FRACTIONAL STATE FEEDBACK CONTROL OF UNDAMPED AND VISCOELASTICALLY-DAMPED STRUCTURES

THESIS

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THESIS

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David L. Yang, B.S.M.E.

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Preface

The purpose of this study was to develop and to demonstrate a control theory, incorporating the feedback of fractional order derivatives of the structural response, for the control of undamped and viscoelastically damped structures. The control theory was developed from the finite element formulation of the structure, and the use of fractional order state feedback provided control authority equal to current structural control schemes with the use of fewer sensors.

I would like to express my deepest gratitude to Lieutenant Colonel Ronald Bagley, my thesis adviser, for his guidance and support throughout this study. I am grateful for his enthusiasm and his confidence in my work, without him this study would not have been possible. I would also like to thank Dr. Brad Liebst and Captain Greg Warhola for their guidance and assistance throughout this research effort. I like to thank my fellow classmates in GA-89D and GA-90M for their friendship and support. Finally, I would like to express my thanks to my mother for the love, guidance, and support she has given me over the years.

In conclusion, I would like to dedicate this work to my father, who continued to offer me encouragement while fighting for his life against cancer. May he rest in peace.

David L. Yang

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Abstract

The purpose of this ~~study~~ thesis is to demonstrate the analysis leading to the development of the fractional order state feedback control theory for structural control of both undamped and viscoelastically-damped structures. It is shown herein that there exists a relation between the traditional state vector that includes structural displacements and velocities, and the fractional state vector which includes fractional derivatives of structural responses. This relation permits the modification of linear quadratic regulator theory to include the application of fractional order states in the feedback control. The application of this theory leads to an alternative form of an observer.

FRACTIONAL ORDER STATE FEEDBACK CONTROL FOR UNDAMPED AND VISCOELASTICALLY-DAMPED STRUCTURES

I. Introduction

Structural control of large space structures has become of interest in recent years due to the application of these structures in areas such as the NASA space station and the Strategic Defense Initiative. The large size and low weight of these structures result in large number of closely spaced, low frequency vibration modes, which require structural control to maintain system stability and performance requirements. Passive damping systems such as damping pads and sophisticated shock absorbers provide only a partial solution to the problem due to their limited control action at low frequencies. The incorporation of active structural control systems with passive damping devices shows potential in improving overall system performance considerably.

The current research effort began with an interest to introduce active control to structures that are viscoelastically damped, in searching for improved system response. The objective of this thesis is to demonstrate the analysis leading to the development of a fractional order state feedback control law, for structures incorporating both passive damping through viscoelastic materials which are modeled by a fractional derivative stress-strain relation and active damping by applied control forces. To achieve this, quadratic optimal control theory is modified. Specifically, linear quadratic regulator theory is modified to include fractional derivatives in the state vector. It is shown herein that there exists a relation between the traditional state vector and the fractional order state vector which permits the incorporation of fractional order states in the feedback control law.

The thesis first reviews the properties of the quadratic optimal control theory and the fractional derivative in the formulation of a viscoelastic constitutive law. The development of the fractional order state feedback control theory is presented in Chapter III for both undamped and viscoelastically-damped structures. Two example problems in Chapter IV illustrate the application of this control theory and demonstrate the solution technique.

II. Brief Overviews of Optimal Control Theory and Fractional Derivatives

Before proceeding with the development of the fractional order state feedback control theory, it is appropriate to introduce the properties of the optimal control theory and fractional derivatives relevant to the following control theory development. Of particular interest are the linear quadratic regulator and the basic fractional derivative viscoelastic constitutive relation. Section 2.1 provides a brief overview on the development of the linear quadratic regular control theory. Section 2.2 provides a general description of the fractional calculus and the development of the fractional derivative constitutive relation for viscoelastic materials.

2.1 Optimal Control Systems Based on Quadratic Performance Index

The concept of control system optimization compromises the selection of a performance index and a control law which yields the optimal control system within the limits imposed by system physical constraints. A performance index is a function whose values indicates how well the actual system performance matches the desired system performance. In many cases, system behavior is optimized by choosing the control vector in such a way that the performance index is minimized (or maximized).

In many practical control systems, it is desired to minimize a measure of the difference between the actual state and the desired state, and to minimize the energy required for the control action. For a given linear system represented by

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1}$$

where

$\underline{x}$ = state vector (n-dimensional real vector)

$\underline{u}$ = control vector (r-dimensional real vector)

A = n × n matrix

B = n × r matrix

the associated quadratic performance index of the system over the time interval $0 \le t \le t_f$ may be written as (1:753)

$$J = \int_0^{t_f} [\zeta(t) - x(t)]^T Q[\zeta(t) - x(t)]\, dt + \lambda\int_0^{t_f} u^T(t)Ru(t)dt \tag{2}$$

where

$\zeta(t)$ = the desired state

$x(t)$ = the actual state

$u(t)$ = the control vector

Q = a positive-definite (or a positive-semidefinite) matrix

R = a positive-definite matrix

The Lagrange multiplier, λ, is a positive constant indicating the weight of control energy with respect to the minimizing errors, and the time interval $0 \le t \le t_f$ is either finite or infinite.

Finding the optimal control law for tne system subject to the performance index given by Eq (2) has a practical significance that the resulting system compromises between minimizing the integral error and minimizing the control energy. For the case where $t_f = \infty$ and the desired final state is the origin, or $\zeta = 0$, the quadratic performance index given by Eq (2) can be expressed as

$$J = \int_0^\infty [x(t)^T Q x(t) + u^T(t) R u(t)] dt \qquad (3)$$

where λ has been included in the positive definite matrix R. This type of control problem is often referred to as a "linear quadratic regular problem" since the final state response is desired to be zero (1:126).

In designing optimal control systems based on the quadratic performance index shown in Eq (3), one is interested in choosing a control vector, u(t), so that the performance index is minimized. The necessary conditions for an optimal solution of the regulator problem may be obtained by using the calculus of variations which may involve the concept of a Hamiltonian (2:180-181). Once the initial and final state conditions have been identified by the state and costate equations, the optimal control law may be determined by solving the two-point boundary-value-problem (2:189-190). The optimal control for the regulator problem is

$$\underline{u} = - R^{-1} B S \underline{x} \qquad (4)$$

where S is the solution of the algebraic matrix Riccati equation:

$$0 = A^T S + SA - SBR^{-1}B^T S + Q \tag{5}$$

Defining the optimal gain matrix as

$$G = R^{-1}B^T S \tag{6}$$

the optimal control can be written as

$$\underline{u} = - G\underline{x} \tag{7}$$

and the state equation can be expressed as

$$\dot{\underline{x}} = A\underline{x} - BG\underline{x} \tag{8}$$

2.2 Fractional Derivatives and Its Application to Viscoelastic Materials

A fractional calculus, in which fractional order derivatives and integrals are defined and studied, is not a very new concept. Although the idea of fractional calculus has been in existence since the late 1600's, the use of fractional calculus has been few as compared to the classical calculus of integer orders. An article written by Ross (3) provides an interesting overview on the historical development of the basic concepts of fractional calculus.

A fractional derivative is a linear operation that generalizes the order of differentiation to fractional values. It is the inverse operation of fractional integration attributed to Riemann and Liouville

(4). The fractional derivative is defined as

$$D^{\alpha}[x(t)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau \qquad (9)$$

for $0 < \alpha < 1$. The fractional derivative operator has the property in the Laplace transform domain that

$$\mathfrak{L}\left[D^{\alpha}[x(t)]\right] = S^{\alpha} \cdot \mathfrak{L}[x(t)] \qquad (10)$$

where

$$\mathfrak{L}[x(t)] = \int_0^{\infty} x(t)\, e^{-st}\, dt \qquad (11)$$

A similar relationship exists for the fractional derivative operator in the Fourier transform domain

$$\mathfrak{F}\left[D^{\alpha}[x(t)]\right] = (i\omega)^{\alpha} \cdot \mathfrak{F}[x(t)] \qquad (12)$$

where

$$\mathfrak{F}[x(t)] = \int_0^{\infty} x(t)\, e^{-i\omega t}\, dt \qquad (13)$$

In fact, the fractional derivative satisfies many of the same properties as the integer derivative, particularly linearity and the composition property

$$D^{\alpha}[ay(t) + bx(t)] = aD^{\alpha}[y(t)] + bD^{\alpha}[x(t)] \quad (14)$$

for a and b are constants, and

$$D^{\alpha}\left[D^{\beta}[x(t)]\right] = D^{\alpha+\beta}[x(t)] \quad (15)$$

The development of the fractional derivative stress-strain constitutive model for viscoelastic materials has its foundation in the early observations made by Nutting and Gemant (6,7,8). Nutting's observations suggested that the stress relaxation property could be modeled by fractional powers of time, while Gemant's observations suggested that the frequency dependent properties of viscoelastic materials could also be modeled by fractional differentials of time. The simultaneous modeling of both relaxation and frequency dependent characteristics of viscoelastic materials, using fractional time differentials, was later proposed by Scott-Blair in 1950's (9). Meanwhile, the use of fractional derivatives was suggested by Caputo for modeling the viscoelastic behavior of geological strata (10,11). He and Minard also showed through experimental data that the fractional derivatives constitutive relationship could be used for some glasses and metals (12). Bagley proposed the use of fractional derivatives in the finite element analysis of viscoelastically damped structures (5). In recent years, Bagley and Torvik have jointly published several papers demonstrating the application of fractional derivatives in viscoelasticity and structural dynamics (13-16). They have also shown that the fractional order models are consistent with thermodynamic constraints (17) and these models have their foundation in molecular theories that predict the macroscopic behavior of viscoelastic materials (18).

The general form of the fractional derivative viscoelastic model put forward by Bagley and Torvik is (14)

$$\sigma(t) + \sum_{m=1}^{M} b_m D^{\beta_m}[\sigma(t)] = E_o \varepsilon(t) + \sum_{n=1}^{N} E_n D^{\alpha_n}[\varepsilon(t)] \qquad (16)$$

where the time-dependent stress fields are related to the time-dependent strain fields through series of fractional derivatives. For many viscoelastic materials, the stress-strain relation can be modeled by retaining only the first fractional derivative term in each series in Eq (16) and the resulting model is

$$\sigma(t) + bD^{\beta}[\sigma(t)] = E_o \varepsilon(t) + E_1 D^{\alpha}[\varepsilon(t)] \qquad (17)$$

The Second Law of Thermodynamics requires that (17)

$$\begin{array}{ll} E_o \geq 0 & E_1 \geq bE_o \\ E_1 > 0 & \alpha = \beta \\ b > 0 & \end{array} \qquad (18)$$

These constraints ensure that energy dissipation rate and stored strain energy are positive.

The stress-strain relation expressed in the Laplace domain is

$$\frac{\sigma(s)}{\epsilon(s)} = \frac{E_o + E_1 S^\alpha}{1 + bS^\alpha} \tag{19}$$

The result is a viscoelastic model with four parameters: b, E_o, E_1, and α. This model has been shown to be very accurate over several decades of frequency for approximately 130 materials (15,16). For some applications where the required frequency range is limited, a three parameter viscoelastic model, where b = 0, is sufficient (21:129). The three parameter model is defined as

$$\frac{\sigma(s)}{\epsilon(s)} = E_o + E_1 S^\alpha \tag{20}$$

and this is the model that will be used in this work.

III. Fractional Order State Feedback Control Theory

The development of the fractional order state feedback control stems from the interest in examining the effect of feedback in controlling viscoelastically damped structures. Bagley and Calico have shown that fractional order state equations can be developed where the state vector includes fractional order time derivatives of structure motion (19). Their work also suggested possible improvement in system performance by feeding back fractional time derivatives of system response. The fractional order state equations for a general system prescribed by Bagley and Calico is (19:493)

$$D^{\alpha}\underline{y} = A\underline{y} - BG\underline{y} \tag{21}$$

for $\underline{y}$ is the fractional order state vector and G is the matrix of effective gain coefficients.

In order to apply fractional order derivatives feedback in classical structural control, it is necessary to develop a control law incorporating the fractional order states. The development of the theory will begin with the formulation of the fractional order equations of motion, which is an extension of the traditional equations of motion derive from the finite element formulation of the structure of interest. A special solution technique is then demonstrated for determining the the eigenstructure of the fractional order equations of motion which includes both integer order and fractional order time derivatives of structural displacements. The eigenvectors of the system are then used to develop the special traditional-state/fractional-state transformation relation, and the result of this transformation is then applied to the development of the control law which incorporates the feedback of fractional order states.

In the application of optimal quadratic control theory for active structural control, more specifically for the linear quadratic regulator problem, it has been shown that the quadratic control theory produces optimal control as linear feedback of the state vector (20:129). The state equations for the structure can be written as

$$\dot{\underline{x}} = A\underline{x} - BG\underline{x} \tag{8}$$

where $\underline{x}$ is the state vector that includes only structural displacements and integer order time derivatives of structural displacements, and G is the optimal gain matrix. It will be shown herein that there exists a relation between the traditional state vector of integer order and the fractional order state vector given by

$$\underline{x} = \Phi\ \underline{y} \tag{22}$$

where Φ is as a state transformation matrix which can be determined from the eigenstructure of the equations of motion posed in an expanded format.

Consider the finite element equations of motion for a structure with N degrees of freedom

$$[M]\{\ddot{w}(t)\} + [K]\{w(t)\} = \{F(t)\} \tag{23}$$

where [M] and [K] are the mass and stiffness matrices of the finite element model, and {F(t)} is the forcing function. In order to develop the state transformation matrix, Φ, it is necessary to first pose Eq (23) as a set of fractional order differential equations and then determine the eigenstructure associated with the fractional order equations of motion. Applying the composition property of the fractional derivative such that

$$D^{\alpha}\left[D^{\beta}\{w(t)\}\right] = D^{\alpha+\beta}\{w(t)\} \tag{24}$$

and Eq (23) can be posed as

$$\begin{aligned} D^{\alpha}[M]\left[D^{2-\alpha} + D^{2-2\alpha} + D^{2-3\alpha} + \cdots + D^{0}\right]\{w(t)\} \\ - [M]\left[D^{2-\alpha} + D^{2-2\alpha} + D^{2-3\alpha} + \cdots + D^{\alpha}\right]\{w(t)\} \\ + [K]D^{0}\{w(t)\} = \{F(t)\} \end{aligned} \tag{25}$$

Eq (25) represents the fractional order equations of motion for the system defined in Eq (23). Here α is chosen to be a fraction of the form 1/n where n is an integer. This is to ensure that velocities will appear in the fractional order state equations and thus allows one to solve the initial value problem where the initial velocity is specified. For systems with zero initial condition, this restriction on α is not required.

The eigenstructure of the fractional order equations of motion, Eq (25), can not be solved using traditional eigenvalue problem solution method because of the inclusion of fractional derivatives. The special solution technique presented here for solving Eq (25) is an extension of the method presented by Foss for determining the solution of an N degree-of-freedom (DOF) viscously damped structure (29:361)

$$[M]\{\ddot{q}(t)\} + [C]\{\dot{q}(t)\} + [K]\{q(t)\} = \{F(t)\} \tag{26}$$

where the damping matrix, [C], is not a linear combination of the mass and stiffness matrices of the system. Foss proposed the use of a set of auxiliary variables and the conversion of a Lagrangian set of N second order differential equations into an equivalent set of 2·N first order differential equations known as Hamilton's canonical equations. This leads to the following equivalent equations of motion:

$$[\tilde{M}]\{\dot{y}(t)\} + [\tilde{K}]\{y(t)\} = \{Y(t)\} \tag{27}$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \tag{28}$$

$$[\tilde{K}] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \tag{29}$$

$$\{y(t)\} = \begin{Bmatrix} \{\dot{q}(t)\} \\ \{q(t)\} \end{Bmatrix} \tag{30}$$

$$\{Y(t)\} = \begin{Bmatrix} \{0\} \\ \{F(t)\} \end{Bmatrix} \tag{31}$$

The lower half of the partitioned matrix equations, Eq (27), is the equations of motion of the system in Eq (26) and the upper half of the matrix equations is satisfied identically. The formulation of Eq (27) has the advantage that its solution can be determined using traditional eigenvalue problem solution method.

To determine the eigenstructure of Eq (25), one can adopt the method presented by Bagley and pose the fractional order equations of motion in terms of two real, square, symmetric matrices in the following format (14:744)

$$D^{\alpha}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{\tilde{F}(t)\} \tag{32}$$

$$[\tilde{M}] = \begin{bmatrix} [0] & [0] & \cdots & [0] & [M] \\ [0] & [0] & \cdots & [M] & [0] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ [0] & [M] & \cdots & [0] & [0] \\ [M] & [0] & \cdots & [0] & [0] \end{bmatrix} \tag{33}$$

$$[\tilde{K}] = \begin{bmatrix} [0] & [0] & \cdots & [0] & -[M] & [0] \\ [0] & [0] & \cdots & -[M] & [0] & [0] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [0] & -[M] & \cdots & [0] & [0] & [0] \\ -[M] & [0] & \cdots & [0] & [0] & [0] \\ [0] & [0] & \cdots & [0] & [0] & [K] \end{bmatrix} \tag{34}$$

$$\{\tilde{w}(t)\} = \begin{Bmatrix} D^{(2-\alpha)}\{w(t)\} \\ D^{(2-2\alpha)}\{w(t)\} \\ \vdots \\ D^{2\alpha}\{w(t)\} \\ D^{\alpha}\{w(t)\} \\ \{w(t)\} \end{Bmatrix} \tag{35}$$

$$\{\tilde{F}(t)\} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \vdots \\ \{0\} \\ \{F(t)\} \end{Bmatrix} \tag{36}$$

This revised form is referred to as the expanded equations of motion where $[\tilde{M}]$ and $[\tilde{K}]$ are the pseudo-mass and pseudo-stiffness matrices respectively (15:922). Note that the lowest set of partitioned matrix equation is the equations of motion given in Eq (25), and that all of the upper sets are satisfied identically. The order of the expanded equations of motion is $2 \cdot N \cdot n$.

Setting $\{\tilde{F}(t)\} = \{0\}$ yields the homogeneous form of Eq (32) and the expanded eigenvalues and eigenvectors for the system may now be solved to satisfy the equation

$$\tilde{\lambda}_j[\tilde{M}]\{\tilde{\Phi}_j\} + [\tilde{K}]\{\tilde{\Phi}_j\} = \{0\} \tag{37}$$

The solutions to this equation can be computed using one of several techniques currently available (22). The eigenvalues appear in complex conjugate pairs and there are N·n pairs of eigenvalues for the system. The eigenvalues may be mapped onto the Riemann surface for the function, $\tilde{\lambda}_j = Z^{1/n}$, consisting of n Riemann sheets (23:902). N pairs of complex conjugate eigenvalues usually appear to each sheet. The eigenvalues on the principal sheet of the Riemann surface, which is the first upper sheet of the n sheets present, represent poles in the system transfer function and they produce sinusoidal motion of the structure at the resonant frequencies of the system. The eigenvectors associated with the eigenvalues on the principal sheet describe the mode shapes of the structure at the modal frequencies. The eigenvalues on other sheets of the Riemann surface represent poles in the system transfer function which produce a monotonically decreasing response of the structure. This monotonically decreasing motion describes the creep and relaxation response of the system. It will be shown in Chapter IV of this thesis that the system response is strongly influenced by the residues of the poles on the principal sheet of the Riemann surface, and one may describe the general response of the system with very good accuracy over an intermediate range of time using only the eigenvalues and eigenvectors associated with those poles on the principal sheet. In the following analysis, only the eigenvalues and eigenvectors associated with the poles on the principal sheet of the Riemann surface are retained for the formulation of the state transformation matrix.

Recalling the expansion theorem in structure modal analysis (24:283) where the system response may be described as a superposition of the normal modes of the system multiplied by corresponding time-dependent generalized coordinates and be written in the form

$$\{q\} = [u]\{\eta\} \tag{38}$$

for

$\{q\}$ = a column vector of the generalized system response

$[u]$ = the modal matrix consisting of the modal vectors arranged in a matrix

$\{\eta\}$ = a column vector of the modal coordinates of the system

Applying the theorem to the expanded system, one may now pose the system response of the structure in the form

$$\{\tilde{w}(t)\} = [\Phi]\{\eta\} \tag{39}$$

where

$$\{\tilde{w}(t)\} = \begin{Bmatrix} D^{(2-\alpha)} \{w(t)\} \\ D^{(2-2\alpha)} \{w(t)\} \\ \vdots \\ D^{2\alpha} \{w(t)\} \\ D^{\alpha} \{w(t)\} \\ \{w(t)\} \end{Bmatrix} \tag{35}$$

$$[\Phi] = [\{\Phi_1\} \ \{\bar{\Phi}_1\} \ \{\Phi_2\} \ \{\bar{\Phi}_2\} \cdots \{\Phi_N\} \ \{\bar{\Phi}_N\}] \tag{40}$$

$$\{\eta\} = \begin{Bmatrix} \eta_1 \\ \bar{\eta}_1 \\ \eta_2 \\ \bar{\eta}_2 \\ \vdots \\ \eta_N \\ \bar{\eta}_N \end{Bmatrix} \tag{41}$$

Notice the expanded response $\{\tilde{w}(t)\}$ includes nodal displacements, velocities, and the fractional derivatives of the nodal displacements for all N degrees of freedom of the finite element model. The modal matrix, $[\Phi]$, consists of the conjugate pairs of modal vectors, $\{\phi_j\}$ and $\{\bar{\phi}_j\}$, associated with the conjugate eigenvalues, λ_j and $\bar{\lambda}_j$, on the principal sheet of the Riemann surface.

From Eq (39) one may extract the necessary information to construct the traditional state vector and the fractional state vector for the system. The traditional state vector may be posed in a form similar to Eq (39)

$$\underline{x} = [\Phi_A]\{\eta\} \tag{42}$$

where

$$\underline{x} = \begin{Bmatrix} \{w(t)\} \\ D^1\{w(t)\} \end{Bmatrix} \tag{43}$$

$[\Phi_A]$ = a matrix consists of row vectors in Eq (40) that are associated with the states defined in Eq (43)

Similarly the fractional state vector may be written as

$$\underline{y} = [\Phi_B]\{\eta\} \tag{44}$$

where

$$\underline{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{2N-2} \\ y_{2N-1} \\ y_{2N} \end{Bmatrix} \tag{45}$$

$[\Phi_B]$ = a matrix consists of row vectors in Eq (40) that are associated with the states defined in Eq (45)

The fractional state vector in Eq (45) is a special subset of the state vector of the expanded equations of motion in Eq (35) and each state of the fractional state vector represents a state of interest in Eq (35). The fractional state vector in Eq (45) has the same dimension as the traditional state vector in Eq (43) and it includes a total of 2·N fractional and integer states. The states of the fractional state vector may be of any integer or fractional order states taken from one structural node or from a combination of several different nodes.

From Eq (44), one may define $\{\eta\}$ as

$$\{\eta\} = [\Phi_B]^{-1}\underline{y} \tag{46}$$

Substitute Eq (46) into Eq (42) yields

$$\underline{x} = [\Phi_A][\Phi_B]^{-1}\underline{y} \tag{47}$$

Define the state transformation matrix, Φ, as

$$\Phi = [\Phi_A][\Phi_B]^{-1} \tag{48}$$

and Eq (47) can be written as

$$\underline{x} = \Phi\ \underline{y} \tag{49}$$

Eq (49) illustrates the relation between the traditional state vector and the fractional state vector in a system, and the state transformation matrix captures the essence of this important relation (19). The task at hand is to utilize the traditional-state/fractional-state relation just presented as the foundation for developing fractional order state feedback control law for both undamped and viscoelastically damped structures. The development of the theory will be presented in the following sections.

3.1 Fractional Order State Feedback Control of Undamped Structures

In this section, the fractional order state feedback control law will be developed for the control of undamped structures. The optimal quadratic regulator control theory serves as the starting point for the development of the new control law. The traditional-state/fractional-state relation is then incorporated into the theory for the formulation of the factional order state feedback control law.

Consider the equations of motion of an N degree of freedom structure under active control

$$[M]\{\ddot{w}(t)\} + [K]\{w(t)\} = [D]\{u(t)\} \tag{50}$$

The equations of motion for the system are similar to Eq (23), except for the forcing function which has been replaced by the product of a matrix, [D], and a vector, {u(t)}. The matrix,[D], describes the location and orientation of the actuators, and the vector, {u(t)}, represents the actuator force. The product of [D] and {u(t)} represents the control force applied to the structure.

The system in Eq (44) may be expressed in state model form

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1}$$

where

$$\underline{x} = \left\{ \begin{matrix} \{w(t)\} \\ \{\dot{w}(t)\} \end{matrix} \right\} \tag{51}$$

$$\dot{\underline{x}} = \left\{ \begin{matrix} \{\dot{w}(t)\} \\ \{\ddot{w}(t)\} \end{matrix} \right\} \quad (52)$$

$$A = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & [0] \end{bmatrix} \quad (53)$$

$$B = \begin{bmatrix} [0] \\ [M]^{-1}[D] \end{bmatrix} \quad (54)$$

and [0] is a zero matrix and [I] is an identity matrix.

Recalling from optimal quadratic regulator control that the optimal for the system is a linear feedback of the state vector

$$\underline{u} = -G\underline{x} \quad (7)$$

for G is the optimal gain matrix for the system derived from the solution of the Riccati equation. Introducing the optimal control into the system and the state equations may be written as

$$\dot{\underline{x}} = A\underline{x} - BG\underline{x} \quad (8)$$

Partition the gain matrix G into two equal size matrices, G_1 and G_2, such that

$$G = [G_1 \,\vdots\, G_2] \quad (55)$$

and incorporate the gains into the finite element model of the structure to obtain

$$[M]\{\ddot{w}(t)\} + [D][G_2]\{\dot{w}(t)\} + ([K] + [D][G_1])\{w(t)\} = \{0\} \tag{56}$$

Note that the feedback control introduces control damping and control stiffness to the structure, as evidenced by the inclusion of $[D][G_2]$ and $[D][G_1]$ in Eq (56).

To determine the modal matrix and the state transformation matrix for the structure under control, first pose the equations of motion, Eq (56), in the fractional format

$$\begin{aligned} &D^{\alpha}[M]\left[D^{2-\alpha} + D^{2-2\alpha} + D^{2-3\alpha} + \cdots + D^{0}\right]\{w(t)\} \\ &\quad - [M]\left[D^{2-\alpha} + D^{2-2\alpha} + D^{2-3\alpha} + \cdots + D^{\alpha}\right]\{w(t)\} \\ &\quad\quad + D^{\alpha}[D][G_2]\left[D^{1-\alpha} + D^{1-2\alpha} + D^{1-3\alpha} + \cdots + D^{0}\right]\{w(t)\} \\ &\quad\quad\quad - [D][G_2]\left[D^{1-\alpha} + D^{1-2\alpha} + D^{1-3\alpha} + \cdots + D^{\alpha}\right]\{w(t)\} \\ &\quad\quad\quad\quad + ([K] + [D][G_1])D^{0}\{w(t)\} = \{0\} \end{aligned} \tag{57}$$

The eigenvalues and eigenvectors of the system in Eq (57) may now be solved using the solution technique of the expanded equations of motion presented in Eq (32)-Eq (36). For the equations of motion are now written as

$$D^{\alpha}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{\tilde{F}(t)\} \tag{32}$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [0] & \cdots & \cdots & \cdots & [0] & [M] \\ [0] & [0] & \cdots & \cdots & \cdots & [M] & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & [D][G_2] \\ [0] & [M] & \cdots & \cdots & [D][G_2] & [0] & [0] \\ [M] & [0] & \cdots & [D][G_2] & \cdots & [0] & [0] \end{bmatrix} \tag{58}$$

$$[\tilde{K}] = \begin{bmatrix} [0] & [0] & \cdots & \cdots & \cdots & [0] & -[M] & [0] \\ [0] & [0] & \cdots & \cdots & \cdots & -[M] & [0] & [0] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -[D][G_2] & \vdots \\ [0] & -[M] & \cdots & \cdots & -[D][G_2] & [0] & [0] & [0] \\ -[M] & [0] & \cdots & -[D][G_2] & \cdots & [0] & [0] & [0] \\ [0] & [0] & \cdots & \cdots & [0] & [0] & & [K]+[D][G_1] \end{bmatrix} \tag{59}$$

and $\{\tilde{w}(t)\}$ is the state vector defined in Eq (35) and $\{\tilde{F}(t)\}$ is a zero vector for this case. Following the procedure identified in Eq (39)-Eq (41), the modal matrix, $[\Phi]$, may be determined for the expanded response of the structure under feedback control. Furthermore the state transformation matrix ,Φ, may be formulated following the steps identified in Eq (42)- Eq (48). Applying the state transformation relation,Eq (49), to the optimal feedback control, Eq (8), and

$$\dot{\underline{x}} = A\underline{x} - BG\Phi\underline{y} \tag{60}$$

This is the control law for active structural control incorporating the feedback of fractional order derivatives of the structure's response. The use of fractional state feedback control permits one to maintain equivalent control authority over the system as the traditional state feedback control with the use of fewer sensors (additional control hardware and circuitries may be required to support the actual operation). The additional information provided by the fractional derivatives of the structural response and the use of the state transformation matrix allows one to observe the motion at all structural nodes of interest, even though only limited sensors are actually used to monitor the system.

To further expand the theory, substitute the state transformation relation into Eq (60) and the control may now be written as

$$\Phi\dot{\underline{y}} = A\Phi\underline{y} - BG\Phi\underline{y} \tag{61}$$

Multiply both sides of Eq (61) by Φ^{-1} and

$$\dot{\underline{y}} = \Phi^{-1}A\Phi\underline{y} - \Phi^{-1}BG\Phi\underline{y} \tag{62}$$

Redefine the elements of Eq (62) as follows

$$A^{*} = \Phi^{-1}A\Phi \tag{63}$$

$$B^{*} = \Phi^{-1}B \tag{64}$$

$$G^{*} = G\Phi \tag{65}$$

and Eq (62) may be written as

$$\dot{\underline{y}} = A^{*}\underline{y} - G^{*}\underline{y} \tag{66}$$

Eq (66) is the fractional order state equations with optimal feedback control. Application of this control law will make possible the controlling of fractional derivatives of the structural response.

3.2 Fractional Order State Feedback Control of Viscoelastically Damped Structures

The next step in the study is to formulate the fractional order state feedback control law for the control of structures that are viscoelastically damped. The development process follows closely to the procedure used in the development of the fractional order state feedback control law for the control of undamped structures. Minor modification was made in the formulation of the finite element model of the viscoelastically damped structures which included the fractional derivative modeling of the viscoelastic damping material.

Consider the three parameter fractional derivative model for a viscoelastic material undergoing uniform, uniaxial deformation

$$\sigma(t) = (E_0 + E_1 D^{\beta})\epsilon(t) \tag{67}$$

where E_0, E_1, and β are the material parameters. Applying this model to the formulation of the finite element equations of motion for a viscoelastically damped structure under control and the equations take the form

$$[M]\{\ddot{w}(t)\} + [K_V]\{D^{\beta}w(t)\} + [K_E]\{w(t)\} = [D]\{u(t)\} \tag{68}$$

where

$[K_V]$ = visco-stiffness matrix

$[K_E]$ = elastic-stiffness matrix

In general the visco-stiffness matrix is not a linear combination of the mass and elastic-stiffness matrices, and the equations of motion, Eq (68), can not be transformed into the traditional first order state equations, Eq (1). In order to apply optimal feedback control to this viscoelastically damped structure, it is necessary to develop an equivalent damping matrix and an equivalent stiffness matrix for the system, thus the equations of motion may then be posed in the familiar form

$$[M]\{\ddot{w}(t)\} + [C^*]\{\dot{w}(t)\} + [K^*]\{w(t)\} = [D]\{u(t)\} \tag{69}$$

and Eq (69) could lead to the formulation of the necessary state equations for structural control.

To develop the equivalent damping and stiffness matrices, one must first solve for the resonance response of the viscoelastically damped structure. Posing the equations of motion in the fractional format

$$
\begin{aligned}
D^{\gamma}[M]\left[D^{2-\gamma} + D^{2-2\gamma} + D^{2-3\gamma} + \cdots + D^{0}\right]\{w(t)\} \\
- [M]\left[D^{2-\gamma} + D^{2-2\gamma} + D^{2-3\gamma} + \cdots + D^{\gamma}\right]\{w(t)\} \\
+ D^{\gamma}[K_V]\left[D^{\beta-\gamma} + D^{\beta-2\gamma} + D^{\beta-3\gamma} + \cdots + D^{0}\right]\{w(t)\} \\
- [K_V]\left[D^{\beta-\gamma} + D^{\beta-2\gamma} + D^{\beta-3\gamma} + \cdots + D^{\gamma}\right]\{w(t)\} \\
+ [K_E]D^{0}\{w(t)\} = \{0\}
\end{aligned}
\tag{70}
$$

where $\gamma = 1/m$ and m is equal to the denominator of β. The modal displacements, modal velocities, and modal accelerations of the structure can then be determined by solving Eq (70) using the solution method of the expanded equations of motion presented in Eq (32)-Eq (36). The expanded equations of motion for Eq (70) are written as

$$
D^{\gamma}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{0\} \tag{71}
$$

where

$$
[\tilde{M}] = \begin{bmatrix}
[0] & [0] & \cdots & \cdots & \cdots & [0] & [M] \\
[0] & [0] & \cdots & \cdots & \cdots & [M] & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & [K_V] \\
[0] & [M] & \cdots & \cdots & [K_V] & [0] & [0] \\
[M] & [0] & \cdots & [K_V] & \cdots & [0] & [0]
\end{bmatrix} \tag{72}
$$

$$
[\tilde{K}] = \begin{bmatrix} [0] & [0] & \cdots & \cdots & \cdots & [0] & -[M] & [0] \\ [0] & [0] & \cdots & \cdots & \cdots & -[M] & [0] & [0] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -[K_V] & \vdots \\ [0] & -[M] & \cdots & \cdots & -[K_V] & [0] & [0] & [0] \\ -[M] & [0] & \cdots & -[K_V] & \cdots & [0] & [0] & [0] \\ [0] & [0] & \cdots & \cdots & \cdots & [0] & [0] & [K_E] \end{bmatrix} \tag{73}
$$

and $\{\tilde{w}(t)\}$ is the state vector defined in Eq (35). Once the expanded eigenvalues and eigenvectors are determined by solving Eq (71), one may proceed to establish the modal matrix, $[\Phi]$, for the uncontrolled system as described in Eq (39)-Eq (41) and the modal displacements, velocities, and accelerations can be extracted from the modal matrix and the expanded eigenvalues on the principal sheet of the Riemann surface.

Given that the equivalent damping and stiffness matrices must generate the same structural response as the original system, one may solve for the matrices provided that the resonant structural response of the original system is known. The homogeneous equations of motion of the equivalent system may be written as

$$
[M]D^2\{w(t)\} + [C^*]D\{w(t)\} + [K^*]\{w(t)\} = \{0\} \tag{74}
$$

Applying the expansion theorem for structural modal analysis (24:283) to Eq (74) and

$$
[M][\phi_a]\{\eta\} + [C^*][\phi_v]\{\eta\} + [K^*][\phi_d]\{\eta\} = \{0\} \tag{75}
$$

Further simplification produces

$$[M][\Phi_a] + [C^*][\Phi_v] + [K^*][\Phi_d] = \{0\} \tag{76}$$

where $[\Phi_a]$, $[\Phi_v]$, and $[\Phi_d]$ are the modal acceleration, velocity, and displacement matrices respectively. These matrices consist of the elements of the modal matrix associated with the known structural nodal acceleration, velocity, and displacement. Since the elements of the modal matrix are complex, Eq (76) may be redefined as two equations

$$[M]\mathrm{Re}[\Phi_a] + [C^*]\mathrm{Re}[\Phi_v] + [K^*]\mathrm{Re}[\Phi_d] = \{0\} \tag{77}$$

$$[M]\mathrm{Im}[\Phi_a] + [C^*]\mathrm{Im}[\Phi_v] + [K^*]\mathrm{Im}[\Phi_d] = \{0\} \tag{78}$$

where $\mathrm{Re}[\Phi_a]$, $\mathrm{Re}[\Phi_v]$, and $\mathrm{Re}[\Phi_d]$ represent the real parts and $\mathrm{Im}[\Phi_a]$, $\mathrm{Im}[\Phi_v]$, and $\mathrm{Im}[\Phi_d]$ represent the imaginary parts of $[\Phi_a]$, $[\Phi_v]$ and $[\Phi_d]$ respectively. Eq (77) and Eq (78) may now be solved for $[C^*]$ and $[K^*]$ where

$$[C^*] = -[M]\left(\mathrm{Re}[\Phi_a] - \mathrm{Im}[\Phi_a]\cdot\mathrm{Im}[\Phi_d]^{-1}\cdot\mathrm{Re}[\Phi_d]\right) \cdot\left(\mathrm{Re}[\Phi_v]-\mathrm{Im}[\Phi_v]\cdot\mathrm{Im}[\Phi_d]^{-1}\cdot\mathrm{Re}[\Phi_d]\right)^{-1} \tag{79}$$

$$[K^*] = -\left([M]\cdot\mathrm{Im}[\Phi_a] + [C^*]\cdot\mathrm{Im}[\Phi_v]\right)\mathrm{Im}[\Phi_d]^{-1} \tag{80}$$

After the equivalent damping and stiffness matrices have been determined for the system, the state equations for the equivalent system, Eq (69), can be developed as

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{81}$$

where

$$A = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K^*] & -[M]^{-1}[C^*] \end{bmatrix} \tag{82}$$

$$B = \begin{bmatrix} [0] \\ [M]^{-1}[D] \end{bmatrix} \tag{83}$$

The optimal gain matrix, G, for feedback control can be determined in the same manner as described in Section 2.1. Incorporating the gain matrix into the system model, and the finite element model of the controlled viscoelastically damped structure takes the form

$$[M]\{\ddot{w}(t)\} + ([C^*] + [D][G_2])\{\dot{w}(t)\} + ([K^*] + [D][G_1])\{w(t)\} = \{0\} \tag{84}$$

To determine the state transformation matrix for the system, one should first pose Eq (84) in the fractional format

$$\begin{aligned} & D^{\alpha}[M]\Big[D^{2-\alpha} + D^{2-2\alpha} + D^{2-3\alpha} + \cdots + D^{0}\Big]\{w(t)\} \\ & - [M]\Big[D^{2-\alpha} + D^{2-2\alpha} + D^{2-3\alpha} + \cdots + D^{\alpha}\Big]\{w(t)\} \\ & + D^{\alpha}([C^*] + [D][G_2])\Big[D^{1-\alpha} + D^{1-2\alpha} + D^{1-3\alpha} + \cdots + D^{0}\Big]\{w(t)\} \\ & - ([C^*] + [D][G_2])\Big[D^{1-\alpha} + D^{1-2\alpha} + D^{1-3\alpha} + \cdots + D^{\alpha}\Big]\{w(t)\} \\ & + ([K^*] + [D][G_1])D^{0}\{w(t)\} = \{0\} \end{aligned} \tag{57}$$

The modal transformation matrix may be formulated by solving the fractional order equations of motion in Eq (85), using the solution method of expanded equations of motion in the manner described in Eq (32)-Eq (36). The expanded equations of motion for Eq (85) are

$$D^{\alpha}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{0\} \tag{86}$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [0] & \cdots & \cdots & \cdots & [0] & [M] \\ [0] & [0] & \cdots & \cdots & \cdots & [M] & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & [C^*]+[D][G_2] \\ [0] & [M] & \cdots & \cdots & [C^*]+[D][G_2] & [0] & [0] \\ [M] & [0] & \cdots & [C^*]+[D][G_2] & \cdots & [0] & [0] \end{bmatrix} \tag{87}$$

$$[\tilde{K}] = \begin{bmatrix} [0] & [0] & \cdots & \cdots & \cdots & [0] & -[M] & [0] \\ [0] & [0] & \cdots & \cdots & \cdots & -[M] & \cdots & [0] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -[C^*]-[D][G_2] & \vdots \\ [0] & -[M] & \cdots & \cdots & -[C^*]-[D][G_2] & [0] & [0] & [0] \\ -[M] & [0] & \cdots & -[C^*]-[D][G_2] & \cdots & [0] & [0] & [0] \\ [0] & [0] & \cdots & \cdots & [0] & [0] & \cdots & [K^*]+[D][G_1] \end{bmatrix} \tag{88}$$

Following the procedure identified in Eq (39)-Eq (48), the state transformation matrix, Φ, can be determined from the system modal matrix, and the control law for the feedback of fractional order states may be written as

$$\dot{\underline{x}} = A\underline{x} - BG\Phi\underline{y} \tag{89}$$

The control law of Eq (89) for the control of viscoelastically damped structures remains unchanged from the control law for the control of undamped structures, only the values of the optimal gain matrix and the state transformation are different.

IV. Numerical Approximation of $D_t^{\alpha}[e^{zt}]$

The purpose of this section is to develop numerical approximation of the fractional derivative of the complex exponential function

$$F(t) = D_t^{\alpha}[e^{zt}] \tag{90}$$

for $0 < \alpha < 1$ and z is a complex number. The numerical approximation of Eq (90) provides the foundation for the investigation of general behavior of structures that are viscoelastically damped and structures that are controlled using fractional order state feedback control theory. It can be shown herein that the behavior of the system is strongly influenced by the poles on the principal sheet of the Riemann surface and the residues of these poles provide good approximation for the general system response.

In this section two numerical approximation expressions for the fractional derivative of the complex exponential function are developed. These expressions are used to approximate the short range and intermediate range behavior of the function in Eq (90). The expressions for the errors associated with these approximations are also developed in the process. Finally these approximations are used to generate estimations for the fractional derivative of the generalized sinusoidal function

$$G(t) = D_t^{\alpha}[e^{zt} + e^{\bar{z}t}] \tag{91}$$

where $e^{\bar{z}t}$ is the complex conjugate of e^{zt}. Recalling from vibration analysis (24:10) that the general response of an underdamped system is

$$X(t) = A_1 e^{zt} + A_2 e^{\bar{z}t} \tag{92}$$

where

$$z = -\zeta\omega_n + i\omega_d = \left(-\zeta + i(1 - \zeta^2)^{1/2}\right) \omega_n = \mathfrak{z}\, \omega_n \tag{93a}$$

$$\bar{z} = -\zeta\omega_n - i\omega_d = \left(-\zeta - i(1 - \zeta^2)^{1/2}\right) \omega_n = \bar{\mathfrak{z}}\, \omega_n \tag{93b}$$

and

$$\mathfrak{z} = \left(-\zeta + i(1 - \zeta^2)^{1/2}\right) \tag{94a}$$

$$\bar{\mathfrak{z}} = \left(-\zeta - i(1 - \zeta^2)^{1/2}\right) \tag{94b}$$

where ζ is the damping ratio of the system. ω_n and ω_d are the natural frequency of the undamped system and the resonant frequency of the damped system respectively. Thus Eq (91) can provide a general description of the fractional derivatives of the response of an underdamped structural mode.

Before proceeding with the development of the approximation for the fractional derivative of the complex exponential function, it is useful to pose the exponential function in terms of the Maclaurin series

$$e^{zt} = \sum_{n=0}^{\infty} \frac{(zt)^n}{n!} = 1 + zt + \frac{(zt)^2}{2!} + \cdots \tag{95}$$

The fractional derivative of the function is

$$D_t^{\alpha}[e^{zt}] = D_t^{\alpha}\left[\sum_{n=0}^{\infty} \frac{(zt)^n}{n!} \right] \tag{96}$$

Taking the Laplace transform of Eq (96) and the fractional derivative of the complex exponential function is seen to be

$$\begin{aligned} \mathfrak{L}\left[D_t^{\alpha}[e^{zt}]\right] &= s^{\alpha}\cdot\mathfrak{L}\left[\sum_{n=0}^{\infty} \frac{(zt)^n}{n!} \right] \\ &= s^{\alpha}\cdot\sum_{n=0}^{\infty} z^n\left(\frac{1}{s^{n+1}}\right) \end{aligned} \tag{97}$$

which can be simplified to be

$$\mathfrak{L}\left[D_t^{\alpha}[e^{zt}]\right] = \sum_{n=0}^{\infty} z^n\left(\frac{1}{s^{n-\alpha+1}}\right) \tag{98}$$

The last step in calculating the derivative in Eq (90) is to calculate the inverse Laplace transform of Eq (98)

$$D_t^{\alpha}[e^{zt}] = \mathfrak{L}^{-1}\left[\sum_{n=0}^{\infty} z^n\left(\frac{1}{s^{n-\alpha+1}}\right)\right]$$

$$= \sum_{n=0}^{\infty} z^n \cdot \mathfrak{L}^{-1}\left(\frac{1}{s^{n-\alpha+1}}\right) \tag{99}$$

and

$$\mathfrak{L}^{-1}\left[\frac{1}{s^{a+1}}\right] = \frac{t^a}{\Gamma(a+1)} \tag{100}$$

Let $a = n - \alpha$ in Eq (100) and apply the inverse transform to Eq (99) then

$$D_t^{\alpha}[e^{zt}] = \sum_{n=0}^{\infty} z^n \frac{t^{(n-\alpha)}}{\Gamma(n-\alpha+1)} \tag{101}$$

Eq (101) is the fractional derivative of a complex exponential function. Express Eq (101) in terms of its Nth partial sum and its remainder

$$D_t^{\alpha}[e^{zt}] = \sum_{n=0}^{N} z^n \frac{t^{(n-\alpha)}}{\Gamma(n-\alpha+1)} + \sum_{n=N+1}^{\infty} z^n \frac{t^{(n-\alpha)}}{\Gamma(n-\alpha+1)} \tag{102}$$

where the first part of Eq (102) is the Nth partial sum of the series and the second part of the equation is the remainder of the series after term N.

Redefine Eq (102) in terms of dimensionless time, $\theta(t)$, where

$$\theta(t) = \omega_n t \tag{103}$$

Eq (102) may then be written as

$$D_t^{\alpha}[e^{\mathfrak{z}\theta(t)}] = \sum_{n=0}^{N} \omega_n^{\alpha}\, \mathfrak{z}^n \frac{\theta^{(n-\alpha)}}{\Gamma(n - \alpha + 1)} + \sum_{n=N+1}^{\infty} \omega_n^{\alpha}\, \mathfrak{z}^n \frac{\theta^{(n-\alpha)}}{\Gamma(n - \alpha + 1)} \qquad (104)$$

The general approximation of the fractional derivative of the complex exponential function can be expressed as the Nth partial sum of the series

$$D_t^{\alpha}[e^{zt}] \approx \sum_{n=0}^{N} z^n \frac{t^{(n-\alpha)}}{\Gamma(n - \alpha + 1)}$$

$$= \sum_{n=0}^{N} \omega_n^{\alpha}\, \mathfrak{z}^n \frac{\theta^{(n-\alpha)}}{\Gamma(n - \alpha + 1)} \qquad (105)$$

and the remainder of the series is the error, E_N, of the approximation

$$E_N = \sum_{n=N+1}^{\infty} z^n \frac{t^{(n-\alpha)}}{\Gamma(n - \alpha + 1)}$$

$$= \sum_{n=N+1}^{\infty} \omega_n^{\alpha}\, \mathfrak{z}^n \frac{\theta^{(n-\alpha)}}{\Gamma(n - \alpha + 1)} \qquad (106)$$

Notice that the error function E_N increases without bound as $\theta(t)$ increases, and the number of terms required to maintain a prescribed degree of accuracy in the solution increases rapidly as $\theta(t)$ increases. Therefore Eq (105) is only valid for the short range approximation of the fractional derivative of the complex exponential function.

To better understand the general behavior of the function, Eq (90), it is necessary to develop an expression that can better portray the overall behavior of the function. The process begins by taking the Laplace transform of Eq (90)

$$\mathfrak{L}\left[D_t^{\alpha}[e^{zt}]\right] = s^{\alpha}\cdot\mathfrak{L}[e^{zt}] \tag{107}$$

The Laplace transform of the exponential function is determined from

$$\mathfrak{L}[e^{zt}] = \int_0^{\infty} e^{zt}e^{-st}\,dt = \frac{1}{s - z} \tag{108}$$

Substitute Eq (108) into Eq (107) and the result is

$$\mathfrak{L}\left[D_t^{\alpha}[e^{zt}]\right] = \frac{s^{\alpha}}{s - z} \tag{109}$$

To determine the fractional derivative of the complex exponential function, one needs to calculate the inverse Laplace transform of Eq (109). Following the definition of inverse Laplace transform integral

$$\mathfrak{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma - i\omega}^{\gamma + i\omega} [F(s)]\, e^{st}\, ds \tag{110}$$

the inverse transform of Eq (109) may be expressed as

$$D_t^{\alpha}[e^{zt}] = \mathfrak{L}^{-1}\left[\frac{s^{\alpha}}{s - z}\right] = \frac{1}{2\pi i}\int_{\gamma - i\omega}^{\gamma + i\omega}\left(\frac{s^{\alpha}}{s - z}\right) e^{st}\, ds \qquad (111)$$

and Eq (111) can be evaluated using the residue theorem from the calculus of complex variables. The closed contour of integration for the inverse transform, in conjunction with the residue theorem, is shown in Figure 1.

The integration contour has six segments and the direction of integration is indicated by the arrows. Segments 3, 4, and 5 are included in the contour because the branch cut of the function $s^{1/m}$ is taken along the negative real axis and the branch point of the same function is taken at the origin of the s plane. The inverse transform, Eq (111), is represented by segment 1 of the contour.

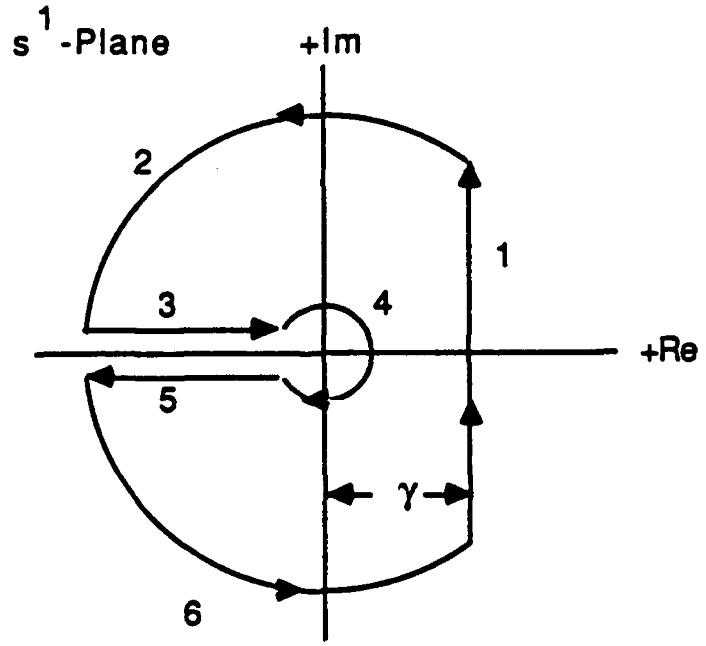

Figure 1. Contour of Integration for Inverse Transform

The residue theorem states that the integral along the contour, divided by $2\pi i$, is equal to the sum of the residues of the poles of the integrand enclosed by the contour (22:865). By including the inverse transform in the integration contour and applying the residue theorem from complex calculus, the inverse transform, Eq (111), may be written as

$$\frac{1}{2\pi i}\int_{\gamma - i\omega}^{\gamma + i\omega}\left(\frac{s^{\alpha}}{s - z}\right) e^{st}\, ds = -\frac{1}{2\pi i}\sum_{k=2}^{6}\int_{k}\left(\frac{s^{\alpha}}{s - z}\right) e^{st}\, ds + \sum_{j} b_j \qquad (112)$$

where the first summation represents the sum of the integrals of the other five segments and the second summation represents the sum of the residues. The residues, b_j, are evaluated by conventional techniques

$$b_j = \lim_{s \to \lambda_j} (s - \lambda_j)[F(s)]e^{st} \qquad (113)$$

provided the limit exists. Under the current analysis, there is one pole at $\lambda_1 = z$ and the residue of this pole enclosed by the contour is

$$b_1 = \lim_{s \to z} (s-z)\left(\frac{s^{\alpha}}{s - z}\right) e^{st} = z^{\alpha}e^{zt} \qquad (114)$$

It has been shown (5:95-99) that the integrals along segments 2, 4, and 6 are equal to zero when the radii of 2 and 6 are increased indefinitely and the radius of 4 is decreased indefinitely. The integrals along segments 3 and 5 are not equal to zero and the sum of the integrals are not equal to zero either. The expression for the inverse transform given by Eq (112) is now

$$\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}\left(\frac{s^{\alpha}}{s-z}\right) e^{st}\,ds = -\frac{1}{2\pi i}\sum_{k=3,5}\int_{k}\left(\frac{s^{\alpha}}{s-z}\right) e^{st}\,ds + z^{\alpha}e^{zt} \qquad (115)$$

and the inverse transform is now the sum of the integrals along the branch cut, segments 3 and 5, plus the residue. The sum of the integrals can be determined as follow

$$-\frac{1}{2\pi i}\sum_{k=3,5}\int_{k}\left(\frac{s^{\alpha}}{s-z}\right) e^{st}\,ds$$

$$= -\frac{1}{2\pi i}\left(\int_{\infty}^{0}\frac{(re^{i\pi})^{\alpha}}{(re^{i\pi}-z)}\, e^{(re^{i\pi}t)}e^{i\pi}\,dr + \int_{0}^{\infty}\frac{(re^{-i\pi})^{\alpha}}{(re^{-i\pi}-z)}\, e^{(re^{-i\pi}t)}e^{-i\pi}\,dr\right)$$

$$= -\frac{1}{2\pi i}\left(-e^{i\pi\alpha}+e^{-i\pi\alpha}\right)\int_{0}^{\infty}\frac{r^{\alpha}e^{-rt}}{r+z}\,dr$$

$$= -\frac{1}{2\pi i}\left(-2i\cdot\sin(\alpha\pi)\right)\int_{0}^{\infty}\frac{r^{\alpha}e^{-rt}}{r+z}\,dr$$

$$= \frac{1}{\pi}\sin(\alpha\pi)\int_{0}^{\infty}\frac{r^{\alpha}e^{-rt}}{r+z}\,dr \qquad (116)$$

Applying the reflection formula of gamma function (27:256), where

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \qquad (117)$$

and Eq (116) can be written as

$$-\frac{1}{2\pi i}\sum_{k=3,5}\int_k\left(\frac{s^\alpha}{s-z}\right)e^{st}\,ds = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty \frac{r^\alpha e^{-rt}}{z(1+r/z)}\,dr \qquad (118)$$

Recalling the geometric series where

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad (119)$$

Let $x = -r/z$ in Eq (119) and apply Watson's lemma (30:21-24) to Eq (118)

$$-\frac{1}{2\pi i}\sum_{k=3,5}\int_k\left(\frac{s^\alpha}{s-z}\right)e^{st}ds \approx \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\frac{1}{z}\,\frac{\Gamma(\alpha+n+1)}{t^{\alpha+n+1}},\ t\to\infty \qquad (120)$$

Now substitute Eq (120) into Eq (115) and the fractional derivative of the complex exponential function can be expressed as

$$D_t^\alpha[e^{zt}] \approx z^\alpha e^{zt} + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\sum_{n=}^{\infty}\frac{1}{z}\,\frac{\Gamma(\alpha+n+1)}{t^{\alpha+n+1}},\ t\to\infty \qquad (121)$$

Eq (121) is the asymptotic expansion representation of the same function in Eq (90). Redefine Eq (121) in terms of dimensionless time, $\theta(t)$, defined in Eq (103), and Eq (121) can be written as

$$D_t^\alpha[e^{\partial\theta(t)}] \approx \omega_n^\alpha\,\partial^\alpha\,e^{\partial\theta} + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\frac{\omega_n^{\alpha+n}\,\Gamma(\alpha+n+1)}{\partial\,\theta^{\alpha+n+1}},\ t\to\infty \qquad (122)$$

The general behavior of the fractional derivative may be approximated as

$$D^{\alpha}[e^{zt}] \approx z^{\alpha} e^{zt}$$

$$= \omega_{n}^{\alpha} \, \mathfrak{z}^{\alpha} \, e^{\mathfrak{z}\theta} \qquad (123)$$

for the intermediate range of time and the summation term represents the remainder, R_n, of the approximation

$$R_n = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{1}{z} \frac{\Gamma(\alpha+n+1)}{t^{\alpha+n+1}}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{\omega_{n}^{\alpha+n} \, \Gamma(\alpha+n+1)}{\mathfrak{z} \, \theta^{\alpha+n+1}} \qquad (124)$$

The remainder function R_n now decreases as $\theta(t)$ increases for this function. Because the exponential term decays faster than the summation term, the long range behavior of the function is influenced more by the remainder term. The effect of the remainder is even more pronounced at low frequency and in short time range. Thus Eq (123) provides only a representation of the intermediate range behavior of the function under investigation.

Given Eq (105) and Eq (123), one may proceed with the evaluation of the approximation of fractional derivative of the generalized sinusoidal function defined by Eq (91). Since z and $\bar{z}$ are complex conjugates, the short range approximation of Eq (91) for systems with zero initial displacement can be written as

$$D_t^{\alpha}[e^{zt} + e^{\bar{z}t}] \approx 2 \cdot \sum_{n=0}^{N} Im(z^n) \frac{t^{(n-\alpha)}}{\Gamma(n - \alpha + 1)} \tag{125}$$

and the approximation in Eq (125) expressed in dimensionless time $\theta(t)$ and normalized by $2 \cdot \omega_n^{\alpha}$ is

$$\frac{1}{2} \omega_n^{-\alpha} D_t^{\alpha}[e^{zt} + e^{\bar{z}t}] \approx \sum_{n=0}^{N} Im\left(\frac{\mathfrak{z}^{\alpha} \theta^{(n-\alpha)}}{\Gamma(n - \alpha + 1)}\right) \tag{126}$$

In the same manner, the intermediate range approximation of Eq (91) can be posed as

$$D_t^{\alpha}[e^{zt} + e^{\bar{z}t}] \approx 2 \cdot Im(z^{\alpha} e^{zt}) \tag{127}$$

and the approximation of Eq (127) expressed in dimensionless time $\theta(t)$ and normalized by $2 \cdot \omega^{\alpha}$ is

$$\frac{1}{2} \omega_n^{-\alpha} D_t^{\alpha}[e^{zt} + e^{zt}] \approx Im(\mathfrak{z}^{\alpha} e^{\mathfrak{z}\theta}) \tag{128}$$

The normalized approximation of the short range and intermediate range behavior for the fractional derivative of the response of an underdamped system, Eq (126) and Eq (128) respectively, with $\alpha = 0.5$, $\zeta = 0.1$, $\omega_n = \pi$ and $N = 60$ are illustrated in Figure 2. Notice that the intermediate range approximation converges onto the exact behavior expression in less than one cycle.

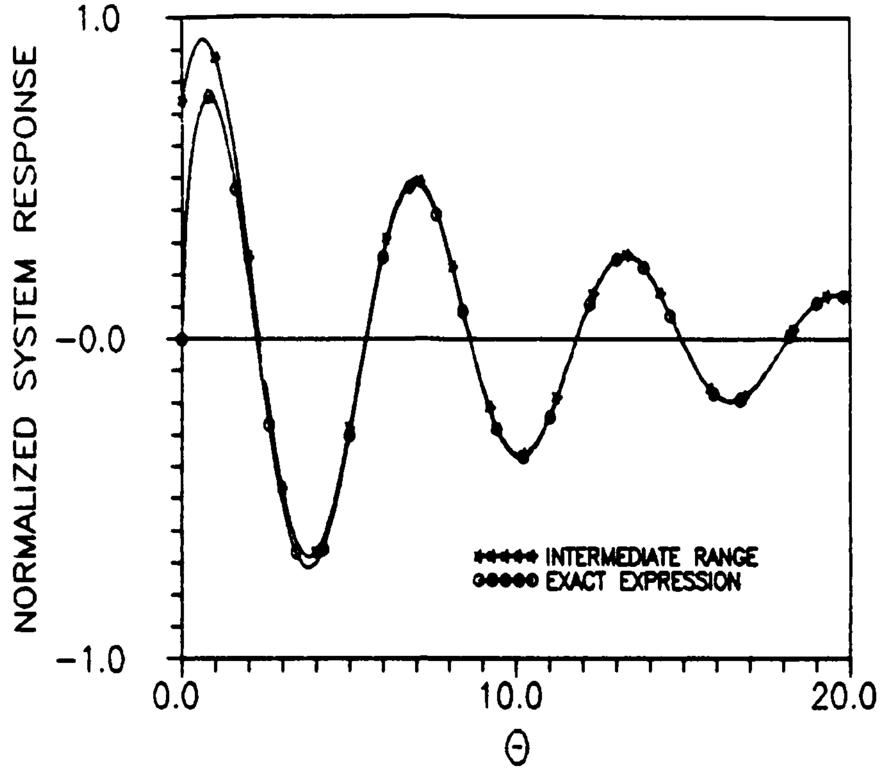

Figure 2. Approximation of Fractional Derivative of Generalized Sinusoidal Function

Based on the analysis just presented, one may conclude that the system response represented by the intermediate range function is a relatively accurate representation of the actual response of the system studied, and the response is strongly influenced by the residues of the poles of the system. This relationship forms the basis of the fractional order state feedback control theory formulation in Chapter III, where only the poles on the principal sheet of the Riemann surface are retained for the development and formulation of the state transformation matrix. Following this analysis, one may formulate the fractional order state feedback control force input based on the system response and the eigenvalues and eigenvectors associated with the poles on the principal sheet of the Riemann surface alone, and still maintain equivalent control authority as the traditional state feedback control with fine control accuracy.

V. Example Problems

To demonstrate the application of the fractional order feedback control theory, the finite element analysis of two simple, controlled, structures are presented in this section. The numerical solution of the examples are computed using matrix manipulation routines in the program PRO-MATLAB (26). The first example is a simple, hypothetical, two degree-of-freedom spring-mass system undergoing longitudinal vibration. Both active closed-loop feedback control and passive viscoelastic damping coupled with active feedback control are examined in this example. The second example is part of a structural control experiment currently being developed at AFIT to demonstrate the applicability of the fractional order state feedback control theory in controlling large flexible space structures. The objective of this example is to develop the optimal gain and state transformation matrices for the control of the modified Advanced Beam Experiment (ABE) (25) using fractional order state feedback.

Example Problem 1

In this first example, the application of fractional order state feedback control theory in structural control will be demonstrated, using numerical analysis of a hypothetical structure. The first part of the example will illustrate the formulation of the expanded equations of motion, the determination of the resonant frequencies and mode shapes, and the composition of the state transformation matrix for an open-loop system.

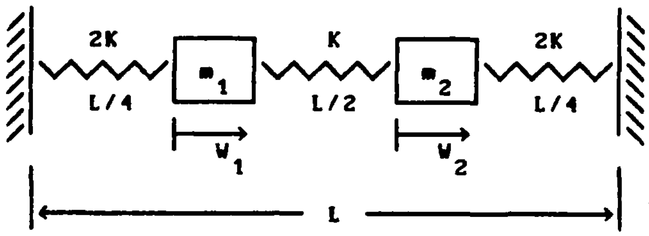

Figure 3. Free Vibrating 2 DOF System

Consider a two degree-of-freedom system, shown in Figure 3, in longitudinal free vibration. The mass and stiffness matrices for the system are defined as

$$[M] = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$$

The general form of the equations of motion for the homogeneous system is

$$[M]\{\ddot{w}(t)\} + [K]\{w(t)\} = \{0\} \tag{129}$$

where $\{w(t)\}$ is the displacement vector defined as

$$\{w(t)\} = \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} \tag{130}$$

The equations of motion for the system under free vibration may be posed as

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \{0\} \tag{131}$$

and the eigenvalues and eigenvectors of the open system can be determined by the EIG command of PRO-MATLAB (26:3.35). These values correspond to the first two modes of vibration. The natural frequencies of the system are

$$\omega_1 = 2.8284$$
$$\omega_2 = 4.0000$$

and the mode shapes of the system are

$$\{w\}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$
$$\{w\}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

The general problem can also be solved by first posing the equations of motion in a fractional order format and solving the fractional order equations using the solution technique of the expanded equations of motion. Assuming $\alpha = 1/2$ and applying Eq (25) in the equation formulation, Eq (129) may be written as

$$D^{1/2}[M]\left[D^{3/2} + D^{1} + D^{1/2} + D^{0}\right]\{w(t)\}$$
$$- [M]\left[D^{3/2} + D^{1} + D^{1/2}\right]\{w(t)\}$$
$$+ [K]D^{0}\{w(t)\} = \{F(t)\} \qquad (132)$$

To solve for the eigenvalues and eigenvectors of Eq (132), follow the solution method of expanded equations of motion described in Eq (32)-Eq (36) and arrange Eq (132) in the expanded format

$$D^{1/2}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{\tilde{F}(t)\} \qquad (133)$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [0] & [0] & [M] \\ [0] & [0] & [M] & [0] \\ [0] & [M] & [0] & [0] \\ [M] & [0] & [0] & [0] \end{bmatrix} \qquad (134)$$

$$[\tilde{K}] = \begin{bmatrix} [0] & [0] & -[M] & [0] \\ [0] & -[M] & [0] & [0] \\ -[M] & [0] & [0] & [0] \\ [0] & [0] & [0] & [K] \end{bmatrix} \qquad (135)$$

$$\{\tilde{w}(t)\} = \begin{Bmatrix} D^{3/2}\{w(t)\} \\ D^{1}\ \{w(t)\} \\ D^{1/2}\{w(t)\} \\ \{w(t)\} \end{Bmatrix} \tag{136}$$

The eigenvalues and eigenvectors of the expanded equations can then be solved using the EIG command of PRO-MATLAB and the eigenvalues are

$$\lambda_{1,2} = -\ 1.4142 \pm 1.4142i$$
$$\lambda_{3,4} = -\ 1.1892 \pm 1.1892i$$
$$\lambda_{5,6} = \ \ 1.4142 \pm 1.4142i$$
$$\lambda_{7,8} = \ \ 1.1892 \pm 1.1892i$$

The eigenvalues with positive real parts may be mapped onto the principal sheet of the Riemann surface for the function, $\tilde{\lambda}_j = s^{1/2}$, consisting of two sheets (23:902). These eigenvalues and the corresponding eigenvectors describe the sinusoidal motion of the structure and form the basis for the development of the state transformation matrix as described in Chapter III of this thesis. The natural frequencies of the system can be determined by squaring the fifth, sixth, seventh, and eighth eigenvalues. This process produces two conjugate pairs

$$\lambda^2_{5,6} = 0.0000 \pm 4.0000i$$
$$\lambda^2_{7,8} = 0.0000 \pm 2.8284i$$

The imaginary parts of each conjugate pair correspond to a natural frequency of the system. The zero in the real part indicates that no damping is present in the structure.

The eigenvectors associated with the eigenvalues identifying system resonant behavior (fifth, sixth, seventh, and eighth) also occur in complex conjugate pairs. They may be posed as a modal matrix of the form, Eq (39), and the modal matrix for the system is

$$[\Phi] =$$

$$\begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ -1.0000 & -1.0000 & 1.0000 & 1.0000 \\ 0.3536 - 0.3536i & 0.3536 + 0.3536i & 0\ 4204 - 0.4204i & 0.4204 + 0.4204i \\ -0.3536 + 0.3536i & -0.3536 - 0.3536i & 0.4202 - 0.4202i & 0.4204 + 0.4204i \\ 0.0000 - 0.2500i & 0.0000 + 0.2500i & 0.0000 - 0.3536i & 0.0000 + 0.3536i \\ 0.0000 + 0.2500i & 0.0000 - 0.2500i & 0.0000 - 0.3536i & 0.0000 + 0.3536i \\ -0.0884 - 0.0884i & -0.0884 + 0.0884i & -0.1487 - 0.1487i & -0.1487 + 0.1487i \\ 0.0884 + 0.0884i & 0.0884 - 0.0884i & -0.1487 - 0.1487i & -0.1487 + 0.1487i \end{bmatrix}$$

The seventh and eighth rows of the matrix represent the mode shape of the system. The fifth and sixth rows are associated with the mode of the $D^{1/2}$ state. The third and fourth rows represent the mode of the D^1 state and the first and second rows represent the mode of the $D^{3/2}$ state of the system.

For this open system, the state equations can be written as

$$\dot{\underline{x}} = A\underline{x} \tag{137}$$

where

$$\underline{x} = \begin{Bmatrix} w_1 \\ w_2 \\ \dot{w}_1 \\ \dot{w}_2 \end{Bmatrix} \tag{138}$$

Recalling from the expansion theorem in structural modal analysis that the state vector may be posed as the product of an associated modal matrix and normal coordinates, the state vector in Eq (138) can be written as

$$\underline{x} = [\phi_A]\{\eta\} \tag{42}$$

where

$$[\phi_A] =$$

$$\begin{bmatrix} -0.0884 - 0.0884i & -0.0884 + 0.0884i & -0.1487 - 0.1487i & -0.1487 + 0.1487i \\ 0.0884 + 0.0884i & 0.0884 - 0.0884i & -0.1487 - 0.1487i & -0.1487 + 0.1487i \\ 0.3536 - 0.3536i & 0.3536 + 0.3536i & 0\ 4204 - 0.4204i & 0.4204 + 0.4204i \\ -0.3536 + 0.3536i & -0.3536 - 0.3536i & 0.4202 - 0.4202i & 0.4204 + 0.4204i \end{bmatrix}$$

for the present system. The row elements of $[\phi_A]$ are those row elements of $[\Phi]$ that correspond to the states of the state vector $\underline{x}$. In a similar manner, the fractional state vector may be written as

$$\underline{y} = [\phi_B]\{\eta\} \tag{44}$$

For the application of one fractional state observer placed at m_1, $\underline{y}$ can be defined as

$$\underline{y} = \begin{Bmatrix} D_1^{3/2} & w_1 \\ D_1 & w_1 \\ D^{1/2} & w_1 \\ & w_1 \\ & w_1 \end{Bmatrix} \tag{139}$$

and

$[\phi_B] =$

$$\begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.3536 - 0.3536i & 0.3536 + 0.3536i & 0\ 4204 - 0.4204i & 0.4204 + 0.4204i \\ 0.0000 - 0.2500i & 0.0000 + 0.2500i & 0.0000 - 0.3536i & 0.0000 + 0.3536i \\ -0.0884 - 0.0884i & -0.0884 + 0.0884i & -0.1487 - 0.1487i & -0.1487 + 0.1487i \end{bmatrix}$$

for this system. Here the row elements of $[\phi_B]$ are those row elements of $[\Phi]$ associated with the state elements identified in $\underline{y}$.

To determine the state transformation matrix, ϕ, between the traditional state vector and the fractional state vector, apply Eq (48) and

$$\phi = [\phi_A][\phi_B]^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1.3017 & -3.3889 & 4.3784 & 0.1716 \\ 0 & 1 & 0 & 0 \\ 4.3784 & -22.9693 & 44.6249 & -38.3410 \end{bmatrix}$$

The state transformation matrix allows one to determine the overall structural response by observing only the fractional and integer states at a_1 and this relationship can be illustrated as

$$\underline{x} = \Phi \underline{y} \tag{49}$$

Eq (49) can be defined for this example as

$$\begin{Bmatrix} w_1 \\ w_2 \\ \dot{w}_1 \\ \dot{w}_2 \end{Bmatrix} = \Phi \begin{Bmatrix} D^{3/2} & w_1 \\ D^{1} & w_1 \\ D^{1/2} & w_1 \\ D & w_1 \\ & w_1 \end{Bmatrix} \tag{140}$$

In the second part of this example, a control force is applied at m_1 for the purpose of implementing feedback control to the system. The system set up is shown in Figure 4.

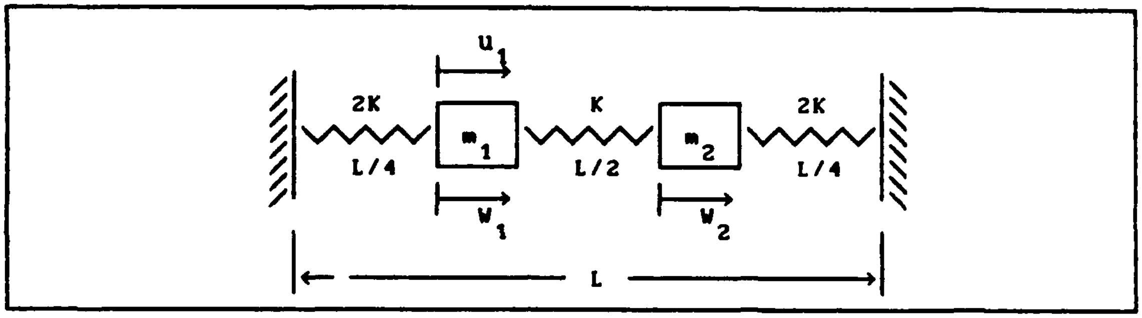

Figure 4. Feedback Controlled 2 DOF System

The equations of motion for the system are

$$[M]\{\ddot{w}(t)\} + [K]\{w(t)\} = [D]\{u(t)\} \tag{141}$$

and

$$[D] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{142}$$

The one entry on the diagonal indicates that the location of control force is at m_1. The state model of the equations of motion may be put in the form

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1}$$

and the elements of the state model are defined in Eq (51), (52), (53), and (54).

The optimal control for the system is determined by minimizing the quadratic performance index

$$J = \frac{1}{2}\int_0^\infty [x(t)^T Qx(t) + u^T(t)Ru(t)]dt \tag{143}$$

where

$$Q = \begin{bmatrix} [K] & [0] \\ [0] & [M] \end{bmatrix} \tag{144}$$

$$R = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \tag{145}$$

Notice that the first term of the integral represents the total energy of the system and the second integral term represents the control energy. The optimal control for the system is

$$\underline{u} = - G\underline{x} \tag{7}$$

where G is the optimal control gain matrix and it is defined as

$$G = R^{-1}BS \tag{4}$$

and S is the solution of the matrix Riccati equation. For the current system the optimal gain matrix is

$$G = [G_1 \vdots G_2]$$
$$= \left[\begin{array}{cc|cc} 0.0149 & -0.0298 & 0.1412 & 0.0002 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Only the first row of the gain matrix is populated because only one control force is applied to the system.

Incorporating the optimal feedback control into the system, the equations of motion may now be written in a homogeneous form

$$[M]\{\ddot{w}(t)\} + [D][G_2]\{\dot{w}(t)\} + ([K] + [D][G_1])\{w(t)\} = \{0\} \qquad (56)$$

The system response can be determined by first posing Eq (56) in the fractional format and then solving the problem using the solution method of the expanded equations of motion described in Eq (32)-Eq (36). For $\alpha = 1/2$, the fractional order equations of motion are

$$\begin{aligned} D^{1/2}[M]\left[D^{3/2} + D^1 + D^{1/2} + D^0\right]\{w(t)\} \\ - [M]\left[D^{3/2} + D^1 + D^{1/2}\right]\{w(t)\} \\ + D^{1/2}[D][G_2]\left[D^{1/2} + D^0\right]\{w(t)\} \\ - [D][G_2]D^{1/2}\{w(t)\} \\ + ([K] + [D][G_1])D^0\{w(t)\} = \{0\} \end{aligned} \qquad (146)$$

and the expanded equations of motion are

$$D^{1/2}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{0\} \tag{147}$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [0] & [0] & [M] \\ [0] & [0] & [M] & [0] \\ [0] & [M] & [0] & [D][G_2] \\ [M] & [0] & [D][G_2] & [0] \end{bmatrix} \tag{148}$$

$$[\tilde{K}] = \begin{bmatrix} [0] & [0] & -[M] & [0] \\ [0] & -[M] & [0] & [0] \\ -[M] & [0] & -[D][G_2] & [0] \\ [0] & [0] & [0] & [K]+[D][G_1] \end{bmatrix} \tag{149}$$

$$\{\tilde{w}(t)\} = \begin{Bmatrix} D^{3/2}\{w(t)\} \\ D^{1}\ \{w(t)\} \\ D^{1/2}\{w(t)\} \\ \{w(t)\} \end{Bmatrix} \tag{136}$$

There are four pairs of complex conjugate eigenvalues to Eq (147) and they are

$$\lambda_{1,2} = -\ 1.4018 \pm 1.4267i$$
$$\lambda_{3,4} = -\ 1.1745 \pm 1.2041i$$
$$\lambda_{5,6} = \ 1.4018 \pm 1.4267i$$
$$\lambda_{7,8} = \ 1.1745 \pm 1.2041i$$

The eigenvalues $\lambda_{5,6}$ and $\lambda_{7,8}$ correspond to the mode two and mode one behaviors of the system. Squaring the eigenvalues generates the respective frequencies for the modes

$$\lambda^2_{5,6} = -\ 0.0706 \pm 4.0000i$$
$$\lambda^2_{7,8} = -\ 0.0706 \pm 2.8284i \tag{150}$$

where the imaginary parts represent the frequencies of the feedback controlled vibration, and the real parts represent the product of the damping ratio and the natural frequencies of the undamped vibration. Notice that damping has been introduced to the system through the optimal feedback control, as evidenced by presence of nonzero real parts of Eq (150).

The modal matrix of this system is formed by grouping the eigenvectors associated with the eigenvalues $\lambda_{5,6}$ and $\lambda_{7,8}$ in a matrix form and

$$[\Phi] =$$

$$\begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ -0.9817 + 0.1387i & -0.9817 - 0.1387i & 0.9890 + 0.0986i & 0.9890 - 0.0986i \\ 0.3504 - 0.3566i & 0.3504 + 0.3566i & 0\ 4151 - 0.4256i & 0.4151 + 0.4256i \\ -0.2945 + 0.3987i & -0.2945 - 0.3987i & 0.4525 - 0.3800i & 0.4525 + 0.3800i \\ -0.0044 - 0.2499i & -0.0044 + 0.2499i & -0.0088 - 0.3533i & -0.0088 + 0.3533i \\ 0.0390 + 0.2447i & 0.0390 - 0.2447i & 0.0261 - 0.3503i & 0.0261 + 0.3503i \\ -0.0907 - 0.0860i & -0.0907 + 0.0860i & -0.1540 - 0.1429i & -0.1540 + 0.1429i \\ 0.1099 + 0.0718i & 0.1009 - 0.0718i & -0.1382 - 0.1565i & -0.1382 + 0.1565i \end{bmatrix}$$

From the modal matrix, one may construct the modal matrices associated with the traditional state vector and the fractional vector of the system, in the same manner as indicated in the first part of this problem. For the current feedback control scheme, the matrices associated with the state vectors are

$$[\phi_A] =$$

$$\begin{bmatrix} -0.0907 - 0.0860i & -0.0907 + 0.0860i & -0.1540 - 0.1429i & -0.1540 + 0.1429i \\ 0.1099 + 0.0718i & 0.1009 - 0.0718i & -0.1382 - 0.1565i & -0.1382 + 0.1565i \\ 0.3504 - 0.3566i & 0.3504 + 0.3566i & 0\ 4151 - 0.4256i & 0.4151 + 0.4256i \\ -0.2945 + 0.3987i & -0.2945 - 0.3987i & 0.4525 - 0.3800i & 0.4525 + 0.3800i \end{bmatrix}$$

and

$$[\phi_B] =$$

$$\begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.3504 - 0.3566i & 0.3504 + 0.3566i & 0\ 4151 - 0.4256i & 0.4151 + 0.4256i \\ -0.0044 - 0.2499i & -0.0044 + 0.2499i & -0.0088 - 0.3533i & -0.0088 + 0.3533i \\ -0.0907 - 0.0860i & -0.0907 + 0.0860i & -0.1540 - 0.1429i & -0.1540 + 0.1429i \end{bmatrix}$$

The state transformation matrix for the system is

$$\Phi = [\Phi_A][\Phi_B]^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1.2696 & -3.2370 & 4.2724 & 0.1719 \\ 0 & 1 & 0 & 0 \\ 4.2675 & -22.1595 & 42.8966 & -37.4050 \end{bmatrix}$$

This state transformation matrix permits one to apply the fractional order state feedback control law to this structure and the control law is defined as

$$\dot{\underline{x}} = A\underline{x} - BG\Phi\underline{y} \qquad (60)$$

where the fractional derivatives of the system response at m_1 are fed back as control input for controlling the structure. Defining the fractional state gain matrix, G^*, as

$$G^* = G\Phi \qquad (151)$$

the state equations for fractional order state feedback control can be rewritten as

$$\dot{\underline{x}} = A\underline{x} - BG^*\underline{y} \qquad (152)$$

and

$$G^* = \left[\begin{array}{cc:cc} -0.0371 & 0.2337 & -0.1198 & 0.0034 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The fractional state gain matrix, G^*, permits the direct feedback of fractional states for the control of the structure. The feedback control force $G^*\underline{y}$ has the same control effect as the traditional state feedback control force $G\underline{x}$.

In the last part of this example, a hypothetical viscoelastic damping pad is introduced at m_2 in addition to the control force at m_1. The system configuration is shown in Figure 5.

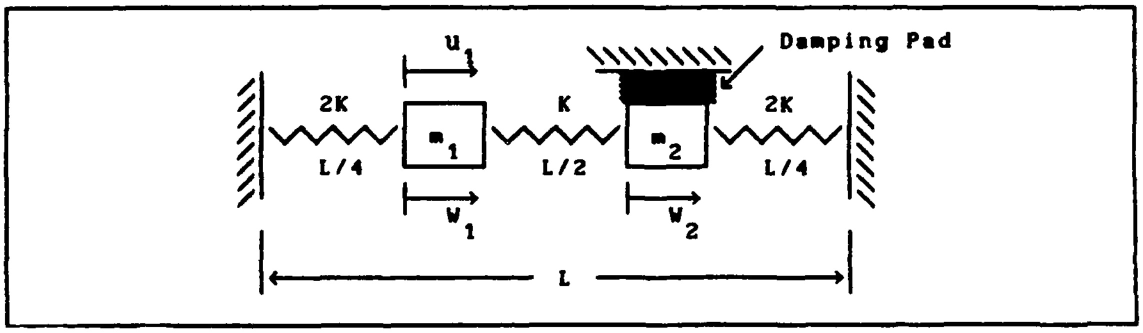

Figure 5. Viscoelastically-Damped 2 DOF System

The equations of motion for the system are now

$$[M]\{\ddot{w}(t)\} + [K_V]\{D^{\beta}w(t)\} + [K_E]\{w(t)\} = [D]\{u(t)\} \tag{68}$$

where

$$[K_V] = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$[K_E] = \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$$

$$\beta = 1/2$$

Before one may apply the optimal control theory, it is necessary to develop the equivalent damping and equivalent stiffness matrices for the system such that the equations of motion may be posed in the integer differential form as

$$[M]\{\ddot{w}(t)\} + [C^*]\{\dot{w}(t)\} + [K^*]\{w(t)\} = [D]\{u(t)\} \tag{69}$$

Following the procedure identified in Section 3.2, the modal acceleration, velocity, and displacement matrices for the homogeneous system may be determined by solving the fractional order equations of motion of the system

$$\begin{aligned} D^{1/2}[M]\left[D^{3/2} + D^{1} + D^{1/2} + D^{0}\right]\{w(t)\} \\ - [M]\left[D^{3/2} + D^{1} + D^{1/2}\right]\{w(t)\} \\ + [K_V]D^{1/2}\{w(t)\} + [K_E]D^{0}\{w(t)\} = \{0\} \end{aligned} \tag{153}$$

through the use of the solution technique of the expanded equations of motion. The expanded equations of motion for Eq (153) are now written as

$$D^{1/2}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{0\} \tag{154}$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [0] & [0] & [M] \\ [0] & [0] & [M] & [0] \\ [0] & [M] & [0] & [0] \\ [M] & [0] & [0] & [K_V] \end{bmatrix} \tag{155}$$

$$[\tilde{K}] = \begin{bmatrix} [0] & [0] & -[M] & [0] \\ [0] & -[M] & [0] & [0] \\ -[M] & [0] & [0] & [0] \\ [0] & [0] & [0] & [K_E] \end{bmatrix} \tag{156}$$

and $\{\tilde{w}(t)\}$ is defined by Eq (136). The modal acceleration, velocity, and displacement matrices: $[\Phi_a]$, $[\Phi_v]$, and $[\Phi_d]$, are then determined from the eigenvalues and the modal matrix of the system posed in Eq (154) and

$$[\Phi_a] = \begin{bmatrix} -1.4128 - 1.3219i & 1.1894 + 1.1979i \\ 1.4141 + 1.4206i & 1.1888 + 1.1277i \end{bmatrix}$$

$$[\Phi_v] = \begin{bmatrix} -0.3274 + 0.3531i & 0.4174 - 0.4204i \\ 0.3520 - 0.3536i & 0.3928 - 0.4200i \end{bmatrix}$$

$$[\Phi_d] = \begin{bmatrix} 0.0883 + 0.0811i & -0.1486 - 0.1454i \\ -0.0884 - 0.0872i & -0.1484 - 0.1368i \end{bmatrix}$$

Applying the above matrices to Eq (79) and (80), the equivalent damping and equivalent stiffness matrices are found to be

$$[C^*] = \begin{bmatrix} 0 & 0 \\ 0.0034 & 0.0387 \end{bmatrix}$$

$$[K^*] = \begin{bmatrix} 6.0000 & -2.0000 \\ -2.0112 & 6.1317 \end{bmatrix}$$

and the resonant frequencies of the viscoelastically damped system are

$$\lambda^2_{5,6} = -\ 0.0183 \pm 4.0177i$$
$$\lambda^2_{7,8} = -\ 0.0204 \pm 2.8495i \qquad (157)$$

By posing the equations of motion in the equivalent form, Eq (69), the state equations can now be developed for determining the optimal gain matrix for the feedback control of this viscoelastically damped structure, following the same procedure identified in Section 3.2. The gain matrix for the present control scheme is

$$G = [G_1 \vdots G_2]$$
$$= \left[\begin{array}{cc|cc} 0.0066 & -0.0048 & 0.1076 & -0.0029 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

With the gain matrix determined, the optimal feedback control is then introduced into the system and the equations of motion are now

$$[M]\{\ddot{w}(t)\} + ([C^*] + [D][G_2])\{\dot{w}(t)\}$$
$$+ ([K^*] + [D][G_1])\{w(t)\} = \{0\} \qquad (84)$$

For $\alpha = 1/2$, the fractional order equations of motion for Eq (84) can be written as

$$
\begin{aligned}
D^{1/2}[M]\left[D^{3/2} + D^{1} + D^{1/2} + D^{0}\right]\{w(t)\} \\
- [M]\left[D^{3/2} + D^{1} + D^{1/2}\right]\{w(t)\} \\
+ D^{1/2}([C^{*}] + [D][G_{2}])\left[D^{1/2} + D^{0}\right]\{w(t)\} \\
- ([C^{*}] + [D][G_{2}])D^{1/2}\{w(t)\} \\
+ ([K^{*}] + [D][G_{1}])D^{0}\{w(t)\} = \{0\} \quad (158)
\end{aligned}
$$

Eq (158) can be solved using the solution method of expanded equations of motion described in Eq (32)-Eq (36) and the expanded equations of motion for Eq (158) are

$$D^{1/2}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{0\} \quad (159)$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [0] & [0] & [M] \\ [0] & [0] & [M] & [0] \\ [0] & [M] & [0] & [C^{*}]+[D][G_{2}] \\ [M] & [0] & [C^{*}]+[D][G_{2}] & [0] \end{bmatrix} \quad (160)$$

$$[\tilde{K}] = \begin{bmatrix} [0] & [0] & -[M] & [0] \\ [0] & -[M] & [0] & [0] \\ -[M] & [0] & -[C^{*}]-[D][G_{2}] & [0] \\ [0] & [0] & [0] & [K^{*}]+[D][G_{1}] \end{bmatrix} \quad (161)$$

and $\{\tilde{w}(t)\}$ is defined by Eq (136). The response of the system can be determined by solving the expanded equations of motion. There are four pairs of complex conjugate eigenvalues to Eq (159)

$$\lambda_{1,2} = -\ 1.4047 \pm 1.4301i$$
$$\lambda_{3,4} = -\ 1.1781 \pm 1.2093i$$
$$\lambda_{5,6} = 1.4047 \pm 1.4301i$$
$$\lambda_{7,8} = 1.1781 \pm 1.2093i$$

The eigenvalues $\lambda_{5,6}$ and $\lambda_{7,8}$ are associated with the modal behaviors of the system and the modal frequencies for the system are

$$\begin{aligned} \lambda^2_{5,6} &= -\ 0.0718 \pm 4.0177i \\ \lambda^2_{7,8} &= -\ 0.0745 \pm 2.8495i \end{aligned} \qquad (162)$$

Notice that there is an increase in damping due to the incorporation of optimal feedback control with passive viscoelastic damping. This is evident by the increase in magnitude for the real parts of Eq (162) compare with the real parts of Eq (157).

The modal matrix for this system is formulated using the same procedure as indicated the first part of this problem and

$$[\Phi] =$$

$$\begin{bmatrix}
-0.9636 - 0.0726i & -0.9636 + 0.0726i & 1.0000 & 1.0000 \\
1.0000 & 1.0000 & 0.9688 + 0.0431i & 0.9688 - 0.0431i \\
-0.3627 + 0.3176i & -0.3627 - 0.3176i & 0\ 4133 - 0.4243i & 0.4133 + 0.4243i \\
0.3496 - 0.3559i & 0.3496 + 0.3559i & 0.4187 - 0.3932i & 0.4187 + 0.3932i \\
-0.0138 + 0.2401i & -0.0138 - 0.2401i & -0.0092 - 0.3507i & -0.0092 + 0.3507i \\
-0.0044 - 0.2488i & -0.0044 + 0.2488i & 0.0062 - 0.3401i & 0.0062 + 0.3401i \\
0.0806 + 0.0888i & 0.0806 - 0.0888i & -0.1526 - 0.1411i & -0.1526 + 0.1411i \\
-0.0901 - 0.0854i & -0.0901 + 0.0854i & -0.1417 - 0.1432i & -0.1417 + 0.1432i
\end{bmatrix}$$

The modal matrices associated with the traditional state vector and the fractional state vector can be formulated from the system modal matrix, following the procedure described in the first part of this problem and they are

$$[\phi_A] =$$

$$\begin{bmatrix} 0.0806 + 0.0888i & 0.0806 - 0.0888i & -0.1526 - 0.1411i & -0.1526 + 0.1411i \\ -0.0901 - 0.0854i & -0.0901 + 0.0854i & -0.1417 - 0.1432i & -0.1417 + 0.1432i \\ -0.3627 + 0.3176i & -0.3627 - 0.3176i & 0\ 4133 - 0.4243i & 0.4133 + 0.4243i \\ 0.3496 - 0.3559i & 0.3496 + 0.3559i & 0.4187 - 0.3932i & 0.4187 + 0.3932i \end{bmatrix}$$

$$[\phi_B] =$$

$$\begin{bmatrix} -0.9636 - 0.0726i & -0.9636 + 0.0726i & 1.0000 & 1.0000 \\ -0.3627 + 0.3176i & -0.3627 - 0.3176i & 0\ 4133 - 0.4243i & 0.4133 + 0.4243i \\ -0.0138 + 0.2401i & -0.0138 - 0.2401i & -0.0092 - 0.3507i & -0.0092 + 0.3507i \\ 0.0806 + 0.0888i & 0.0806 - 0.0888i & -0.1526 - 0.1411i & -0.1526 + 0.1411i \end{bmatrix}$$

The state transformation matrix for the current system is

$$\phi = [\phi_A][\phi_B]^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1.2821 & -3.2777 & 4.2951 & 0.1951 \\ 0 & 1 & 0 & 0 \\ 4.2822 & -22.5219 & 43.7790 & -38.3175 \end{bmatrix}$$

This transformation matrix may now be introduced into the state equations to incorporate the feedback of fractional order states, and the new state equations are

$$\dot{\underline{x}} = A\underline{x} - BG^{*}\underline{y} \tag{152}$$

and

$$G^{*} = \left[\begin{array}{cc:cc} -0.0186 & 0.1887 & -0.1476 & 0.1168 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The formulation of the fractional state gain matrix permits the direct application of fractional state feedback for the control of structure that is viscoelastically damped. The control theory remains unchanged from the undamped case to the viscoelastically damped case, while the composition of the gain matrices are different due to the variation in structural components.

Example Problem 2

This example is part of an ongoing research effort here at AFIT to demonstrate the application of fractional order state feedback control theory in the active control of large flexible space structures. The objective of this example is to develop the optimal gain matrix and the state transformation matrix for implementing fractional order state feedback control on the modified Advanced Beam Experiment (ABE).

The modified ABE consists of a six-foot beam, with the attached end welded to the support structure and a circular mounting plate attached to the free end of the beam (see Figure 6). The physical properties of the structure is listed in Table 1. A complete description of the modified ABE setup can be found in Reference (25). For the purpose of this analysis, it is assumed that only the planar bending modes about the Z-axis would be present in any response and no other bending or torsional modes would be excited. A twenty two degree-of-freedom finite element model of the inverted cantilever beam is developed for this analysis. The circular plate is modeled as a point mass and a mass moment of inertia combination at the free end of the beam. The attached end connection of the beam is modeled as a linear spring and a rotational spring combination to better represent the flexible boundary condition. The two-spring model of the attached end also allows further fine tuning of the ABE model to match the measured baseline characteristics of the beam.

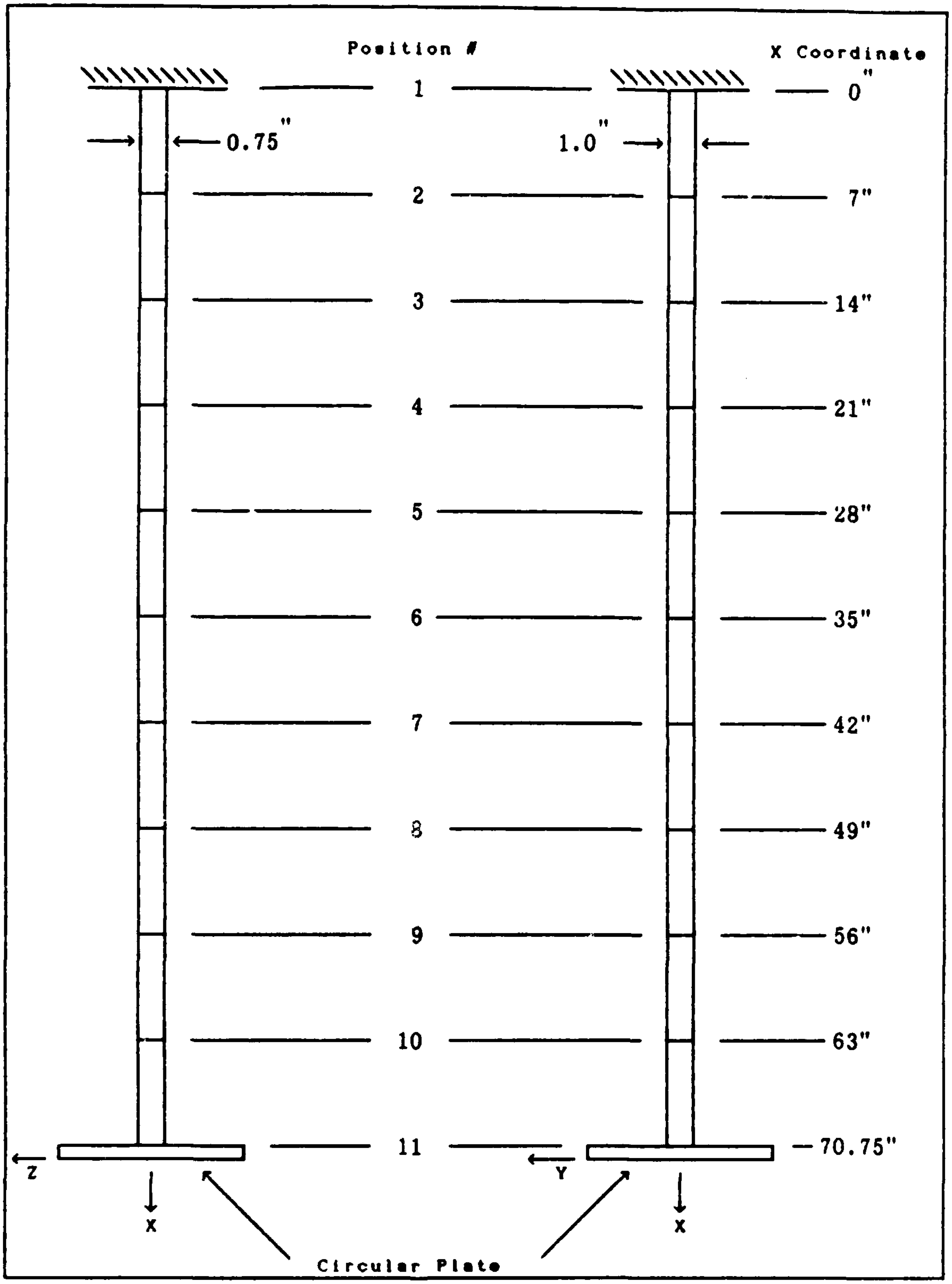

Figure 6. ÀBE Structure Configuration

Table 1. Structure Physical Properties

Material : Aluminum

Property Description	Value	Unit
Beam Length (L)	70.75	in
Y Cross-Section Width (a)	1.01	in
Z Cross-Section Width (b)	0.758	in
Cross-Section Area (A)	0.7656	in^2
Young's Modulus (E)	10.8×10^6	lbf/in^2
Shear Modulus (G)	4.1×10^6	lbf/in^2
Beam Density (ρ)	2.591×10^{-4}	$lbf \cdot sec^2/in^4$
Beam mass (m_b)	1.403×10^{-2}	$lbf \cdot sec^2/in$
Y Moment of Inertia (I_y)	3.667×10^{-2}	in^4
Z Moment of Inertia (I_z)	6.508×10^{-2}	in^4
Torsional Moment of inertia (K)	7.913×10^{-2}	in^4
Polar Moment of Inertia (I_{mx})	1.865×10^{-3}	$lbf \cdot sec^2/in$
Plate Diameter (d)	12.0	in
Plate Thickness (t)	1.0	in
Plate Mass (m_p)	2.847×10^{-2}	$lbf \cdot sec^2/in$
Plate X Mass Moment of Inertia (I_{Dmx})	0.5125	$lbf \cdot sec^2 \cdot in$
Plate Y-Z Mass Moment of Inertia (I_{Dm})	0.2562	$lbf \cdot sec^2 \cdot in$

The element stiffness matrix for the model is (27:391)

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12L & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12L & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \qquad (163)$$

and the element mass matrix is (27:392)

$$[M] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \qquad (164)$$

The natural frequencies for the first two bending modes of the finite element model are

$$\omega_1 = 12.355 \text{ rad/sec} = 1.966 \text{ Hz}$$

$$\omega_2 = 147.24 \text{ rad/sec} = 23.435 \text{ Hz}$$

and they are in close agreement with the measured resonances of the final ABE configuration (25:41)

$$\omega_{1,meas} = 1.97 \text{ Hz}$$

$$\omega_{2,meas} = 23.43 \text{ Hz}$$

In this example, it is desired to apply the fractional order state feedback control for controlling the first two bending modes of the beam. To begin the analysis of this problem, the bending motion of the beam is measured with accelerometers at positions 2, 5, 8, and 11. With traditional state feedback control, the translational displacements and velocities at these four nodes are fed back to the controller at position 5 to provide control force input for controlling the vibration of the beam (see Figure 7). Since the structural control of the beam is based on the modal response at the four nodal positions, it is necessary to determine the optimal gain matrix based on these responses.

The equations of motion for the system can be expressed as

$$[M]\{\ddot{w}(t)\} + [K]\{w(t)\} = [D]\{u(t)\} \tag{165}$$

where

$$\{w(t)\} = \begin{Bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \\ \vdots \\ y_{11} \\ \theta_{11} \end{Bmatrix} \tag{166}$$

[M] and [K] are the mass and stiffness matrices of the finite element model, and $\{w(t)\}$ is a displacement vector that includes both translational and angular displacements of the beam at the modes. The product of [D] and [u(t)] represents the control force applied to the system. In this case [D] is a 22 × 22 zero matrix except for an entry of one at the matrix position D(9,9) to represent the control force applied at nodal position 5 of the beam.

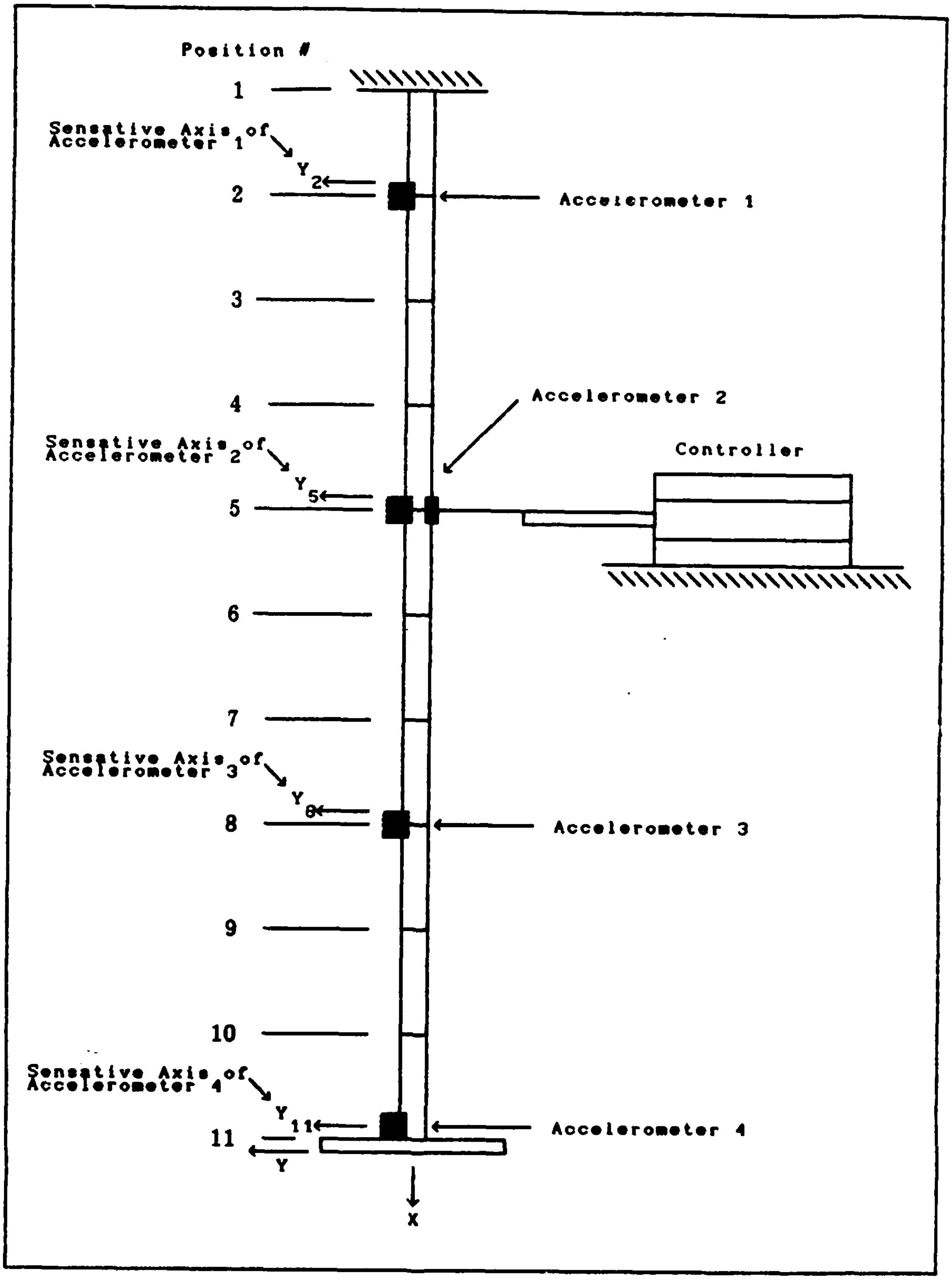

Figure 7. ABE Configuration for Traditional State Feedback

The analysis of the gain matrix begins with the formulation of the state equations of the system with optimal control

$$\dot{\underline{x}} = A\underline{x} - BG\underline{x} \qquad (167)$$

where

$$\underline{x} = \begin{Bmatrix} \{w(t)\} \\ \{\dot{w}(t)\} \end{Bmatrix} \qquad (168)$$

$\underline{x}$ is the state vector of the system with 44 states, that includes both displacement and velocity at all degrees of freedom. [A] and [B] are constant matrices defined by Eq (53) and (54). G is the optimal gain matrix for the present system with full state feedback control, and it is derived from the solution of the Riccati equation. For the quadratic performance index

$$J = \frac{1}{2}\int_0^\infty [x(t)^T Qx(t) + u^T(t)Ru(t)]dt \qquad (169)$$

where

$$Q = \begin{bmatrix} [K] & [0] \\ [0] & [M] \end{bmatrix} \qquad (170)$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (170)$$

the optimal gain matrix, G, for full state feedback is a 22 × 44 matrix with one row of gain elements at row 9 and no entries at other rows. This corresponds to the full state feedback control setup with one control force applied in the direction of the translational displacement at node 5 of the beam. The gain elements at row 9 of the full state feedback gain matrix are

G(9,1:44) =

columns 1 through 8

[0.3374 -2.7632 -0.6534 0.5310 -0.9246 -0.4154 -3.9616 -8.0019

columns 10 through 16

9.6753 1.1421 -2.4241 8.3482 -0.4293 4.1572 0.9121 7.4925

columns 19 through 24

-3.9078 -2.4949 0.2251 0.4790 1.3739 -2.6400 0.0122 -0.0046

columns 28 through 32

-0.0087 -0.0028 -0.0096 0.0011 -0.0103 -0.0111 0.1595 0.0207

columns 37 through 40

0.0298 0.0058 0.0119 0.0075 0.0189 0.0367 -0.0413 -0.0406

columns 41 through 44

-0.0086 0.0255 0.2232 0.1307]

Formulating the gain matrix $G = [G_1 \vdots G_2]$ and apply the feedback control to the system, the equations of motion of the beam may now be written as

$$[M]\{\ddot{w}(t)\} + [D][G_2]\{\dot{w}(t)\} + ([K] + [D][G_1])\{w(t)\} = \{0\} \qquad (56)$$

The system response of the full state feedback controlled beam can be determined by solving Eq (56) in first order form

$$D^1[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{0\} \tag{172}$$

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [M] \\ [M] & [D][G_2] \end{bmatrix} \tag{173}$$

$$[\tilde{K}] = \begin{bmatrix} -[M] & [0] \\ [0] & [K]+[D][G_1] \end{bmatrix} \tag{174}$$

$$\{\tilde{w}(t)\} = \begin{Bmatrix} D^1\{w(t)\} \\ \{w(t)\} \end{Bmatrix} \tag{175}$$

and the frequencies for the first four modes of vibration are

$$\begin{aligned} \omega_1 &= -\ 0.9725 \pm 12.355i \\ \omega_2 &= -\ 7.1178 \pm 147.24i \\ \omega_3 &= -\ 5.0212 \pm 381.56i \\ \omega_4 &= -\ 3.4060 \pm 638.53i \end{aligned} \tag{176}$$

The damping effect introduced by the feedback control are evident as shown by the negative real parts of the system responses.

Since, for this analysis, the motion of the beam is only represented by the translational displacements and velocities at the nodal positions 2, 5, 8, and 11, the application of full state feedback control using the complete 44 states of the system is not very practical. It is desirable to develop an equivalent control force that has the same effect as the full state feedback control while using a reduced set of states, $\underline{x}_R$, and $\underline{x}_R$ is defined in this case as

$$\underline{x}_R = \begin{Bmatrix} y_2 \\ y_5 \\ y_8 \\ y_{11} \\ \dot{y}_2 \\ \dot{y}_5 \\ \dot{y}_8 \\ \dot{y}_{11} \end{Bmatrix} \tag{177}$$

It can be shown that there exists a relation between the full state vector, $\underline{x}$, and the reduced-state vector, $\underline{x}_R$, through a state reduction transformation matrix, $[\phi_R]$.

Recalling from structural modal analysis that the system response may be expressed as a product of the modal matrix and the modal coordinate vector. Express the full state vector, $\underline{x}$, and the reduced state vector, $\underline{x}_R$, in terms of the their respective modal matrix and the modal coordinates

$$\underline{x} = [\phi'] \{\eta\} \tag{178}$$

$$\underline{x}_R = [\phi''] \{\eta\} \tag{179}$$

where $[\phi']$ and $[\phi'']$ are the associated modal matrices for $\underline{x}$ and $\underline{x}_R$, respectively. The row elements of $[\phi']$ and $[\phi'']$ are those row elements of the full modal matrix, $[\Phi]$, of the full state feedback control system in Eq (172) associated with the states in $\underline{x}$ and $\underline{x}_R$. The full modal matrix, $[\Phi]$, is a square matrix consist of conjugate eigenvectors of the system in Eq (172). It is desired for the equivalent control force to produce the same control effect over the first four modes of vibration as the full state feedback control force, thus only those eigenvectors of the full modal matrix, $[\Phi]$, that are associated with the first four modes are used in the formulation of $[\phi']$ and $[\phi'']$, where $[\phi']$ is a 44×8 matrix and $[\phi'']$ is a 8×8 matrix. From Eq (179) one may define $\{\eta\}$ as

$$\{\eta\} = [\phi'']^{-1}\underline{x}_R \tag{180}$$

Substitute Eq (180) into Eq (178) yields

$$\underline{x} = [\phi'][\phi'']^{-1}\underline{x}_R \tag{181}$$

Defining the state reduction transformation matrix, $[\phi_R]$, as

$$[\phi_R] = [\phi'][\phi'']^{-1} \tag{182}$$

and Eq (181) may be posed as

$$\underline{x} = [\phi_R]\underline{x}_R \tag{183}$$

and Eq (183) illustrates the relation between the full state vector and the reduced state vector for the first four modes of vibration.

Substitute Eq (183) into Eq(167) and the state equations can be written as

$$\dot{\underline{x}} = A\underline{x} - BG[\Phi_R]\underline{x}_R \tag{184}$$

Define the reduced gain matrix as

$$G_R = G[\Phi_R] \tag{185}$$

and Eq (184) can be written as

$$\dot{\underline{x}} = A\underline{x} - BG_R\underline{x}_R \tag{186}$$

where G_R is a 22 × 8 matrix with one row of gain elements at row 9. Eq (186) is the state equations with an equivalent control force being applied in the translational direction at nodal position 5, and the equivalent control force is developed based on the states of the reduced state vector defined in Eq (177). The gain elements at row 9 of the reduced gain matrix of Eq (185) are

$G_R(9,1:8)$ =

columns 1 through 8

[-1.1477 2.4025 -1.6402 0.4945 -0.0482 0.2196 -0.0371 0.2224] and the gain elements represent the gains to be applied to the states of the reduced state vector in Eq (177) for generating an equivalent control force for controlling the first four modes of vibration.

To introduce the gains for the reduced state vector into the state equations, Eq (167), one simply expands G_R columnwise by incorporating zero column vectors into those columns that correspond to the states of the full state vector not considered for the feedback. This expansion process results in a gain matrix, G_{RE}, with the same gain elements from the reduced gain matrix expanded across row 9 of the matrix G_{RE}. The gain elements at row 9 of the matrix G_{RE} are

$G_{RE}(9,1:44)$ =

columns 1 through 8

[0.0 0.0 -1.1477 0.0 0.0 0.0 0.0 0.0

columns 9 through 16

2.4025 0.0 0.0 0.0 0.0 0.0 -1.6402 0.0

columns 17 through 24

0.0 0.0 0.0 0.0 0.4945 0.0 0.0 0.0

columns 25 through 32

-0.0482 0.0 0.0 0.0 0.0 0.0 0.2196 0.0

columns 33 through 40

0.0 0.0 0.0 0.0 -0.0371 0.0 0.0 0.0

columns 41 through 44

0.0 0.0 0.2224 0.0]

The state equations for the system with reduced state feedback is

$$\dot{\underline{x}} = A\underline{x} - BG_{RE}\underline{x} \tag{187}$$

and the product of $G_{RE}\underline{x}$ has the same control effect on the first four modes as $G_R\underline{x}_R$. Formulate the new gain matrix as $G_{RE} = [G_{RE1} \vdots G_{RE2}]$, the equations of motion for the system with reduced feedback control can be written as

$$[M]\{\ddot{w}(t)\} + [D][G_{RE2}]\{\dot{w}(t)\} + ([K] + [D][G_{RE1}])\{w(t)\} = \{0\} \tag{188}$$

The system response may be determined by solving the expanded equations of motion using the procedure identified in Eq (172)-(175), and the frequencies for the first four modes of vibration are

$$\begin{aligned} \omega_1 &= - 0.9725 \pm 12.355i \\ \omega_2 &= - 7.1178 \pm 147.24i \\ \omega_3 &= - 5.0212 \pm 381.56i \\ \omega_4 &= - 3.4060 \pm 638.53i \end{aligned} \tag{189}$$

Notice that the frequencies for the first four modes in Eq (189), using the reduced state feedback control, are identical to the response of the full state feedback control in Eq (176).

At this point of the analysis, it is desired to introduce fractional order state feedback control in place of the reduced state feedback control. The accelerometers at positions 2, 5, 8, and 11 are replaced by two accelerometers at positions 5 and 8, plus the associated electronic fractional order differentiator circuits of order $\alpha = 1/2$. The fractional derivatives of the translational displacements at these two nodes, together with their displacements and velocities, are fed back to the controller to provide control force input for controlling the structure (see Figure 8). The application of the fractional order state feedback control theory in this example will reduce the sensor requirement from four to two, while still maintaining the same control authority over the system as the traditional state feedback control.

To introduce fractional order state feedback control to the beam, one needs to solve the fractional order equations of motion of the system in the expanded format and formulate the state transformation matrix following the procedure identified in Eq (32)-Eq (48). For $\alpha = 1/2$, the fractional order equations of motion of Eq (188) are

$$\begin{aligned} D^{1/2}[M]\left[D^{3/2} + D^{1} + D^{1/2} + D^{0}\right]\{w(t)\} \\ - [M]\left[D^{3/2} + D^{1} + D^{1/2}\right]\{w(t)\} \\ + D^{1/2}[D][G_{RE2}]\left[D^{1/2} + D^{0}\right]\{w(t)\} \\ - [D][G_{RE2}]D^{1/2}\{w(t)\} \\ + ([K] + [D][G_{RE1}])D^{0}\{w(t)\} = \{0\} \end{aligned} \qquad (190)$$

and Eq (190) can be solved using the solution method of the expanded equations of motion described in Eq (32)-Eq (36). The expanded equations of motion for Eq (190) are

$$D^{1/2}[\tilde{M}]\{\tilde{w}(t)\} + [\tilde{K}]\{\tilde{w}(t)\} = \{0\} \qquad (191)$$

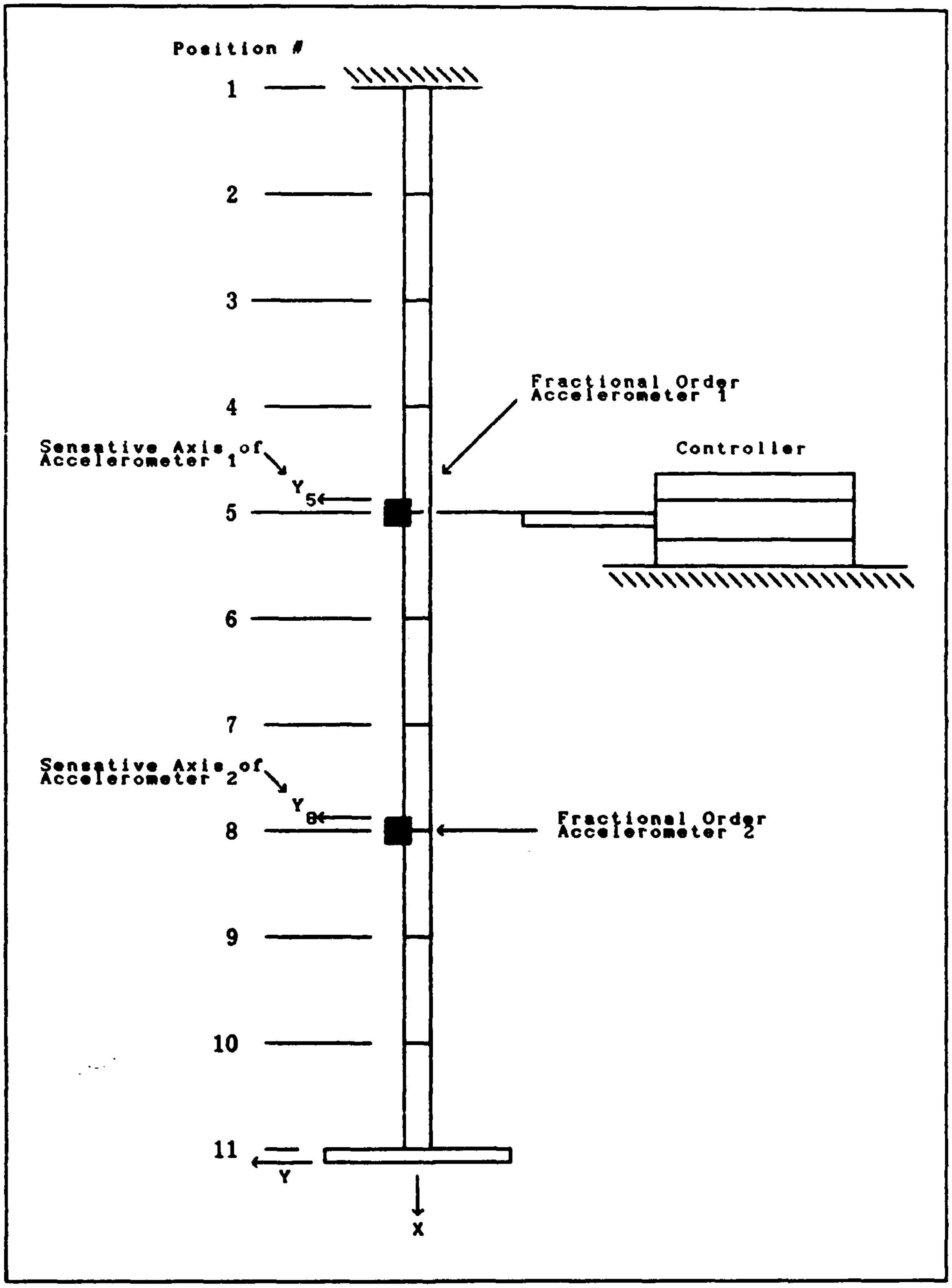

Figure 8. ABE Configuration for Fractional State Feedback

where

$$[\tilde{M}] = \begin{bmatrix} [0] & [0] & [0] & [M] \\ [0] & [0] & [M] & [0] \\ [0] & [M] & [0] & [D][G_{RE2}] \\ [M] & [0] & [D][G_{RE2}] & [0] \end{bmatrix} \tag{192}$$

$$[\tilde{K}] = \begin{bmatrix} [0] & [0] & -[M] & [0] \\ [0] & -[M] & [0] & [0] \\ -[M] & [0] & -[D][G_{RE2}] & [0] \\ [0] & [0] & [0] & [K]+[D][G_{RE1}] \end{bmatrix} \tag{193}$$

$$\{\tilde{w}(t)\} = \begin{Bmatrix} D^{3/2}\{w(t)\} \\ D^{1}\ \{w(t)\} \\ D^{1/2}\{w(t)\} \\ \{w(t)\} \end{Bmatrix} \tag{194}$$

and $\{w(t)\}$ for this system is defined in Eq (166). The modal matrix of the system, $[\Phi]$, is composed of the complex conjugate pairs of eigenvectors associated with the conjugate eigenvalues on the principal sheet of the Riemann surface. Since only the first four modes of vibration are controlled by the equivalent control force, only the eigenvectors correspond to the first four modes of vibration will be used in formulating $[\Phi]$ and the resulting matrix is 88 × 8. It is desired to determine the state transformation matrix, Φ, for the application of feeding back the fractional states of responses at nodal positions 5 and 8. Invoke the expansion theorem and the full state vector may be written as

$$\underline{x} = [\Phi_A]\{\eta\} \tag{195}$$

for

$$\underline{x} = \begin{Bmatrix} \{w(t)\} \\ \{\dot{w}(t)\} \end{Bmatrix} \tag{196}$$

$[\Phi_A]$ = a 44 × 8 matrix consist of row vectors of $[\Phi]$ associated with the states of the full state vector $\underline{x}$

The fractional state vector may be written in the form

$$\underline{y} = [\Phi_B]\{\eta\} \tag{197}$$

for

$$\underline{y} = \begin{Bmatrix} D^{3/2}\begin{Bmatrix} y_5 \\ y_8 \end{Bmatrix} \\ D^{1}\begin{Bmatrix} y_5 \\ y_8 \end{Bmatrix} \\ D^{1/2}\begin{Bmatrix} y_5 \\ y_8 \end{Bmatrix} \\ \begin{Bmatrix} y_5 \\ y_8 \end{Bmatrix} \end{Bmatrix} \tag{198}$$

$[\Phi_B]$ = a 8 × 8 matrix consist of row vectors of $[\Phi]$ associated with the states of the fractional state vector $\underline{y}$

The state transformation matrix, Φ, is defined as

$$\Phi = [\Phi_A][\Phi_B]^{-1} \tag{199}$$

and the full state vector can be defined as

$$\underline{x} = \Phi\underline{Y} \tag{200}$$

Substituting Eq (200) into the state equations for reduced state feedback, Eq (187) and

$$\dot{\underline{x}} = A\underline{x} - BG_{RE}\Phi\underline{Y} \tag{201}$$

Redefine the gain matrix for Eq (201) as

$$G^*_{RE} = G_{RE}\Phi \tag{202}$$

and substitute Eq (202) into Eq (201) then

$$\dot{\underline{x}} = A\underline{x} - BG^*_{RE}\underline{Y} \tag{203}$$

Eq (203) is the state equations for the feedback of reduced fractional order state control, where G^*_{RE} is a 22 × 8 matrix with one row of gain elements at row 9. The gain element at row 9 of G^*_{RE} are

$G^*_{RE}(9,1:8)$ =

columns 1 through 8

[-0.0226 0.0177 1.1198 -0.5565 -15.9949 9.6436 100.1118 -49.1084]

This gain matrix permits the fractional states from positions 5 and 8 to be fed back for the controlling of the first four vibrational modes of the structure.

Theoretically, the the control effect provided the product of the gain matrix G^*_{RE} and the fractional states observed at positions 5 and 8 for controlling the first four modes of vibration of the beam is equivalent to the control effect provided by the product of the gain matrix G_{RE} and the traditional states observed at positions 2, 5, 8, and 11. But physically only the first two modes of vibration of the beam can be accurately controlled by this control system. This is due to the limitation of the finite element model of the beam, of which only the first two modes of vibration can be modeled with relative accuracy.

The next step in this experiment is to incorporate the gain matrix G_{RE} to the fractional order state feedback control of the beam and physically demonstrate the validity of the control theory. This validation process is currently underway at AFIT and the results of this experiment will become available in the near future.

VI. Conclusion

The fractional order state feedback control theory is developed by incorporating the state transformation matrix into the traditional state-space equations of the structure under control. The state transformation matrix between the traditional states and the fractional states can be formulated by establishing the associated modal matrices from the eigenstructures of the expanded equations of motion of the system. The optimal control gains are determined by solving the Riccati equation for the linear quadratic regular problem. The gains are then combined with the state transformation matrix to form the necessary fractional state gains for the fractional order state feedback control for the system.

The use of fractional order state feedback control appears to hold value in providing an alternative form of an observer which permits one to use fewer sensors at limited locations, while still be able to provide the equivalent observability and control authority over the structure as the traditional state feedback control scheme.

The next step in the development of fractional order state feedback control is to test the theory by experiment. The current research effort on the modified ABE and the development of a prototype sensor for fractional derivative motion should provide insight on the application of the theory in real-time control situation. Additional research efforts are still needed in determining the optimal number of sensors required, the optimal sensor/controller locations on the structure being controlled, and the optimal fractional states for feedback structural control in order to further expand the application of the theory. Furthermore, advanced electronic fractional order differentiator circuitries are needed to advance the development and performance of the fractional order state control system.

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Abstract

The purpose of this study is to demonstrate the analysis leading to the development of the fractional order state feedback control theory for structural control of both undamped and viscoelastically-damped structures. It is shown herein that there exists a relation between the traditional state vector that includes structural displacements and velocities, and the fractional state vector which includes fractional derivatives of structural responses. This relation permits the modification of linear quadratic regulator theory to include the application of fractional order states in the feedback control. The application of this theory leads to an alternative form of an observer.