonekar, Shi, and Tempo [1989], we have discovered that the robustness margin can be a **discontinuous** function of the problem data. Although the ramifications of this result will not be clear for at least a few months, it certainly seems likely that this will open a new research direction robust stability analysis. In particular, it raises the issue of 'conditioning properties' of robust stability margin problems.

2.7 Infinite-dimensional and sistributed systems

In Gu, Khargonekar, and Lee [1988, 1989] we have developed a new approach to the problem of finding finite-dimensional approximations for (possibly unstable) infinite-dimensional systems which converge in the L_∞ norm. Our approach uses the FFT algorithm and singular value decompositions and seems to be computationally less demanding than the approaches based on the Hankel operator theory. Indeed, we have implented these numerical algorithms on an IBM PC, and applied them to some simple examples, where they seem to give very good results. They may also have applications in digital filter design.

In Georgiou and Khargonekar [1989], we have developed a new algorithm for spectral factorization of matrix valued functions using the Nevanlinna-Pick theory. At this point, the computational potential of this algorithm is unexplored and it remains a subject for future investiagtion.

List of publications of P. P. Khargonekar and his group supported by Air Force Office of Scientific Research Contract no. AFOSR- 88-0020

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Part 2 of Final Report for the AFOSR Contract AFOSR-88-0020 entitled Generalized Interpolation Theory for the Design and Analysis of Robust Feedback Systems: Work of Allen Tannenbaum and His Group

May 11, 1990

1 Introduction

Referring to the list of publications below in Section 7 since the start of funding on October 1, 1987, Tannenbaum and collaborators have produced 22 refereed journal papers which have appeared or been accepted for publication (papers 1-22), 4 refereed papers which will appear as a book chapters (papers 23-26), 14 conference papers (27-40). All of the conference papers were either refereed or invited. One paper (paper 41) has appeared in which Tannenbaum's Ph. D. student supported by AFOSR-88-0020, Hitay Ozbay, appears as sole author. Finally a monograph, *Feedback Control Theory* (written in collaboration with John Doyle and Bruce Francis, and to be published by MacMillan) on robust control was begun under AFOSR-88-0020 and will be completed under AFOSR-90-0024.

A complete list of these papers (including the monograph) is given in Section 7. This constitutes the list of papers produced under AFOSR-88-0020 by Tannenbaum and various collaborators. The grant ended March 31, 1990 after two and a half years of funding. The research program is being continued under AFOSR-90-0024.

Several of the later papers were also written during the intersection of AFOSR-88-0020 and the new contract AFOSR-90-0024, and so both contract numbers appear on these papers.

Hitay Ozbay completed his Ph.D. in August 1989. Since then two graduate students Brian Schipper (masters) and Evgeny Berzon (Ph.D.) have been supported on the contract on a half time basis. Brian Schipper will complete his masters degree work in the spring of 1990, and then Berzon will receive full time support on the new contract AFOSR-90-0024.

We now sketch the work done by A. Tannenbaum and collaborators on this contract. All the papers which have not been sent in the past will be sent to AFOSR under separate cover. (Many of the papers on the list have already been sent to AFOSR in the past.) In our summary, we will highlight a few of the key results and concepts. All the details may be found of course in the papers listed in Section 7.

2 H^∞ Control of Distributed Systems

During this contract, the H^∞ problem in its most general form (the **four block problem or standard problem**) has been solved for arbitrary lumped and stable distributed systems [8], [11], [14]. (As this contract is ending, the solution is being extended to unstable distributed systems with a finite number of unstable poles. This work will be described in our yearly report for AFOSR-90-0024.)

In solving such problems, we have introduced a new class of operators called **skew Toeplitz,** which seem ideally suited for studying H^∞ design problems, especially for distributed parameter systems [1], [3], [5], [7], [13]. These operator theoretic methods are quite powerful since they allow one to synthesize optimal and sub-optimal compensators just using the input/output operators. An important new feature of this approach is that the complexity of the computations only depends on the the weighting matrices (modelling the disturbances) and not on the plant (which may be distributed). Since the weighting matrices are typically taken to be rational of low order, this approach seems to fit in very well even in the finite dimensional case for plants with large state spaces (e.g., finite element models of large space flexible structures.). Computer implementations for our procedures have been worked out at at the Systems Research Center of Honeywell, Minneapolis, and the University of Minnesota. Actually, for problems even involving finite dimensional systems our algorithm seemed to converge much more quickly than competing computational schemes.

We now would like to describe briefly some of the ideas behind our approach. Full details can of course be found in the references given above. We want to indicate how operator theoretic methods may be employed to solve a very general case of the **standard** or **four block problem** in H^∞ design valid for a large class of distributed, i.e., infinite dimensional systems.

The motivations for studying the H^∞ optimization in systems theory lie in the most natural problems of control engineering such as robust stabilization, sensitivity minimization, and model matching. It can be shown that, in the sense of H^∞ optimality, these problems are equivalent, and can be formulated as one ***standard problem.***

Now it is quite well-known that an optimal solution of the standard problem can be reduced to finding the singular values and vectors of a certain operator (the so-called **four block operator**) which will be defined below. Depending on the specific problem considered, the corresponding four block operator can be simplified to a 2-block (mixed sensitivity) or a 1-block (sensitivity) operator.

Besides appearing in the most general H^∞ synthesis problems, the four block operators also have a number of intriguing mathematical properties in the sense that they are natural extensions of both the Hankel and Toeplitz operators. For this reason they fit into the skew Toeplitz framework developed in [5]. Once again for the full details of our arguments and details about the skew Toeplitz theory applied to this problem we refer the reader to above references. Here we will just consider the four block problem for single input / single output systems.

More precisely, invoking the Youla parametrization and employing standard manipulations involving inner-outer factorizations, for a large class of distributed systems we may reduce the standard problem mentioned above to the following mathematical one. Let $w, f, g, h, \in H^\infty$, where w, f, g, h are rational and m is nonconstant inner. (All of our Hardy spaces will be defined on the unit disc D in the standard way.) Set

$$\mu := \inf\{\| \begin{bmatrix} w - mq & f \\ g & h \end{bmatrix} \|_\infty : q \in H^\infty\}. \tag{1}$$

Then we want to give an algorithm for calculating the quantity μ, and for finding the corresponding $q_{opt} \in H^\infty$, i.e., q_{opt} is such that

$$\mu := \| \begin{bmatrix} w - mq_{opt} & f \\ g & h \end{bmatrix} \|_\infty$$

Note that for $f = g = h = 0$, this reduces to the classical Nehari problem.

Our program will be to identify μ as the norm of a certain "four block operator," and then give a determinantal formula for its computation.

2.1 The Four Block Operator

We will now define the *four block operator*. We set $H(m) := H^2 \ominus mH^2$, $L(m) := L^2 \ominus mH^2$, and we let $P_{H(m)} : H^2 \to H(m)$, $P_{L(m)} : L^2 \to L(m)$ denote the corresponding orthogonal projections. Let $S : H^2 \to H^2$ denote unilateral shift, $T : H(m) \to H(m)$ the compression of S, and let $U : L^2 \to L^2$ denote bilateral shift, with $T(m) : L(m) \to L(m)$ the compression of U. Then for $w, f, g, h \in H^\infty$ rational, we set

$$A := \begin{bmatrix} P_{L(m)} w(S) & P_{L(m)} f(U) \\ g(S) & h(U) \end{bmatrix}.$$

Note that

$$A = \begin{bmatrix} w(T)P_{H(m)} & f(T(m))P_L(m) \\ g(S) & h(U) \end{bmatrix}$$

(Clearly $A : H^2 \oplus L^2 \rightarrow L(m) \oplus L^2$.)

Using the commutant lifting theorem, one may prove the following:

Proposition 1 *Notation as above. Then* $\|A\| = \mu$.

Thus in order to solve the four block problem we are required to compute the norm of the operator A.

In order to do this, we will first need to identify the essential norm of A (denoted by $\|A\|_e$). σ_e will denote the essential spectrum, and $A(\overline{D})$ will stand for the set of analytic functions on D which are continuous on the closed disc $\overline{D}$. We can now state the following result [11]:

Theorem 1 *Notation as above. Let* $w, f, g, h \in A(\overline{D})$*, and set*

$$\alpha := \max\{\| \begin{bmatrix} w(\zeta) & f(\zeta) \\ g(\zeta) & h(\zeta) \end{bmatrix} \| : \zeta \in \sigma_e(T)\} \quad (2)$$

$$\beta := \max\{\| \begin{bmatrix} 0 & 0 \\ g(\zeta) & h(\zeta) \end{bmatrix} \| : \zeta \in \partial D\} \quad (3)$$

$$\gamma := \sup\{\| \begin{bmatrix} f(\zeta) \\ h(\zeta) \end{bmatrix} \| : \zeta \in \partial D\}. \quad (4)$$

Then

$$\|A\|_e = \max(\alpha, \beta, \gamma). \quad (5)$$

2.2 Singular System

In this section, we will study the invertibility of certain skew Toeplitz operators [5] which occur as the basic elements in our procedure for computing the norm and singular values of the four block operator. We will show that the calculation of the singular values of the four block operator A amounts to inverting two ordinary Toeplitz operators, and essentially inverting an associated skew Toeplitz operator. The Fredholm conditions on the invertibility of the skew Toeplitz operator (which is essentially invertible), and the coupling between the various systems (expressed as **matching conditions**) constitutes a certain linear system of equations called the **singular system** which allows one to determine the invertibility of A.

Let $\rho > \max(\alpha, \beta, \gamma)$. Note that when $\|A\| > \|A\|_e$, $\|A\|^2$ is an eigenvalue of AA^*. By slight abuse of notation, ζ will denote a complex variable as well as an element of ∂D (the unit circle). The context will always make the meaning clear. Of course, if $\zeta \in \partial D$, then $\overline{\zeta} = 1/\zeta$.

We take w, f, g, h to be rational, and so we can express $w = a/q$, $f = b/q$, $g = c/q$, $h = d/q$, where a, b, c, d, q are polynomials of degree $\leq n$. Then we have that

$$A := \begin{bmatrix} P_{L(m)}(\frac{a}{q})(S) & P_{L(m)}(\frac{b}{q})(U) \\ (\frac{c}{q})(S) & (\frac{d}{q})(U) \end{bmatrix}.$$

Now ρ^2 is an eigenvalue of AA^* if and only if

$$\begin{bmatrix} \rho^2 q(T(m))q(T(m))^* & 0 \\ 0 & \rho^2 q(U)q(U)^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \tag{6}$$

$$\begin{bmatrix} P_{L(m)}a(S) & P_{L(m)}b(U) \\ c(S) & d(U) \end{bmatrix} \begin{bmatrix} a(S)^*P & Pc(U)^* \\ b(U)^*P_L(m) & d(U)^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

for some non-zero

$$\begin{bmatrix} u \\ v \end{bmatrix} \in L(m) \oplus L^2$$

where $P : L^2 \to H^2$ denotes orthogonal projection.

Set

$$u_+ := Pu, \;\; u_- := (I - P)u$$

and

$$v_+ := Pv, \;\; v_- := (I - P)v, \;\; v_{++} := (I - P_{H(m)})v.$$

Then we can write (6) equivalently as

$$\begin{bmatrix} \rho^2 q(T(m))q(T(m))^* - b(T(m))b(T(m))^* & -b(T(m))P_{L(m)}d(U)^* \\ -d(U)b(T(m))^* & \rho^2 q(U)q(U)^* - d(U)d(U)^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{7}$$

$$- \begin{bmatrix} a(T)a(T)^* & a(T)P_{H(m)}c(S)^* \\ c(S)a(T)^* & c(S)c(S)^* \end{bmatrix} \begin{bmatrix} u_+ \\ v_+ \end{bmatrix} = 0$$

Let $V := U^*|L^2 \ominus H^2$. If we apply $(I - P)$ to both rows of (7), we see that the basic block operator applied to

$$\begin{bmatrix} u_- \\ v_- \end{bmatrix}$$

is

$$C_- \begin{bmatrix} u_- \\ v_- \end{bmatrix} := \tag{8}$$

$$\begin{bmatrix} \rho^2 q(V^*)q_*(V) - b(V^*)b_*(V) & -b(V^*)d_*(V) \\ -d(V^*)b_*(V) & \rho^2 q(V^*)q_*(V) - d(V^*)d_*(V) \end{bmatrix} \cdot$$

$$\begin{bmatrix} u_- \\ v_- \end{bmatrix}.$$

Next applying $(I - P_{H(m)})$ to both rows of (7), we see that the basic operator applied to v_{++} is

$$C_{++}\overline{m}v_{++} := P\{(\rho^2|q|^2 - |c|^2 - |d|^2)\overline{m}v_{++}\}. \tag{9}$$

Finally, applying $P_{H(m)}$ to (7), we derive that the basic operator applied to

$$\begin{bmatrix} u_+ \\ P_{H(m)}v_+ \end{bmatrix}$$

is

$$C_+ \begin{bmatrix} u_+ \\ P_{H(m)}v_+ \end{bmatrix} := \tag{10}$$

$$\begin{bmatrix} \rho^2 q(T)q(T)^* - b(T)b(T)^* - a(T)a(T)^* & -b(T)P_{H(m)}d(S)^* \\ -d(T)b(T)^* & \rho^2 q(T)q(T)^* - d(T)d(T)^* - c(T)c(T)^* \end{bmatrix} \cdot$$

$$\begin{bmatrix} u_+ \\ P_{H(m)}v_+ \end{bmatrix}.$$

The operators C_-, C_{++}, C_+ are all skew Toeplitz (see [5] for the precise definition). In [7], [8], [11], [14] we show how to invert C_- and C_{++} under the assumption $\rho > \|A\|_e$, and how to essentially invert C_+.

Based on the inversion formulae of C_-, C_{++}, and the essential inversion of the skew Toeplitz operator C_+, the singular system can be constructed and the following theorem can be proved:

Theorem 2 *There exists an explicitly computable* $5n \times 5n$ *Hermitian matrix* $M(\rho)$ *such that* $\overline{\rho} > \max\{\alpha, \beta, \gamma\}$ *is a singular value of the four block operator* A *if and only if*

$$\det M(\overline{\rho}) = 0$$

Remarks. See [11] for the precise formula for $M(\rho)$, and [14], [38] for the coresponding formula in the MIMO case. Notice that the size of matrix $M(\rho)$ only depends on the MacMillan degree of the weighting filters and not on the plant. Hence we can solve the H^∞ problem for distributed systems. Computer code for carrying out the whole procedure has been written at the Systems Research Center of Honeywell.

2.3 Optimal Compensators

The above method also gives a way of computing the optimal compensator in a given four block problem. Indeed, from the above determinantal formula one can compute the Schmidt pair ψ, η corresponding to the singular value $s := \|A\|$ when $s > \|A\|_e$. We will indicate how one derives the optimal interpolant (and thus the opimal compensator) from these Schmidt vectors. In order to do this, notice

$$A\psi = s\eta$$

Thus, there exists $q_{opt} \in H^\infty$ with

$$(w - q_{opt})\psi_1 + f\psi_2 = s\eta_1$$

$$g\psi_1 + h\psi_2 = s_k\eta_2$$

where

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

One can show, that $\psi_1 \neq 0$, so that

$$q_{opt} = w - \frac{s\eta_1 - f\psi_2}{\psi_1}$$

Note from q_{opt}, using the Youla parametrization, we can derive the corresponding optimal controller in a given systems design problem. See also [25] for an extension of the theory of Adamjan-Arov-Krein (valid for the Hankel operator) to the singular values of the four block operator and their relationship to more general interpolation and distance problems.

2.4 Suboptimal Compensators

We would like to describe now our techniques for constructing suboptimal compensators for distributed plants. Indeed, using the one-step dilation procedure, we have given a way of explicitly parametrizing all the suboptimal controllers for such generalized interpolation problems [10]. We have also studied in this context finite dimensional suboptimal controllers for infinite dimensional systems in [18], [34], [39]. Such techniques are of course very important in the practical implementation of distributed controllers.

For simplicity, we will consider, the sensitivity minimization problem. Thus we are required to find to find an internally stabilizing controller C such that the following optimum performance is achieved

$$\inf_{C \text{ stabilizing}} \|W(1+PC)^{-1}\|_\infty =: \mu,$$

where P is the plant to be controlled and W is the weight modelling the disturbances. Assuming that the weight is an outer function and the plant P is stable we can transform this problem to a Nehari problem (in the usual way of first invoking the Youla parametrization for the controller $C = Q_c(1-PQ_c)^{-1}$, $Q_c \in H^\infty$, $(1-PQ_c) \neq 0$; and then finding an inner/outer factorization for $P = mP_o$, where m is inner and P_o is outer):

$$\mu = \inf_{Q \in H^\infty} \|W - mQ\|_\infty.$$

Now given a tolerance $\epsilon > 0$, we say that C_ϵ is *suboptimal*, (or *approximately optimal*), with tolerance ϵ, if it internally stabilizes the system and satisfies the bound

$$\|W(1+PC_\epsilon)^{-1}\|_\infty \leq \mu + \epsilon =: \rho.$$

We are thus led to consider the following problem: given $\rho \geq \mu$, find the set of all $Q \in H^\infty$ such that

$$\|W - mQ\|_\infty \leq \rho. \tag{11}$$

(Notice that from such a suboptimal interpolant Q, we can solve for the corresponding suboptimal controller.)

In [10], we give a parametrization of all such suboptimal interpolants. Indeed, suppose that the weight is rational: $W(z) = p(z)/q(z)$ where $p(z) = p_0 + zp_1 + \ldots + z^n p_n$ and $q(z) = q_0 + zq_1 + \ldots + z^n q_n$, (i.e., n is the maximum of the degrees of p and q, so some of the above coefficients may well be zero). Let S denote the unilateral shift on H^2 and define the space $H(m) = H^2 \ominus mH^2$. Then *the compressed shift* associated with $H(m)$ is defined as $T := P_{H(m)}S|_{H(m)}$, where $P_{H(m)}$ denotes orthogonal projection.

First, consider the optimal case: $\rho = \mu$. The optimal interpolant Q_{opt}, which makes $\|B_{opt}\|_\infty = \rho$, where

$$B_{opt} = W - mQ_{opt},$$

can be computed using Sarason's theorem which states that

$$\mu = \|W(T)\|, \quad W(T) := p(T)q(T)^{-1}.$$

The *essential norm* can be defined as follows:

$$\|W(T)\|_e = \sup\{|W(\zeta)| \ : \ \zeta \text{ singular point of } m\}.$$

Assuming that $\mu > \|W(T)\|_e$, we have that $W(T)$ attains its norm at a singular value $\rho = \mu$. In this case there exists a singular vector h_o, for the *skew Toeplitz* operator

$$A_\rho := \rho^2 q(T)q(T)^* - p(T)p(T)^*$$

(* denotes adjoint) which makes

$$A_\rho h_o = 0.$$

The vector h_o can be computed explicitly from the problem data $W = p/q$ and m in terms of a determinantal formula; see [7], [10], and [18]. Then B_{opt} can be found via Sarason's result as

$$B_{opt} = \rho^2 \frac{q(T)^* h_o}{p(T)^* h_o}.$$

Let us now consider the case where: $\rho > \mu$. It is obvious that in this case A_ρ is invertible and its inverse can be computed explicitly; again, the formula is given in [10]. This is going to be used in the characterization of all the suboptimal solutions $Q_s \in H^\infty$ which make

$$\|W - mQ_s\|_\infty \leq \rho. \tag{12}$$

This characterization is obtained using the one step extension procedure of Adamjan-Arov-Krein. Here we want to summarize the method briefly. Set $m_v(z) := zm(z)$ and let T_v denote the compression of S to $H(m_v) =$

$H(m) \oplus \mathbf{C}z$. For $\alpha \in \mathbf{C}$ fixed, the problem of finding $B_{opt}(z,\alpha) = (W - \alpha m - m_v Q_{opt}^{\alpha})(z)$ such that

$$\|B_{opt}(\cdot,\alpha)\|_{\infty} = \|(W - \alpha m)(T_v)\| = \rho$$

can be solved using the technique described above for the optimal case. From one step extension theory, we know that the set of all such $\alpha \in \mathbf{C}$ form a circle, say Γ. Furthermore, the equation of Γ can be explicitly calculated. Then the set of all suboptimal solutions $Q_s \in H^{\infty}$ satisfying (12) is obtained in terms of $B_{opt}(z, \phi(u))$:

$$W - mQ_s = B_{opt}(z, \phi(u)),$$

where $\phi(z)$ is a linear fractional map taking the unit circle to Γ, and $u \in H^{\infty}$, $\|u\|_{\infty} \leq 1$ is the free parameter. The explicit characterization is as follows. Set

$$g_1 := (\rho^2 q(T) P_{H(m)} q(S)^* - p(T) P_{H(m)} p(S)^*) m,$$
$$g_2 := q_0 p(T)(1 - m\, m(0)),$$

and

$$h_1 := A_{\rho}^{-1} g_1, \quad h_2 := A_{\rho}^{-1} g_2.$$

For a given $\alpha \in \Gamma$ define

$$h_{\alpha}(z) := m(z) - h_1(z) - \overline{\alpha} h_2(z),$$

and

$$B(z,\alpha) := \frac{\rho^2 q(S)^* h_{\alpha}}{p(S)^* h_{\alpha} - \overline{\alpha} q_0}.$$

Then we have the following result.

Theorem 3 *The set of all functions of the form*

$$B(z) = W(z) - m(z) Q_s(z)$$

with $Q_s \in H^{\infty}$, such that $\|B\|_{\infty} \leq \rho$, is given by

$$\{B(z,\alpha) = \frac{\rho^2 q(S)^* h_{\alpha}}{p(S)^* h_{\alpha} - \overline{\alpha} q_0} |_{\overline{\alpha} = \eta + r/u}, u \in H^{\infty} \|u\|_{\infty} \leq 1\}$$

where r and η are certain explicitly computable constants. (See [10] and [18] for the formulae.)

From the above parametrization can obtain the structure of all suboptimal H^∞ controllers. Using the notation of Theorem 3, we set $B_\alpha(z) := B(z,\alpha)$. We can find the controller from $C = Q_c(1 - PQ_c)^{-1}$, the Youla parametrization, where Q_c is such that

$$B_\alpha = W - PWQ_c.$$

Therefore,

$$C = P^{-1}(B_\alpha^{-1}W - 1).$$

We now study B_α,

$$B_\alpha(z) = \frac{\rho^2\tilde{q}(z)h_\alpha(z) - \rho^2 h_q(z)}{\tilde{p}(z)h_\alpha(z) - h_p(z) - q_0 z^n\overline{\alpha}}$$

where $h_q(z)$ and $h_p(z)$ are polynomials of degree $\leq n-1$ and $\tilde{q}(z) = z^n q(z^{-1})$, similarly $\tilde{p}(z) = z^n p(z^{-1})$. Then,

$$P_oC = \frac{1}{m}\left(\frac{\tilde{p}(z)h_\alpha(z) - h_p(z) - q_0 z^n\overline{\alpha}}{\rho^2\tilde{q}(z)h_\alpha(z) - \rho^2 h_q(z)}\ \frac{p(z)}{q(z)} - 1\right)$$

$$= \frac{1}{m}\left(\frac{-\lambda(z)h_\alpha(z) - p(z)h_p(z) + \rho^2 q(z)h_q(z) - q_0 z^n p(z)\overline{\alpha}}{(\rho^2\tilde{q}(z)h_\alpha(z) - \rho^2 h_q(z))q(z)}\right),$$

where $\lambda(z) = \rho^2\tilde{q}(z)q(z) - \tilde{p}(z)p(z)$. Recall that $h_\alpha(z) = m(z) - h_1(z) - \overline{\alpha}h_2(z)$. It is easy to see from the inversion of the skew Toeplitz operator A_ρ, that h_1 and h_2 have the following form:

$$h_1(z) = \frac{f_1(z) + m(z)F_1(z)}{\lambda(z)},$$

and

$$h_2(z) = \frac{f_2(z) + m(z)F_2(z)}{\lambda(z)}$$

for some f_1, F_1, f_2, F_2 polynomials of degree $\leq 2n$. This leads us to the following expression:

$$P_oC = \left(\frac{-\lambda(z)}{\rho^2 q(z)\tilde{q}(z)}\right)\frac{G_u(z)}{1 + m(z)G_u(z)}$$

where

$$G_u(z) = \tilde{q}(z)\frac{F_\alpha(z) - \lambda(z)}{\tilde{q}(z)f_\alpha(z) + h_q(z)\lambda(z)}$$

$$F_\alpha(z) := F_1(z) + \overline{\alpha} F_2(z), \text{ and } f_\alpha(z) := f_1(z) + \overline{\alpha} f_2(z).$$

and

$$\overline{\alpha} = \frac{r}{u} + \eta.$$

Note that

$$\frac{-\lambda(z)}{\rho^2 q(z)\tilde{q}(z)} = \frac{p(z)\tilde{p}(z)}{\rho^2 q(z)\tilde{q}(z)} - 1 = \frac{W(z)W(z^{-1})}{\rho^2} - 1.$$

We summarize the above formulae with the following.

Corollary 1 *The set of all controllers which internally stabilize the plant P, and satisfy the bound*

$$\|W(1+PC)^{-1}\|_\infty \leq \rho$$

for $\rho \geq \mu$*, have the form*

$$C(z) = \left(\frac{W(z)W(z^{-1})}{\rho^2} - 1\right) \frac{G_u(z)}{1 + m(z)G_u(z)} P_o^{-1}(z)$$

$u \in H^\infty$, $\|u\|_\infty \leq 1$, *where* $G_u(z)$ *is a linear fractional transformation in the free parameter* u*:*

$$G_u(z) = \frac{\varphi_1(z) + \varphi_2(z)u}{\varphi_3(z) + \varphi_4(z)u}$$

with $\varphi_1, \ldots, \varphi_4$ *polynomials of degree* $\leq 3n$*. They can be computed explicitly from the equations given in [10] and [18] via* f_1, F_1, f_2, F_2, r *and* η*.* □

In [18], [34], [39], we discuss how to employ these methods in finding finite dimensional suboptimal controllers for distributed systems. In particular, in [34] these techniques are used for the mixed sensitivity design of a flexible beam modelled by the Euler-Bernoulli equation. (This work is being continued under AFOSR-90-0024, and a full journal paper is being prepared on this subject.)

2.5 Other Results in H^∞ Control

We have also verified a formula which combines the state and frequency approaches to H^∞-optimal design in the one block (sensitivity minimization) case [2]. Indeed, the formula utilizes the fact that the weighting filter is taken to be rational while the plant may be distributed. Work on our new contract AFOSR-90-0024 is being continued to extend this to the more general two and four block frameworks. This type of formula combines the advantages of both the state space (Ricatti equation) and frequency domain (input/output) approaches to H^∞ theory.

Moreover in [25], we have given a precise interpolation theoretic description of the singular values of the four block operator generalizing the classical results of Adamjan-Arov-Krein. We believe this type of result should prove very useful in approximation theory (along the lines of the present applications of the Adamjan-Arov-Krein result).

A general description of our interpolation operator theoretic approach to distributed parameter systems is given in [4] and [24]. A monograph being written jointly by J. Doyle, B. Francis, and A. Tannenbaum (which was started under AFOSR-88-0020 and which will be completed under AFOSR-90-0024), *Feedback Control Theory* (to be published by MacMillan), describes from a basic point of view some of the uses of interpolation techniques in robust control theory.

3 Nonlinear Systems

A large portion of the work on AFOSR-88-0020 has been devoted to robust nonlinear design. This work is being carried over into AFOSR-90-0024 as well. We believe that we now have a viable nonlinear generalization of the (linear) H^∞ theory in both the weighted sensitivity minimization (one block) and mixed sensitivity (two block) frameworks. This is based on a **iterative commutant lifting theorem** [12], [31], [35] which gives an explicit design procedure for nonlinear systems and captures the H^∞-control problem for a large class of nonlinear plants. We have also defined a notion of *rationality* for nonlinear systems, and we have proven that the iterative commutant lifting procedure produces rational controllers (in this nonlinear sense) if we start from rational data [9]. We have thus been able to write computer code for this procedure along the lines that was done for the four block problem using the theory of skew Toeplitz operators.

We would like to explain a bit our approach to nonlinear H^∞. We will concentrate below on the one block problem. Extensions have already gone through to the mixed sensitivity (two block) problem.

3.1 Analytic Input/Output Mappings

In order to carry out our extension of H^∞ synthesis theory to nonlinear systems, we will need to first discuss a few standard results about analytic mappings on Hilbert spaces.

Let G and H denote complex Hilbert spaces. Set

$$B_{r_o}(G) := \{g \in G : \|g\| < r_o\}$$

(the open ball of radius r_o in G about the origin). Then we say that a mapping $\phi : B_{r_o}(G) \to H$ is *analytic* if the complex function $(z_1, \dots, z_n) \mapsto \langle (z_1 g_1 + \dots + z_n g_n), h\rangle$ is analytic in a neighborhood of $(1, 1, \dots, 1) \in \mathbf{C}^n$ as a function of the complex variables $z_1, \dots, z_n$ for all $g_1, \dots, g_n \in G$ such that $\|g_1 + \dots + g_n\| < r_o$, for all $h \in H$, and for all $n > 0$. (Note that we denote the Hilbert space norms in G and H by $\|\ \ \|$ and the inner products by $\langle , \rangle$.)

We will now assume that $\phi(0) = 0$. It is easy to see that if $\phi : B_{r_o}(G) \to H$ is analytic, then ϕ admits a convergent Taylor series expans ion, i.e.

$$\phi(g) = \phi_1(g) + \phi_2(g, g) + \cdots + \phi_n(g, \cdots, g) + \cdots$$

where $\phi_n : G \times \cdots \times G \to H$ is an n-linear map.

Clearly, without loss of generality we may assume that the n-linear map $(g_1, \cdots, g_n) \to \phi(g_1, \cdots, g_n)$ is symmetric in the arguments $g_1, \cdots, g_n$. For ϕ a Volterra series, ϕ_n is basically the n^{th}-Volterra kernel.

Now set

$$\hat{\phi}_n(g_1 \otimes \cdots \otimes g_n) := \phi_n(g_1, \cdots, g_n).$$

Then $\hat{\phi}_n$ extends in a unique manner to a dense set of $G^{\otimes n} := G \otimes \dots \otimes G$ (tensor product taken n times). Notice by $G^{\otimes n}$ we mean the Hilbert space completion of the algebraic tensor product of the G's. Clearly if $\hat{\phi}_n$ has finite norm on this dense set, then $\hat{\phi}_n$ extends by continuity to a bounded linear operator $\hat{\phi}_n : G^{\otimes n} \to H$. By abuse of notation, we will set $\phi_n := \hat{\phi}_n$.

We now make the following definitions.

Definitions 1.

(i) Notation as above. By a *majorizing sequence* for the holomorphic map ϕ, we mean a positive sequence of numbers α_n $n = 1, 2, \dots$ such that $\|\phi_n\| < \alpha_n$

for $n \geq 1$. Suppose that $\rho := \limsup \alpha_n{}^1/n < \infty$. Then it is completely standard that the Taylor series expansion of ϕ converges at least on the ball $B_r(G)$ of radius $r = 1/\rho$.

(ii) If ϕ admits a majorizing sequence as in (i), then we will say that ϕ is *majorizable*.

3.2 Nonlinear Control Problem

We will describe in this section the control problem in which we are interested. First, we will need to consider the precise kind of input/output operator we will be considering.

We assume that all of the operators we consider are causal and majorizable. $H^2(\mathbf{C^k})$ denotes the standard Hardy space of $\mathbf{C^k}$-valued functions on the unit circle (k may be infinite, i.e., in this case $\mathbf{C^k}$ is replaced by h^2, the space of one-sided square summable sequences). We now make the following definition:

Definition 2.

Let $S : H^2(\mathbf{C^k}) \rightarrow H^2(\mathbf{C^k})$ denote the canonical unilateral right shift. Then we say an input/output operator ϕ is *locally stable* if it is causal and majorizable, $\phi(0) = 0$, and if there exists an $r > 0$ such that $\phi : B_r(H^2(\mathbf{C^k})) \rightarrow H^2(\mathbf{C^k})$ with $S\phi = \phi \circ S$ on $B_r(H^2(\mathbf{C^k}))$. We set

$$C_l := \{\text{space of locally stable operators}\}.$$

Since the theory we are considering is local, the notion of local stability is sufficient for all of the applications we have in mind.

The theory we are about to give holds for all plants which admit coprime locally stable factorizations. However, for simplicity we will assume that our plant is also locally stable. Accordingly, let P, W denote locally stable operators, with W invertible. As before, P represents the plant, and W the weight or filter on the set of disturbances whose energy is bounded by a fixed constant. Now we say that the feedback compensator C *locally stabilizes* the closed loop if the operators $(I + P \circ C)^{-1}$ and $C \circ (I + P \circ C)^{-1}$ are well-defined and locally stable. One can show that C locally stabilizes the closed loop if and only if

$$C = \hat{q} \circ (I - P \circ \hat{q})^{-1} \tag{13}$$

for some $\hat{q} \in C_l$. Notice then that the weighted sensitivity $(I + P \circ C)^{-1} \circ W$ can be written as $W - P \circ q$, where $q := \hat{q} \circ W$. (Since W is invertible, the data q and $\hat{q}$ are equivalent.) In this context, we will call such a q, a *compensating parameter*. Note that from the compensating parameter q, we get a locally stabilizing compensator C via the formula (13).

The problem we would like to solve here, is a version of the classical disturbance attenuation problem. This of course corresponds to the "minimization" of the "sensitivity" $W - P \circ q$ taken over all locally stable q. In order to formulate a precise mathematical problem, we need to say in what sense we want to minimize $W - P \circ q$. This we will do in the next section where we will propose a notion of "sensitivity minimization" which we seems quite natural to analytic input/output operators.

3.3 Nonlinear Sensitivity Function

We now define a fundamental object, namely a nonlinear version of *sensitivity*. We will see that while the optimal H^∞ sensitivity is a real number in the linear case, the measure of performance which seems to be more natural in this nonlinear setting is a certain function defined in a real interval. This new kind of performance criterion is one of the keys concepts developed in [9] and [12].

In order to define our notion of sensitivity, we will first have to partially order germs of analytic mappings. All of the input/output operators here will be locally stable. We also follow here our convention that for given $\phi \in C_l$, ϕ_n will denote the bounded linear map on the tensor space $(H^2(\mathbf{C}^{\mathbf{k}}))^{\otimes n}$ associated to the n-linear part of ϕ which we also denote by ϕ_n (and which we always assume without loss of generality is symmetric in its arguments). The context will always make the meaning of ϕ_n clear.

We can now state the following definitions:

Definitions 3.

(i) For $W, P, q \in C_l$ (W is the weight, P the plant, and q the compensating parameter), we define the *sensitivity function* $S(q)$,

$$S(q)(\rho) := \sum_{n=1} \rho^n \|(W - P \circ q)_n\|$$

for all $\rho > 0$ such that the sum converges. Notice that for fixed P and W, for each $q \in C_l$, we get an associated sensitivity function.

(ii) We write $S(q) \preceq S(\tilde{q})$, if there exists a $\rho_o > 0$ such that $S(q)(\rho) \leq S(\tilde{q})(\rho)$ for all $\rho \in [0, \rho_o]$. If $S(q) \preceq S(\tilde{q})$ and $S(\tilde{q}) \preceq S(q)$, we write $S(q) \cong S(\tilde{q})$. This means that $S(q)(\rho) = S(\tilde{q})(\rho)$ for all $\rho > 0$ sufficiently small, i.e., $S(q)$ and $S(\tilde{q})$ are equal as germs of functions.

(iii) If $S(q) \preceq S(\tilde{q})$, but $S(\tilde{q}) \not\preceq S(q)$, we will say that q *ameliorates* $\tilde{q}$. Note that this means $S(q)(\rho) < S(\tilde{q})(\rho)$ for all $\rho > 0$ sufficiently small.

Now with Definitions 3, we can define a notion of "optimality" relative to the sensitivity function:

Definitions 4.

(i) $q_o \in C_l$ is called *optimal* if $S(q_o) \preceq S(q)$ for all $q \in C_l$.

(ii) We say $q \in C_l$ is *optimal with respect to its n-th term* q_n, if for every n-linear $\hat{q}_n \in C_l$, we have

$$S(q_1 + \ldots + q_{n-1} + q_n + q_{n+1} \ldots) \preceq S(q_1 + \ldots + q_{n-1} + \hat{q}_n + q_{n+1} + \ldots).$$

If $q \in C_l$ is optimal with respect to all of its terms, then we say that it is *partially optimal.*

3.4 Iterative Commutant Lifting Method

In this section, we discuss the main construction of this paper from which we will derive both partially optimal and optimal compensators relative to the sensitivity function given in Definitions 3 above. As before, P will denote the plant, and W the weighting operator, both of which we assume are locally stable. As in the linear case, we always suppose that P_1 is an isometry, i.e. P_1 is **inner.**

We begin by noting the following key relationship:

$$(W - P \circ q)_k = W_k - \sum_{1 \leq j \leq k} \sum_{i_1 + \ldots + i_j = k} P_j(q_{i_1} \otimes \ldots \otimes q_{i_j})$$

Note that once again for ϕ locally stable, ϕ_n denotes the n-linear part of ϕ, as well as the associated linear operator on the appropriate tensor space.

We are now ready to formulate the *iterative commutant lifting procedure.* Let $\Pi : H^2(\mathbf{C^k}) \to H^2(\mathbf{C^k}) \ominus P_1 H^2(\mathbf{C^k})$ denote orthogonal projection. Using the linear commutant lifting theorem (CLT), we may choose q_1 such that

$$\|W_1 - P_1 q_1\| = \|\Pi W_1\|.$$

Now given this q_1, we choose (using CLT) q_2 such that

$$\|W_2 - P_2(q_1 \otimes q_1) - P_1 q_2\| = \|\Pi(W_2 - P_2(q_1 \otimes q_1))\|.$$

Inductively, given $q_1, \ldots, q_{n-1}$, set

$$A_n := (W_n - \sum_{2 \le j \le n} \sum_{i_1 + \ldots + i_j = n} P_j(q_{i_1} \otimes \ldots \otimes q_{i_j}))$$

for $n \ge 2$. Then from the CLT, we may choose q_n such that

$$\|A_n - P_1 q_n\| = \|\Pi A_n\|. \tag{14}$$

We now come to the key point on the convergence of the iterative commutant lifting method.

Proposition 2 *With the above notation, let $q^{(1)} := q_1 + q_2 + \ldots$. Then $q^{(1)} \in C_l$.*

Note that given any $q \in C_l$, we can apply the iterative commutant lifting procedure to $W - P \circ q$. Now set

$$S_\Pi(q)(\rho) := \sum_{n=1} \rho^n \|\Pi(W - P \circ q)_n\|.$$

Clearly, $S_\Pi(q) \le S(q)$ (as functions). We can now state the following result whose proof is immediate from the above discussion:

Proposition 3 *Given $q \in C_l$, there exists $\tilde{q} \in C_l$, such that $S(\tilde{q}) \equiv S_\Pi(q)$. Moreover $\tilde{q}$ may be constructed from the iterated commutant lifting procedure.*

Moreover, we easily have the following result:

Proposition 4 *q is partially optimal if and only if $S(q) \cong S_\Pi(q)$.*

Finally, from the above discussion, we can prove:

Theorem 4 *For given P and W as above, any $q \in C_l$ is either partially optimal or can be ameliorated by a partially optimal compensating parameter.*

It is important to emphasize that a partially optimal compensating parameter need not be optimal in the sense of Definition 4(i). Basically, what we have shown here is that using the iterated commutant lifting procedure, we can ameliorate any given design. The question of optimality will be considered next.

3.5 Optimal Nonlinear Compensators

We will mention now some of our results about optimal compensators. Basically, we want to give conditions when the the iterated commutant lifting procedure leads to an optimal design. (We have seen that it always leads to a partially optimal one.) We begin with the following very general result from [12]:

Theorem 5 *There always exist optimal compensators.*

In fact, in several very important cases, we can use the iterated commutant lifting procedure to explicitly construct optimal compensators. We now quote one of these cases from [12]:

Theorem 6 *Let P and W be SISO, locally stable, with the linear part of P rational. Then the partially optimal compensating parameter q_{opt} constructed by the iterated commutant lifting procedure is optimal.*

In collaboration with colleagues at Honeywell, we have been applying these methods to certain systems with input saturations (e.g., a saturated double integrator) in order to attenuate bounded energy output disturbances [35]. Much more work is being performed in this direction which will be part of AFOSR-90-0024.

3.6 Nonlinear Beurling-Lax Theory

There were several theoretical results in the area of nonlinear systems during the AFOSR-88-0020 funding which we mention briefly. These were based on a **nonlinear Beurling-Lax-Halmos theorem** [6]. The linear Beurling-Lax-Halmos theorem characterizes the shift-invariant closed subspaces of a complex separable Hilbert space H.

In [6], we formulate a theory relative to locally shift-invariant nonlinear analytic manifolds in Hilbert space, and develop the corresponding representation theory. We should note the proof of the nonlinear Beurling-Lax-Halmos theorem makes use of Poincaré-Dulac type changes of variable from the theory of nonlinear ordinary differential equations in the parameter space of the manifold as well as the commutant lifting theorem.

These ideas allow us to give nonlinear analogues of the classical notion of inner-outer factorization, and should be applicable to certain problems in nonlinear realization theory, and a number of nonlinear optimization problems. They also lead to a new linearization result for very general nonlinear input/output operators, as well as nonlinear analogues of the classical Hankel and Toeplitz operators. (In AFOSR-90-0024, we will be exploring the possibilities of globalizing this theory.)

4 Multivariable Gain Margin Problem

We have found an analytic procedure for solving the multivariable gain margin problem, based on a novel interpolation scheme which we call *spectral Nevanlinna-Pick interpolation* [19], [21], [22], [27], [37]. This involves matrix interpolation in which one bounds the spectral radius, and not the norm of the interpolants as in ordinary Nevanlinna-Pick theory. (Ordinary Nevanlinna-Pick is precisely the type of mathematical problem that arises in standard H^∞ synthesis.) The spectral problem has been solved in both its matricial and tangential (directional) forms. We would like to give a few details about these ideas now.

A key observation is that the standard one block H^∞ optimal control problem in the finite dimensional case amounts to Nevanlinna-Pick interpolation (both for SISO and MIMO systems). Indeed, the connection between interpolation theory and certain questions involving LTI, finite dimensional plants is the simple fact that the problem of internal stabilization reduces to Lagrange-type interpolation for such plant. The basis of H^∞ optimization theory is that for certain design problems, one is required to interpolate on the unit disc (or equivalently, the right half plane) by analytic matrix-valued functions of bounded norm, which is precisely the Nevanlinna-Pick interpolation problem.

For SISO systems gain margin optimization works in the same way in that via a conformal equivalence, one can transform the question of finding an interpolating function whose range is a certain simply-connected subset of the complex plane to one whose range is the unit disc, and hence once more derive an interpolation problem of the Nevanlinna-Pick kind. Now for MIMO systems, we have shown that if one wants to play the same game with multivariable generalizations of gain margin, one derives interpolation not with a norm constraint but with a spectral radius constraint.

4.1 Internal Stability and Interpolation

We would first like to briefly discuss how the problem of internal stabilization for LTI, finite dimensional plants reduces to one of interpolation.

Let $P(s)$ denote a $p \times m$ LTI finite dimensional plant and $C(s)$ an $m \times p$ internally stabilizing compensator. In the usual way we define the *sensitivity function* to be $S(s) := (I + P(s)C(s))^{-1}$. Then invoking the standard coprime factorizations we get that

$$S(s) = L - UZ$$

where L and U are completely determined by $P(s)$, U is an inner matrix-valued function, and the "free parameter" $Z \in RH^{\infty}_{m \times p}$ (= the space of $m \times p$ matrices with entries which are real rational functions bounded in the right half plane H). Now one wants to compute

$$\inf\{\|S\|_{\infty} : C \text{ stabilizing compensator}\}.$$

As is well-known this problem can either be reduced to one of tangential Nevanlinna-Pick interpolation, or matricial Nevanlinna-Pick interpolation via the tranformation

$$U^{alg} S = U^{alg} L - (\det U) I Z,$$

where U^{alg} denotes the algebraic adjoint of U, and I the identity matrix.

With these preliminary remarks made we are ready to formulate our control problem which will lead to the spectral Nevanlinna-Pick interpolation. Let $P(s)$ be a $p \times m$ MIMO finite dimensional plant. Consider the following family of plants

$$P_k(s) := \{kP(s) : k \in K\}$$

$$K := \{k \in \mathbf{C} : k = 1 + s, |s| \leq r\}.$$

Then we may show that $C(s)$ internally stabilizes the closed loop for the family $P_k(s)$, $k \in K$ if and only if there exists a rational matrix-valued function $S(s)$ which is analytic and bounded in the right half plane such that $S : \hat{H} \to G$ and which moreover satisfies standard Nevanlinna-Pick interpolation conditions, where $\hat{H}$ denotes the closed right half plane $\cup \{\infty\}$, and

$$G := \{p \times p \text{ matrices } M : \det(I + kM) \neq 0, k \in K\}.$$

But now it is easy to construct a conformal equivalence (which is a linear fractional transformation) $\phi : G \to \Omega$ where Ω denotes the space of $p \times p$ matrices with spectral radius less than one. From standard conformal mapping theory, the interpolation constraints on S may be tranformed to similar constraints on $\phi \circ S : \hat{H} \to \Omega$. In other words, we have a Nevannlinna-Pick type problem in which instead of bounding the norm of the interpolant we bound its spectral radius.

We should also add that a similar trick works for real parameter variations ($k \in [a,b]$ a real interval with $0 < a < 1 < b$), and we can even take parameter variation spaces of the form

$$K := \{\text{diag}(k_1, \ldots, k_p)\}$$

where the k_j's may be either real or complex (μ in the 1×1 block case).

4.2 Solution of the Spectral Nevanlinna-Pick Problem

We now outline our solution to the spectral Nevanlinna-Pick interpolation problem. For simplicity, we will consider just the matricial case. (The tangential case is worked out in [21].) Following standard mathematical practice we work on the unit disc D rather than (its conformal equivalent) the right half plane H. Accordingly, we are given n distinct points $z_1, \ldots, z_n \in D$, and n, $p \times p$ matrices $F_1, \ldots, F_n$. We want necesssary and sufficient conditions for the existence of a rational matrix-valued function $F(z)$, analytic in the unit disc such that

$$F(z_i) = F_i$$

for $i = 1, \ldots, n$, and such that

$$\sup\{sr(F(z)) : z \in \bar{D}\} < 1, \tag{15}$$

where for a matrix Q, $sr(Q)$ denotes its spectral radius. We can now state the following theorem:

Theorem 7 *With the above notation, $F(z)$ exists if and only if there exist invertible $p \times p$ matrices M_i for $i = 1, \ldots, n$ such that*

$$\left[\frac{I - M_i F_i M_i^{-1}(M_j F_j M_j^{-1})^*}{1 - z_i \bar{z}_j}\right]_{1 \leq i,j \leq n} > 0. \tag{16}$$

We would now like to discuss a bit the derivation of Theorem 7 which will also give us a numerical scheme for the theorem's implementation. Let $\mathbf{H}$ denote a separable Hilbert space. Then a *contraction* $A : \mathbf{H} \to \mathbf{H}$ is a linear bounded operator such that $\|A\| \leq 1$. Now given two contractions T and A on $\mathbf{H}$, we set

$$\rho_T(A) := \inf\{\|MAM^{-1}\| : M \text{ is invertible and } MT = TM\}.$$

The quantity $\rho_T(A)$ is called the *T-spectral radius of A*.

Now given $m \in H^\infty$ an all-pass (inner) rational function, set $H(m) := H^2 \ominus mH^2$, and $\mathbf{H} := H(m) \oplus \ldots \oplus H(m)$ (direct sum of p copies of $H(m)$). For S the unilateral shift on H^2, and for $P_{H(m)} : H^2 \to H(m)$ orthogonal projection, let $S(m) := P_{H(m)} S | H(m)$ denote the compressed shift. We set $T := S(m) \oplus \ldots \oplus S(m)$, and $U := S \oplus \ldots \oplus S$ on $\mathbf{K} := H^2 \oplus \ldots \oplus H^2$ (direct sums of p copies). We can state the following spectral version [19] of the commutant lifting theorem:

Theorem 8 *Taking T, U, $\mathbf{H}$, $\mathbf{K}$ as just defined, let $A : \mathbf{H} \to \mathbf{H}$ be any contraction such that $AT = TA$. Then we have that*

$$\rho_T(A) :=$$

$$\inf\{sr(B) : B \text{ is a commuting dilation of } A, \text{ i.e., } BU = UB,\ P_{\mathbf{H}} B | \mathbf{H} = A P_{\mathbf{H}}\},$$

where $P_{\mathbf{H}} : \mathbf{K} \to \mathbf{H}$ denotes orthogonal projection.

Remarks. First of all, note that Theorem 7 may be derived as a corollary of Theorem 8 by taking

$$m(z) := \prod_{i=1}^{n} \frac{z - z_i}{1 - \bar{z}_i z}.$$

Next in [19], it is shown that we can always infimize over rational B. Moreover, under the hypotheses of Theorem 8 (in which we are dealing with finite block diagonal matrices), one can use gradient search procedures to compute "scalings" M arbitrarily close to the optimal. Then if one invertibly dilates such a (nearly) optimal M to an analytic matrix $M(z)$, one is required to check the validity of matrix inequality (16) for $M_i := M(z_i)$ $(i = 1, \ldots, n)$, i.e., via these scalings we are reduced to a classical matricial Nevanlinna-Pick problem.

Actually we have now even found a way to apply such methods to the **structured singular value** (in which considers much more general structured multiplicative perturbation models as in the work of Doyle and Safonov), which may lead to an analytic procedure for performing μ-synthesis. This will be described in our yearly report for AFOSR-90-0024.

5 Results in Interpolation Theory

We have also made some strides for understanding the minimal entropy solutions for a number of interpolation problems [15], [20]. Indeed, in studying the spectral properties of the four block operator and its relation to classical interpolation theory, we have been led to a strong version of the classical **Parrott's theorem**. Parrott's theorem is one of the key matrix extension results and has found numerous uses in control theory as well as signal processing. Our strengthened version in a certain sense picks out an extension which is "opposite" to the famous maximal entropy or central solution to such extension problems. This solution has a natural physical interpretation in the **waves through multi-layered media** context, and we are thus quite excited about some of the applied implications of our result. From the more theoretical side, this result can be used in generalizing some beautiful results of Adamjan-Arov-Krein on the connection of the singular values of the Hankel operator to optimal interpolation by functions with a presribed number of poles on the unit disc to the four block operator of H^∞ control.

Further, based on the strong Parrott theorem we have proven a strong version of the commutant lifting theorem [20] which leads to an explicit parametrization of minimal entropy solutions in dilation theory. This has already been applied to the classical Nevanlinna-Pick and Caratheodory interpolation problems.

6 Other Results

During the course of AF0SR-88-0020, we have have also obtained some new results on the asymptotic stability of time-varying systems [16]. Some work has also been done by the Tannenbaum's Ph.D. student, Hitay Ozbay, on L^1 optimization [41]. (Ozbay showed that the problem of minimizing the effect of the worst disturbance on the output where the signals have finite one-norm is equivalent to the problem with bounded disturbances.)

Finally, we returned to the topic of the studying the properties of global families of systems [23]. This is an area which we plan to consider in some depth in AF0SR-90-0024. Indeed, previously our approach was quite algebraic and geometric (as exhibited in [23]). We now plan to use some of the analytical operator theoretic techniques described above in studying this important problem.

7 Papers of Allen Tannenbaum and Collaborators under AFOSR-88-0020

(All the authors are listed in **alphabetical order** on the following papers.)

1. "On the Nehari problem for a certain class of L^∞-functions appearing in control, II" (with C. Foias), *Journal of Functional Analysis* **81** (1988), 207-218.

2. "Weighted sensitivity minimization: general plants in H^∞ and rational weights" (with T. Lypchuk and M. Smith), *Linear Algebra and its Applications* **109** (1988), 71-90.

3. "Optimal sensitivity theory for multivariate distributed plants" (with C. Foias), *International Journal of Control* **47** (1988), 985-992.

4. "Generalized interpolation theory in control" (with B. Francis), *Mathematical Intelligencer* **10** (1988), 48-53.

5. "On skew Toeplitz operators, I" (with H. Bercovici and C. Foias), *Operator Theory: Advances and Applications* **29** (1988), 21-45.

6. "A Poincaré-Dulac approach to a nonlinear Beurling-Lax-Halmos theorem" (with J. Ball, C. Foias, and J. W. Helton), *Journal of Math. Analysis and Applications* **139** (1989), 496-514.

7. "Some explicit formulae for the singular values of certain Hankel operators with factorizable symbol" (with C. Foias and G. Zames), *SIAM J. Mathematical Analysis* **19** (1988), 1081-1091.

8. "On the four block problem, I," (with C. Foias), *Operator Theory: Advances and Applications* **32** (1988), 93-112.

9. "Iterated commutant lifting for systems with rational symbol" (with C. Foias), *Operator Theory: Advances and Applications* **41** (1989), 255-277.

10. "On the parametrization of the suboptimal solutions in generalized interpolation" (with C. Foias), *Linear Algebra and its Applications* **124** (1989), 145-164.

11. "On the four block problem, II: the singular system" (with C. Foias), *Integral Equations and Operator Theory* **11** (1988), 726-767.

12. "Weighted sensitivity optimization for nonlinear systems" (with C. Foias), *SIAM Journal on Control and Optimization* **27** (1989), 842-860.

13. "Some remarks on optimal interpolation" (with C. Foias), *Systems and Control Letters* **11** (1988), 259-264.

14. "Skew Toeplitz approach to the H^∞-control of multivariate distributed systems" (with H. Ozbay), to appear in *SIAM J. Control and Optimization.*

15. "A strong Parrott theorem" (with C. Foias), *Proceedings of the American Mathematical Society* **106** (1989), 777-784.

16. "Control of slowly time-varying linear systems" (with E. Kamen and P. Khargonekar), *IEEE Transactions Automatic Control* **AC-34** (1989), 1283-1285.

17. "Four block problem : stable weights and rational weightings" (with P. Khargonekar and H. Ozbay), *International Journal of Control* **50** (1989), 1013-1023.

18. "On the structure of suboptimal H^∞ controllers in the sensitivity minimization problem for distributed stable plants" (with H. Ozbay), to appear in *Automatica.*

19. "A spectral commutant lifting theorem" (with H. Bercovici and C. Foias), to appear in *Transactions of the AMS.*

20. "On certain minimal entropy extensions appearing in dilation theory" (with C. Foias and A. Frazho), to appear in *Linear Algebra and Its Applications.*

21. "On spectral tangential Nevanlinna-Pick interpolation" (with H. Bercovici and C. Foias), to appear in *Journal of Math. Analysis and Applications.*

22. "The invariant subspaces of a uniform Jordan operator" (with H. Bercovici), to appear in *Journal Math. Analysis and Applications.*

23. "Invariant theory and families of systems," to appear in a volume dedicated to Rudolf Kalman on his 60-th birthday.

24. "Operator theoretic methods in robust control" (with C. Foias), to appear in *Signal Processing and Control,* IMA Series on Applied Mathematics, Springer-Verlag, 1990.

25. "On the singular values of the four block operator and certain generalized interpolation problems" (with C. Foias), in *Analysis and Partial Differential Equations,* edited by Cora Sadosky, Marcel Dekker, New York (1989), 483-493.

26. "On the synthesis of H^∞ optimal controllers for infinite dimensional plants" (with H. Ozbay), to appear in *New Trends and Applications in Distributed Parameter Control Systems,* Marcel Dekker, New York 1990.

27. "Spectral Nevanlinna-Pick interpolation theory," Proceedings of IEEE Conference on Decision and Control, Los Angeles, 1987, 1635-1638.

28. "On the spectra and invertibility of a certain class of operators in control" (with C. Foias), Proceedings of IEEE Conference on Decision and Control, Los Angeles, 1987, 1338-1342.

29. "The four block problem for distributed systems" (with C. Foias), Proceedings of the IEEE Conference on Decision and Control, Austin, Texas, 1988, 993-998.

30. "Spectral radius interpolation and robust control" (with H. Bercovici and C. Foias), Proceedings of the 28th IEEE Conference on Decision and Control, 1989, Tampa, Florida, 916-918.

31. "Remarks on H^∞-optimization of multivariate distributed systems," Proceedings of the IEEE Conference on Decision and Control, Austin, Texas, 1988, 985-987.

32. "Nonlinear H^∞ theory" (with C. Foias), to appear in *MTNS Proceedings.*

33. "Standard problem for distributed systems" (with C. Foias), to appear in *MTNS Proceedings.*

34. "Robust control design for a flexible beam using a distributed parameter H^∞ method" (with K. Lenz, B. Morton, H. Ozbay, J. Turi), Proceedings of the IEEE 28-th Conference on Decision and Control, Tampa, Florida, 1989, 2673-2678. 1989.

35. "On the nonlinear mixed sensitivity problem" (with D. Enns, C. Foias, T. Georgiou, M. Jackson, B. Schipper), Proceedings of the IEEE 28-th Conference on Decision and Control, Tampa, Florida, 1989, 986-990.

36. "Interpolation theory in robust control" (with H. Bercovici and C. Foias), to appear in IFAC Proceedings, 1990.

37. "Spectral variants of the Nevanlinna-Pick interpolation problem" (with H. Bercovici and C. Foias), *MTNS Proceedings.*

38. "A solution to the standard H^∞ problem for multivariable distributed systems" (with H. Ozbay), Proceedings of IEEE 28-th Conference on Decision and Control, Tampa, Florida, 1989, 1444-1446.

39. "On approximately optimal H^∞ controllers for distributed systems" (with H. Ozbay), Proc. of IEEE 28-th Conference on Decision and Control, Tampa, Florida, 1989, 1454-1460.

40. "Controller design for unstable distributed plants," to appear in the Proceedings of the ACC, May 1990.

41. (This paper supported under AFOSR-88-0020 was done solely by Hitay Ozbay.)

"L^1 optimal control," *IEEE Trans. Automatic Control* **AC-29**, August 1989.

Monograph

J. Doyle, B. Francis, and A. Tannenbaum, *Feedback Control Theory,* to be published by MacMillan.

Students of A. Tannenbaum Supported by AFOSR-88-0020

1. Hitay Ozbay (completed Ph.D. in August 1989)

2. Brian Schipper (half time support for masters degree)

3. Evgeny Berzon (half time support on AFOSR-88-0020 and half time on AFOSR-90-0024; he will go to full time support on AFOSR-90-0024 in the spring of 1990 for his Ph.D.)