NSWC TR 89-109

SEMIANALYTIC SATELLITE THEORY

BY
CLAUS OESTERWINTER
STRATEGIC SYSTEMS DEPARTMENT

JULY 1989

NAVAL SURFACE WARFARE CENTER
Dahlgren, Virginia 22448-5000 • Silver Spring, Maryland 20903-5000

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FOREWORD

The work described in this report was performed as research and development funded by the SLBM Research and Analysis Division (K40) and the Space and Surface Systems Division (K10) of the Strategic Systems Department.

The development of this satellite theory resulted from the joint efforts of Russell H. Lyddane, Charles J. Cohen, and the author. The expertise of Drs. Lyddane and Cohen strongly affected the approach taken in this task, and their subsequent vigilance provided a continuous flow of improvements to almost all sections of the theory.

Much useful information was extracted from Wayne D. McClain's two reports CSC/TR-77/6010 and CSC/TR-78/6001, the documentation of a semianalytic satellite theory developed with Paul J. Cefola and others. We also had several discussions with Dr. Cefola during his visits to NSWC; his knowledge of the subject matter is extensive. At NSWC, Dr. Armido R. DiDonato's expertise in numerical analysis proved most helpful when numerical difficulties arose.

This report was reviewed by J. Ralph Fallin, Head, Space and Surface Systems Division.

Approved by:

R. L. SCHMIDT, Head
Strategic Systems Department

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INTRODUCTION

The K10-Division at NSWC has many years of experience in computing satellite orbits by the famed "Cowell" numerical integration methods. For orbits close to the Earth, a stepsize of 60 seconds is typical. Increasing accuracy requirements entailed ever larger gravity fields, and orbit computation by numerical integration became more expensive.

In 1978, NSWC asked Lyddane to develop a semianalytic theory, restricted to the zonal harmonics, but including Moon and Sun as perturbing accelerations. Lyddane promptly produced an elegant solution, choosing the nonsingular Poincaré elements as primary variables. His work was published as an NSWC TR in 1984 (Reference 2). As Lyddane points out, such a solution should be at least two orders of magnitude faster than the customary Cowell integration.

Subsequently the decision was made to develop a more complete semianalytic theory here at NSWC. Lyddane and Cohen offered their support, a most welcome event. The major feature of the mathematical model was the addition of the tesseral gravity field, with particular attention given to the notoriously troublesome resonance perturbations. This phase of the work has now been completed, and the computer program is undergoing checkout.

In assembling the program, the tesseral algorithm was merged with Lyddane's operational zonal routine. Since the latter is documented in Reference 2, this report will describe the effects of the tesseral field and explain how various types of terms, especially those with small divisors, are treated.

TESSERAL HARMONICS IN POINCARÉ VARIABLES

THE DISTURBING FUNCTION

In this section, the disturbing function for the tesseral gravity field will be derived in terms of slightly modified Poincaré orbital elements.

The Earth's potential, as adopted by the International Astronomical Union (Reference 1), is written

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_E}{r}\right)^n P_{nm}(\sin\beta)\left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right]\right\} \tag{1}$$

For the sake of brevity in subsequent derivations, we write the potential in the form

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{n}\sum_{m=0}^{n}\left(\frac{R_E}{r}\right)^n P_{nm}(\cos\theta)\left[\gamma_{nm}e^{im\lambda} + \bar{\gamma}_{nm}e^{-im\lambda}\right]\right\} \tag{2}$$

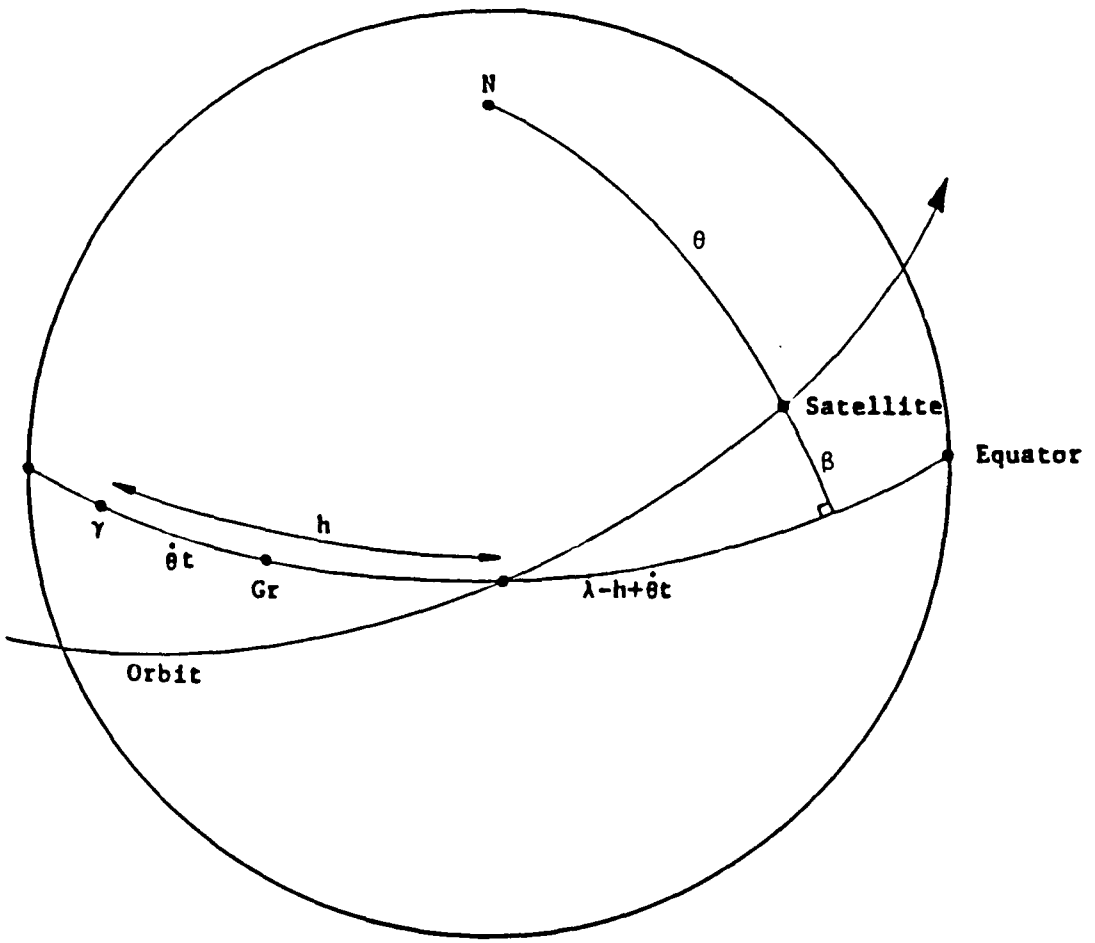

Figure 1. Orbital geometry

In the equations and figure above,

μ = $k^2 M$
k = Gaussian constant
m = mass of Earth
R_E = equatorial radius of Earth
n = degree of tesseral harmonic
m = order of tesseral harmonic
P_{nm} = associated Legendre function
λ = longitude from Greenwich
β = latitude (declination)
θ = colatitude
C_{nm}, S_{nm} = unnormalized gravity coefficients
$\gamma_{nm}, \overline{\gamma}_{nm}$ = complex form of C, S and its conjugate
i = $\sqrt{-1}$
$\dot{\theta}$ = inertial rotation rate of Earth
$\dot{\theta}t$ = GMST = Greenwich Mean Sidereal Time

A comparison of (1) and (2) yields the relations

$$\left.\begin{array}{ll} C_{nm} = \gamma_{nm} + \overline{\gamma}_{nm} & \gamma_{nm} = \frac{1}{2}(C_{nm} - i\, S_{nm}) \\ S_{nm} = i(\gamma_{nm} - \overline{\gamma}_{nm}) & \overline{\gamma}_{nm} = \frac{1}{2}(C_{nm} + i\, S_{nm}) \\ C_{n0} = \gamma_{n0} + \overline{\gamma}_{n0} = -J_n & \end{array}\right\} \tag{3}$$

Let the Hamiltonian be

$$F = F_0 + R \tag{4}$$

where

$$F_0 = \frac{\mu^2}{2L^2} \tag{5a}$$

$$R = \frac{\mu}{r} \sum_{n=2}^{n} \sum_{m=0} \left(\frac{R_E}{r}\right)^n P_{nm}(\cos\theta) \left[\gamma_{nm} e^{im\lambda} + \overline{\gamma}_{nm} e^{-im\lambda}\right] \tag{5b}$$

Since the disturbing function R contains the time, the Hamiltonian F is no longer a constant. This feature does not affect the validity of our theory. The Delaunay and Poincaré variable L is defined by

$$L = \sqrt{\mu a}$$

where

$$a = \text{semimajor axis}$$

In preparation for subsequent developments, the disturbing function will be clad in a differenct form. First we note that

$$P_{nm}(x) = (1 - x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}$$

and introduce

$$\widetilde{P}_{nm}(x) = \frac{d^m P_n(x)}{dx^m}$$

In our application, this yields

$$P_{nm}(\cos\theta) = (\sin\theta)^m \widetilde{P}_{nm}(\cos\theta) \tag{6}$$

Define

$$\begin{aligned} T_{nm}(\theta, \lambda) &= P_{nm}(\cos\theta) e^{im\lambda} \\ &= \widetilde{P}_{nm}(\cos\theta) \left[\sin\theta e^{i\lambda}\right]^m \end{aligned} \tag{7}$$

Also substitute

$$\frac{\mu}{r}\left(\frac{R_E}{r}\right)^n = \frac{\mu^{n+2}}{L^{2n+2}} R_E^n \left(\frac{a}{r}\right)^{n+1} \tag{8}$$

Finally, let

$$R_{nm} = \frac{\mu^{n+2}}{L^{2n+2}} R_E^n \left(\frac{a}{r}\right)^{n+1} T_{nm}(\theta, \lambda) \tag{9}$$

With these modifications, the disturbing function becomes

$$R = \sum_{n=2} \sum_{m=0}^{n} \left[\gamma_{nm} R_{nm} + \overline{\gamma}_{nm} \overline{R}_{nm}\right]$$

Since the zonal harmonics have already been done by Lyddane (Reference 2), we will separate R accordingly and write

$$R = \sum_{n=2} C_{n0} R_{n0} + \sum_{n=2} \sum_{m=1}^{n} \left[\gamma_{nm} R_{nm} + \overline{\gamma}_{nm} \overline{R}_{nm}\right] \tag{10}$$

Next we wish to express the surface harmonics in terms of the orbital elements $u = f + g$, h, and I. They are

u = argument of latitude
f = true anomaly
h = longitude of ascending node
I = inclination

In doing so, we follow the work of Allan (Reference 3) who appeals to Izsak (Reference 4). Thus

$$T_{nm}(\theta,\lambda) = \sum_{p=0}^{n} F_{nmp}(I)\, e^{i[(n-2p)u+m(h-\dot{\theta}t)]}$$

with the inclination function

$$F_{nmp}(I) = i^{n-m}\frac{(n+m)!}{2^n p!(n-p)!}\sum_{n=k_1}^{k_2}(-1)^k\binom{2n-2p}{k}\binom{2p}{n-m-k}c^{3n-m-2p-2k}s^{m-n+2p+2k}$$

where

$$k_1 = \tfrac{1}{2}(n-m-2p+|n-m-2p|)$$

$$k_2 = \tfrac{1}{2}(3n-m-2p-|n+m-2p|)$$

$$c = \cos\frac{I}{2} \qquad s = \sin\frac{I}{2} \tag{11}$$

Let us introduce

$$\nu = n-m-2p$$

and a slightly modified inclination function

$$F_{nmp}(c,s) = \frac{(n+m)!}{2^n p!(n-p)!}\sum_{k=k_1}^{k_2}(-1)^k\binom{2n-2p}{k}\binom{2p}{n-m-k}c^{2n+\nu-2k}s^{-\nu+2k} \tag{12}$$

where

$$k_1 = \tfrac{1}{2}(\nu + |\nu|)$$

$$k_2 = \tfrac{1}{2}(2n + \nu - |2m + \nu|)$$

Then T_{nm} may be written

$$T_{nm}(\theta, \lambda) = i^{n-m} \sum_{p=0}^{n} F_{nmp}(c, s)\, e^{i[(n-2p)u + m(h - \dot{\theta}t)]}$$

or, putting

$$i^{n-m} = e^{i(n-m)\pi/2},$$

$$T_{nm}(\theta, \lambda) = \sum_{p=0}^{n} F_{nmp}(c, s)\, e^{i\psi_{nmp}} \tag{13}$$

where

$$\psi_{nmp} = (n - 2p)u + mh - m\dot{\theta}t + (n - m)\frac{\pi}{2} \tag{14}$$

Next, working toward Poincaré variables, we express ψ_{nmp} in terms of h and $u + h$ for direct orbits and of h and $u - h$ for retrograde orbits. Accordingly, we write

$$T_{nm} = \sum_{p=0}^{n} T_{nmp}$$

where

$$T_{nmp} = F_{nmp}(c,s)\, e^{i[(n-2p)(u+h)-(n-m-2p)h-m\dot{\theta}t+(n-m)\pi/2]} \tag{15}$$

for direct orbits and

$$T_{nmp} = F_{nmp}(c,s)\, e^{i[(n-2p)(u-h)+(n+m-2p)h-m\dot{\theta}t+(n-m)\pi/2]} \tag{16}$$

for retrograde orbits.

For direct equatorial orbits, $s = 0$ and $c = 1$; for retrograde equatorial orbits, the values interchange, becoming $s = 1$ and $c - 0$. One would therefore look for symmetry in the exponents of c and s to overcome the indeterminancy of h in equatorial orbits, both when $c = 0$ and $s = 0$. It turns out the symmetry is there if F_{nmp} is compared with $F_{nm,n-p}$ and then R_{nmp} with $R_{nm,n-p}$. Thus for $F_{nm,n-p}$, if we replace the index of summation k by $k' = n - m - k$, we may write

$$\begin{aligned} F_{nm,n-p}(c,s) &= \frac{(n+m)!}{2^n(n-p)!p!} \sum_{k'=k'_1}^{k'_2} (-1)^{n-m-k'} \binom{2p}{n-m-k'} \binom{2n-2p}{k'} c^{-\nu+2k'} s^{2n+\nu-2k'} \\ &= (-1)^{n-m} F_{nmp}(s,c) \end{aligned} \tag{17}$$

since

$$k'_1 = n - m - k_2(n,m,n-p) = k_1(n,m,p)$$

and

$$k'_2 = n - m - k_1(n,m,n-p) = k_2(n,m,p)$$

For the retrograde orbit now consider $T_{nm,n-p}$. From (16), we have

$$T_{nm,n-p} = F_{nm,n-p}(c,s)\, e^{i[-(n-2p)(u-h)-(n-2p)\dot{\theta}t-\nu h+(n-m-2p)\dot{\theta}t+(n-m)\pi/2]}$$

$$= (-1)^{n-m} F_{nmp}(s,c)\, e^{-i[(n-2p)(u-h)+\nu h+m\dot{\theta}t]} e^{i(n-m)\pi/2}$$

after using (17). Since

$$(-1)^{n-m} e^{i(n-m)\pi/2} = e^{-i(n-m)\pi/2}$$

we obtain

$$T_{nm,n-p} = F_{nmp}(s,c)\, e^{-i[(n-2p)(u-h)+\nu h+m\dot{\theta}t+(n-m)\pi/2]}$$

This form for retrograde orbits is conveniently close to the form for direct orbits, (15), and henceforth (18), together with

$$T_{nm} = \sum_{p=0}^{n} T_{nm,n-p}, \tag{19}$$

will be used for retrograde orbits. We will return to the effects of summing on p vs $n-p$ when we later compare R_{nmpq} and $R_{nm,n-p,-q}$, in (39).

Now let

$$\epsilon = \begin{cases} +1 & \text{for } I < \frac{\pi}{2}\pm \qquad \text{(direct orbits)} \\ -1 & \text{for } I > \frac{\pi}{2}\pm \qquad \text{(retrograde orbits)} \end{cases}$$

and also

$$h' = \epsilon h \qquad H' = \epsilon H$$
$$c' = \sqrt{\frac{1}{2}\left(1 + \frac{H'}{G}\right)} \qquad s' = \sqrt{\frac{1}{2}\left(1 - \frac{H'}{G}\right)} \tag{21}$$

Let's also define a new function

$$\psi'_{nmp} = \epsilon\left[\left(n-2p\right)\left(u + h' - \epsilon\dot{\theta}t\right) - \left(n-m-2p\right)\left(h' - \epsilon\dot{\theta}t\right) + \left(n-m\right)\frac{\pi}{2}\right]$$

or

$$\psi'_{nmp} = \epsilon\left[\left(n-2p\right)\left(u + h'\right) - \nu h' - \epsilon m\dot{\theta}t + \left(n-m\right)\frac{\pi}{2}\right] \tag{22}$$

With these preparations, we obtain a single equation for T_{nm} good in both the direct and retrograde cases:

$$T_{nm} = \sum_{p=0}^{n} F_{nmp}\left(c', s'\right)\epsilon^{i\psi'_{nmp}} \tag{23}$$

Let's verify this assertion:

$\underline{\epsilon = +1}$

$$T_{nm} = \sum_{p=0}^{n} F_{nmp}\left(c, s\right)\, e^{i\left[(n-2p)(u+h) - \nu h - m\dot{\theta}t + (n-m)\frac{\pi}{2}\right]}$$

which does agree with 15.

$\underline{\epsilon = -1}$

$$T_{nm} = \sum_{p=0}^{n} F_{nmp}(s,c)\; e^{-i\left[(n-2p)(u-h)+\nu h+m\dot{\theta}t+(n-m)\frac{\pi}{2}\right]}$$

which is, indeed, (18).

Now recall R_{nm}, given by (9):

$$R_{nm} = \frac{\mu^{n+2}}{L^{2n+2}} R_E^n \left(\frac{a}{r}\right)^{n+1} T_{nm}(\theta,\lambda)$$

This becomes

$$R_{nm} = \frac{\mu^{n+2}}{L^{2n+2}} R_E^n \left(\frac{a}{r}\right)^{n+1} \sum_{p=0}^{n} F_{nmp}(c',s')\; e^{i\epsilon\left[(n-2p)(g+h')-\nu h'-\epsilon m\dot{\theta}t+(n-m)\frac{\pi}{2}\right]}\; e^{i\epsilon(n-2p)f}$$

From this expression we extract the short-periodic content and expand the latter in a Fourier series in the mean anomaly l with Hansen coefficients $X(\beta)$:

$$\left(\frac{a}{r}\right)^{n+1} e^{i\epsilon(n-2p)f} = \sum_{q=-\infty}^{\infty} X_{n-2p+q}^{-n-1,n-2p}(\beta)\; e^{i\epsilon(n-2p+q)l} \tag{25}$$

where

$$\beta = \frac{e}{1+\sqrt{1-e^2}} \tag{26}$$

The e in (26), as well as below, is the eccentricity. Since the base of the natural log always involves i, confusion should not arise.

Also note that the validity of (25) for $\epsilon = -1$ is easily established by taking it to be the complex conjugate of the $\epsilon = +1$ case.

If we write

$$R_{nm} = \sum_{p=0}^{n} \sum_{q=-\infty}^{\infty} R_{nmpq} \tag{27}$$

we obtain

$$\begin{aligned} R_{nmpq} &= \frac{\mu^{n+2}}{L^{2n+2}} R_E^n F_{nmp}(c', s')\, e^{i\epsilon\left[(n-2p)(g+h')-\nu h'-\epsilon m\dot{\theta}t+(n-m)\frac{\pi}{2}\right]} \\ &\quad * X_{n-2p+q}^{-n-1,n-2p}(\beta)\, e^{i\epsilon(n-2p+q)l} \\ &= \frac{\mu^{n+2}}{L^{2n+2}} R_E^n F_{nmp}(c', s')\, X_{n-2p+q}^{-n-1,n-2p}(\beta)\, e^{i\psi'_{nmpq}} \end{aligned} \tag{28}$$

where

$$\psi'_{nmpq} = \epsilon\left[(n-2p+q)\,l + (n-2p)(g+h') - \nu h' - \epsilon m\dot{\theta}t + (n-m)\frac{\pi}{2}\right] \tag{28a}$$

In preparation for introducing the Poincaré variables, rearranging terms yields

$$\psi'_{nmpq} = \epsilon\left[(n-2p+q)(l+g+h') - q(g+h') - \nu h' - \epsilon m\dot{\theta}t + (n-m)\frac{\pi}{2}\right] \tag{29}$$

Our Poincaré variables, modified to accommodate all inclinations, are defined by

$$\left.\begin{array}{ll} L & \lambda' = l+g+h \\ \xi = \sqrt{2(L-G)}\cos(g+h') & \eta = -\sqrt{2(L-G)}\sin(g+h') \\ \sigma = \sqrt{2(G-H')}\cos h' & \tau = \sqrt{2(G-H')}\sin h' \end{array}\right\} \tag{30}$$

Also anticipating further developments, we define the functions

$$\left.\begin{aligned} \widetilde{F}_{nmp}(c',s') &= \frac{1}{(s')^{|\nu|}} F_{nmp}(c',s') \\ \widetilde{X}^{-n-1,n-2p}_{n-2p+q}(\beta) &= \frac{1}{\beta^{|q|}} X^{-n-1,n-2p}_{n-2p+q}(\beta) \end{aligned}\right\} \tag{32}$$

Then, in (28),

$$\begin{aligned} F_{nmp}(c',s')\, X^{-n-1,n-2p}_{n-2p+q}(\beta)\, e^{i\psi'_{nmpq}} &= (s')^{|\nu|}\, \widetilde{F}_{nmp}(c',s')\, \beta^{|q|}\, \widetilde{X}^{-n-1,n-2p}_{n-2p+q}(\beta) \\ &\quad * e^{i\epsilon\left[(n-2p+q)\lambda' - q(g+h') - \nu h' - \epsilon m\dot{\theta}t + (n-m)\frac{\pi}{2}\right]} \\ &= \widetilde{F}_{nmp}(c',s')\, \widetilde{X}^{-n-1,n-2p}_{n-2p+q}(\beta)\, (s')^{|\nu|} \\ &\quad * e^{-i\epsilon\nu h'} \beta^{|q|} e^{-i\epsilon q(g+h')} \\ &\quad * e^{i\epsilon\left[(n-2p+q)\lambda' - \epsilon m\dot{\theta}t + (n-m)\frac{\pi}{2}\right]} \\ &= \widetilde{F}_{nmp}(c',s')\, \widetilde{X}^{-n-1,n-2p}_{n-2p+q}(\beta) \left[s' e^{-i\epsilon h' sgn\,\nu}\right]^{|\nu|} \\ &\quad * \left[\beta e^{-i\epsilon(g+h')sgn\,q}\right]^{|q|} \\ &\quad * e^{i\epsilon\left[(n-2p+q)\lambda' - \epsilon m\dot{\theta}t + (n-m)\frac{\pi}{2}\right]} \end{aligned} \tag{33}$$

For the remaining steps, as well as for the computer program algorithm, the following auxiliary relations will be useful:

$$\left.\begin{aligned} 2(L-G) &= \xi^2 + \eta^2 & 2(G-H') &= \sigma^2 + \tau^2 \\ \sqrt{1-e^2} &= 1 - \frac{1}{2}\frac{\xi^2+\eta^2}{L} & G &= L - \frac{1}{2}(\xi^2+\eta^2) \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} \epsilon &= \sqrt{\frac{\xi^2+\eta^2}{L} - \frac{1}{4}\left(\frac{\xi^2+\eta^2}{L}\right)^2} \qquad \beta = \sqrt{\frac{\xi^2+\eta^2}{4L-(\xi^2+\eta^2)}} \\ & \qquad\qquad\qquad\qquad s' = \sqrt{\frac{\sigma^2+\tau^2}{4L-2(\xi^2+\eta^2)}} \end{aligned}\right\} \tag{35}$$

Then

$$\begin{aligned} s' e^{ieh' sgn\,\nu} &= s' \left[e^{-ih'}\right]^{\epsilon\, sgn\,\nu} \\ &= s' \left[\cos h' - i \sin h'\right]^{\epsilon\, sgn\,\nu} \\ &= s' \left[\cos h' - i\,\epsilon\, sgn\,\nu \sin h'\right] \\ &= \sqrt{\frac{\sigma^2+\tau^2}{4L-2(\xi^2+\eta^2)}} \quad \left[\frac{\sigma + i\,\epsilon\, sgn\,\nu\,\tau}{\sqrt{\sigma^2+\tau^2}}\right] \end{aligned}$$

or

$$s' e^{ieh' sgn\,\nu} = \frac{\sigma + (i\,\epsilon\, sgn\,\nu)\,\tau}{\sqrt{4L-2(\xi^2+\eta^2)}} \tag{36}$$

Quite similarly we obtain

$$\beta e^{ie(g+h')sgn\,q} = \frac{\xi + (i\,\epsilon\, sgn\,q)\,\eta}{\sqrt{4L-(\xi^2+\eta^2)}} \tag{37}$$

Substituting (36) and (37), we reach teh following form for R_{nmpq}, free of small divisors:

$$R_{nmpq} = \frac{\mu^{n+2}}{L^{2n+2}} R_E^n \widetilde{F}_{nmp}(c', s') \left[\frac{\sigma + i\,\epsilon\,\tau\, sgn\,\nu}{\sqrt{4L - 2(\xi^2 + \eta^2)}} \right]^{|\nu|} \widetilde{X}_{n-2p+q}^{-n-1,n-2p}(\beta)$$

$$* \left[\frac{\xi + i\,\epsilon\,\eta\, sgn\, q}{\sqrt{4L - 2(\xi^2 + \eta^2)}} \right]^{|q|} e^{i\epsilon\left[(n-2p+q)\lambda' + (n-m)\frac{\pi}{2}\right]} e^{-im\dot{\theta}t} \qquad (38)$$

We note that $R_{nm}(\epsilon = -1) = R_{nm}(\epsilon = +1)$ but, because of a shift in the summation indices p and q, we have $R_{nmpq}(\epsilon = -1) \neq R_{nmpq}(\epsilon = +1)$. To see the effect of the shift, we return to (28):

$$R_{nmpq}(\epsilon = -1) = \frac{\mu^{n+2}}{L^{2n+2}} R_E^n F_{nmp}(s, c)\, X_{n-2p+q}^{-n-1,n-2p}(\beta)$$

$$* e^{-i\left[(n-2p+q)l + (n-2p)(g-h) + \nu h + m\dot{\theta}t + (n-m)\frac{\pi}{2}\right]}$$

$$R_{nm,n-p,-q}(\epsilon = +1) = \frac{\mu^{n+2}}{L^{2n+2}} R_E^n F_{nm,n-p}(s, c)\, X_{-(n-2p+q)}^{-n-1,-(n-2p)}(\beta)$$

$$* e^{i\left[(-n+2p-q)l + (-n+2p)(g+h) + (n+m-2p)h - m\dot{\theta}t + (n-m)\frac{\pi}{2}\right]}$$

using (17)

$$= \frac{\mu^{n+2}}{L^{2n+2}} R_E^n (-1)^{n-m} F_{nmp}(s, c)\, X_{n-2p+q}^{-n-1,n-2p}(\beta)$$

$$* e^{i\left[-(n-2p+q)l - (n-2p)(g-h) - \nu h - m\dot{\theta}t + (n-m)\frac{\pi}{2}\right]}$$

using $(-1)^{n-m} = e^{i(n-m)\pi}$

$$= \frac{\mu^{n+2}}{L^{2n+2}} R_E^n F_{nmp}(s,c)\, X^{-n-1,n-2p}_{n-2p+q}(\beta)$$

$$* e^{i\left[(n-2p+q)l+(n-2p)(g-h)+\nu h+m\dot{\theta}t-(n-m)\frac{\pi}{2}\right]} e^{i2(n-m)\frac{\pi}{2}}$$

implies

$$R_{nmpq}(\epsilon=-1) = R_{nm,n-p,-q}(\epsilon=+1) \tag{39}$$

Note that use was made of

$$X^{-n-1,n-2p}_{n-2p+q} = X^{-n-1,-(n-2p)}_{-(n-2p+q)}$$

which follows from (25) and its complex conjugate.

SUMMARY OF RESULTS

We recapitulate showing the Hamiltonian framework and the definitions of the variables that enter R_{nmpq}. The canonical variables are:

$$\left.\begin{array}{ll} L & \lambda' = l+g+h \\ \xi = \sqrt{2(L-G)}\cos(g+h') & \eta = -\sqrt{2(L-G)}\sin(g+h') \\ \sigma = \sqrt{2(G-H')}\cos h' & \tau = -\sqrt{2(G-H')}\sin h' \end{array}\right\} \tag{40}$$

Note that

$$H' = \epsilon H \qquad h' = \epsilon h$$

where

$$\epsilon = \begin{cases} +1 & \text{for } I \leq \frac{\pi}{2}\pm \\ -1 & \text{for } I \geq \frac{\pi}{2}\pm \end{cases}$$

Once set, ϵ is never changed during a run. The Hamiltonian is

$$F = F_0 + R \tag{41}$$

where

$$F_0 = \frac{\mu^2}{2L^2} \tag{42a}$$

$$R = \sum_{n=2} C_{n0} R_{n0} + \sum_{n=2} \sum_{m=1}^{n} \left[\gamma_{nm} R_{nm} + \bar{\gamma}_{nm} \bar{R}_{nm}\right] \tag{42b}$$

$$R_{nm} = \sum_{p=0}^{n} \sum_{q=-\infty}^{\infty} R_{nmpq}$$

$$R_{nmpq} = \frac{\mu^{n+2}}{L^{2n+2}} R_E^n \tilde{F}^J_{nmp}(c', s') \left[\frac{\sigma + (i\,\epsilon\, sgn\,\nu)\,\tau}{\sqrt{4L - 2(\xi^2 + \eta^2)}}\right]^{|\nu|} \tilde{\chi}^{-n-1,n-2p}_{n-2p+q}(\beta)$$

$$* \left[\frac{\xi + (i\,\epsilon\, sgn\, q)\,\eta}{\sqrt{4L - (\xi^2 + \eta^2)}}\right]^{|q|} e^{i\epsilon\left[(n-2p+q)\lambda' + (n-m)\frac{\pi}{2}\right]} e^{-im\dot{\theta}t} \tag{43}$$

Note that

$$\sqrt{4L - 2(\xi^2 + \eta^2)} = 2\sqrt{G}$$

$$\sqrt{4L - (\xi^2 + \eta^2)} = \sqrt{2(L + G)}$$

$$\widetilde{F}^J_{nmp}(c', s') = (-1)^\delta \frac{(n - r')!(n + r')!}{2^n (n - m)! p! (n - p)!} (c')^b P^{a,b}_{n-r'}(x') \tag{44}$$

where

$$r = n - 2p \qquad \delta = \frac{1}{2}(a' - a)$$

$$a' = m - r \qquad r' = \frac{1}{2}(a + b)$$

$$a = |m - r| \qquad c' = \sqrt{1 - \frac{\sigma^2 + \tau^2}{4L - 2(\xi^2 + \eta^2)}}$$

$$b = |m + r| \qquad x' = 2(c')^2 - 1$$

Compute

$$P^{a,b}_0(x') = 1$$

$$P^{a,b}_1(x') = \frac{1}{2}(\rho + 1)x' + \frac{1}{2}(a - b)$$

where

$$\rho = a + b + 1$$

$$P_d^{a,b}(x') = \frac{1}{2d(d+\rho-1)(2d+\rho-3)}\Big\{(2d+\rho-2)$$
$$* \left[(2d+\rho-3)(2d+\rho-1)\,x' + (a^2-b^2)\right] P_{d-1}^{a,b}(x')$$
$$- 2(d+a-1)(d+b-1)(2d+\rho-1)\,P_{d-2}^{a,b}(x')\Big\} \tag{45}$$

Starting with $d = 2$, continue until $d = n - r'$.

$$\widetilde{X}_{n-2p+q}^{-n-1,n-2p}(\beta) = (-1)^{|q|}\,\frac{(1+\beta^2)^n}{(1-\beta^2)^{2n-1}}\sum_{j=0}^{\infty} M_s N_t \beta^{2j} \tag{46}$$

where

$$\beta = \sqrt{\frac{\xi^2+\eta^2}{4L-(\xi^2+\eta^2)}} = \frac{e}{1+\sqrt{1-e^2}}$$

$$\left.\begin{aligned} M_s &= (-1)^s \sum_{v=0}^{s} \binom{\alpha-v}{s-v}\frac{b^v}{v!} \\ N_t &= (-1)^t \sum_{w=0}^{t} \binom{\gamma-w}{t-w}\frac{(-b)^w}{w!} \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} s &= j + \frac{1}{2}(|q|+q) \\ t &= j + \frac{1}{2}(|q|-q) \\ \alpha &= 2n-2p+q-1 \\ \gamma &= 2p-q-1 \\ b &= (n-2p+q)\sqrt{1-e^2} \end{aligned}\right\} \tag{48}$$

Alternatively, M_s *and* N_t may be computed using Lyddane's relations:

Starting with

$$\left.\begin{aligned} M(0,\alpha;b) &= 1 \\ M(s,s-1;b) &= (-1)^s \frac{b^s}{s!} \end{aligned}\right\} \tag{49}$$

compute

$$M_s = M(s,\alpha;b) = M(s,\alpha-1;b) - M(s-1,\alpha-1;b) \tag{50}$$

$$N_t = N(t,\gamma;b) \equiv M(t,\gamma;-b) \tag{51}$$

The last line means that N is obtained similarly to M, but writing -b for b.

To truncate the summation in (46), let

$$A_j = |M_s N_t| \beta^{2j} \tag{52}$$

Terminate when

$$A_j + A_{j+1} < max(A_k) \cdot \tau_x, \qquad k \leq j, \tag{53}$$

where τ_x is an input.

EQUATIONS OF MOTION

THE DERIVATION

The basic equations of motion for our canorical Poincaré variables take the well-known form

$$\left.\begin{aligned} \dot{L} &= \frac{\partial F}{\partial \lambda'} & \dot{\lambda}' &= -\frac{\partial F}{\partial L} \\ \dot{\xi} &= \frac{\partial F}{\partial \eta} & \dot{\eta} &= -\frac{\partial F}{\partial \xi} \\ \dot{\sigma} &= \frac{\partial F}{\partial \tau} & \dot{\tau} &= -\frac{\partial F}{\partial \sigma} \end{aligned}\right\} \tag{54}$$

where F is the Hmailtonian, developed in the previous section. In this section the partial derivatives indicated in (54) will be derived.

As mentioned before, the central force field and the zonal harmonics

$$\frac{\mu^2}{2L^2} + \sum_{n=2} C_{n0} R_{n0}$$

have already been treated by Lyddane (Reference 2). Hence, we will address here the tesseral part

$$\Delta F_{2T} = R_{2T}, \tag{55}$$

where the subscript 2 designates that all terms are of order two or higher. Following (42b), we write

$$R_{2T} = \sum_{n=2} \sum_{m=1}^{n} \left[\gamma_{nm} R_{nm} + \overline{\gamma}_{nm} \overline{R}_{nm}\right] \tag{56}$$

Also from the previous section, namely (43), each individual term in (56) is

$$R_{nmpq} = \frac{\mu^{n+2}}{L^{2n+2}} \widetilde{F}^{J}_{nmp}(c', s') \left[\frac{\sigma + (i\,\epsilon\, sgn\,\nu)\,\tau}{\sqrt{4L - 2(\xi^2 + \eta^2)}} \right]^{|\nu|} \widetilde{X}^{-n-1,n-2p}_{n-2p+q}(\beta)$$

$$* \left[\frac{\xi + (i\,\epsilon\, sgn\, q)\,\eta}{\sqrt{4L - (\xi^2 + \eta^2)}} \right]^{|q|} e^{i\epsilon\left[(n-2p+q)\lambda' + (n-m)\frac{\pi}{2}\right]} e^{-im\dot{\theta}t} \tag{57}$$

where

$$\left.\begin{aligned} (s')^2 &= \frac{\sigma^2 + \tau^2}{4L - 2(\xi^2 + \eta^2)} \qquad c' = \sqrt{1 - (s')^2} \\ \beta &= \sqrt{\frac{\xi^2 + \eta^2}{4L - 2(\xi^2 + \eta^2)}} \end{aligned}\right\} \tag{58}$$

Introducing short-hand notation for (57), we write

$$R = \widetilde{F}\, B^{|\nu|}\, \widetilde{X} C^{|q|}\, D \tag{59}$$

where

$$\left.\begin{aligned} A &= A(L) & \widetilde{X} &= \widetilde{X}(L, \xi, \eta) \\ \widetilde{F} &= \widetilde{F}(L, \xi, \eta, \sigma, \tau) & C &= C(L, \xi, \eta) \\ B &= B(L, \xi, \eta, \sigma, \tau) & D &= D(\lambda', t) \end{aligned}\right\} \tag{60}$$

Now let y designate any one of our Poincaré variables. Then eventually

$$\frac{\partial R}{\partial y} = \frac{R}{A}\frac{\partial A}{\partial y} + \frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial y} + |\nu|\frac{R}{B}\frac{\partial B}{\partial y} + \frac{R}{\widetilde{X}}\frac{\partial \widetilde{X}}{\partial y} + |q|\frac{R}{C}\frac{\partial C}{\partial y} + \frac{R}{D}\frac{\partial D}{\partial y} \tag{61}$$

Hence, we need the partials of the factors in (60) with respect to their Poincaré arguments. In above sequence,

$$A = \frac{\mu^{n+2} R_E^n}{L^{2n+2}} \tag{61a}$$

which yields

$$\frac{\partial A}{\partial L} = -(2n+2)\frac{A}{L} \tag{62}$$

Next

$$\widetilde{F} = -(-1)^{\delta} \frac{(h-r')!(n+r')!}{2^n (n-m)! p! (n-p)!} (c')^b P_{n-r'}^{a,b}(x') \tag{63}$$

$$\left.\begin{array}{ll} r = n - 2p & \delta = \dfrac{1}{2}(a'-a) \\ a' = m - r & r' = \dfrac{1}{2}(a+b) \\ a = |m-r| & \rho = a+b+1 \\ b = |m+r| & x' = 2(c')^2 - 1 \\ (s')^2 = \dfrac{\sigma^2+\tau^2}{4L - 2(\xi^2+\eta^2)} & c' = \sqrt{1-(s')^2} \end{array}\right\} \tag{64}$$

Also note

$$4L - 2(\xi^2 + \eta^2) = 4G \tag{64a}$$

We also need

$$P_0^{a,b}(x') = 1 \tag{65}$$

$$P_1^{a,b}(x') = \frac{1}{2}(\rho+1)\,x' + \frac{1}{2}(a-b)$$

$$P_d^{a,b}(x') = \frac{1}{2d(d+\rho-1)(2d+\rho-3)}\Big\{(2d+\rho-2)$$

$$* \left[(2d+\rho-3)(2d+\rho-1)\,x' + \left(a^2-b^2\right)\right] P_{d-1}^{a,b}(x')$$

$$- 2(d+a-1)(d+b-1)(2d+\rho-1)\,P_{d-2}^{a,b}(x')\Big\} \tag{66}$$

For convenience, let's temporarily drop a and b from P, the prime from c', s', and x', and define

$$K = (-1)^{\delta}\frac{(n-r')!\,(n+r')!}{2^n\,(n-m)!p!\,(n-p)!}$$

Then

$$\widetilde{F} = Kc^b P_{n-r'}(x) \tag{67}$$

implies

$$\frac{\partial \widetilde{F}}{\partial y} = \frac{d\widetilde{F}}{dc}\frac{\partial c}{\partial y} \tag{68}$$

where y is any one of our six Poincaré elements. Using (67), first do

$$\frac{d\widetilde{F}}{dc} = Kbc^{b-1}P_{n-r'}(x) + Kc^{b}\frac{dP_{n-r'}(x)}{dx}\frac{dx}{dc} \tag{69}$$

But since

$$x = 2c^2 - 1,$$

$$\frac{dx}{dc} = 4c,$$

and (69) becomes

$$\frac{d\widetilde{F}}{dc} = (-1)^{\delta}\frac{(n-r')!(n+r')!}{2^n(n-m)!p!(n-p)!}\left[c^{b-1}\right]\left[bP_{n-r'}(x') + 4c^2\frac{dP_{n-r'}(x)}{dx}\right] \tag{70}$$

In order to obtain $\frac{dP}{dx}$, write (66) in the abbreviated form

$$P_d(x) = \frac{1}{a_1}\left\{a_2[a_3x + a_4]P_{d-1}(x) - a_5P_{d-2}(x)\right\}$$

implies

$$\frac{dP_d(x)}{dx} = \frac{1}{a_1}\left\{a_2\left[a_3x + a_4\right]\frac{dP_{d-1}(x)}{dx} - a_5\frac{dP_{d-2}(x)}{dx}\right\} + \frac{a_2a_3}{a_1}P_{d-1}(x)$$

or

$$\frac{dP_d(x)}{dx} = \frac{1}{2d(d+\rho-1)(2d+\rho-3)}\Big\{(2d+\rho-2)$$
$$* \left[(2d+\rho-3)(2d+\rho-1)x + (a^2-b^2)\right]\frac{dP_{d-1}(x)}{dx}$$
$$-2(d+a-1)(d+b-1)(2d+\rho-1)\frac{dP_{d-2}(x)}{dx}$$
$$+\frac{(2d+\rho-1)(2d+\rho-2)}{2d(d+\rho-1)}P_{d-1}(x)\Big\} \tag{71}$$

From (65) we derive

$$\frac{dP_0(x)}{dx} = 0$$
$$\frac{dP_1(x)}{dx} = \frac{1}{2}(\rho+1) \tag{72}$$

Note that (71) has the form of (66), with P replaced by $\frac{dP}{dx}$, plus an additional term. The recurrence loops for P and $\frac{dP}{dx}$ should probably be computed simultaneously.

Now turn to

$$\frac{\partial c}{\partial y}.$$

Since we have now introduced s in addition to c, in (64), let's make use of

$$\frac{\partial c}{\partial y} = \frac{-1}{2c}\frac{\partial}{\partial y}(s^2) \tag{73}$$

The five partial drivatives are found to be

$$\begin{aligned} \frac{\partial c}{\partial L} &= \frac{s^2}{2cG} & \frac{\partial c}{\partial \xi} &= \frac{-\xi s^2}{2cG} \\ \frac{\partial c}{\partial \sigma} &= \frac{-\sigma}{4cG} & \frac{\partial c}{\partial \eta} &= \frac{-\eta s^2}{2cG} \\ \frac{\partial c}{\partial \tau} &= \frac{-\tau}{4cG} \end{aligned} \tag{74}$$

The final results for $\widetilde{F}$ are:

$$\left.\begin{aligned} \frac{\partial \widetilde{F}}{\partial L} &= \frac{(s')^2}{2c'G}\frac{d\widetilde{F}}{dc'} \\ \frac{\partial \widetilde{F}}{\partial \xi} &= (-\xi)\frac{(s')^2}{2c'G}\frac{d\widetilde{F}}{dc'} \\ \frac{\partial \widetilde{F}}{\partial \eta} &= (-\eta)\frac{(s')^2}{2c'G}\frac{d\widetilde{F}}{dc'} \\ \frac{\partial \widetilde{F}}{\partial \sigma} &= \frac{-\sigma}{4c'G}\frac{d\widetilde{F}}{dc'} \\ \frac{\partial \widetilde{F}}{\partial \tau} &= \frac{-\tau}{4c'G}\frac{d\widetilde{F}}{dc'} \end{aligned}\right\} \tag{75}$$

where

$$\frac{\partial \widetilde{F}}{\partial c'} = (-1)^{\delta} \frac{(n-r')!\,(n+r')!}{2^n\,(n-m)!p!\,(n-p)!} \left[(c')^{b-1}\right]$$

$$* \left[bP^{a,b}_{n-r'}(x') + 4(c')^2 \frac{dP^{a,b}_{n-r'}(x')}{dx'} \right] \tag{76}$$

with

$P^{a,b}_{n-r'}(x')$ defined by (65) and (66),

$\dfrac{dP^{a,b}_{n-r'}(x')}{dx'}$ defined by (71) and (72) .

The next factor is B. With later developments in mind, write it as follows:

$$B = \frac{\sigma + (i\,\epsilon\, sgn\,\nu)\,\tau}{\sqrt{4L - 2(\xi^2 + \eta^2)}}$$

$$= \frac{\sigma + (i\,\epsilon\, sgn\,\nu)\,\tau}{2\sqrt{G}} \tag{77}$$

From (77), we immediately obtain

$$\left.\begin{aligned}\frac{\partial B}{\partial \sigma} &= \frac{1}{2\sqrt{G}} \\ \frac{\partial B}{\partial \tau} &= \frac{(i\,\epsilon\, sgn\,\nu)}{2\sqrt{G}}\end{aligned}\right\} \tag{78}$$

For the other three, write

$$B = \frac{1}{2}\left[\sigma + (i\,\epsilon\, sgn\,\nu)\,\tau\right] G^{-1/2}$$

Hence

$$\frac{\partial B}{\partial y} = \frac{\partial B}{\partial G}\frac{\partial G}{\partial y} \tag{79}$$

First get

$$\frac{\partial B}{\partial G} = -\frac{1}{2}\frac{\left[\sigma + (i\,\epsilon\, sgn\,\nu)\,\tau\right]}{2\sqrt{G}G}$$

or

$$\frac{\partial B}{\partial G} = \frac{-B}{2G} \tag{80}$$

and (79) becomes

$$\frac{\partial B}{\partial y} = \frac{-B}{2G}\frac{\partial G}{\partial y} \tag{81}$$

Since

$$G = L - \frac{1}{2}\left(\xi^2 + \eta^2\right)$$

$$\frac{\partial G}{\partial L} = 1$$

$$\frac{\partial G}{\partial \xi} = -\xi$$

$$\frac{\partial G}{\partial \eta} = -\eta$$

Therefore

$$\left.\begin{aligned} \frac{\partial B}{\partial L} &= \frac{-B}{2G} \\ \frac{\partial B}{\partial \xi} &= \frac{\xi B}{2G} \\ \frac{\partial B}{\partial \eta} &= \frac{\eta B}{2G} \end{aligned}\right\} \tag{82}$$

Let's turn to

$$\tilde{X} = \tilde{X}\left(L, \xi, \eta\right)$$

Using (46) through (48), we write

$$\tilde{X} = E \sum_{j=0}^{\infty} M_s N_t \beta^{2j} \tag{83}$$

where

$$E = (-1)^{|q|} \frac{(1+\beta^2)^n}{(1-\beta^2)^{2n-1}}$$

$$\beta = \frac{e}{1+\sqrt{1-e^2}}$$

or

$$\beta^2 = \frac{\xi^2+\eta^2}{4L-(\xi^2+\eta^2)}$$

$$\left.\begin{aligned} M_s &= (-1)^s \sum_{v=0}^{s} \binom{\alpha-v}{s-v} \frac{b^v}{v!} \\ N_t &= (-1)^t \sum_{w=0}^{t} \binom{\gamma-w}{t-w} \frac{(-b)^w}{w!} \end{aligned}\right\} \tag{84}$$

$$\left.\begin{aligned} s &= j + \frac{1}{2}(|q|+q) \\ t &= j + \frac{1}{2}(|q|-q) \\ \alpha &= 2n-2p+q-1 \\ \gamma &= 2p-q-1 \end{aligned}\right\} \tag{85}$$

$$\begin{aligned} b &= (n-2p+q)\left[1-\frac{1}{2L}(\xi^2+\eta^2)\right] \\ &= (n-2p+q)\sqrt{1-e^2} \end{aligned} \tag{86}$$

Writing termporarily

$$Q = n - 2p + q,$$

other useful relations are

$$b = Q\frac{1-\beta^2}{1+\beta^2} \qquad e = \frac{2\beta}{1+\beta^2} \tag{87}$$

Now we proceed to obtain

$$\frac{\partial \widetilde{X}}{\partial y} = \frac{\partial \widetilde{X}}{\partial \beta^2}\frac{\partial \beta^2}{\partial y}, \qquad y = L, \xi, \eta \tag{88}$$

Starting with (83):

$$\begin{aligned}\frac{d\widetilde{X}}{d\beta^2} = \frac{dE}{d\beta^2}\sum_{j=0}^{\infty} M_s N_t \beta^{2j} + E\sum_{j=0}^{\infty}\left(\frac{dM_s}{d\beta^2}N_t + M_s\frac{dN_t}{d\beta^2}\right)(\beta^2)^j \\ + E\sum_{j=0}^{\infty} M_s N_t j\,(\beta^2)^{j-1}\end{aligned} \tag{89}$$

The derivatives in (89) are as follows. First

$$\frac{dE}{d\beta^2} = \left[\frac{n}{(1+\beta^2)} + \frac{2n-1}{(1-\beta^2)}\right]E \tag{90}$$

Next

$$\frac{dM_s}{d\beta^2} = (-1)^s \sum_{v=1}^{s} \binom{\alpha - v}{s - v} \frac{v b^{v-1}}{v!} \frac{db}{d\beta^2}$$

$$\frac{dN_t}{d\beta^2} = (-1)^t \sum_{w=1}^{t} \binom{\gamma - w}{t - w} \frac{w(-b)^{w-1}(-1)}{w!} \frac{db}{d\beta^2} \tag{91}$$

Since

$$b = Q\frac{1 - \beta^2}{1 + \beta^2}$$

$$\frac{db}{d\beta^2} = \frac{-2Q}{(1 - \beta^2)^2} \tag{92}$$

(91) becomes

$$\frac{dM_s}{d\beta^2} = \frac{-2Q}{(1 + \beta^2)^2} (-1)^s \sum_{v=1}^{s} \binom{\alpha - v}{s - v} \frac{b^v}{v!} \frac{v}{b}$$

$$\frac{dN_t}{d\beta^2} = \frac{+2Q}{(1 + \beta^2)^2} (-1)^t \sum_{w=1}^{t} \binom{\gamma - w}{t - w} \frac{(-b)^w}{w!} \frac{w}{(-b)}$$

or, since

$$\frac{Q}{b} = \frac{1 + \beta^2}{1 - \beta^2},$$

$$\left.\begin{aligned} \frac{dM_s}{d\beta^2} &= \frac{-2}{1-\beta^4}(-1)^s \sum_{v=1}^{s} \binom{\alpha-v}{s-v} \frac{b^v}{v!} v \\ \frac{dN_t}{d\beta^2} &= \frac{-2}{1-\beta^4}(-1)^t \sum_{w=1}^{t} \binom{\gamma-w}{t-w} \frac{(-b)^w}{w!} w \end{aligned}\right\} \tag{93}$$

Substitution of (90) and (93) into (89) yields

$$\begin{aligned} \frac{d\widetilde{X}}{d\beta^2} &= \left[\frac{n}{(1+\beta^2)} + \frac{2n-1}{1-\beta^2}\right](-1)^{|q|} \frac{(1+\beta^2)^n}{(1-\beta^2)^{2n-1}} \sum_{j=0}^{\infty} M_s N_t \beta^{2j} \\ &+ (-1)^{|q|} \frac{(1+\beta^2)^n}{(1-\beta^2)^{2n-1}} \sum_{j=0}^{\infty} \left[\frac{-2}{1-\beta^4}(-1)^s \binom{\alpha-1}{s-v} \frac{b^v}{v!} v N_t \right. \\ &\left. + M_s \frac{-2}{1-\beta^4}(-1)^t \sum_{w=1}^{t} \binom{\gamma-w}{t-w} \frac{(-b)^w}{w!} w\right] \beta^{2j} \\ &+ (-1)^{|q|} \frac{(1+\beta^2)^n}{(1-\beta^2)^{2n-1}} \sum_{j=0}^{\infty} M_s N_t j \beta^{2j-2} \end{aligned} \tag{94}$$

Let now

$$\left.\begin{aligned} (Mv) &= (-1)^s \sum_{v=1}^{s} \binom{\alpha-v}{s-v} \frac{b^v}{v!} v \\ (Nw) &= (-1)^t \sum_{w=1}^{t} \binom{\gamma-w}{t-w} \frac{(-b)^w}{w!} w \end{aligned}\right\} \tag{95}$$

Then

$$\frac{d\tilde{X}}{d\beta^2} = \left[\frac{n}{(1+\beta^2)} + \frac{2n-1}{1-\beta^2}\right]\tilde{X} - 2(-1)^{|q|}\frac{(1+\beta^2)^{n-1}}{(1-\beta^2)^{2n}}\sum_{j=0}^{\infty}[M_s(Nw)$$

$$+ (Mv)N_t\,]\beta^{2j} + (-1)^{|q|}\frac{(1+\beta^2)^n}{(1-\beta^2)^{2n-1}}\sum_{j=1}^{\infty}M_sN_tj\beta^{2j-2} \qquad (96)$$

Next we differentiate

$$\beta^2 = \frac{\xi^2+\eta^2}{4L-\xi^2-\eta^2}$$

with respect to L, ξ, and η. It will be useful to note that

$$(\xi^2+\eta^2) = 2(L-G)$$

$$4L-(\xi^2+\eta^2) = 2(L+G)$$

Then

$$\left.\begin{aligned}\frac{\partial\beta^2}{\partial L} &= \frac{2(G-L)}{(L+G)^2}\\ \frac{\partial\beta^2}{\partial\xi} &= \frac{2\xi L}{(L+G)^2}\\ \frac{\partial\beta^2}{\partial\eta} &= \frac{2\eta L}{(L+G)^2}\end{aligned}\right\} \qquad (97)$$

Using (88) and (97), we finally obtain

$$\left.\begin{aligned} \frac{\partial \widetilde{X}}{\partial L} &= \frac{2(G-L)}{(L+G)^2}\frac{\partial \widetilde{X}}{\partial \beta^2} \\ \frac{\partial \widetilde{X}}{\partial \xi} &= \frac{2\xi L}{(L+G)^2}\frac{\partial \widetilde{X}}{\partial \beta^2} \\ \frac{\partial \widetilde{X}}{\partial \eta} &= \frac{2\eta L}{(L+G)^2}\frac{\partial \widetilde{X}}{\partial \beta^2} \end{aligned}\right\} \tag{98}$$

with $\partial \widetilde{X} / \partial \beta^2$ given by (96).

An alternative path to obtain the partials given in (98) uses the recurrence relations proposed by Lyddane. Instead of (84), which uses potentially troublesome binomial coefficients, we write

$$\left.\begin{aligned} M_s &\equiv M(s, \alpha; b) = M(s, \alpha - 1; b) - M(s-1, \alpha - 1; b) \\ N_t &\equiv N(t, \gamma; b) \equiv M(t, \gamma; -b) \end{aligned}\right\} \tag{99}$$

The recurrence (99) uses the starting values

$$\left.\begin{aligned} M(0, \alpha; b) &= 1 \\ M(s, s-1; b) &= (-1)^s \frac{b^s}{s!} \end{aligned}\right\} \tag{100}$$

Equations (99) state that N_t may be obtained using the algorithm for M_s by substituting t and γ for s and α and by writing $-b$ in place of $+b$.

Dropping a few subscripts, we copy from Lyddane

$$\frac{dM}{db} = -M(s-1,\, \alpha-1;\, b)$$

$$\frac{dN}{db} = +M(t-1,\, \gamma-1;\, -b)$$

implies

$$\frac{dM}{d\beta^2} = \frac{dM}{db}\frac{db}{d\beta^2} = \frac{+2Q}{(1+\beta^2)^2} M(s-1,\, \alpha-1;\, b)$$

$$\frac{dN}{d\beta^2} = \frac{-2Q}{(1+\beta^2)^2} M(t-1,\, \gamma-1;\, -b) \qquad (101)$$

We now return to (89) and substitute (90) and (101):

$$\frac{d\widetilde{X}}{d\beta^2} = \left[\frac{n}{(1+\beta^2)} + \frac{2n-1}{(1-\beta^2)}\right]\widetilde{X} + 2(-1)^{|q|}(n-2p+q)\frac{(1+\beta^2)^{n-2}}{(1-\beta^2)^{2n-1}}\sum_{j=0}^{\infty}\left[M(s-1,\, \alpha-1;\, b)\right.$$

$$\left. * M(t,\, \gamma;\, -b) - M(s,\, \alpha;\, b)\, M(t-1,\, \gamma-1;\, -b)\right]\beta^{2j}$$

$$+ (-1)^{|q|}\frac{(1+\beta^2)^{n}}{(1-\beta^2)^{2n-1}}\sum_{j=1}^{\infty} M(s,\, \alpha;\, b)\, M(t,\, \gamma;\, -b)\,(j\,\beta^{2j-2}) \qquad (102)$$

This equation takes the place of (96) in (98).

The fifth factor in the disturbing function is

$$C = \frac{\xi + (i\,\epsilon\, sgn\, q)\,\eta}{\sqrt{4L - (\xi^2+\eta^2)}} \qquad (103)$$

where

$$4L - \left(\xi^2 + \eta^2\right) = 2\left(L + G\right)$$

The three partial derivatives of C become

$$\left.\begin{aligned} \frac{\partial C}{\partial L} &= \frac{-C}{(L+G)} \\ \frac{\partial C}{\partial \xi} &= \frac{1}{\sqrt{2(L+G)}}\left[1 + \frac{\xi C}{\sqrt{2(L+G)}}\right] \\ \frac{\partial C}{\partial \eta} &= \frac{1}{\sqrt{2(L+G)}}\left[(i\,\epsilon\, sgn\, q) + \frac{\eta C}{\sqrt{2(L+G)}}\right] \end{aligned}\right\} \tag{104}$$

Finally, with (57) and (60),

$$\begin{aligned} D &= e^{i\,\epsilon\left[(n-2p+q)\lambda' + (n-m)\frac{\pi}{2}\right]} e^{-i\,m\,\dot{\theta}\,t} \\ \frac{dD}{d\lambda'} &= i\,\epsilon\,(n-2p+q)D \end{aligned} \tag{105}$$

SUMMARY OF RESULTS

Throughout this section, the abbreviation R was used for the individual term $R_{n\,m\,p\,q}$. As in (59), we write

$$R = A\,\widetilde{F}\,B^{|\nu|}\,\widetilde{X}\,C^{|q|}\,D \tag{106}$$

where

$$\left.\begin{aligned}
A &= \frac{\mu^{n+2} R_E^n}{L^{2n+2}} && = A(L) \\
\widetilde{F} &= (-1)^{\delta} \frac{(n-r')!\,(n+r')!}{2^n\,(n-m)!p!\,(n-p)!} (c')^b\, P_{n-r'}^{a,b}(x') && = \widetilde{F}(L, \xi, \eta, \sigma, \tau) \\
B &= \frac{\sigma + (i\,\epsilon\, sgn\,\nu)\,\tau}{\sqrt{4L - 2(\xi^2 + \eta^2)}} && = B(L, \xi, \eta, \sigma, \tau) \\
\widetilde{X} &= (-1)^{|q|} \frac{(1+\beta^2)^n}{(1-\beta^2)^{2n-1}} \sum_{j=0}^{\infty} M_s N_t \beta^{2j} && = \widetilde{X}(L, \xi, \eta) \\
C &= \frac{\xi + (i\,\epsilon\, sgn\, q)\,\eta}{\sqrt{4L - (\xi^2 + \eta^2)}} && = C(L, \xi, \eta) \\
D &= e^{i\,\epsilon\left[(n-2p+q)\lambda' + (n-m)\frac{\pi}{2}\right]} e^{-i\,m\,\dot{\theta}\,t} && = D(\lambda')
\end{aligned}\right\} \tag{107}$$

From (61) and (107) it follows that the final partials of R with respect to the Poincaré variables are

$$\left.\begin{aligned}
\frac{\partial R}{\partial L} &= \frac{R}{A}\frac{dA}{dL} + \frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial L} + |\nu|\frac{R}{B}\frac{\partial B}{\partial L} + \frac{R}{\widetilde{X}}\frac{\partial \widetilde{X}}{\partial L} + |q|\frac{R}{C}\frac{\partial C}{\partial L} \\
\frac{\partial R}{\partial \nu} &= \frac{R}{\widetilde{F}}\frac{\partial \widetilde{X}}{\partial \xi} + |\nu|\frac{R}{B}\frac{\partial B}{\partial \xi} + \frac{R}{\widetilde{X}}\frac{\partial \widetilde{X}}{\partial \xi} + |q|\frac{R}{C}\frac{\partial C}{\partial \xi} \\
\frac{\partial R}{\partial \eta} &= \frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial \eta} + |\nu|\frac{R}{B}\frac{\partial B}{\partial \eta} + \frac{R}{\widetilde{X}}\frac{\partial \widetilde{X}}{\partial \eta} + |q|\frac{R}{C}\frac{\partial C}{\partial \eta} \\
\frac{\partial R}{\partial \sigma} &= \frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial \sigma} + |\nu|\frac{R}{B}\frac{\partial B}{\partial \sigma} \\
\frac{\partial R}{\partial \tau} &= \frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial \tau} + |\nu|\frac{R}{B}\frac{\partial B}{\partial \tau} \\
\frac{\partial R}{\partial \lambda'} &= \frac{R}{D}\frac{dD}{d\lambda'}
\end{aligned}\right\} \tag{108}$$

From the preceding pages, we find

$$\frac{R}{A}\frac{dA}{dL} = -(2n+2)\frac{R}{L} \tag{109}$$

$$\left.\begin{aligned}
\frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial L} &= \frac{R}{\widetilde{F}}\frac{(s')^2}{2\,c'\,G}\frac{d\widetilde{F}}{dc'} \\
\frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial \xi} &= \frac{R}{\widetilde{F}}(-\xi)\frac{(s')^2}{2\,c'\,G}\frac{d\widetilde{F}}{dc'} \\
\frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial \eta} &= \frac{R}{\widetilde{F}}(-\eta)\frac{(s')^2}{2\,c'\,G}\frac{d\widetilde{F}}{dc'} \\
\frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial \sigma} &= \frac{R}{\widetilde{F}}\frac{-\sigma}{4\,c'\,G}\frac{d\widetilde{F}}{dc'} \\
\frac{R}{\widetilde{F}}\frac{\partial \widetilde{F}}{\partial \tau} &= \frac{R}{\widetilde{F}}\frac{-\tau}{4\,c'\,G}\frac{d\widetilde{F}}{dc'}
\end{aligned}\right\} \tag{110}$$

where

$$\frac{\partial \widetilde{F}}{\partial c'} = (-1)^{\delta}\frac{(n-r')!(n+r')!}{2^n(n-m)!p!(n-p)!}\left[(c')^{b-1}\right]$$

$$* \left[bP^{a,b}_{n-r'}(x') + 4(c')^2\frac{dP^{a,b}_{n-r'}(x')}{dx'}\right] \tag{111}$$

and

$P^{a,b}_{n-r'}(x')$ obtained with (65) and (66),

$\dfrac{dP^{a,b}_{n-r'}(x')}{dx'}$ obtained with (71) and (72).

$$\left.\begin{aligned}
|\nu|\frac{R}{B}\frac{\partial B}{\partial L} &= \frac{-|\nu|\,R}{2G} \\
|\nu|\frac{R}{B}\frac{\partial B}{\partial \xi} &= \frac{|\nu|\,\xi\,R}{2G} \\
|\nu|\frac{R}{B}\frac{\partial B}{\partial \eta} &= \frac{|\nu|\,\eta\,R}{2G} \\
|\nu|\frac{R}{B}\frac{\partial B}{\partial \sigma} &= \frac{|\nu|\,R}{2\sqrt{G}B} \\
|\nu|\frac{R}{B}\frac{\partial B}{\partial \tau} &= \frac{|\nu|(i\,\epsilon\,sgn\,\nu)R}{2\sqrt{G}B}
\end{aligned}\right\} \tag{112}$$

Test if $|\nu| = 0$. If yes, put all (112) to zero and skip (118b).

$$\left.\begin{aligned}
\frac{R}{\widetilde{X}}\frac{\partial \widetilde{X}}{\partial L} &= \frac{R}{\widetilde{X}}\frac{2(G-L)}{(L+G)^2}\frac{d\widetilde{X}}{d\beta^2} \\
\frac{R}{\widetilde{X}}\frac{\partial \widetilde{X}}{\partial \xi} &= \frac{R}{\widetilde{X}}\frac{2\xi}{(L+G)^2}\frac{d\widetilde{X}}{d\beta^2} \\
\frac{R}{\widetilde{X}}\frac{\partial \widetilde{X}}{\partial \eta} &= \frac{R}{\widetilde{X}}\frac{2\eta\,L}{(L+G)^2}\frac{d\widetilde{X}}{d\beta^2}
\end{aligned}\right\} \tag{113}$$

where

$$\frac{d\widetilde{X}}{d\beta^2} = \left[\frac{n}{(1+\beta^2)} + \frac{2n-1}{(1-\beta^2)}\right]\widetilde{X} - 2(-1)^{|q|}\frac{(1+\beta^2)^{n-1}}{(1-\beta^2)^{2n}}\sum_{j=0}^{\infty}\left[M_s\,(Nw) + (Mv)\,N_t\right]\beta^{2j}$$

$$+ (-1)^{|q|}\frac{(1+\beta^2)^{n}}{(1-\beta^2)^{2n-1}}\sum_{j=1}^{\infty}M_s N_t\left[j\beta^{2j-2}\right] \tag{114}$$

with M_s, N_t, (Mv), and (Nw) given by (84) and (95), or where

$$\frac{d\widetilde{X}}{d\beta^2} = \left[\frac{n}{(1+\beta^2)} + \frac{2n-1}{(1-\beta^2)}\right]\widetilde{X} + 2(-1)^{|q|}\frac{(1+\beta^2)^{n-2}}{(1-\beta^2)^{2n-1}}$$

$$* \sum_{j=0}^{\infty} \left[M(s-1,\alpha-1;b)M(t,\gamma;-b) - M(s,\alpha;b)M(t-1,\gamma-1;-b)\right]\beta^{2j}$$

$$+ (-1)^{|q|}\frac{(1+\beta^2)^{n}}{(1-\beta^2)^{2n-1}} \sum_{j=1}^{\infty} M(s,\alpha;b)M(t,\gamma;-b)\left[j\beta^{2j-2}\right] \tag{115}$$

with all M defined by (99) and (100).

$$\left.\begin{aligned} |q|\frac{R}{C}\frac{\partial C}{\partial L} &= \frac{-|q|R}{(L+G)} \\ |q|\frac{R}{C}\frac{\partial C}{\partial \xi} &= \frac{R}{C}\frac{|q|}{\sqrt{2(L+G)}}\left[1 + \frac{\xi C}{\sqrt{2(L+G)}}\right] \\ |q|\frac{R}{C}\frac{\partial C}{\partial \eta} &= \frac{R}{C}\frac{|q|}{\sqrt{2(L+G)}}\left[(i\,\epsilon\, sgn\, q) + \frac{\eta C}{\sqrt{2(L+G)}}\right] \end{aligned}\right\} \tag{116}$$

If $|q| = 0$, put all (116) to zero and skip (118d).

$$\frac{R}{D}\frac{dD}{d\lambda'} = i\,\epsilon(n-2p+q)R \tag{117}$$

All ingredients required in the preceding pages are defined within this document; they will be compiled again in the final algorithm.

Expressions such as $R/\widetilde{F}$ are convenient short-hand notation. Since they involve potential zero divisors, they should be computed as follows:

$$\frac{R}{\widetilde{F}} = AB^{|\nu|}\widetilde{X}C^{|q|}D \tag{118a}$$

$$\frac{R}{B} = A\widetilde{F}B^{|\nu|-1}\widetilde{X}C^{|q|}D \qquad \text{skip if } |\nu| = 0 \tag{118b}$$

$$\frac{R}{\widetilde{X}} = A\widetilde{F}B^{|\nu|}C^{|q|}D \tag{118c}$$

$$\frac{R}{C} = A\widetilde{F}B^{|\nu|}\widetilde{X}C^{|q|-1}D \qquad \text{skip if } |q| = 0 \tag{118d}$$

With the aid of (108), we can now compute the six partial derivatives which we will label

$$\left(\frac{\partial R}{\partial y}\right)_{nmpq}, \qquad y = L,\ \xi,\ \sigma,\ \lambda',\ \eta,\ \tau \tag{119}$$

Compute

$$\left(\frac{\partial R}{\partial y}\right)_{nm} = \sum_{p=0}^{n}\sum_{q=-\infty}^{\infty}\left(\frac{\partial R}{\partial y}\right)_{nmpq}, \qquad y = L, \ldots \tag{120}$$

$$\frac{\partial}{\partial y}\triangle F_{2T} = \sum_{n=2}^{n}\sum_{m=1}^{n}\left[\gamma_{nm}\left(\frac{\partial R}{\partial y}\right)_{nm} + \text{complex conjugate}\right], \quad y = L, \ldots \tag{121}$$

Finally,

$$\begin{aligned}
\dot{L}_{2T} &= \frac{\partial \triangle F_{2T}}{\partial \lambda'} \qquad & \dot{\lambda}'_{2T} &= -\frac{\partial \triangle F_{2T}}{\partial L} \\
\dot{\xi}_{2T} &= \frac{\partial \triangle F_{2T}}{\partial \eta} \qquad & \dot{\eta}_{2T} &= -\frac{\partial \triangle F_{2T}}{\partial \xi} \\
\dot{\sigma}_{2T} &= \frac{\partial \triangle F_{2T}}{\partial \tau} \qquad & \dot{\tau}_{2T} &= -\frac{\partial \triangle F_{2T}}{\partial \sigma}
\end{aligned} \tag{122}$$

Combine (122) with Lyddane's equation of motion. The later may be found in Reference 2, page A-26, Fortran lines 0648 to 0653.

THE ANALYTIC SOLUTION

PERTURBATIONS IN DELAUNAY VARIABLES

Each Delaunay variable can be written in the form

$$L = L_0 + \delta L \tag{123}$$

where δL is the perturbation due to the tesseral field. The Hamiltonian is, for one term in n, m, p, q,

$$\begin{aligned} F &= F_0 + F_{2T} \\ &= \frac{\mu^2}{2L^2} + \left(\gamma R + \overline{\gamma}\overline{R}\right)_{nmpq} \end{aligned} \tag{124}$$

For brevity, the subscript n, m, p, q will be dropped for now. The familiar canonical equations of motion are

$$\left.\begin{array}{ll} \dot{L} = F_l & \dot{l} = -F_L \\ \dot{G} = F_g & \dot{g} = -F_G \\ \dot{H}' = F_{h'} & \dot{h}' = -F_{H'} \end{array}\right\} \tag{125}$$

We take the time-derivative of (123), apply (125), and substitute (124). Since $\dot{L}_0 = 0$, it follows that

$$\left.\begin{aligned} \delta\dot{L} &= \left(\gamma R_l + \overline{\gamma}\overline{R}_l\right) \\ \delta\dot{G} &= \left(\gamma R_g + \overline{\gamma}\overline{R}_g\right) \\ \delta\dot{H}' &= \left(\gamma R_{h'} + \overline{\gamma}\overline{R}_{h'}\right) \end{aligned}\right\} \tag{126}$$

However, for l we find

$$\begin{aligned} \dot{l} &= \dot{l}_0 + \delta\dot{l} \\ &= -F_L \\ &= -F_{0L} - \left(\gamma R_L + \overline{\gamma}\overline{R}_L\right) \\ &= -\left[\left(F_{0L}\right)_{L_0} + F_{0LL}\delta L\right] - \left(\gamma R_L + \overline{\gamma}\overline{R}_L\right) \end{aligned}$$

Since

$$\dot{l}_0 = -\left(F_{0L}\right)_{L_0}$$

there follows

$$\left.\begin{aligned} \delta\dot{l} &= -\left(\gamma R_L + \overline{\gamma}\overline{R}_L\right) - 3\frac{\mu^2}{L^4}\delta L \\ \delta\dot{g} &= -\left(\gamma R_G + \overline{\gamma}\overline{R}_G\right) \\ \delta\dot{h}' &= -\left(\gamma R_{H'} + \overline{\gamma}\overline{R}_{H'}\right) \end{aligned}\right\} \tag{127}$$

The disturbing function may be written in the form

$$R = C\left(L, G, H'\right) e^{i\psi'\left(l, g, h'\right)} \tag{128}$$

so that

$$R_l = i\psi_l' R \tag{128a}$$

Substitution into (126) yields

$$\delta\dot{L} = i\psi_l' \left(\gamma R - \overline{\gamma}\overline{R}\right)$$

which integrates to

$$\left.\begin{aligned} \delta L &= \psi_l' \frac{\gamma R + \overline{\gamma}\overline{R}}{\dot{\psi}'} \\ \delta G &= \psi_g' \frac{\gamma R + \overline{\gamma}\overline{R}}{\dot{\psi}'} \\ \delta H' &= \psi_{h'}' \frac{\gamma R + \overline{\gamma}\overline{R}}{\dot{\psi}'} \end{aligned}\right\} \tag{129}$$

For reasons to be seen later, these equatins will be put into a different form. Because of (128a), we write

$$\gamma R + \overline{\gamma}\overline{R} = \frac{-i}{\psi_l} \left(\gamma R_l - \overline{\gamma}\overline{R_l}\right)$$

and similar equations for g and h'. Then (129) becomes

$$\left.\begin{aligned} \delta L &= \frac{-i}{\dot{\psi}'} \left(\gamma R_l - \overline{\gamma}\overline{R}_l\right) \\ \delta G &= \frac{-i}{\dot{\psi}'} \left(\gamma R_g - \overline{\gamma}\overline{R}_g\right) \\ \delta H' &= \frac{-i}{\dot{\psi}'} \left(\gamma R_{h'} - \overline{\gamma}\overline{R}_{h'}\right) \end{aligned}\right\} \tag{130}$$

In order to solve (127), substitute the first of (129) for δL:

$$\delta\dot{l} = -\left(\gamma R_L + \overline{\gamma}\overline{R}_L\right) - 3\frac{\mu^2}{L^4}\psi_l'\frac{\gamma R + \overline{\gamma}\overline{R}}{\dot{\psi}'}$$

This integrates to

$$\left.\begin{aligned} \delta l &= -\frac{\gamma R_L - \overline{\gamma}\overline{R}_L}{i\dot{\psi}'} - 3\frac{\mu^2}{L^4}\psi_l'\frac{\gamma R - \overline{\gamma}\overline{R}}{i(\dot{\psi}')^2} \\ \delta g &= -\frac{\gamma R_G - \overline{\gamma}\overline{R}_G}{i\dot{\psi}'} \\ \delta h' &= -\frac{\gamma R_{H'} - \overline{\gamma}\overline{R}_{H'}}{i\dot{\psi}'} \end{aligned}\right\} \tag{131}$$

Above equations require ψ_l' and $\dot{\psi}'$. Write (28a) in the form

$$\psi' = \epsilon\left[Ql + (n-2p)g + mh' - \epsilon\, m\,\dot{\theta}\, t + (n-m)\frac{\pi}{2}\right] \tag{132}$$

There follows

$$\psi_l' = \epsilon Q \tag{133}$$

and

$$\dot{\psi}' = \epsilon\left[Q(n_0 + \dot{l}_2) + (n-2p)\dot{g}_2 + m\dot{h}_2' - \epsilon\, m\,\dot{\theta}\right] \tag{134}$$

where

$$Q = n - 2p + q.$$

Upon substitution of (133) into (131), the tesseral perturbations in Delaunay variables become

$$\left.\begin{aligned}
\delta L &= \frac{-i}{\dot{\psi}'}\left(\gamma R_l - \overline{\gamma}\overline{R}_l\right) \\
\delta G &= \frac{-i}{\dot{\psi}'}\left(\gamma R_g - \overline{\gamma}\overline{R}_g\right) \\
\delta H' &= \frac{-i}{\dot{\psi}'}\left(\gamma R_{h'} - \overline{\gamma}\overline{R}_{h'}\right) \\
\delta l &= \frac{i}{\dot{\psi}'}\left(\gamma R_L - \overline{\gamma}\overline{R}_L\right) + i3\frac{\mu^2}{L^4}\frac{\epsilon Q}{(\dot{\psi}')^2}\left(\gamma R - \overline{\gamma}\overline{R}\right) \\
\delta g &= \frac{i}{\dot{\psi}'}\left(\gamma R_G - \overline{\gamma}\overline{R}_G\right) \\
\delta h' &= \frac{i}{\dot{\psi}'}\left(\gamma R_{H'} - \overline{\gamma}\overline{R}_{H'}\right)
\end{aligned}\right\} \qquad (135)$$

PERTURBATIONS IN POINCARÉ VARIABLES

Expressing the tesseral perturbations in terms of Poincaré variables follows a suggestion by Cohen.

Let

$$X = (L,\, G,\, H',\, l,\, g,\, h')^T$$

and

$$Y = (L,\, \xi,\, \sigma,\, \lambda',\, \eta,\, \tau)^T$$

Then (135) may be written as

$$X = X^0 + U\left(\frac{\gamma R - \overline{\gamma}\overline{R}}{i\dot{\psi}'}\right)_X \tag{136}$$

$$U = \begin{pmatrix} O_3 & I_3 \\ -I_3 & O_3 \end{pmatrix}$$

Upon making the canonical transformation to Poincaré variables, we obtain

$$\begin{aligned} Y(X) &= Y\left(X^0\right) + Y_{X^T} U\left(\frac{\gamma R - \overline{\gamma R}}{i\dot{\psi}'}\right)_X \\ &= Y^0 + Y_{X^T} U Y_X^T \left(\frac{\gamma \widetilde{R}(Y) - \overline{\gamma \widetilde{R}(Y)}}{i\widetilde{\dot{\psi}}'(Y)}\right)_Y \\ &= Y^0 + U\left(\frac{\gamma \widetilde{R} - \overline{\gamma \widetilde{R}}}{i\widetilde{\dot{\psi}}'}\right)_Y \end{aligned} \tag{137}$$

where the temporary tilde emphasizes that $\widetilde{R}$ and $\widetilde{\dot{\psi}}'$ are now functions of the Poincaré elements.

Note that

$$\dot{\psi}'(X) = \widetilde{\dot{\psi}}(Y) \tag{138}$$

and

$$\left(\frac{\widetilde{\gamma R} - \overline{\widetilde{\gamma R}}}{i}\right)_Y = 2\ Im\left(\widetilde{\gamma R}_Y\right), \tag{139}$$

namely twice the imaginary part of $\widetilde{\gamma R}_Y$. Substitution of (139) into (137) yields the final expressions

$$\left.\begin{aligned}
\Delta L &= \frac{2}{\dot{\psi}'}\ Im(\gamma R_{\lambda'}) \\
\Delta \xi &= \frac{2}{\dot{\psi}'}\ Im(\gamma R_{\eta}) \\
\Delta \sigma &= \frac{2}{\dot{\psi}'}\ Im(\gamma R_{\tau}) \\
\Delta \lambda' &= \frac{-2}{\dot{\psi}'}\ Im(\gamma R_L) - \frac{6\epsilon Q}{(\dot{\psi}')^2}\frac{\mu^2}{L^4}\ Im(\gamma R) \\
\Delta \eta &= \frac{-2}{\dot{\psi}'}\ Im(\gamma R_{\xi}) \\
\Delta \tau &= \frac{-2}{\dot{\psi}'}\ Im(\gamma R_{\sigma})
\end{aligned}\right\} \tag{140}$$

where $\dot{\psi}'$ is obtained by evaluation of (134). The various partial derivatives of the disturbing function R may be calculated from (108).

PARTIAL DERIVATIVES OF PSI DOT

The prime used in the derivations above flagged those variables which were "modified" Poincaré elements or functions thereof. This prime will be dropped temporarily so that it may be available for a different purpose.

As seen in the last section, we may write

$$\Delta P = U \left[\frac{2\ Im(\gamma R)}{\dot{\psi}} \right]_P , \tag{141}$$

were now

$$P^T = (L, \xi, \sigma, \lambda, \eta, \tau).$$

Equation (141) yields

$$\Delta P = U \left[\frac{2}{\dot{\psi}}\ Im(\gamma R_P) - \frac{2}{\dot{\psi}^2}\ Im(\gamma R)\, \dot{\psi}_P \right] \tag{142}$$

When the first term in (142) was developed, the assumption was made that the second term is negligible. This premise proved erroneous. The missing terms will now be derived, following Cohen's approach.

Let us write (134) in the form

$$\dot{\psi} = \dot{\psi}^0 + \Delta\dot{\psi},$$

where

$$\dot{\psi}^0 = \epsilon Q n_0 - m\dot{\theta}$$

and

$$\Delta\dot{\psi} = -F_{1_x T}\, \psi_y. \tag{143}$$

Here

$$x = \begin{pmatrix} L \\ G \\ \epsilon H \end{pmatrix}, \qquad y = \begin{pmatrix} l \\ g \\ \epsilon h \end{pmatrix}$$

$$F_1 = \frac{\mu^4 C_{20} R_E^2}{4L^3 G^3} \left(3\frac{H^2}{G^2} - 1 \right)$$

and

$$\psi = \epsilon \left[Ql + (Q-q)\, g + \epsilon m\, (h-\theta) + (n-m)\, \frac{\pi}{2} \right]$$

In preparation for subsequent steps, we make a canonical transformation to the variables

$$x' = \begin{pmatrix} L \\ L-G \\ G-\epsilon H \end{pmatrix}, \qquad y' = \begin{pmatrix} \lambda \\ -g-\epsilon h \\ -\epsilon h \end{pmatrix}$$

and obtain

$$F_1' = \frac{\mu^4 C_{20} R_E^2}{4L^6} \left[\frac{L}{L-(L-G)} \right]^3 \left[3 \left(\frac{L-(L-G)-(G-\epsilon H)}{L-(L-G)} \right)^2 - 1 \right]$$

$$\psi' = \epsilon \left[Q\lambda + q\,(-g-\epsilon h) + (n-m-2p)\,(-\epsilon h) - \epsilon m\theta + (n-m)\, \frac{\pi}{2} \right]$$

$$\triangle \dot{\psi}' = -\epsilon Q F_{1,L}' - \epsilon q F_{1,L-G}' - \epsilon \nu F_{1,G-\epsilon H}' \tag{144}$$

where

$$\nu = n - m - 2p.$$

The leading part of $\dot{\psi}_P$ is

$$\dot{\psi}_P^0 = \epsilon Q n_{0,P} = -\frac{3}{L}\epsilon Q n_0 \delta_{L,P},$$

where δ is the Kronecker delta. For $\Delta\dot{\psi}_P$, we use

$$\begin{aligned} F'_{1,x'PT} &= F'_{1,x'x'T}\, x'_{PT} \\ &= F'_{1x'x'T} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & \eta & 0 \\ 0 & 0 & \sigma & 0 & 0 & \tau \end{pmatrix} \end{aligned} \tag{145}$$

For the sake of readability, we abbreviate

$$L - G = \alpha$$

$$G - \epsilon H = \beta$$

Then

$$F_1' = KL^{-3}(L-\alpha)^{-3}\left[3\left(\frac{L-\alpha-\beta}{L-\alpha}\right)^2 - 1\right]$$

where

$$K = \frac{1}{4}\mu^4 C_{20} R_E^2$$

Then the required partial derivatives become

$$\left.\begin{aligned}\Delta\dot{\psi}_{(L,\lambda)} &= -(1,0)\,\epsilon\left(QF'_{1L,L} + qF'_{1L,\alpha} + \nu F'_{1L,\beta}\right)\\ \Delta\dot{\psi}_{(\xi,\eta)} &= -(\xi,\eta)\,\epsilon\left(QF'_{1L,\alpha} + qF'_{1\alpha,\alpha} + \nu F'_{1\alpha,\beta}\right)\\ \Delta\dot{\psi}_{(\sigma,\tau)} &= -(\sigma,\tau)\,\epsilon\left(QF'_{1L,\beta} + qF'_{1\alpha,\beta} + \nu F'_{1\beta,\beta}\right)\end{aligned}\right\} \tag{146}$$

After several pages of algebra, the six partials in (146) may be written

$$\frac{1}{K}F'_{1L,L} = 6L^{-5}(L-\alpha)^{-5}\left[2(2L-\alpha)^2 - L(L-\alpha)\right]\left[3\left(\frac{L-\alpha-\beta}{L-\alpha}\right)^2 - 1\right]$$

$$-\,6L^{-4}(L-\alpha)^{-7}\,\beta\left[3(5L-2\alpha)(L-\alpha-\beta) - L(L-\alpha)\right] \tag{147a}$$

$$\frac{1}{K}F'_{1L,\alpha} = -3L^{-4}(L-\alpha)^{-5}(7L-3\alpha)\left[3\left(\frac{L-\alpha-\beta}{L-\alpha}\right)^2 - 1\right]$$

$$+\,6L^{-4}(L-\alpha)^{-7}\,\beta\left[3(4L-\alpha)(L-\alpha-\beta) - L(L-\alpha)\right] \tag{147b}$$

$$\frac{1}{K}F'_{1L,\beta} = 18L^{-4}(L-\alpha)^{-5}(2L-\alpha)(L-\alpha-\beta)$$

$$+\,6L^{-3}(L-\alpha)^{-6}(L-\alpha-2\beta) \tag{147c}$$

$$\frac{1}{K}F'_{1\alpha,\alpha} = 12L^{-3}(L-\alpha)^{-5}\left[3\left(\frac{L-\alpha-\beta}{L-\alpha}\right)^2 - 1\right]$$

$$-\,6L^{-3}(L-\alpha)^{-6}\,\beta\left[9\left(\frac{L-\alpha-\beta}{L-\alpha}\right) - 1\right] \tag{147d}$$

$$\frac{1}{K}F'_{1\alpha,\beta} = -18L^{-3}(L-\alpha)^{-5}\left(\frac{L-\alpha-\beta}{L-\alpha}\right)$$

$$-6L^{-3}(L-\alpha)^{-6}(L-\alpha-2\beta) \qquad (147e)$$

$$\frac{1}{K}F'_{1\beta,\beta} = 6L^{-3}(L-\alpha)^{-5} \qquad (147f)$$

where K, α, and β were defined earlier. However, α and β may also be computed from

$$\left.\begin{aligned}\alpha &= \frac{1}{2}(\xi^2+\eta^2)\\ \beta &= \frac{1}{2}(\sigma^2+\tau^2)\end{aligned}\right\} \qquad (148)$$

With the abbreviations

$$\left.\begin{aligned}D_1 &= \epsilon\left(QF'_{1L,L}+qF'_{1L,\alpha}+\nu F'_{1L,\beta}\right)\\ D_2 &= \epsilon\left(QF'_{1L,\alpha}+qF'_{1\alpha,\alpha}+\nu F'_{1\alpha,\beta}\right)\\ D_3 &= \epsilon\left(QF'_{1L,\beta}+qF'_{1\alpha,\beta}+\nu F'_{1\beta,\beta}\right)\end{aligned}\right\} \qquad (149)$$

equations (146) become

$$\left.\begin{aligned}\Delta\dot{\psi}_L &= -D_1 & \Delta\dot{\psi}_\lambda &= 0\\ \Delta\dot{\psi}_\xi &= -\xi D_2 & \Delta\dot{\psi}_\eta &= -\eta D_2\\ \Delta\dot{\psi}_\sigma &= -\sigma D_3 & \Delta\dot{\psi}_\tau &= -\tau D_3\end{aligned}\right\} \qquad (150)$$

with all members known.

Finally, we define

$$FAC = \frac{2}{\dot{\psi}}$$

$$FAD = \frac{FAC \cdot Im(\gamma R)}{\dot{\psi}}$$

and write the increment to Equations (135) in vector form:

$$DP \begin{pmatrix} \eta \\ \xi \\ \tau \\ \sigma \\ \lambda \\ \mathrm{L} \end{pmatrix} = FAD \begin{pmatrix} \triangle \dot{\psi}_\xi \\ -\triangle \dot{\psi}_\eta \\ \triangle \dot{\psi}_\sigma \\ -\triangle \dot{\psi}_\tau \\ \triangle \dot{\psi}_L \\ 0 \end{pmatrix} \tag{151}$$

RESONANCE

CLASSIFICATION BY PERIOD

Equation (28a) may be written as

$$\psi = (n - 2p + q)l + (n - 2p)(g + h) - \nu h - \epsilon m\dot{\theta}t + (n - m)\frac{\pi}{2}, \tag{152}$$

if we disregard ϵ and subscripts for the time being.

Reintroducing the abbreviations

$$\left.\begin{aligned} Q &= n - 2p + q \\ r &= n - 2p \end{aligned}\right\} \tag{153}$$

the time derivative of (123) becomes

$$\dot{\psi} = Q(n_0 + \dot{l}_2) - m\dot{\theta} + r\dot{g}_2 + m\dot{h}_2 \tag{154}$$

where the quantities $\dot{l}_2$, $\dot{g}_2$, and $\dot{h}_2$ are the first-order secular rates of these elements, due to $J_2 = -C_{20}$. Formulae for the rates are found throughout the pertinent literature as, for example, in Brouwer's (Reference 5) celebrated 1959 paper:

$$\left.\begin{aligned} \dot{l}_2 &= \frac{1}{2}C\left(1 - 3\cos^2 I\right)\sqrt{1-e^2} \\ \dot{g}_2 &= \frac{1}{2}C\left(1 - 5\cos^2 I\right) \\ \dot{h}_2 &= C\cos I \end{aligned}\right\} \tag{155}$$

where

$$C = \frac{-1.5J_2R_E^2}{(1-e^2)^2}\frac{\mu^4}{L^7}$$

Equation (154) is used to examine the frequency of each possible combination m, Q, r. While the period of $\dot{\psi}$ is short for most terms, it may become large for some values of Q and m. When the period is long , the associated perturbations become large, and the term may be labeled resonant. To be more specific, a term is

$$\left.\begin{aligned} \text{short-periodic if} \quad & \frac{\dot{\theta}}{\tau_S} \leq |\dot{\psi}| \\ \text{in shallow resonance if} \quad & \frac{\dot{\theta}}{\tau_D} \leq |\dot{\psi}| < \frac{\dot{\theta}}{\tau_S} \\ \text{in deep resonance if} \quad & |\dot{\psi}| < \frac{\dot{\theta}}{\tau_D} \end{aligned}\right\} \tag{156}$$

Theoretical considerations suggest that $\tau_D = 15$ days and $\tau_S = 0.5$ days are reasonable initial trial values. However, these two quantities are inputs to the program, to be set according to the application at hand. Since the inequalities (156) are a bit hard to visualize, the following should help clarify the procedure: if the critical period $T_{\dot{\psi}}$

$15^d < T_{\dot{\psi}}$,	the term is integrated numerically,
$0.5^d < T_{\dot{\psi}} \leq 15^d$,	the term is evaluated analytically,
$T_{\dot{\psi}} \leq 0.5^d$,	the term is ignored, except for a few cases of low degree and order.

Since $m = 1$ will only be of interest in dealing with geostationary orbits, we will consider all m from 2 to n_{max}. For each such m and each r in $(-n_{max}, n_{max})$, obtain the resonant indices Q as follows.

Let

$$\alpha = \frac{m\left(\dot{\theta} - \dot{h}_2\right) - r\dot{g}_2}{n_0 + \dot{l}_2}$$

so that

$$Q = \alpha + \frac{\dot{\psi}}{n_0 + \dot{l}_2}$$

Also, let

$$\beta = \frac{\dot{\theta}}{n_0 + \dot{l}_2}$$

Consider now Figure 2.

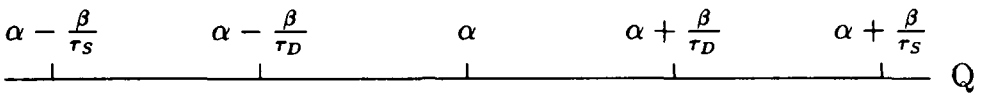

Figure 2. Resonance Conditions for Q

If there is an integer Q in the interval $\alpha - \beta/\tau_D$ to $\alpha + \beta/\tau_D$, we are dealing with deep resonance. If there are any other integers Q in $(\alpha - \beta/\tau_S, \alpha - \beta/\tau_D)$ or in $(\alpha + \beta/\tau_D, \alpha + \beta/\tau_S)$, they lead to shallow resonance.

Assuming, once again, that $\tau_S = 0.5$, the spread in possible Q_S is 4β. If $4\beta < 1$, there can be, at most, one Q. This situation may occur for orbits with periods of less than six hours. If $1 \leq 4\beta < 2$, there can be, at most, two Qs. Such conditions may arise for orbits with periods from six to 12 hours.

We have now identified all resonant terms with indices m, Q, r. Since the algorithm operates in n, m, p, q-space, proceed as follows:

n assumes all values from n_{min} to n_{max}. All

m have been identified; they must also obey $m \leq n$.

p $= (n - r)/2$, but only if $n - r$ is even, and

q $= Q - r$.

The computer will write two files. One will contain all n, m, p, q combinations identified as shallow resonance, the other all those that are in deep resonance.

THE AMPLITUDE TEST

Although any given term may have a period T_ψ large enough to classify it as resonant, its amplitutde can still render it insignificant. Since a first-order theory neglects second order perturbation by design, it serves no purpose to carry terms of order two and smaller.

Cohen developed a test in which the estimated amplitude in the mean anomaly (or longitude) is compared to a tolerance of order two. Considering Equations (135) or (140), he approximates the amplitude as

$$Ampl\left|\left(\frac{\gamma A\overline{F}X}{C_N\dot{\psi}}\right)_L\right| \simeq \left|\frac{\gamma A}{C_N\dot{\psi}}\left(\left(\frac{3Qn_0}{L\dot{\psi}}-\frac{2_n+2}{L}\right)\overline{F}X+s_L\overline{F}_SX+\beta_L\overline{F}X_\beta\right)\right| \qquad (157)$$

where the disturbing function R is defined in (106) and (107), and where C_N is the normalization coefficient

$$C_N=\left[\frac{(n-m)!\,(2n+1)\,(2-\delta_{om})}{(n+m)!}\right]^{\frac{1}{2}}$$

A is defined in (61a), $\overline{F}$ is the normalized inclination function

$$\overline{F}=C_Ns^{|\nu|}\widetilde{F}$$

with the latter given by (44). Moreover, X is the classical Hansen coefficient

$$X=\beta^{|q|}\widetilde{X},$$

where

$$\beta=\frac{e}{1+\sqrt{1-e^2}}$$

and $\widetilde{X}$ defined by (46). The two products of partial derivatives in (157) are found to be

$$s_L\overline{F}_s=\frac{-2C_Ns^{|\nu|}}{4L-(\xi^2+\eta^2)}\left[|\nu|\,\widetilde{F}-\frac{s^2}{c}\frac{d\widetilde{F}}{dc}\right] \qquad (158)$$

and

$$\beta_L X_\beta = \frac{-2\beta^{|q|}}{4L - (\xi^2 + \eta^2)} \left[|q| \, \widetilde{X} + 2\beta^2 \frac{d\widetilde{X}}{d\beta^2} \right] \tag{159}$$

The two derivatives $d\widetilde{F}/dc$ and $d\widetilde{X}/d\beta^2$ have been defined by (76) and (96), with (102) as an alternative.

If the amplitude (157) for any given term in n, m, p, q exceeds the tolerance

$$tol = \left[\frac{J_2}{2(1-e^2)^2} \left(\frac{R_E}{a} \right)^2 \right], \tag{160}$$

it is retained as a resonant perturbation and calculated in accord with its classification. If not, the term is dropped.

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APPENDIX A

THE FUNCTIONS $\widetilde{F}$ and $\widetilde{X}$

The literature on orbital mechanics contains a vast collection of differenct expressions for the inclination function and the Hansen coefficient. Some of the classical forms of these functions are presented in terms of summations over products of binomial coefficients. Such expressions are often numerically unstable, and many significant figures are lost when calculations are made for large values of n and m, the degree and order of a tesseral term.

In order to overcome this difficulty, and to reduce execution time, a variety of recurrence relations have been developed and published. Cohen and Lyddane perused the literature and eventually developed serveral methods suitable for our requirements. Given below are the presently operational versions of $\widetilde{F}$ and $\widetilde{X}$. Although Lyddane has derived algorithms more elegant than our current $\widetilde{X}$, we will defer publication until numerical tests are complete.

THE INCLINATION FUNCTION $\widetilde{F}$

The current elegant form is due to Cohen who credits McClain (Reference 6), DiDonato, and Lyddane with the inspiration. Copying Equation (44),

$$\widetilde{F}^{J}_{nmp}(c',s') = (-1)^{\delta}\,\frac{(n-r')!\,(n+r)!}{2^{n}(n-m)!p!(n-p)!}\,(c')^{b}\,P^{a,b}_{n-r'}(x') \tag{161}$$

The Jacobi polynomials $P^{a,b}_{n-r'}$ are calculated from the recurrence relation (45), beginning with $P^{a,b}_{2}$ as function of $P^{a,b}_{0}$ and $P^{a,b}_{1}$. Rather than derive (161), we will demonstrate that this expression always reduces to a well-known form of the inclination function. Recall that

$$r = n - 2p \qquad \delta = \frac{1}{2}(a' - a)$$

$$a' = m - r \qquad r' = \frac{1}{2}(a + b)$$

$$a = |m - r| \qquad x' = 2(c')^{2} - 1$$

$$b = |m + r| \qquad c' = \left[1 - \frac{\sigma^{2} + \tau^{2}}{4L - 2(\xi^{2} + \eta^{2})}\right]^{\frac{1}{2}}$$

The Joacobi polynomial is defined by

$$P^{a,b}_{n-r(x')} = \sum_{j=0}^{n-r'} (-1)^{n-r'-j} \binom{n-r'+a}{j} \binom{n-r'+b}{n-r'-j} (s')^{2(n-r'-j)} (c')^{2j}$$

Now let

$$j = n - \frac{1}{2}(m - r + b) - k$$

then

$$k_2 = k(j = 0) = \frac{1}{2}(2n + \nu - |2m + \nu|)$$

since

$$\nu = n - m - 2p,$$

and

$$k_1 = k(j = n - r') = \frac{1}{2}(\nu + |\nu|)$$

Then

$$P^{a,b}_{n-r'} = \sum_{k=k_1}^{k_2} (-1)^{\frac{1}{2}(m-r-a)+k} \binom{n + \frac{1}{2}(a-b)}{\frac{1}{2}(m-r+a)+k} \binom{n - \frac{1}{2}(a-b)}{\frac{1}{2}(m-r-a)+k}$$

$$* (s')^{2(k-k_1)} (c')^{2n+\nu-b-2k}$$

Substituting this in $\widetilde{F}^J_{nmp}$,

$$\widetilde{F}^J_{nmp}(c', s') = \frac{1}{2^n (n-m)!} \frac{(n-r')!(n+r')!}{p!(n-p)!} \sum_{k=k_1}^{k_2} (-1)^k \binom{n + \frac{1}{2}(a-b)}{\frac{1}{2}(m-r+a)+k}$$

$$* \binom{n - \frac{1}{2}(a-b)}{\frac{1}{2}(m-r-a)+k} (s')^{2(k-k_1)} (c')^{2n+\nu-2k}$$

Let C^J_{nmpk} be the coefficient of $(s')^{2(k-k_1)} (c')^{2n+\nu-2k}$.

We will substitute for a, b in terms of m, r in three cases:

(1) $r \geq m$. Then

$$a = r - m, \qquad b = r + m, \qquad \frac{1}{2}(a+b) = r, \qquad \frac{1}{2}(a-b) = -m$$

$$C^J_{nmpk} = \frac{(-1)^k}{2^n(n-m)!}\frac{(n-r)!(n+r)!}{p!(n-p)!}\frac{(n-m)!(n+m)!}{k!(n-m-k)!(m-r+k)!(n+r-k)!}$$

$$= \frac{(-1)^k(n+m)!}{2^n p!(n-p)!}\binom{n+r}{k}\binom{n-r}{n-m-k}$$

(2) $m \geq r \geq -m$. Then

$$a = m - r, \qquad b = m + r, \qquad \frac{1}{2}(a+b) = m, \qquad \frac{1}{2}(a-b) = -r$$

$$C^J_{nmpk} = \frac{(-1)^k}{2^n(n-m)!}\frac{(n-m)!(n+m)!}{p!(n-p)!}\binom{n-r}{m-r+k}\binom{n+r}{k}$$

$$= \frac{(-1)^k(n+m)!}{2^n p!(n-p)!}\binom{n+r}{k}\binom{n-r}{n-m-k}$$

(3) $-m \geq r$. Then

$$a = m - r, \qquad b = -m - r, \qquad \frac{1}{2}(a+b) = -r, \qquad \frac{1}{2}(a-b) = m$$

$$C^J_{nmpk} = \frac{(-1)^k}{2^n(n-m)!}\frac{(n+r)!(n-r)!}{p!(n-p)!}\frac{(n+m)!(n-m)!}{(m-r+k)!(n+r-k)!k!(n-m-k)!}$$

$$= \frac{(-1)^k(n+m)!}{2^n p!(n-p)!}\binom{n+r}{k}\binom{n-r}{n-m-k}$$

In all three cases, C^J_{nmpk} has the same form in m,r. Replacing r by $n-2p$, we find

$$\tilde{F}^J_{nmp}(c',s) = \frac{(n+m)!}{2^n p!(n-p)!} \sum_{k=k_1}^{k_2} (-1)^k \binom{2n-2p}{k} * \binom{2p}{n-m-k} (s')^{2(k-k_1)} (c')^{2n+\nu-2k} \tag{162}$$

where

$$k_1 = \frac{1}{2}(\nu + |\nu|)$$

$$k_2 = \frac{1}{2}(2n + \nu - |2m + \nu|)$$

With the aid of Equation (32), (162) is seen to match Equation (12). Q.E.D.

THE HANSEN COEFFICIENT $\tilde{X}$

This function is defined by Equation (46). Its principal components,

$$M_s = (-1)^s \sum_{v=0}^{s} \binom{\alpha - v}{s - v} \frac{b^v}{v!} \tag{163}$$

$$N_t = (-1)^t \sum_{w=0}^{t} \binom{\gamma - w}{t - w} \frac{(-b)^w}{w!} \tag{164}$$

are numerically unattractive. Lyddane proposed to calculate these quantities by the compact recurrence relations shown in Equations (49) through (51).

To derive Lyddane's expressions, first define the function

$$M(s,\alpha;b) = (-1)^s \sum_{v=0}^{s} \binom{\alpha - v}{s - v} \frac{b^v}{v!} \tag{165}$$

Substitute the relation

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$$

which yields

$$\begin{aligned} M(s,\alpha;b) &= (-1)^s \sum_{v=0}^{s} \left[\binom{\alpha - 1 - v}{s - v} + \binom{\alpha - 1 - v}{s - 1 - v} \right] \frac{b^v}{v!} \\ &= M(s,\alpha-1;b) + (-1)(-1)^{s-1} \sum_{v=0}^{s-1} \binom{\alpha - 1 - v}{s - 1 - v} \frac{b^v}{v!} \\ &= M(s,\alpha-1;b) - M(s-1,\alpha-1;b) \end{aligned}$$

which is, indeed, Equation (50). Since N_t is identical in form to M_s, except for the signum of b, we may write

$$N_t \equiv M(t,\gamma;-b).$$

In other words, the relations developed for M can be used to evaluate N by simply replacing b by $-b$.

In the derivation of the equations of motion, derivatives of the disturbing function are formed which, in due course, require differentiation of M_s and N_t. With the aid of (165), we write

$$\frac{d}{db}M(s,\alpha;b) = (-1)^s \sum_{v=0}^{s} \binom{\alpha - v}{s - v} \frac{v b^{v-1}}{v!}$$

$$= (-1)^s \sum_{v=1}^{s} \binom{\alpha - v}{s - v} \frac{b^{v-1}}{(v-1)!}$$

$$= -(-1)^s \sum_{v=0}^{s-1} \binom{\alpha - v - 1}{s - v - 1} \frac{b^v}{v!}$$

or

$$\frac{d}{db}M(s,\alpha;b) = -M(s-1,\alpha-1;b) \tag{166}$$

In a similar fashion we obtain

$$\frac{d}{db}N(t,\gamma;b) = M(t-1,\gamma-1;-b) \tag{167}$$

The derivatives (166) and (167) are subsequently used in Equation (102), namely the formula for $d\widetilde{X}/d\beta^2$.

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The equations of motion for artificial Earth satellites are solved by semianalytic techniques. The force model includes an arbitrary gravity field for the Earth as well as Moon and Sun. All tesseral harmonics are examined for the presence of resonance. Any resonant term is classified according to its period and amplitude. Those exhibiting deep resonance (long-periodic) are integrated numerically. Shallow resonance terms (intermediate period) are evaluated analytically. Additional tests decide whether short-periodic terms are discarded or calculated analytically. Primary variables of the theory are the non-singular Poincaré elements.

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