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Determination of Noise Field Directionality Directly from Spatial Correlation for Linear, Planar, and Volumetric Arrays

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Preface

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The spatial correlation between two points of an array is given by a two-dimensional integral in terms of the noise field directionality. Depending on the dimensionality of the array, this integral equation can be partially solved, to yield explicit expressions for the noise field directionality in terms of a multi-dimensional Fourier transform. In particular, for a linear array, a one-dimensional collapsed field distribution can be determined; for a planar array, the sum of symmetrically-arriving rays can be solved for; and for a volumetric array, the complete field can be found. The effects of finite length and discrete arrays on the estimate of the noise field directionality are also considered.

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LIST OF SYMBOLS

θ	polar angle, figure 1
ϕ	azimuthal angle, figure 1
f	temporal-frequency
$N_f(\theta,\phi)$	noise field directionality, (1)
τ_1	time of arrival at location x_1,y_1,z_1, (2)
c	speed of propagation, (2)
$H_1(f)$	transfer function, (3)
λ	wavelength = c/f, (3)
x,y,z	separation distances, (5)
$G_f(x,y,z)$	spatial correlation at temporal-frequency f, (6)
$G_f(z)$	spatial correlation for linear array, (7)
$\overline{N}_f(\theta)$	integrated noise field directionality, (8)
δ	delta function, (9)
Δ	spacing in z, (12)
M	size of fast Fourier transform, (13),(14),(29),(30)
$\circledast$	convolution, (15a)
$w(z)$	weighting, (15b)
$W(u)$	window, (15c)

$G_f(x,y)$	spatial correlation for planar array, (16)
$\tilde{N}_f(\theta,\phi)$	sum of symmetrically-arriving rays, (17)
$I(u,v)$	two-dimensional Fourier transform of $G_f(x,y)$, (18)
$\arg(z)$	argument of complex number z, (21)
C_1	circle of radius 1 at origin, (23)
F	auxiliary function, (24)
Δ_x, Δ_y	spacings in x,y, (28)
N	size of fast Fourier transform, (29),(30)
$I(u,v,w)$	three-dimensional Fourier transform of $G_f(x,y,z)$, (36)
$q(z)$	weighting in z, (36)
$Q(t)$	Fourier transform of q(z), (37)
s	auxiliary function, (42)
F_1, F_2	auxiliary functions, (44)
w_1, w_2	two distinct values of w, (46),(47)
$Q_n(\pm)$	abbreviated notation, (48)
D	denominator, (49)
L_z	effective length of weighting q(z), figure 2
$I(\pm)$	abbreviated notation, (55)
θ'	complementary angle = $\pi - \theta$, (56)
$I'(\pm)$	abbreviated notation, (57)

DETERMINATION OF NOISE FIELD DIRECTIONALITY DIRECTLY FROM SPATIAL CORRELATION FOR LINEAR, PLANAR, AND VOLUMETRIC ARRAYS

INTRODUCTION

When an array is located in a homogeneous stationary noise field, measurement of the crosscorrelations between all pairs of separated elements, at each temporal-frequency of interest, is the most general second-order statistical information that can be extracted. These spatial correlations depend upon the directionality of the surrounding noise field, which is the primary quantity of interest here. Instead of beamforming the element outputs, for example, and trying to suppress the inherent sidelobes by proper weighting procedures, we want to avoid any preconceived notions about data processing and go directly from the spatial correlation to the noise field directionality in as direct and simple a manner as possible.

However, because the noise field directionality is a two-dimensional function of polar and azimuthal angles, some inherent loss or condensation of information takes place with a linear array and, to a much lesser extent, with a planar array. Nevertheless, we want to preserve and extract the maximum amount of information about the noise field directionality, consistent with the dimensionality of the array employed, and to minimize the amount of data processing required.

We begin by assuming the array to be an infinite continuous line in the one-dimensional case, and solve the integral equation for the integrated (or collapsed) noise field directionality, at each temporal-frequency, in terms of the spatial correlation along the line. Then, we discretize the line, so as to be an equi-spaced array, and determine the effect that this limitation has upon the estimated directionality. Finally, we investigate the smoothing that is caused by the practical requirement that any physical array must have finite length. Thus, the facts that the spatial correlation will never be available on a continuum, nor for infinite separations, are included in the analysis.

A similar procedure is pursued for the two-dimensional case, where the planar array is presumed to have equal spacings Δ_x and Δ_y in the x and y dimensions, respectively. Again, the aliasing effects are considered, as well as the limitation of having to employ a finite-size planar array. Finally, in the three-dimensional case, where the problem is overdetermined, a plausible and efficient procedure for collapsing the surplus information is presented, although it is recognized that an unlimited number of alternatives exist.

Although it was stated that the noise field directionality is of interest, this does not preclude the presence of plane-wave arrivals, that is, additive signals or interferences in the background. In fact, the examples are specifically of that type, for these can be considered as the fundamental building blocks of a general noise field.

Some related results on this problem of restoring the noise field directionality from the spatial correlation are given in [1,2,3,4], but limited to the line array. Specifically, [1] gave a least squares approach, starting from a discrete finite-length array. However, ill-conditioning of the simultaneous linear equations for the noise field directionality precluded its use for more than approximately ten elements. This ill-conditioning is circumvented here by deferring the discretization until after the integral equation is solved; this procedure for the line array was first given in [4].

CHARACTERIZATION OF NOISE FIELD

Let $N_f(\theta,\phi)$ be the intensity of the homogeneous stationary noise field at temporal-frequency f, arriving from direction θ,ϕ, where $0 \le \theta \le \pi$, $-\pi < \phi \le \pi$; see figure 1. The amount of power received in solid angle $d\theta\, d\phi\, \sin\theta$ about θ,ϕ is

$$d\theta\, d\phi\, \sin\theta\, N_f(\theta,\phi). \tag{1}$$

We call $N_f(\theta,\phi)$ the noise field directionality; the product $\sin\theta\, N_f(\theta,\phi)$ could be called the plane-wave density.

$0 \le \theta \le \pi$

$-\pi < \phi \le \pi$

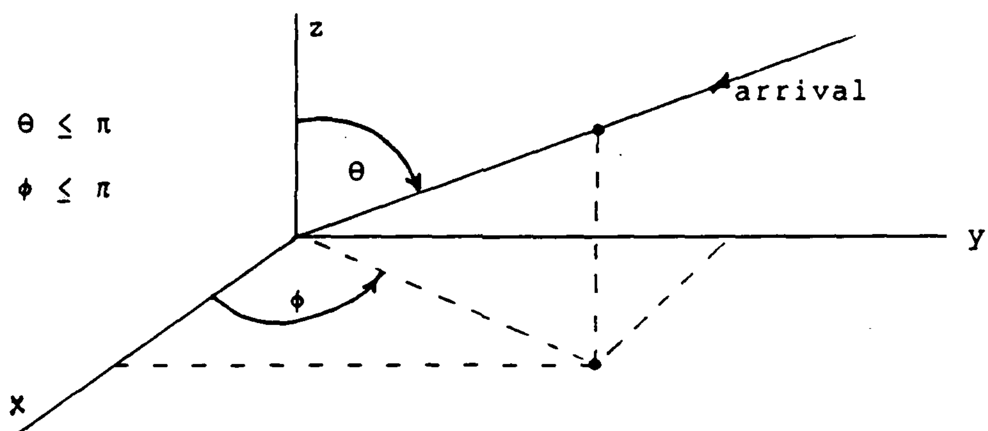

Figure 1. Coordinate System

Consider general field point x_1, y_1, z_1. Then if the time of arrival at the origin, of the component from direction θ,ϕ, is zero, then the time of arrival at x_1, y_1, z_1 is

$$\tau_1 = -\frac{1}{c}\,(x_1 \sin\theta \cos\phi + y_1 \sin\theta \sin\phi + z_1 \cos\theta), \tag{2}$$

where c is the speed of propagation. Therefore, the transfer function at x_1, y_1, z_1 applied to the arrival from direction θ,ϕ is

$$H_1(f) = \exp(-i2\pi f\tau_1) =$$

$$= \exp\left(i\frac{2\pi}{\lambda}\,(x_1 \sin\theta \cos\phi + y_1 \sin\theta \sin\phi + z_1 \cos\theta)\right), \qquad (3)$$

where wavelength $\lambda = c/f$.

The elemental contribution to the crosscorrelation between this arrival at x_1,y_1,z_1 and x_2,y_2,z_2, at temporal-frequency f, is then

$$d\theta\, d\phi\, \sin\theta\, N_f(\theta,\phi)\, H_1(f)\, H_2(f)^* = d\theta\, d\phi\, \sin\theta\, \times$$

$$\times\, N_f(\theta,\phi)\, \exp\left(i\frac{2\pi}{\lambda}\,(x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta)\right), \qquad (4)$$

where separations

$$x = x_1 - x_2,$$
$$y = y_1 - y_2,$$
$$z = z_1 - z_2. \qquad (5)$$

If the arrivals from different directions are uncorrelated, the spatial correlation (at frequency f) between two points separated by x,y,z is then given by integrating over all angular space,

$$G_f(x,y,z) = \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi\, \sin\theta\, N_f(\theta,\phi)\, \times$$

$$\times \exp\left(i\frac{2\pi}{\lambda}\,(x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta)\right). \qquad (6)$$

The problem of interest is: given spatial correlation $G_f(x,y,z)$ versus x,y,z (or restricted slices of $G_f(x,y,z)$), solve for noise field directionality $N_f(\theta,\phi)$ (or smoothed versions of

$N_f(\theta,\phi)$). That is, invert integral equation (6) for noise field directionality $N_f(\theta,\phi)$ or for whatever can be determined. There are three cases that must be distinguished, namely, linear, planar, and volumetric arrays.

LINEAR ARRAY

It is most convenient mathematically to locate the line array along the z axis, that is, $x = y = 0$. Then the exponential in (6) is independent of ϕ, and (6) reduces to*

$$G_f(z) = \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \ \sin\theta \ N_f(\theta,\phi) \ \exp\left(i\frac{2\pi}{\lambda} z \cos\theta\right) =$$

$$= \int_0^{\pi} d\theta \ \sin\theta \ \overline{N}_f(\theta) \ \exp\left(i\frac{2\pi}{\lambda} z \cos\theta\right) , \qquad (7)$$

where

$$\overline{N}_f(\theta) = \int_{-\pi}^{\pi} d\phi \ N_f(\theta,\phi) \quad \text{for } 0 \le \theta \le \pi \qquad (8)$$

is the integrated or averaged noise field directionality, and $G_f(z)$ is the one-dimensional spatial correlation at separation z along the line, both functions evaluated at frequency f. $G_f(z)$ is the only second-order function that can be measured (or estimated) from the line array, and $\overline{N}_f(\theta)$ is the only field function that can be determined. There is no possibility of undoing the integration of (8); this is a mathematical representation of the inherent conical symmetry of response of a linear

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*The case where the line array is located on the x axis is treated in appendix A.

array. It is also one reason for choosing the line array to lie along the $\theta = 0$ axis, since all the two-dimensional field information is conveniently collapsed into a one-dimensional function of θ alone. See appendix A for the problems associated with choosing a different coordinate system.

SOLUTION OF INTEGRAL EQUATION

To solve integral equation (7) for noise field directionality $\overline{N}_f(\theta)$, consider the following:

$$\int_{-\infty}^{+\infty} dz \, \exp\left(-i\frac{2\pi}{\lambda} u \, z\right) G_f(z) =$$

$$= \int_0^{\pi} d\theta \, \sin\theta \, \overline{N}_f(\theta) \int_{-\infty}^{+\infty} dz \, \exp\left(-i\frac{2\pi}{\lambda} z \, (u - \cos\theta)\right) =$$

$$= \int_0^{\pi} d\theta \, \sin\theta \, \overline{N}_f(\theta) \, \lambda \, \delta(u - \cos\theta), \tag{9}$$

where δ is the delta function. Now let $t = \cos\theta$, which is a one-to-one transformation for $0 \le \theta \le \pi$, to get

$$\int_{-\infty}^{+\infty} dz \, \exp\left(-i\frac{2\pi}{\lambda} u \, z\right) G_f(z) = \lambda \int_{-1}^{1} dt \, \overline{N}_f(\mathrm{acos}(t)) \, \delta(u - t) =$$

$$= \left\{ \begin{array}{ll} \lambda \, \overline{N}_f(\mathrm{acos}(u)) & \text{for } |u| < 1 \\ 0 & \text{for } |u| > 1 \end{array} \right\}. \tag{10}$$

That is,

$$\bar{N}_f(\mathrm{acos}(u)) = \frac{1}{\lambda} \int_{-\infty}^{+\infty} dz \, \exp\left(-i\frac{2\pi}{\lambda} u \, z\right) G_f(z) \quad \text{for } |u| < 1, \qquad (11a)$$

or

$$\bar{N}_f(\theta) = \frac{1}{\lambda} \int_{-\infty}^{+\infty} dz \, \exp\left(-i\frac{2\pi}{\lambda} \cos\theta \, z\right) G_f(z) \quad \text{for } 0 \le \theta \le \pi. \qquad (11b)$$

Here, acos is the principal value inverse cosine function. Compare (11b) with starting point (7).

Thus, given the spatial correlation $G_f(z)$ for all possible separations z along the line array, the integrated noise field directionality $\bar{N}_f$ is available via a single one-dimensional Fourier transform.

EXAMPLE

An example is informative at this point. Let

$$\overline{N}_f(\theta) = \delta(\theta - \theta_o), \quad 0 < \theta_o < \pi.$$

Then the spatial correlation is, from (7),

$$G_f(z) = \sin\theta_o \exp\left(i\frac{2\pi}{\lambda} z \cos\theta_o\right).$$

Observe that as $\theta_o \to 0$ or π, that is, endfire of the line array, the strength of this quantity decays to zero, due to the $\sin\theta$ term in the area element in (1). Substitution of correlation $G_f(z)$ into (11b) yields noise field directionality

$$\overline{N}_f(\theta) = \sin\theta_o \, \delta(\cos\theta - \cos\theta_o) \quad \text{for } 0 \leq \theta \leq \pi.$$

Now the delta function here is located at $\theta = \theta_o$ and has area $1/\sin\theta_o$. Thus, $\overline{N}_f(\theta)$ is $\delta(\theta - \theta_o)$, as it should be; however, the trigonometric form shows $\overline{N}_f(\theta)$ as the product of two terms, the first of which tends to zero as $\theta_o \to 0$ or π, and the second of which has an area that tends to infinity as $\theta_o \to 0$ or π. This behavior will re-occur in the following investigations.

We have employed the following useful property above: if $g(x)$ has an isolated zero at x_o, then in the neighborhood of x_o,

$$\delta\left(g(x)\right) = \delta\left(g'(x_o)\ (x-x_o)\right) = \frac{1}{|g'(x_o)|}\,\delta(x - x_o) \ .$$

That is, the area of the delta function at x_o is equal to the reciprocal absolute slope of the argument at x_o, if nonzero.

DISCRETE INFINITE-LENGTH ARRAY

If samples of spatial correlation $G_f(z)$ at increment Δ in z are available, an approximation to (11a) is afforded, for $|u| < 1$, by

$$\overline{N}_f(\mathrm{acos}(u)) \approx \frac{\Delta}{\lambda} \sum_{n=-\infty}^{+\infty} \exp\left(-i\frac{2\pi}{\lambda}\, u\, \Delta\, n\right) G_f(\Delta\, n), \tag{12}$$

the right-hand side of which has period λ/Δ in u. Since the integrated noise field directionality in (11a) is defined on an interval of length 2, that is, $-1 < u < 1$, aliasing will occur in approximation (12) unless $\Delta < \lambda/2$. Thus, the spacing Δ, between samples of $G_f(z)$, must be less than a half-wavelength at the temporal-frequency f of interest. This is presumed true henceforth.

Now if u is restricted to the values

$$u_m = \frac{m}{M}\frac{\lambda}{\Delta} \quad \text{for} \quad -\frac{M}{2} \le m \le \frac{M}{2} - 1, \tag{13}$$

which cover a full period, there follows, for $\left|\frac{m}{M}\frac{\lambda}{\Delta}\right| \le 1$,

$$\overline{N}_f\left(\mathrm{acos}\left(\frac{m}{M}\frac{\lambda}{\Delta}\right)\right) \approx \frac{\Delta}{\lambda} \sum_{n=-\infty}^{+\infty} \exp(-i2\pi mn/M)\, G_f(\Delta\, n). \tag{14a}$$

The sum on the right-hand side can be accomplished via an M-point fast Fourier transform when collapsing is employed [5; p.5]. The resultant angles at which $\overline{N}_f(\theta)$ is available are

$$\theta_m = \mathrm{acos}\left(\frac{m}{M}\frac{\lambda}{\Delta}\right), \quad \text{or} \quad \cos\theta_m = \frac{m}{M}\frac{\lambda}{\Delta} \quad \text{for} \; -\frac{M}{2} \le m \le \frac{M}{2} - 1, \tag{14b}$$

provided that $\frac{|m|}{M} \frac{\lambda}{\Delta} \le 1$. These values are equally spaced in $\cos\theta$ space.

The right-hand side of (12) can be rewritten in the form [5; pp. 3-4]

$$\frac{1}{\lambda} \int_{-\infty}^{+\infty} dz \exp\left(-i\frac{2\pi}{\lambda} u z\right) G_f(z) \Delta \sum_{n=-\infty}^{+\infty} \delta(z - \Delta n) =$$

$$= \overline{N}_f(\operatorname{acos}(u)) \oplus \sum_{n=-\infty}^{+\infty} \delta\left(u - \frac{n\lambda}{\Delta}\right) = \sum_{n=-\infty}^{+\infty} \overline{N}_f\left(\operatorname{acos}\left(u - \frac{n\lambda}{\Delta}\right)\right), \quad (15a)$$

where $\oplus$ denotes convolution. The separation of these aliased lobes (for $n \neq 0$) is λ/Δ on the u scale; then, since the extent of $\overline{N}_f(\operatorname{acos}(u))$ is 2 on the u scale, overlapped aliasing lobes do not occur if $\Delta < \lambda/2$. This is a mathematical back-up to the claim under (12).

DISCRETE FINITE-LENGTH ARRAY

The effect of a finite-length array can easily be incorporated by modifying (15a), so as to include weighting w(z). Then, we have, for the estimated noise field directionality,

$$\frac{1}{\lambda}\int_{-\infty}^{+\infty} dz \, \exp\left(-i\frac{2\pi}{\lambda} u \, z\right) G_f(z) \, \Delta \sum_{n=-\infty}^{+\infty} \delta(z - \Delta \, n) \, w(z) =$$

$$= \bar{N}_f(\mathrm{acos}(u)) \oplus \sum_{n=-\infty}^{+\infty} W\left(u - \frac{n\lambda}{\Delta}\right), \tag{15b}$$

where window

$$W(u) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} dz \, \exp\left(-i\frac{2\pi}{\lambda} u \, z\right) w(z). \tag{15c}$$

Thus, not only is the noise field directionality aliased at separations λ/Δ in u, but, in addition, it is smoothed by window W. Sampling, per se, does not distort the estimated directionality, if done finely enough, that is, $\Delta < \lambda/2$. However, the finite length of the array always causes smearing, with a window width of the order of λ/L_z, where L_z is the effective length of weighting w(z).

PLANAR ARRAY

It is now most advantageous mathematically to locate the planar array in the x,y plane, that is, at $z = 0$. Then the exponential in (6) is independent of $\cos\theta$, and (6) reduces to

$$G_f(x,y) = \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \, \sin\theta \, N_f(\theta,\phi) \exp\left(i\frac{2\pi}{\lambda} \sin\theta \, (x \cos\phi + y \sin\phi)\right) =$$

$$= \int_0^{\pi/2} d\theta \int_{-\pi}^{\pi} d\phi \, \sin\theta \, \tilde{N}_f(\theta,\phi) \exp\left(i\frac{2\pi}{\lambda} \sin\theta \, (x \cos\phi + y \sin\phi)\right), \quad (16)$$

where

$$\tilde{N}_f(\theta,\phi) = N_f(\theta,\phi) + N_f(\pi - \theta,\phi)$$

$$\text{for} \quad 0 \le \theta \le \pi/2, \; -\pi < \phi \le \pi. \quad (17)$$

$\tilde{N}_f$ is the sum of the elemental components in symmetrically-arriving rays on opposite sides of the planar array; recall that $\theta = \pi/2$ now corresponds to the plane of the array. Spatial correlation $G_f(x,y)$ is the only function that can be measured (or estimated) from the planar array, and $\tilde{N}_f(\theta,\phi)$ is the only field directionality function that can be determined. There is no possibility of undoing the summation of (17); this is a mathematical representation of the inherent two-sided symmetric response of a planar array. It is also one reason for choosing the planar array to lie along the $\theta = \pi/2$ plane, since the totality of the two-dimensional field information is conveniently collapsed into a one-sided function of θ, that is, $0 \le \theta \le \pi/2$.

SOLUTION OF INTEGRAL EQUATION

Consider the two-dimensional Fourier transform of (16),

$$I(u,v) \equiv \iint_{-\infty}^{+\infty} dx\, dy\, \exp\left(-i\frac{2\pi}{\lambda}(ux + vy)\right) G_f(x,y) =$$

$$= \int_0^{\pi/2} d\theta \int_{-\pi}^{\pi} d\phi\, \sin\theta\, \tilde{N}_f(\theta,\phi)\, \lambda^2\, \delta(u - \sin\theta\cos\phi)\, \delta(v - \sin\theta\sin\phi). \qquad (18)$$

Let

$$\alpha = \sin\theta\cos\phi,\ \beta = \sin\theta\sin\phi \quad \text{for } 0 \le \theta \le \pi/2,\ -\pi < \phi \le \pi. \qquad (19)$$

These relations can be inverted by using

$$\alpha + i\beta = \sin\theta\exp(i\phi), \qquad (20)$$

to give

$$\sin\theta = |\alpha + i\beta| = \left(\alpha^2 + \beta^2\right)^{\frac{1}{2}}, \qquad \phi = \arg(\alpha + i\beta). \qquad (21)$$

Thus, (19) is a one-to-one two-dimensional transformation in the ranges $0 \le \theta \le \pi/2$, $-\pi < \phi \le \pi$ allowed in (18). From (19) and (21), the Jacobian is

$$\frac{\partial(\alpha,\beta)}{\partial(\theta,\phi)} = \begin{vmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \end{vmatrix} =$$

$$= \sin\theta\cos\theta = \left(\alpha^2 + \beta^2\right)^{\frac{1}{2}} \left(1 - \alpha^2 - \beta^2\right)^{\frac{1}{2}}. \qquad (22)$$

Substitution of these results in (18) yields

$$I(u,v) = \lambda^2 \iint_{C_1} d\alpha\, d\beta\, F(\alpha,\beta) \left(1 - \alpha^2 - \beta^2\right)^{-\frac{1}{2}} \delta(u-\alpha)\, \delta(v-\beta) =$$

$$= \left\{ \begin{array}{ll} \lambda^2\, F(u,v) \left(1 - u^2 - v^2\right)^{-\frac{1}{2}} & \text{for } u^2 + v^2 < 1 \\ 0 & \text{otherwise} \end{array} \right\}, \qquad (23)$$

where C_1 is a circle of radius 1 located at the origin, and

$$F(\alpha,\beta) = \tilde{N}_f(\mathrm{asin}(|\alpha + i\beta|),\ \arg(\alpha + i\beta)). \qquad (24)$$

Here, asin is the principal value inverse sine function. From (23), (24), and (18), the noise field directionality is

$$\tilde{N}_f(\mathrm{asin}(|u + iv|),\ \arg(u + iv)) =$$

$$= \frac{\left(1 - u^2 - v^2\right)^{\frac{1}{2}}}{\lambda^2} \iint_{-\infty}^{+\infty} dx\, dy\, \exp\left(-i\frac{2\pi}{\lambda}(ux + vy)\right) G_f(x,y)$$

$$\text{for } u^2 + v^2 < 1. \qquad (25)$$

An alternative form is available by letting

$$u = \sin\theta \cos\phi,\ v = \sin\theta \sin\phi \quad \text{for } 0 \le \theta \le \pi/2,\ -\pi < \phi \le \pi, \qquad (26)$$

namely

$$\tilde{N}_f(\theta,\phi) = \frac{\cos\theta}{\lambda^2} \iint_{-\infty}^{+\infty} dx\, dy\, \exp\left(-i\frac{2\pi}{\lambda}\sin\theta\,(x\cos\phi + y\sin\phi)\right) G_f(x,y)$$

$$\text{for } 0 \le \theta \le \pi/2,\ -\pi < \phi \le \pi. \qquad (27)$$

It is interesting to compare this form with starting result (16). An alternative, when the planar array lies in the $y = 0$ plane, is given in appendix B.

BEHAVIOR NEAR PLANE OF ARRAY

At first sight, the presence of the $\cos\theta$ term in (27) would appear to be a problem for $\theta \approx \pi/2$, which is the plane of the array. However, the following example illustrates what is happening; let

$$\tilde{N}_f(\theta,\phi) = \delta(\theta - \theta_o)\ \delta(\phi - \phi_o) \quad \text{for } 0 < \theta_o < \pi/2,\ -\pi < \phi_o \leq \pi.$$

Then (16) yields spatial correlation

$$G_f(x,y) = \sin\theta_o \exp\left(i\frac{2\pi}{\lambda} \sin\theta_o\ (x \cos\phi_o + y \sin\phi_o)\right).$$

The strength of this quantity tends to zero as $\theta_o \to 0$. Substitution of this $G_f(x,y)$ in (27) yields noise field directionality

$$\tilde{N}_f(\theta,\phi) = \sin\theta_o \cos\theta\ \delta(\sin\theta \cos\phi - \sin\theta_o \cos\phi_o) \times$$
$$\times\ \delta(\sin\theta \sin\phi - \sin\theta_o \sin\phi_o).$$

By use of the property

$$\delta(ax + by)\ \delta(cx + dy) = \frac{\delta(x)\ \delta(y)}{|ad - bc|},$$

it may be shown that $\tilde{N}_f(\theta,\phi)$ is $\delta(\theta - \theta_o)\ \delta(\phi - \phi_o)$, as expected; however, the trigonometric form shows $\tilde{N}_f$ as the product of two terms, the first of which tends to 0 as $\theta_o \to 0$ or $\pi/2$, and the second of which has impulses with area which tends to infinity as $\theta_o \to 0$ or $\pi/2$. Thus, the $\sin\theta_o$ and $\cos\theta$ terms are not a problem since they are compensated by multiplicative terms; however, they may lead to inaccuracies in numerical computation.

DISCRETE INFINITE-LENGTH ARRAY

The form in (25) gives the noise field directionality sum $\tilde{N}_f$, defined in (17), as a double Fourier transform of the two-dimensional spatial correlation function $G_f(x,y)$. If samples of $G_f(x,y)$ at increments Δ_x in x and Δ_y in y, respectively, are available, an approximation to (25) is afforded, for $u^2 + v^2 < 1$, by

$$\tilde{N}_f(\operatorname{asin}(|u + iv|), \arg(u + iv)) \approx \frac{\left(1 - u^2 - v^2\right)^{\frac{1}{2}}}{\lambda^2} \Delta_x \Delta_y \times$$

$$\times \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \exp\left(-i\frac{2\pi}{\lambda} (u \Delta_x k + v \Delta_y j)\right) G_f(\Delta_x k, \Delta_y j). \qquad (28)$$

The summation on the right-hand side of (28) has periods λ/Δ_x in u, and λ/Δ_y in v. Since the sum $\tilde{N}_f$ is defined within the circle $u^2 + v^2 < 1$, overlapped aliasing lobes will occur in (28) unless $\Delta_x < \lambda/2$ and $\Delta_y < \lambda/2$; that is, the spacings between samples of $G_f(x,y)$ must be less than a half-wavelength at the temporal-frequency f of interest. We presume this to be true henceforth.

Now if we restrict u and v in (28) to the values

$$u_m = \frac{m}{M} \frac{\lambda}{\Delta_x} \quad \text{for} \quad -\frac{M}{2} \le m \le \frac{M}{2} - 1,$$

$$v_n = \frac{n}{N} \frac{\lambda}{\Delta_y} \quad \text{for} \quad -\frac{N}{2} \le n \le \frac{N}{2} - 1, \qquad (29)$$

both of which cover full periods in u and v, respectively, there follows

$$\tilde{N}_f(\mathrm{asin}(|u_m + iv_n|),\ \arg(u_m + iv_n)) \approx \frac{\left(1 - u_m^2 - v_n^2\right)^{\frac{1}{2}}}{\lambda^2} \times$$

$$\times\ \Delta_x\ \Delta_y \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \exp(-i2\pi mk/M - i2\pi nj/N)\ G_f(\Delta_x\ k,\ \Delta_y\ j), \qquad (30)$$

provided that

$$|u_m + iv_n| = \left|\frac{m}{M}\frac{\lambda}{\Delta_x} + i\ \frac{n}{N}\frac{\lambda}{\Delta_y}\right| \leq 1. \qquad (31)$$

The double sum in (30) can be accomplished as an $M\times N$ two-dimensional fast Fourier transform, when collapsing is employed [5; p.5]. The resultant angles at which noise field directionality $\tilde{N}_f(\theta,\phi)$ is available are

$$0 \leq \theta_{mn} = \mathrm{asin}\left|\frac{m}{M}\frac{\lambda}{\Delta_x} + i\ \frac{n}{N}\frac{\lambda}{\Delta_y}\right| \leq \frac{\pi}{2}\ ,$$

$$-\pi < \phi_{mn} = \arg\left(\frac{m}{M}\frac{\lambda}{\Delta_x} + i\ \frac{n}{N}\frac{\lambda}{\Delta_y}\right) \leq \pi, \qquad (32)$$

or

$$\sin\theta_{mn} = \left|\frac{m}{M}\frac{\lambda}{\Delta_x} + i\ \frac{n}{N}\frac{\lambda}{\Delta_y}\right| =$$

$$= \left[\left(\frac{m}{M}\frac{\lambda}{\Delta_x}\right)^2 + \left(\frac{n}{N}\frac{\lambda}{\Delta_y}\right)^2\right]^{\frac{1}{2}}\ , \qquad (33a)$$

where

$$-\frac{M}{2} \leq m \leq \frac{M}{2} - 1\ , \qquad -\frac{N}{2} \leq n \leq \frac{N}{2} - 1\ , \qquad (33b)$$

but remembering that (31) must remain true.

The right-hand side of (28) can be re-written in the form [5; pp. 3-4]

$$\frac{\left(1 - u^2 - v^2\right)^{\frac{1}{2}}}{\lambda^2} \iint_{-\infty}^{+\infty} dx\ dy\ \exp\left(-i\frac{2\pi}{\lambda}(ux + vy)\right) G_f(x,y) \times$$

$$\times\ \Delta_x \sum_{k=-\infty}^{+\infty} \delta(x - \Delta_x k)\ \Delta_y \sum_{j=-\infty}^{+\infty} \delta(y - \Delta_y j) =$$

$$= \tilde{N}_f(\operatorname{asin}(|u + iv|), \arg(u + iv)) \oplus$$

$$\overset{u}{\oplus} \sum_{k=-\infty}^{+\infty} \delta\left(u - \frac{k\lambda}{\Delta_x}\right) \overset{v}{\oplus} \sum_{j=-\infty}^{+\infty} \delta\left(v - \frac{j\lambda}{\Delta_y}\right) . \tag{34a}$$

The separations of the aliased lobes (for $(k,j) \neq (0,0)$) are λ/Δ_x on the u scale and λ/Δ_y on the v scale. Then, since the extent of the noise field directionality $\tilde{N}_f$ is $u^2 + v^2 < 1$, overlapped aliasing lobes do not occur if $\Delta_x < \lambda/2$ and $\Delta_y < \lambda/2$. This is a quantitative restatement of the claims made in the sequel to (28).

DISCRETE FINITE-LENGTH ARRAY

The effect of finite lengths in the x and y directions can be incorporated by modifying (34a) so as to include weighting $w(x,y)$. Then, we have, for the estimated noise field directionality,

$$\frac{\left(1 - u^2 - v^2\right)^{\frac{1}{2}}}{\lambda^2} \iint\limits_{-\infty}^{+\infty} dx\, dy\, \exp\left(-i\frac{2\pi}{\lambda}(ux + vy)\right) G_f(x,y)\, \times$$

$$\times\, \Delta_x \sum_{k=-\infty}^{+\infty} \delta(x - \Delta_x k)\, \Delta_y \sum_{j=-\infty}^{+\infty} \delta(y - \Delta_y j)\, w(x,y) =$$

$$= \tilde{N}_f(\operatorname{asin}(|u + iv|), \arg(u + iv)) \overset{uv}{\circledast} \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} W\left(u - \frac{k\lambda}{\Delta_x},\ v - \frac{j\lambda}{\Delta_y}\right), \tag{34b}$$

where window

$$W(u,v) = \frac{1}{\lambda^2} \iint\limits_{-\infty}^{+\infty} dx\, dy\, \exp\left(-i\frac{2\pi}{\lambda}(ux + vy)\right) w(x,y)\ . \tag{34c}$$

Thus, not only is the noise field directionality aliased at separations λ/Δ_x in u and λ/Δ_y in v, but, in addition, it is smoothed by window W. Sampling alone does not distort the estimated directionality if done with $\Delta_x < \lambda/2$ and $\Delta_y < \lambda/2$; see (34a). However, the finite lengths of the array always smears, with window widths of the order of λ/L_x in u and λ/L_y in v, where L_x and L_y are the effective lengths of weighting $w(x,y)$ in x and y, respectively.

VOLUMETRIC ARRAY

We now have the full version (6):

$$G_f(x,y,z) = \int_0^\pi d\theta \int_{-\pi}^{\pi} d\phi \ \sin\theta \ N_f(\theta,\phi) \ \times$$

$$\times \ \exp\left(i\frac{2\pi}{\lambda} \left(x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta\right)\right). \qquad (35)$$

However, since noise field directionality N_f is a function of two variables, while spatial correlation G_f has three arguments, some of the information in G_f is superfluous and must be reduced or collapsed in some fashion.

SOLUTION OF INTEGRAL EQUATION

We begin by defining triple Fourier transform

$$I(u,v,w) \equiv \iiint_{-\infty}^{+\infty} dx \ dy \ dz \ q(z) \ \exp\left(-i\frac{2\pi}{\lambda} (ux + vy + wz)\right) G_f(x,y,z) =$$

$$= \int_0^\pi d\theta \int_{-\pi}^{\pi} d\phi \ \sin\theta \ N_f(\theta,\phi) \ \lambda^2 \ \delta(u - \sin\theta \cos\phi) \ \delta(v - \sin\theta \sin\phi) \ \times$$

$$\times \ Q(w - \cos\theta), \qquad (36)$$

where we use a weighting $q(z)$ on the z variable, and define

$$Q(t) = \int_{-\infty}^{+\infty} dz \ \exp(-i2\pi tz/\lambda) \ q(z). \qquad (37)$$

(If $q(z) = 1$ for all z, then $Q(t) = \lambda \ \delta(t)$.)

We now break the right-hand side of (36) into two parts according to

$$\int_0^\pi d\theta \int_{-\pi}^\pi d\phi = \int_0^{\pi/2} d\theta \int_{-\pi}^\pi d\phi + \int_{\pi/2}^\pi d\theta \int_{-\pi}^\pi d\phi \,, \tag{38}$$

and in each region, we make the change of variable used in (19) et seq., namely

$$\alpha = \sin\theta \cos\phi, \quad \beta = \sin\theta \sin\phi. \tag{39}$$

Then $\phi = \arg(\alpha + i\beta)$, while

$$\theta = \left\{ \begin{array}{lll} \operatorname{asin}(|\alpha + i\beta|) & \text{for} & 0 \le \theta \le \pi/2 \\ \pi - \operatorname{asin}(|\alpha + i\beta|) & \text{for} & \pi/2 \le \theta \le \pi \end{array} \right\} \tag{40}$$

and

$$\cos\theta = \left\{ \begin{array}{lll} \left(1 - \alpha^2 - \beta^2\right)^{1/2} & \text{for} & 0 \le \theta \le \pi/2 \\ -\left(1 - \alpha^2 - \beta^2\right)^{1/2} & \text{for} & \pi/2 \le \theta \le \pi \end{array} \right\}. \tag{41}$$

Define, for future use,

$$s(\alpha,\beta) = \left(1 - \alpha^2 - \beta^2\right)^{1/2} \quad \text{for } \alpha^2 + \beta^2 \le 1. \tag{42}$$

Using these results in (36), there follows

$$I(u,v,w) = \lambda^2 \iint_{C_1} d\alpha\, d\beta\, s(\alpha,\beta)^{-1}\, F_1(\alpha,\beta)\, \delta(u-\alpha)\, \delta(v-\beta)\, Q(w-s(\alpha,\beta))$$

$$+ \lambda^2 \iint_{C_1} d\alpha\, d\beta\, s(\alpha,\beta)^{-1}\, F_2(\alpha,\beta)\, \delta(u-\alpha)\, \delta(v-\beta)\, Q(w+s(\alpha,\beta)), \tag{43}$$

where C_1 is a circle of radius 1 located at the origin, and

$$\left.\begin{aligned} F_1(\alpha,\beta) &= N_f(\mathrm{asin}(|\alpha + i\beta|),\ \arg(\alpha + i\beta)) \\ F_2(\alpha,\beta) &= N_f(\pi - \mathrm{asin}(|\alpha + i\beta|),\ \arg(\alpha + i\beta)) \end{aligned}\right\}\ \text{for } |\alpha + i\beta| \le 1. \tag{44}$$

Evaluating the integrals in (43), we have

$$I(u,v,w) = \lambda^2\, s(u,v)^{-1} \Big[F_1(u,v)\, Q(w - s(u,v)) + + F_2(u,v)\, Q(w + s(u,v))\Big] \quad \text{for } u^2 + v^2 < 1. \tag{45}$$

SIMULTANEOUS EQUATIONS

If we evaluate the triple Fourier transform in (36) at two different values of w, we have

$$\frac{s(u,v)}{\lambda^2} I(u,v,w_1) = F_1(u,v)\, Q(w_1 - s(u,v)) + F_2(u,v)\, Q(w_1 + s(u,v)) \tag{46}$$

and

$$\frac{s(u,v)}{\lambda^2} I(u,v,w_2) = F_1(u,v)\, Q(w_2 - s(u,v)) + F_2(u,v)\, Q(w_2 + s(u,v)). \tag{47}$$

Also, if we define

$$\begin{aligned} Q_n(+) &= Q(w_n + s(u,v)) , \\ Q_n(-) &= Q(w_n - s(u,v)) , \end{aligned} \tag{48}$$

and denominator

$$D = Q_1(-)\, Q_2(+) - Q_1(+)\, Q_2(-), \tag{49}$$

the solutions to (46) and (47) are

$$\left.\begin{aligned} F_1(u,v) &= \frac{s(u,v)}{\lambda^2 D}\Big(Q_2(+)\ I(u,v,w_1) - Q_1(+)\ I(u,v,w_2)\Big) \\ F_2(u,v) &= \frac{s(u,v)}{\lambda^2 D}\Big(Q_1(-)\ I(u,v,w_2) - Q_2(-)\ I(u,v,w_1)\Big) \end{aligned}\right\} \begin{array}{l}\text{for} \\ u^2 + v^2 < 1,\end{array} \tag{50}$$

provided that $D \neq 0$. Function s is defined in (42).

There is a great deal of leeway in these solutions. Namely, $Q(t)$ in (37) is arbitrary, and the values w_1 and w_2 are arbitrary as well; the only restriction is that D in (49) not be zero. In the special case where weighting $q(z)$ in (36) is real and even, then $Q(t)$ in (37) is also real and even; we presume this to be the case henceforth. If we then choose $w_2 = -w_1$, (49) becomes

$$D = Q^2(w_1 - s(u,v)) - Q^2(w_1 + s(u,v)). \tag{51}$$

If the effective length of weighting $q(z)$ is L_z, a *representative* plot of $Q^2(t)$ is displayed in figure 2. For small s, a good location for w_1 is at the point where $Q^2(t)$ has its maximum slope. For larger s, a value for w_1 near s would guarantee a large value for D in (51).

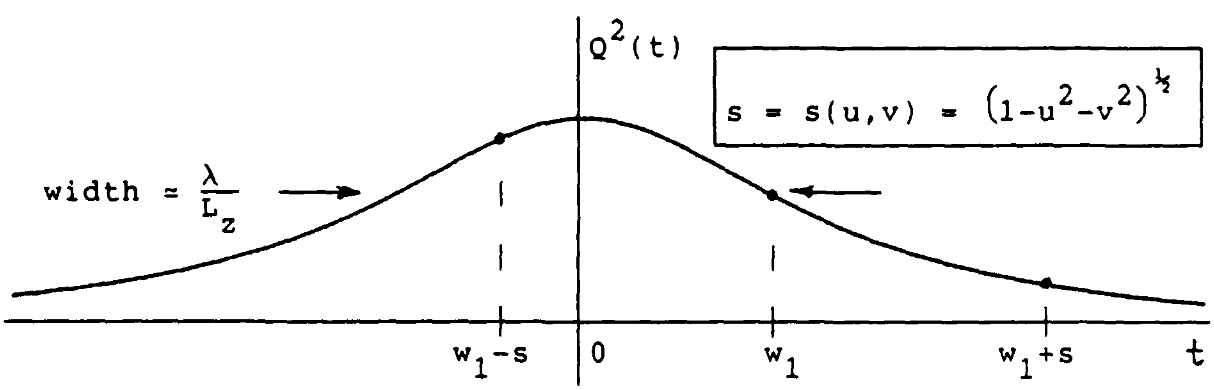

Figure 2. Window Function Q^2

ANGULAR REPRESENTATIONS

If we make the substitutions

$$u = \sin\theta \cos\phi, \; v = \sin\theta \sin\phi \quad \text{for } 0 \le \theta \le \pi/2, \; -\pi < \phi \le \pi, \tag{52}$$

then (42), (44), and (51) yield

$$\left.\begin{aligned} s(\sin\theta \cos\phi, \sin\theta \sin\phi) &= \cos\theta \\ F_1(\sin\theta \cos\phi, \sin\theta \sin\phi) &= N_f(\theta,\phi) \\ F_2(\sin\theta \cos\phi, \sin\theta \sin\phi) &= N_f(\pi - \theta,\phi) \\ D = Q^2(w_1 - \cos\theta) &- Q^2(w_1 + \cos\theta) \end{aligned}\right\} \begin{gathered} \text{for} \\ 0 \le \theta \le \pi/2, \; -\pi < \phi \le \pi. \end{gathered} \tag{53}$$

Then (50) becomes

$$N_f(\theta,\phi) = \frac{\cos\theta}{\lambda^2} \frac{Q(w_1 - \cos\theta)\, I(+) - Q(w_1 + \cos\theta)\, I(-)}{Q^2(w_1 - \cos\theta) - Q^2(w_1 + \cos\theta)}, \tag{54a}$$

$$N_f(\pi - \theta,\phi) = \frac{\cos\theta}{\lambda^2} \frac{Q(w_1 - \cos\theta)\, I(-) - Q(w_1 + \cos\theta)\, I(+)}{Q^2(w_1 - \cos\theta) - Q^2(w_1 + \cos\theta)}, \tag{54b}$$

$$\text{for } 0 \le \theta \le \pi/2, \; -\pi < \phi \le \pi,$$

where we define

$$I(\pm) = I(\sin\theta \cos\phi, \sin\theta \sin\phi, \pm w_1) = \iiint_{-\infty}^{+\infty} dx\, dy\, dz\, q(z) \times$$

$$\times \exp\left(-i\frac{2\pi}{\lambda} (x \sin\theta \cos\phi + y \sin\theta \sin\phi \pm z\, w_1)\right) G_f(x,y,z), \tag{55}$$

upon use of (36). We repeat that these results for the noise field directionality apply only for $Q(t)$ real and even; otherwise, $Q(t)$ and w_1 are arbitrary.

An alternative form to (54b) is available, if desired, by the substitution $\theta' = \pi - \theta$, namely

$$N_f(\theta',\phi) = \frac{\cos\theta'}{\lambda^2} \frac{Q(w_1 + \cos\theta')\, I'(-) - Q(w_1 - \cos\theta')\, I'(+)}{Q^2(w_1 - \cos\theta') - Q^2(w_1 + \cos\theta')}$$

$$\text{for} \quad \pi/2 \le \theta' \le \pi, \; -\pi < \phi \le \pi, \tag{56}$$

where

$$I'(\pm) = I(\sin\theta' \cos\phi,\, \sin\theta' \sin\phi,\, \pm w_1). \tag{57}$$

The most extensive calculation required here is that given by (55); rewriting it differently,

$$I(u,v,\pm w_1) = \iint_{-\infty}^{+\infty} dx\, dy\, \exp\left(-i\frac{2\pi}{\lambda}(ux + vy)\right) \times$$

$$\times \int_{-\infty}^{+\infty} dz\, \exp\left(-i\frac{2\pi}{\lambda}(\pm w_1)z\right) q(z)\, G_f(x,y,z). \tag{58}$$

The innermost integral, the Fourier transform on z, only needs to be accomplished for the two values $\pm w_1$, whereas the outer integrals must be done for ranges of u and v. This is the collapsing operation alluded to under (35). On the other hand, the inner integral must be repeated for every x,y value of interest;

nevertheless, (58) is not as difficult as a three-dimensional Fourier transform.

An example of this procedure for the volumetric array is carried out in appendix C; it illustrates the care that must be taken with respect to the θ variable in (54).

SUMMARY

The noise field directionality for the cases of one-, two-, and three-dimensional arrays have been solved for, explicitly, in terms of the appropriate spatial correlation available in each case. In the one-dimensional case, only the polar directionality can be determined, while in the two-dimensional case, the sum of symmetrically arriving rays on both sides of the planar array can be evaluated. For the three-dimensional case, all ambiguity can be eliminated, but the overdetermined nature of the problem requires some collapsing of information and leaves many options to consider. For example, one could let the volumetric array be a thin-shelled sphere; however, the resulting two-dimensional integral equation for the noise field directionality cannot be solved explicitly. The attractive feature of large stacked planar arrays is that it permits the use of Fourier transforms and, therefore, an explicit expression for the noise field directionality in terms of the three-dimensional spatial correlation. Also, Fourier transforms are efficiently evaluated by the use of fast Fourier transforms.

In this investigation, we have presumed exact knowledge of the spatial correlation $G_f(z)$ or $G_f(x,y)$ or $G_f(x,y,z)$, depending on the dimensionality of the array employed. In practice, G_f must be estimated from measurements made from a physical array; in this case, maximum advantage should be taken of the stationarity and homogeneity of the noise field. Thus, for a line array of equi-spaced elements, $G_f(n\Delta)$ should be estimated from all the

available pairs of elements that have separation $n\Delta$ in space and over the total available observation time that data have been recorded on all elements.

A comparison [6] is underway between the methods of this report and the Fourier series method given in [4], at least for the line array. Results are similar, but not identical; in particular, the aliasing of the Fourier series method is more severe than for the Fourier integral approach.

APPENDIX A. ALTERNATIVE LOCATION OF LINEAR ARRAY

If we locate the line array along the x axis, that is, $y = z = 0$, then (6) reduces to

$$G_f(x) = \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \, \sin\theta \, N_f(\theta,\phi) \, \exp\left(i\frac{2\pi}{\lambda} x \sin\theta \cos\phi\right) =$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi} d\phi \, \sin\theta \, \vec{N}_f(\theta,\phi) \, \exp\left(i\frac{2\pi}{\lambda} x \sin\theta \cos\phi\right), \quad \text{(A-1)}$$

where

$$\vec{N}_f(\theta,\phi) = N_f(\theta,\phi) + N_f(\pi - \theta,\phi) + N_f(\theta,-\phi) + N_f(\pi - \theta,-\phi) \quad \text{(A-2)}$$

for $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi$. Therefore, Fourier transform

$$I(u) \equiv \int_{-\infty}^{+\infty} dx \, \exp\left(-i\frac{2\pi}{\lambda} u \, x\right) G_f(x) =$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi} d\phi \, \sin\theta \, \vec{N}_f(\theta,\phi) \, \lambda \, \delta(u - \sin\theta \cos\phi). \quad \text{(A-3)}$$

Now let

$$s = \sin\theta, \quad t = \cos\phi, \quad \text{(A-4)}$$

which are one-to-one transformations in the ranges allowed in integral (A-3). Then

$$I(u) = \lambda \int_0^1 \frac{ds}{\left(1 - s^2\right)^{\frac{1}{2}}} \int_{-1}^1 \frac{dt}{\left(1 - t^2\right)^{\frac{1}{2}}} \vec{N}_f(a\sin(s), a\cos(t))\ s\ \delta(u-ts). \tag{A-5}$$

The innermost integral on t yields

$$\left\{ \begin{array}{ll} \dfrac{\vec{N}_f(a\sin(s),\ a\cos(u/s))}{\left(1 - u^2/s^2\right)^{\frac{1}{2}}} & \text{for } |u| < s \\ 0 & \text{for } |u| > s \end{array} \right\}, \tag{A-6}$$

thereby giving

$$I(u) = \lambda \int_{|u|}^1 \frac{ds\ s}{\left(1 - s^2\right)^{\frac{1}{2}} \left(s^2 - u^2\right)^{\frac{1}{2}}} \vec{N}_f(a\sin(s),\ a\cos(u/s)) \quad \text{for } |u| < 1. \tag{A-7}$$

This integral equation for noise field directionality $\vec{N}_f$ is more general than Abel's integral equation, because limit u is also involved in one of the arguments of $\vec{N}_f$. We have been unable to simplify (A-7) and extract any simple descriptor of the noise field directionality analogous to (8). Placing the linear array along the y axis, instead, encounters the same problem.

APPENDIX B. ALTERNATIVE LOCATION OF PLANAR ARRAY

Suppose the planar array lies in the x,z plane, that is, $y = 0$. Then (6) becomes

$$G_f(x,z) = \int_0^\pi d\theta \int_{-\pi}^{\pi} d\phi \, \sin\theta \, N_f(\theta,\phi) \, \exp\left(i\frac{2\pi}{\lambda}(x \sin\theta \cos\phi + z \cos\theta)\right) =$$

$$= \int_0^\pi d\theta \int_0^\pi d\phi \, \sin\theta \, \underline{N}_f(\theta,\phi) \, \exp\left(i\frac{2\pi}{\lambda}(x \sin\theta \cos\phi + z \cos\theta)\right) , \quad \text{(B-1)}$$

where

$$\underline{N}_f(\theta,\phi) = N_f(\theta,\phi) + N_f(\theta,-\phi) \quad \text{for } 0 \le \theta \le \pi, \ 0 \le \phi \le \pi. \quad \text{(B-2)}$$

Then

$$I(u,v) \equiv \iint_{-\infty}^{+\infty} dx \, dz \, \exp\left(-i\frac{2\pi}{\lambda} (ux + vz)\right) G_f(x,z) =$$

$$= \int_0^\pi d\theta \int_0^\pi d\phi \, \sin\theta \, \underline{N}_f(\theta,\phi) \, \lambda^2 \, \delta(u - \sin\theta \cos\phi) \, \delta(v - \cos\theta). \quad \text{(B-3)}$$

Now let

$$s = \sin\theta \cos\phi, \quad t = \cos\theta, \quad \text{(B-4)}$$

for which the Jacobian is

$$\frac{\partial(s,t)}{\partial(\theta,\phi)} = \sin^2\theta \sin\phi = \left(1 - t^2\right)^{\frac{1}{2}} \left(1 - s^2 - t^2\right)^{\frac{1}{2}}. \quad \text{(B-5)}$$

Then

$$I(u,v) = \lambda^2 \iint_{C_1} ds\, dt \, \frac{\underline{N}_f\left(\operatorname{acos}(t), \operatorname{acos}\left(s\left(1-t^2\right)^{-\frac{1}{2}}\right)\right)}{\left(1 - s^2 - t^2\right)^{\frac{1}{2}}} \, \delta(u-s)\, \delta(v-t) =$$

$$= \left\{ \begin{array}{ll} \dfrac{\lambda^2}{\left(1 - u^2 - v^2\right)^{\frac{1}{2}}} \underline{N}_f\left(\operatorname{acos}(v),\ \operatorname{acos}\left(\dfrac{u}{\left(1-v^2\right)^{\frac{1}{2}}}\right)\right) & \text{for } u^2 + v^2 < 1 \\ 0 \quad \text{otherwise} & \end{array} \right\} . \tag{B-6}$$

Thus, we have the explicit representation for the noise field directionality,

$$\underline{N}_f\left(\operatorname{acos}(v),\ \operatorname{acos}\left(\frac{u}{\left(1-v^2\right)^{\frac{1}{2}}}\right)\right) = \frac{\left(1 - u^2 - v^2\right)^{\frac{1}{2}}}{\lambda^2} \iint_{-\infty}^{+\infty} dx\, dz \, \times$$

$$\times \exp\left(-i\frac{2\pi}{\lambda}(ux + vz)\right) G_f(x,z) \quad \text{for} \quad u^2 + v^2 < 1 . \tag{B-7}$$

If we now let

$$u = \sin\theta \cos\phi, \quad v = \cos\theta, \tag{B-8}$$

this becomes

$$\underline{N}_f(\theta,\phi) = \frac{\sin\theta \sin\phi}{\lambda^2} \iint_{-\infty}^{+\infty} dx\, dz \exp\left(-i\frac{2\pi}{\lambda}(x \sin\theta \cos\phi + z \cos\theta)\right) \times$$

$$\times G_f(x,z) \quad \text{for} \quad 0 \le \theta \le \pi,\ 0 \le \phi \le \pi. \tag{B-9}$$

This is a viable alternative to (27). Compare with starting result (B-1).

APPENDIX C. EXAMPLE FOR VOLUMETRIC ARRAY

Let the noise field directionality be given by

$$N_f(\theta,\phi) = \delta(\theta-\theta_o)\ \delta(\phi-\phi_o), \quad 0 < \theta_o < \pi,\ -\pi < \phi_o \leq \pi. \qquad \text{(C-1)}$$

Notice that arrival angle θ_o can range over an interval of length π. We distinguish two cases:

$$\begin{aligned} &A: \quad 0 < \theta_o < \pi/2, \\ &B: \quad \pi/2 < \theta_o < \pi. \end{aligned} \qquad \text{(C-2)}$$

From (35), the three-dimensional spatial correlation is

$$G_f(x,y,z) = \sin\theta_o \exp\left(i\frac{2\pi}{\lambda}\,(x \sin\theta_o \cos\phi_o + \right.$$
$$\left. + y \sin\theta_o \sin\phi_o + z \cos\theta_o)\right) . \qquad \text{(C-3)}$$

The problem addressed here is the reestablishment of (C-1) by means of the solution procedure given in (54)-(57). Recall that $Q(t)$ is real and even.

First, substituting (C-3) in (55), there follows

$$I(\pm) = \sin\theta_o\ \lambda^2\ \delta(\sin\theta \cos\phi - \sin\theta_o \cos\phi_o) \times$$
$$\times\ \delta(\sin\theta \sin\phi - \sin\theta_o \sin\phi_o)\ Q(\pm w_1 - \cos\theta_o) . \qquad \text{(C-4)}$$

Now, when we recall that θ is limited to $(0,\pi/2)$ in (54), the delta functions in (C-4) are located at

$$\begin{aligned} &A: \quad \theta = \theta_o\ , \quad \phi = \phi_o\ , \\ \text{or} \quad &B: \quad \theta = \pi - \theta_o,\ \phi = \phi_o\ . \end{aligned} \qquad \text{(C-5)}$$

By means of the two-dimensional transformation employed in (19)-(22), we find that

$$\text{A:}\quad I(\pm) = \frac{\lambda^2}{\cos\theta_o}\,\delta(\theta - \theta_o)\,\delta(\phi - \phi_o)\,Q(w_1 \mp \cos\theta_o)\ ,$$

$$\text{B:}\quad I(\pm) = \frac{\lambda^2}{|\cos\theta_o|}\,\delta(\theta - \pi + \theta_o)\,\delta(\phi - \phi_o)\,Q(w_1 \mp \cos\theta_o)\ . \tag{C-6}$$

Substitution of (C-6) in the numerator of (54a) yields

$$\text{A:}\quad \cos\theta\,\frac{\lambda^2}{\cos\theta_o}\,\delta(\theta - \theta_o)\,\delta(\phi - \phi_o)\,C(\theta,\theta_o)\ ,$$

$$\text{B:}\quad \cos\theta\,\frac{\lambda^2}{|\cos\theta_o|}\,\delta(\theta - \pi + \theta_o)\,\delta(\phi - \phi_o)\,C(\theta,\theta_o)\ , \tag{C-7}$$

where

$$C(\theta,\theta_o) = Q(w_1-\cos\theta)\,Q(w_1-\cos\theta_o) - Q(w_1+\cos\theta)\,Q(w_1+\cos\theta_o). \tag{C-8}$$

But since

$$C(\theta_o,\theta_o) = Q^2(w_1 - \cos\theta_o) - Q^2(w_1 + \cos\theta_o)\ ,$$

$$C(\pi-\theta_o,\theta_o) = 0, \tag{C-9}$$

we find that

$$N_f(\theta,\phi) = \left\{ \begin{array}{rl} \delta(\theta - \theta_o)\,\delta(\phi - \phi_o) & \text{for case A} \\ 0 & \text{for case B} \end{array} \right\}. \tag{C-10}$$

On the other hand, substitution of (C-6) in the numerator of (54b) yields

$$\text{A:} \quad \cos\theta \, \frac{\lambda^2}{\cos\theta_o} \, \delta(\theta - \theta_o) \, \delta(\phi - \phi_o) \, D(\theta,\theta_o) \, ,$$

$$\text{B:} \quad \cos\theta \, \frac{\lambda^2}{|\cos\theta_o|} \, \delta(\theta - \pi + \theta_o) \, \delta(\phi - \phi_o) \, D(\theta,\theta_o) \, , \tag{C-11}$$

where

$$D(\theta,\theta_o) = Q(w_1-\cos\theta) \, Q(w_1+\cos\theta_o) - Q(w_1+\cos\theta) \, Q(w_1-\cos\theta_o) . \tag{C-12}$$

But since

$$D(\theta_o,\theta_o) = 0,$$

$$D(\pi-\theta_o,\theta_o) = Q^2(w_1+\cos\theta_o) - Q^2(w_1-\cos\theta_o) \, , \tag{C-13}$$

we find that

$$N_f(\pi-\theta,\phi) = \left\{ \begin{array}{ll} 0 & \text{for case A} \\ \delta(\theta-\pi+\theta_o) \, \delta(\phi-\phi_o) & \text{for case B} \end{array} \right\} . \tag{C-14}$$

This last case could be written in a form similar to (56) as

$$N_f(\theta',\phi) = \delta(\theta'-\theta_o) \, \delta(\phi-\phi_o) \quad \text{for} \quad \pi/2 < \theta' < \pi . \tag{C-15}$$

In any event, (C-10) and (C-14) confirm starting result (C-1) for the noise field directionality.

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