SYSTEMS OPTIMIZATION LABORATORY
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305-4022

A Monotone Complementarity Problem in Hilbert Space

by
Jen-Chih Yao

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A Monotone Complementarity Problem in Hilbert Space

Jen-Chih Yao

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Department of Operations Research

Stanford University

Stanford, CA 94305-4022

Abstract

An existence theorem for a complementarity problem involving a weakly coercive *monotone mapping over an arbitrary closed convex cone in a real Hilbert space is* established.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let K be a nonempty subset of H and f be a mapping from K into H. f is said to be *weakly coercive* if

$$\langle x, f(x) \rangle \longrightarrow \infty \qquad \text{as } \|x\| \longrightarrow \infty \text{ and } x \in K.$$

f is said to be *monotone* if

$$\langle x - y, f(x) - f(y) \rangle \geq 0 \qquad \text{for all } x, y \in K.$$

f is said to be *strictly monotone* if the above inequality is strict whenever x and y are distinct. f is said to be *strongly monotone* if there exists a positive number c such that

$$\langle x-y, f(x)-f(y)\rangle \geq c\|x-y\|^2 \qquad \text{for all } x, y \in K.$$

A subset K of a real Hilbert space H is said to be a *cone* if $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$. Let K be a closed convex cone in H and dual cone K^*, that is,

$$K^* = \{u \in H \mid \langle u, x\rangle \geq 0, \ \forall\, x \in K\}.$$

The *complementarity problem* (CP) is to find $x \in K$ such that

$$f(x) \in K^* \quad \text{and} \quad \langle x, f(x)\rangle = 0. \tag{1}$$

Problem (1) was formulated by Karamardian [7] and has been extensively studied in the literature. See, e.g., [2, 4, 5, 6, 7, 8] and the references therein. The purpose of this paper is to prove an existence theorem for a complementarity problem involving a weakly coercive monotone mapping over an arbitrary closed convex cone in a real Hilbert space. The main result extends some existing results; the method of the proof of this main result is to consider the family of finite-dimensional subspaces by using the known results for finite-dimensional spaces and to show that a certain net of solutions from such subspaces converges to a solution to CP.

2. The Main Result

Now we prove the main result.

Theorem 2.1. *Let K be a closed convex cone in the real Hilbert space H. Let f be a weakly coercive monotone mapping from K into H which is continuous on $K \cap U$ for any* *finite-dimensional subspace U of H. Then there exists $x \in K$ such that*

$$f(x) \in K^* \text{ and } \langle x, f(x)\rangle = 0.$$

Proof. Let U be any finite-dimensional subspace of H with $K \cap U \neq \emptyset$ and let P_U be the orthogonal projection of H onto U. Let $f_U = P_U f$ be the composition of P_U and f. Since the adjoint P_U^* of P_U is itself, we have

$$\lim_{\|u\| \to \infty,\ u \in K \cap U} \langle u, f_U(u)\rangle = \infty.$$

Therefore f_U is weakly coercive on $K \cap U$. By [1, Corollary 2.3], there exists $x_U \in K \cap U$ such that

$$\langle u - x_U, f(x_U)\rangle \geq 0 \qquad \text{for all } u \in K \cap U. \tag{2}$$

Then by [7, Lemma 3.1], we have

$$f_U(x_U) \in (K \cap U)^* \text{ and } \langle x_U, f(x_U)\rangle = 0. \tag{3}$$

Let Λ be the family of all finite-dimensional subspaces U of H with $K \cap U \neq \emptyset$ and $K_U = \{x_V \mid U \subset V \in \Lambda\}$. Since f is weakly coercive, it follows from (3) that there exists a constant $r > 0$ so that $K_U \subset \bar{B}_r$ for all $U \in \Lambda$ where $\bar{B}_r$ is the closure of the ball with center at 0 and radius r. For $U \in \Lambda$, let $\bar{K_U}^{\omega}$ be the weak closure of K_U. Then the family $\{\bar{K_U}^{\omega} \mid U \in \Lambda\}$ has the finite intersection property. Indeed, for $U, V \in \Lambda$, let $W \in \Lambda$ be such that $U \cup V \subset W$. Then $K_U \cap K_V \supset K_W \neq \emptyset$. Since $\bar{B}_r$ is weakly compact and $\bar{K_U}^{\omega} \subset \bar{B}_r$ for all $U \in \Lambda$, it follows that $\bigcap_{U \in \Lambda} \bar{K_U}^{\omega} \neq \emptyset$.

Let $x \in \bigcap_{U \in \Lambda} \bar{K_U}^{\omega}$. Suppose $u \in K$ is arbitrary and let $U \in \Lambda$ contain u. Since K_U is bounded and $x \in \bar{K_U}^{\omega}$, there exists a sequence $\{x_n\} \subset K_U$ which converges to x weakly. Since f is monotone, by (2) we have

$$\langle u - x_n, f(u)\rangle \geq 0 \qquad \text{for all } n.$$

Since $\langle u - \cdot, f(u)\rangle$ is weakly continuous, we have

$$\langle u - x, f(u)\rangle \geq 0 \qquad \text{for all } u \in K.$$

For any $u \in K$ and any $0 < t \leq 1$, let $u_t = tu + (1-t)x$. By substituting u_t into (2), we have

$$\langle u_t - x, f(u_t)\rangle \geq 0 \qquad \text{for all } 0 < t \leq 1. \tag{4}$$

Letting t approach 0 in (4), we get

$$\langle u - x, f(x)\rangle \geq 0 \qquad \text{for all } u \in K. \tag{5}$$

By (5) and [7, Lemma 3.1], it follows that

$$f(x) \in K^* \text{ and } \langle x, f(x)\rangle = 0.$$

The next corollary follows from Theorem 2.1 directly.

Corollary 2.2. *Let K be a closed convex cone in the real Hilbert space H. Let f be a mapping from K into H which is continuous on $K \cap U$ for any finite-dimensional subspace U of H. Then there exists a unique $x \in K$ such that*

$$f(x) \in K^* \textit{ and } \langle x, f(x)\rangle = 0$$

under each of the following conditions:

1. *f is strictly monotone and weakly coercive,*

2. *f is strongly monotone.*

We note that Corollary 2.2.2 extends a result of Nanda and Nanda [8, Theorem] where f is assumed to be strongly monotone and Lipschitzian.

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An existence theorem for a complementarity problem involving a weakly coercive monotone mapping over an arbitrary closed convex cone in a real Hilbert space is established.