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CENTER FOR STOCHASTIC PROCESSES

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TWO BARRIERS PROBLEM FOR CONTINUOUSLY DIFFERENTIABLE PROCESSES

by

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198. T. Hsing. On the extreme order statistics for a stationary sequence. July 87. *Stochastic Proc. Appl.* 29, 1988, 155-169.

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Durbin has presented a compact formula for the first passage density of a Gaussian process, which is locally like Brownian motion, to a smooth barrier. In previous works, we have extended the formula to the case of processes which are smooth functions of a continuously differentiable Gaussian vector process and to more general kinds of first passage time problems, so called marked crossings. In the present paper we obtain similar results for the first passage density in presence of a second absorbing barrier and use it to construct upper and lower bounds for the first passage, rainflow cycle amplitude, zerocrossing wave-length and amplitude densities. Numerical examples illustrate the results. (KR)

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TWO BARRIERS PROBLEM FOR CONTINUOUSLY DIFFERENTIABLE PROCESSES

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Abstract. Durbin has presented a compact formula for the first passage density of a Gaussian process, which is locally like Brownian motion, to a smooth barrier. In previous works, we have extended the formula to the case of processes which are smooth functions of a continuously differentiable Gaussian vector process and to more general kinds of first passage time problems, so called marked crossings. In the present paper we obtain similar results for the first passage density in presence of a second absorbing barrier and use it to construct upper and lower bounds for the first passage, rainflow cycle amplitude, zerocrossing wave-length and amplitude densities. Numerical examples illustrate the results.

Keywords:. crossings, fatigue, first passage density, rainflow cycle, two barrier problem, wave-height, wave-length

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1. INTRODUCTION

Suppose that $y(s), s \geq 0$, is a continuously differentiable process. Let $u(s)$ and $v(s)$, $u(s) \geq v(s)$. be continuously differentiable barriers. Assume that the process starts between the barriers, i.e. $v(0) \leq y(0) \leq u(0)$ and if $y(0) = u(0)$ or $y(0) = v(0)$ then $y'(0) < u'(0)$, $y'(0) > v'(0)$, respectively.

In this paper, we are interested in the densities of the absorption times T_u, T_v of the y-process in the barriers u, v, respectively. More precisely, let T'_u, T'_v be the first passage times to the barriers u, v, respectively. Then the absorption times T_u, T_v are defined as follows

$$T_u = \begin{cases} T'_u & \text{if } T'_u < T'_v, \\ +\infty & \text{otherwise,} \end{cases}$$

$$T_v = \begin{cases} T'_v & \text{if } T'_v < T'_u, \\ +\infty & \text{otherwise.} \end{cases}$$

Our main result, presented in Section 2, states that the densities of T_u, T_v can be expressed in terms of conditional expectations in the following way

$$f_{T_u}(t) = E\left[I_{(0,t)}(y)\,(y'(t) - u'(t))^+ \,\middle|\, y(t) = u(t)\right] f_{y(t)}(u(t)), \tag{1}$$

$$f_{T_v}(t) = E\left[I_{(0,t)}(y)\,(y'(t) - v'(t))^- \,\middle|\, y(t) = v(t)\right] f_{y(t)}(v(t)),$$

where $x^+ = max(0, x)$, $x^- = max(0, -x)$ and $I_{(0,t)}(y)$ is the indicator function defined equal to 1 if the sample path does not cross the barriers u, v prior to time t and equal to 0 otherwise. The formula (1) is an extension of Durbin's formula for the first passage density [3, 10], which can be obtained by replacing the lower barrier v by $-\infty$. Since the indicator $I_{(0,t)}$ is a function of the whole sample path of the y-process, the expectations in (1) are difficult to evaluate exactly. However, we shall use (1), in Section 3, to construct upper- and lower-bounds for the densities of T_u, T_v.

Several important applications are related to the two absorbing barrier problem. Two are discussed in Section 4. The first one, which arises in oceanography, is the evaluation of the joint distribution of wave-length and amplitude of random waves; i.e. the joint distribution of the difference in time between the upcrossing of the mean sea level and the following downcrossing of this

level, and the highest value of the sea in this interval, (see Section 4.2). In the second application discussed in Section 4.3, we give approximations for the distribution of the so called rainflow cycle amplitude. (The Rainflow method was developed in fatigue analysis to describe a load process.)

2. Basic theorem

We begin with a definition of a class of processes for which (1) holds.

Definition 1. *Let $y(s)$ be a continuously differentiable process. Assume that there exists a k-variate continuously differentiable process $\Delta(s)$, and a random variable X, independent of $\Delta(s)$, with bounded and continuous density function such that*

$$y(s) = F(s, X, \Delta(s)), \qquad s > 0, \tag{2}$$

where F is a continuously differentiable mapping from $R^+ \times R \times R^k$ into R. For a fixed $t > 0$, the y-process will be called **decomposable at t** *if there exists $\epsilon > 0$ such that for all s, $|s - t| < \epsilon$, and all $\mathbf{z} \in R^k$, $F(s, \cdot, \mathbf{z})$ is one-to-one. Further, if the y-process is decomposable at almost all t then y is called* **decomposable.**

The class of decomposable processes is quite large and contains for example: Gaussian processes, functions of Gaussian vector processes, Slepian model processes, the sum of a Gaussian process and any independent continuously differentiable process, etc. An example of a class of processes which are not decomposable are processes which are "deterministic"on some interval, e.g. $y(s) = g(s)$. $g(s)$ is a continuously differentiable function.

We turn now to the definition of the particular version of conditional expectations used in (1). Assume that y is a decomposable process at t. Denote by $p_{t,\mathbf{z}}$ an inverse mapping to $F(t, \cdot, \mathbf{z})$ (2), i.e.

$$F(t, p_{t,\mathbf{z}}(r), \mathbf{z}) = r, \tag{3}$$

where $r \in R$. Let y_r be the following process

$$y_r(s) = F(s, p_{t,\Delta(t)}(r), \Delta(s)). \tag{4}$$

LEMMA 2. *Assume that the y-process is decomposable at t. If h is a nonnegative measurable functional defined on y, then for any* $r \in R$

$$E[h(y)|y(t) = r]f_{y(t)}(r) = E\big[h(y_r) \cdot f(r|\Delta(t))\big], \tag{5}$$

where the process y_r *is defined by (4), and the function* $f(r|z)$ *is given by*

$$f(r|z) = \Big|\frac{\partial p_{t,z}(r)}{\partial r}\Big| \cdot f_X\big(p_{t,z}(r)\big). \tag{6}$$

The function p is defined by (3) and f_X *is the density of X.*

PROOF: The lemma follows from Fubini's theorem.

Obviously, for many processes y, the decomposition (2) is not unique. In that case one can choose the decomposition (2), which gives the most convenient expression for the process y_r (4). For example, if y is a zero-mean Gaussian process, with $Var(y(t)) > 0$, the most natural decomposition is

$$y(s) = y(t)b_t(s) + \Delta_t(s), \tag{7}$$

where $\Delta_t(s)$ is a zero-mean Gaussian residual process independent of $y(t)$ with covariance function $r_t(s_1, s_2) = Cov(y(s_1), y(s_2)|y(t))$, and $b_t(s) = Cov(y(s), y(t))/Var(y(t))$. Now, by (4), the process y_r is defined by $y_r(s) = r \cdot b_t(s) + \Delta_t(s)$. Since, $p_{t,z}(r) = r$, the formula (5) can be written as

$$E[h(y)|y(t) = r]f_{y(t)}(r) = E\big[h(y_r)\big]f_{y(t)}(r).$$

In general, $h(y_r)$ and $f(r|\Delta(t))$ in (5), are dependent random variables. We shall illuminate this by using a different decomposition (7), e.g.

$$y(s) = y(0)b_0(s) + \Delta_0(s).$$

Suppose $b_0(t) \neq 0$, then, by (3), $p_{t,z}(r) = \frac{r-z}{b_0(t)}$ and the process y_r is given by

$$y_r(s) = \frac{r - \Delta_0(t)}{b_0(t)}b_0(s) + \Delta_0(s).$$

Finally, the conditional expectation (5) can be written as

$$E[h(y)|y(t) = r]f_{y(t)}(r) = E\left[h(y_r)\frac{1}{|b_0(t)|}f_{y(0)}\left(\frac{r - \Delta_0(t)}{b_0(t)}\right)\right].$$

THEOREM 3. *Assume that the process y is decomposable at t. If $E[|y'(t)||y(t) = u(t)] < +\infty$. $E[|y'(t)||y(t) = v(t)] < +\infty$, see (5), then the densities of T_u, T_v, are finite and are given by by (1).*

PROOF: The proof is similar to the proof of Theorem 2 in [10].

3. BOUNDS FOR THE ABSORPTION TIMES DENSITIES

3.1 Introduction. In this section we present upper- and lower-bounds for the density of T_u; the density of T_v can be treated similarly.

For a fixed time point s, denote by $I(y;s)$ the indicator function defined equal to 1 if $v(s) < y(s) < u(s)$ and equal to 0 otherwise. In the same way, for a vector of time points $\mathbf{s} = (s_1, \ldots, s_n)$. $0 < s_i < t$, let $I(y; s_1, \ldots, s_n)$ be the following indicator

$$I(y; s_1, \ldots, s_n) = \prod_{i=1}^{n} I(y; s_i). \tag{8}$$

Since, for any vector $\mathbf{s}$, $I_{(0,t)}(y) \leq I(y;\mathbf{s})$, an upper bound for the density of T_u can be obtained by replacing in (1) the indicator $I_{(0,t)}$ by $I(y;\mathbf{s})$, i.e.

$$f_u^+(t;n) = E\left[I(y; s_1, \ldots, s_n)(y'(t) - u'(t))^+ \middle| y(t) = u(t)\right] f_{y(t)}(u(t)). \tag{9}$$

However, it is in general difficult to give useful lower bounds for the indicator $I_{(0,t)}(y)$ in (1), and therefore formula (1) is not useful in the construction of lower bounds for the density of T_u. Hence, as in [13], we prove, in Theorem 4, a second formula for the density of T_u, which can be used to construct lower bounds. The approach taken in [13] leads to very general lower bounds, but the numerical effort to evaluate these lower bounds is much bigger than that corresponding to the upper bound. By some further restriction on the residual process Δ in (1), e.g. when Δ is a Gaussian process, we can construct lower bounds of the same complexity as the upper bounds

(9). In addition, by (2), the process y is a function of the process Δ and $y'(t)$ is a function of $\Delta(t), \Delta'(t)$, and hence, in order to evaluate numerically the upper bound (9), the joint density of $\Delta(s)$, $\Delta(t), \Delta'(t)$ must be given in an explicit form. Thus, from this point on, we assume that Δ is a zero-mean continuously differentiable vector valued **Gaussian** process.

3.2 A second formula for the density of T_u. Let Δ be a zero-mean continuously differentiable vector valued Gaussian process. For a fixet $t > 0$, consider the following sequence of random vectors

$$\zeta^{(0)} = (\Delta(t), \Delta'(t)), \tag{10}$$

$$\vdots$$

$$\zeta^{(n)} = (\Delta(t), \Delta'(t), \Delta(S_1), \ldots, \Delta(S_n)),$$

where S_i, $0 < S_i < t$, $i = 1, \ldots, n$, are **random times**, such that each S_i is a function of $\zeta^{(i-1)}$ alone, i.e. if $\zeta^{(i-1)} = z^{(i-1)}$, then $S_i(z^{(i-1)}) = s_i$, where $0 < s_i < t$ is a fixed time point.

In the following we use the decomposition of the process Δ into the conditional expectation on $\zeta^{(n)}$ and the residual process, i.e.

$$\Delta(s) = E[\Delta(s)|\zeta^{(n)}] + \Delta_n(s). \tag{11}$$

In [11] Lemma 9, we gave an explicit formula for the conditional expectation in (11) and proved that there exists a one-to-one transformation of $\zeta^{(n)}$ to a vector $\tilde{\zeta}^{(n)}$, say, of independent standard Gaussian variables. Since the transformation is a bijection, one can equivalently use in (11) $\zeta^{(n)}$ or $\tilde{\zeta}^{(n)}$. Consequently, in the following, we shall assume that $\zeta^{(n)}$ is transformed to a vector of independent standard Gaussian variables. Generally, for random times S_i, the conditional expectation in (11) is a nonlinear function of $\zeta^{(n)}$, and the residual process Δ_n is dependent on $\zeta^{(n)}$; however, given values of $\zeta^{(n)}$, Δ_n is Gaussian. This is a simple consequence of the definition of the vector $\zeta^{(n)}$ and the random times $S_1, \ldots, S_n$ (10), since, given $\zeta^{(n)} = z^{(n)}$, $S_i(z^{(n-1)}) = s_i$, $i = 1, \ldots, n$, are fixed time points, and hence the conditional distribution of $\Delta(s)$ given $\zeta^{(n)} = z^{(n)}$ is Gaussian

with mean zero and covariance

$$r(s,t;z^{(n-1)}) = Cov\big(\Delta(s),\Delta(t)|\Delta(t),\Delta'(t),\Delta(s_1),\ldots,\Delta(s_n)\big). \tag{12}$$

Let y be a decomposable process, i.e. $y(s) = F(s,X,\Delta(s))$, see Definition 1, where Δ is a vector valued Gaussian process. Assume that the vector of random times $S_1,\ldots,S_n$ in (10) is given, and let $\zeta^{(n)}$ be the corresponding vector (10). Denote by $\Delta_n(s|z^{(n)})$, $\mathbf{z}^{(n)} = (\mathbf{r},\mathbf{r}_1,\mathbf{z}_1,\ldots,\mathbf{z}_n)$, a zero-mean Gaussian process with covariance function (12). Consider the process $y(s|z^{(n)})$ defined as follows

$$y(s|z^{(n)}) = F\big(s,X,E[\Delta(s)|\zeta^{(n)} = \mathbf{z}^{(n)}] + \Delta_n(s|z^{(n)})\big). \tag{13}$$

Since X,Δ are independent $y(s|z^{(n)}) \stackrel{D}{=} y(s)|\zeta^{(n)} = \mathbf{z}^{(n)}$, (where $\stackrel{D}{=}$ denotes equility in distribution).

In order to use (5) with $y(s)$ replaced by $y(s|z^{(n)})$, let $y_{u(t)}(s|z^{(n)})$ be the process (4), i.e.

$$y_{u(t)}(s|z^{(n)}) = F\big(s,\quad ^{\cdot}u(t)),E[\Delta(s)|\zeta^{(n)} = \mathbf{z}^{(n)}] + \Delta_n(s|z^{(n)})\big), \tag{14}$$

where $p_{t,\mathbf{r}}(u(\iota))$ is defined by (3). e, in (14), we are using that $E[\Delta(t)|\zeta^{(n)} = z^{(n)}] = \mathbf{r}$ and $\Delta_n(t|z^{(n)}) = 0$.

We turn now to the second formula for the T_u-density.

THEOREM 4. *Let $y(s)$ be a decomposable process, i.e. $y(s) = F(s,X,\Delta(s))$, as in Definition 1, where Δ is a k-dimensional zero-mean Gaussian process. Further, assume that there exist $k-1$ continuously differentiable mappings from $R^+ \times R \times R^k$ into R, $(F_1,\ldots,F_{k-1})$, such that for all $x \in R$ and all s, $0 < s < t$, except possibly for a finite number of s,*

$$\mathbf{F}(s,x,\cdot) = \Big(F(s,x,\cdot),F_1(s,x,\cdot),\ldots,F_{k-1}(s,x,\cdot)\Big) \tag{15}$$

is a one-to-one mapping from R^k into R^k. Let $u(s),v(s)$, $u(s) > v(s)$, be continuously differentiable barriers. If for all s, $0 < s < t$, the following conditional expectations are finite

$$E[|y'(t)y'(s)||y(t) = u(t),y(s) = u(s)] < +\infty, \qquad E[|y'(t)y'(s)||y(t) = u(t),y(s) = v(s)] < +\infty,$$

then

$$
\begin{aligned}
(16)\qquad f_{T_u}(t) = {} & E\big[(y'(t)-u'(t))^{+} I(y;\mathbf{S}) \big| y(t)=u(t)\big] f_{y(t)}(u(t)) \\
& - \int_0^t E\Big[(y'(s)-u'(s))^{+}(y'(t)-u'(t))^{+} I(y;\mathbf{S}) I_{(0,s)}(y) \big| y(t)=u(t), \\
& \qquad\qquad y(s)=u(s)\Big] f(u(t),u(s))\,ds \\
& - \int_0^t E\Big[(y'(s)-v'(s))^{-}(y'(t)-u'(t))^{+} I(y;\mathbf{S}) I_{(0,s)}(y) \big| y(t)=u(t), \\
& \qquad\qquad y(s)=v(s)\Big] f(u(t),v(s))\,ds,
\end{aligned}
$$

where $\mathbf{S} = S_1,\ldots,S_n$ *are random times defined by (10) and* f *is a joint density of* $y(t), y(s)$. *The indicator function* $I(y,S_1,\ldots,S_n)$ *is given by* $(\mathcal{E})$, $I_{(0,s)}(y)$ *is the indicator function defined equal to 1 if the sample path of* y *does not cross the barriers* u, v *prior to time* s *and equal to 0 otherwise and* $x^{+} = max(0,x)$, $x^{-} = max(0,-x)$.

PROOF: Since $y(s|\mathbf{z}^{(n)})$ (13), $\mathbf{z}^{(n)} = (\mathbf{r},\mathbf{r}_1,\mathbf{z}_1,\ldots,\mathbf{z}_n)$, is decomposable, then, by Theorem 3, the conditional density of T_u given $\zeta^{(n)} = \mathbf{z}^{(n)}$ is defined by

$$
f_{T_u|\zeta^{(n)}}(t|\mathbf{z}^{(n)}) = E\Big[I_{(0,t)}\big(y(\cdot|\mathbf{z}^{(n)})\big)\big(y'(t|\mathbf{z}^{(n)}) - u'(t)\big)^{+} \big| y(t|\mathbf{z}^{(n)}) = u(t)\Big] f_{y(t|\mathbf{z}^{(n)})}(u(t)).
$$

By Lemma 2, the density of $T_u|\zeta^{(n)}$ is given by

$$
(17)\qquad f_{T_u|\zeta^{(n)}}(t|\mathbf{z}^{(n)}) = E\big[I_{(0,t)}\big(y_{u(t)}(\cdot|\mathbf{z}^{(n)})\big)\big]\cdot\big(y'_{u(t)}(t|\mathbf{z}^{(0)}) - u'(t)\big)^{+}\cdot f(u(t)|\mathbf{r}),
$$

where $y_{u(t)}$ is given by (14) and $f(u(t)|\mathbf{r})$ is defined by (6). We are also using that $y'_{u(t)}(t|\mathbf{z}^{(n)}) = y'_{u(t)}(t|\mathbf{z}^{(0)})$ is a constant variable dependent only on $\mathbf{z}^{(0)} = (\mathbf{r},\mathbf{r}_1)$.

Now, for all i, $i = 1,\ldots,n$, $\Delta_n(s_i|\mathbf{z}^{(n)}) = 0$, where $s_1,\ldots,s_n$ are the values of random times $S_1,\ldots,S_n$ given $\zeta^{(n)} = \mathbf{z}^{(n)}$, $\mathbf{z}^{(n)} \in R^{(n+2)k}$, and hence we are allowed to multiply the expectation in (17) by the indicator $I(y_{u(t)}(\cdot|\mathbf{z}^{(n)}); s_1,\ldots,s_n)$ (8), which is a function of $\mathbf{z}^{(n)}$. Further, for all $\mathbf{z}^{(n)}$, we have

$$
(18)\qquad E[I_{(0,t)}(y_u)] = 1 - \int_0^t f_{T_u(y_u)}(s)\,ds,
$$

where $y_u(\cdot) = y_{u(t)}(\cdot|\mathbf{z}^{(n)})$ (14).

Now, since $\Delta_n(\cdot|\mathbf{z}^{(n)})$ is a k-dimensional Gaussian process and (15) holds, one can prove, see Theorem 2 in [10], that the absorptions times $T_u(y_u), T_v(y_u)$ are given by (1). Finally, by combining (1), (17-18), multiplying (17) by the density of $\boldsymbol{\zeta}^{(n)}$ and integrating out $\mathbf{z}^{(n)}$, we obtain (16).

3.3 Bounds for the density of T_u. Since $0 \le I_{(0,s)}(y) \le 1$, then, for $n \ge 1$, we have the following upper and lower bounds for the density of T_u

$$
\begin{aligned}
(19) \qquad f_u^+(t;n) =& E\big[(y'(t)-u'(t))^+ I(y;\mathrm{S})\big|y(t)=u(t)\big] f_{y(t)}(u(t)) \\
f_u^-(t;n) =& f_u^+(t;n) - \int_0^t E\Big[(y'(s)-u'(s))^+ (y'(t)-u'(t))^+ I(y;\mathrm{S})\big|y(t)=u(t), \\
& \qquad\qquad y(s)=u(s)\Big] f(u(t),u(s))\,ds \\
& - \int_0^t E\Big[(y'(s)-v'(s))^- (y'(t)-u'(t))^+ I(y;\mathrm{S})\big|y(t)=u(t), \\
& \qquad\qquad y(s)=v(s)\Big] f(u(t),v(s))\,ds.
\end{aligned}
$$

Further, for $n = 0$, the bounds $f_u^+(t;0)$, $f_u^-(t;0)$ are obtained by replacing in (19) the indicator $I(y;\boldsymbol{S})$ by 1.

We turn now to the problem of choosing the vector S. Obviously, for any fixed t and n, the best choice of S is that which minimizes the upper bound $f_u^+(t;n)$, or the difference between the bounds $f_u^+(t;n) - f_u^-(t;n)$, see (19). However, since these procedures lead to complicated optimization problems, we propose a simpler recursive procedure.

We begin with some simple properties of the bounds f_u^+, f_u^- (19). Assume that we have selected a vector of random times $\boldsymbol{S} = (S_1,\ldots,S_n)$, and let $\boldsymbol{\zeta}^{(n)}$ be a random vector defined by (10). In order to simplify notation, we shall denote the conditional process $y_{u(t)}(\cdot|\mathbf{z}^{(n)})$ in (14) by y_u. Now, similarly as in (17), we can write the density of T_u as follows

$$
\begin{aligned}
(20) \qquad f_{T_u}(t) &= \int f_{T_u|\boldsymbol{\zeta}^{(n)}}(t|\mathbf{z}^{(n)}) f_{\boldsymbol{\zeta}^{(n)}}(\mathbf{z}^{(n)})\,d\mathbf{z}^{(n)} \\
&= \int E\big[I_{(0,t)}(y_u)\big] I(y_u;s_1,\ldots,s_n) f(z^{(0)}) f_{\boldsymbol{\zeta}^{(n)}}(\mathbf{z}^{(n)})\,d\mathbf{z}^{(n)},
\end{aligned}
$$

where $y_u(s) = y_{u(t)}(s|z^{(n)})$, $s_i = S_i(z^{(i-1)})$ are the values of S_i for $\zeta^{(i-1)} = z^{(i-1)}$ and $f(z^{(0)})$ is defined by

$$f(z^{(0)}) = \left(y'_{u(t)}(t|z^{(0)}) - u'(t)\right)^+ f(u(t)|\mathbf{r}).$$

Further, $f_{\zeta^{(n)}}$ is the density of $\zeta^{(n)}$. Consequently, the bounding problem of the density of T_u is reduced to the construction of an upper and lower bound for the expectation

$$E[I_{(0,t)}(y_u)] = P(v(s) < y_{u(t)}(s|z^{(n)}) < u(s) \text{ for all } s, 0 < s < t). \tag{21}$$

Let P_0^+, P_0^- be the following upper and lower-bounds for the probability (21)

$$\begin{aligned} P_0^+(t; y_u) &= 1, \\ P_0^-(t; y_u) &= 1 - \int_0^t E\left[\left(y'_u(s) - u'(s)\right)^+ \middle| y_u(s) = u(s)\right] f_{y_u(s)}(u(s))\, ds \\ &\quad - \int_0^t E\left[\left(y'_u(s) - v'(s)\right)^- \middle| y_u(s) = v(s)\right] f_{y_u(s)}(v(s))\, ds, \end{aligned} \tag{22}$$

where the lower bound P_0^- is obtained using (1) and (18). Now, by replacing the expectation in (20) by the upper and lower bound $P_0^+(t; y_u)$, $P_0^-(t; y_u)$, we obtain the bounds $f_u^+(t; n)$, $f_u^-(t; n)$ (19). respectively, generated by the vector $\zeta^{(n)} = \left(\Delta(t), \Delta'(t), \Delta(S_1), \ldots, \Delta(S_n)\right) = \left(\zeta^{(0)}, \zeta_1, \ldots, \zeta_n\right)$.

In order to obtain more accurate bounds $f_u^+(t; n+1), f_u^-(t; n+1)$, we have to choose an additional random time S_{n+1}, $0 < S_{n+1} < t$, which is a function of $\zeta^{(n)}$. Note, that the optimal strategy is to select the whole new vector $\zeta^{(n+1)}$, so that $f_u^+(t; n+1)$ is minimized. However, here we are restricting ourselves to recursive selection procedures of S_{n+1}, i.e. we add S_{n+1} to the old vector $\boldsymbol{S}$.

Now, assume that we have selected recursively k additional random times $S_{n+1}, \ldots, S_{n+k}$ and let $\zeta_{n+1}, \ldots, \zeta_{n+k}$ be the vector $(\Delta(S_{n+1}), \ldots, \Delta(S_{n+k}))$ transformed to iid. standard Gaussian variables. (The selection procedure will be given later in this subsection.) The vector $(\zeta_{n+1}, \ldots, \zeta_{n+k})$

generates new bounds for the probability (21), P_k^+, P_k^+, say, defined as follows

$$(23)\qquad P_k^+(t;y_u) = E\big[I(y_u;S_{n+1},\dots,S_{n+k})\big|\zeta^{(n)}=z^{(n)}\big]$$

$$P_k^-(t;y_u) = P_k^+(t;y_u) - \int_0^t E\Big[\big(y_u'(s)-u'(s)\big)^+ I(y_u;S_{n+1},\dots,S_{n+k})\big|\zeta^{(n)}=z^{(n)},$$
$$y_u(s)=u(s)\Big] f_{y_u(s)}(u(s))\,ds$$
$$- \int_0^t E\Big[\big(y_u'(s)-v'(s)\big)^- I(y_u;S_{n+1},\dots,S_{n+k})\big|\zeta^{(n)}=z^{(n)},$$
$$y_u(s)=v(s)\Big] f_{y_u(s)}(v(s))\,ds.$$

Once again, by replacing the expectation in (20) by the upper and lower bound $P_k^+(t;y_u), P_k^-(t;y_u)$ we obtain the bounds $f_u^+(t;n+k)$, $f_u^-(t;n+k)$ (19), respectively, generated by the vector $\zeta^{(n+k)}$.

In the following lemma, we give a recursive formula for the bounds (23).

LEMMA 5. *The upper and lower bounds (23) $P_k^+(y_u;t), P_k^-(y_u;t)$, $k>0$, for the probability (21), satisfy the following recursive formula*

$$(24)\qquad P_k^+\big(t;y_{u(\cdot)}(\cdot|z^{(n)})\big) = \int P_{k-1}^+\big(t;y_{u(t)}(\cdot|z^{(n+1)})\big)\cdot I\big(y_{u(t)}(\cdot|z^{(n+1)});s_{n+1}\big) f_{\zeta_{n+1}}(z)\,dz,$$
$$P_k^-\big(t;y_{u(t)}(\cdot|z^{(n)})\big) = \int P_{k-1}^-\big(t;y_{u(t)}(\cdot|z^{(n+1)})\big)\cdot I\big(y_{u(t)}(\cdot|z^{(n+1)});s_{n+1}\big) f_{\zeta_{n+1}}(z)\,dz,$$

where $s_{n+1}=S_{n+1}(z^{(n)})$ is the value of S_{n+1} for $\zeta^{(n)}=z^{(n)}$ and $z^{(n+1)}=(z^{(n)},z)$.

PROOF: We prove the lemma only for the upper bound P_k^+. A full proof is only notationally more complicated. By additional conditioning on ζ_{n+1} in (23) and using (8),(13), $P_k^+(t;y_u)$ can be written as

$$P_k^+(t;y_{u(t)}(\cdot|z^{(n)})) = \int E\big[I\big(y_{u(t)}(\cdot|z^{(n)});S_{n+1},\dots,S_{n+k}\big)\big|\zeta^{(n)}=z^{(n)},\zeta_{n+1}=z\big] f_{\zeta_{n+1}}(z)\,dz$$
$$= \int E\big[I\big(y_{u(t)}(\cdot|z^{(n+1)});S_{n+2},\dots,S_{n+k}\big)\big|\zeta^{(n+1)}=z^{(n+1)}\big]$$
$$I\big(y_{u(t)}(\cdot|z^{(n+1)});s_{n+1}\big) f_{\zeta_{n+1}}(z)\,dz,$$

showing (24).

Observe that in the recursive definition of the bounds $P_k^+(t;y_u)$, $P_k^-(t;y_u)$ (24), we have assumed that the random times $S_{n+1},\dots,S_{n+k}$ are given. Consequently, in order to use (20) and (24) to

calculate $f_u^+(t;n+k)$, $f_u^+(t;n+k)$, we have to define a recursive procedure to choose the random time S_{n+1} as a function of $\zeta^{(n)}$. In addition, since $y_{u(t)}(s|z^{(n)}) = F\big(s, p_{t,\mathbf{r}}(u(t)), E[\Delta(s)|\zeta^{(n)} = \mathbf{z}^{(n)}] + \Delta_n(s|z^{(n)})\big)$, where the only random component is a zero-mean vector valued Gaussian process $\Delta_n(s|z^{(n)})$, then, once the random time $S_{n+1}(z^{(n)}) = s_{n+1}$ is chosen, one can easily obtain the distribution of the process $y_{u(t)}(s|z^{(n+1)})$ by calculating $\Delta_n(s|z^{(n)})|\Delta_n(s_{n+1}|z^{(n)}) = z$.

We turn now to the presentation of the procedure for choosing the random times $\boldsymbol{S}$ in (19).

P:. *Step 1: choose the time S_1, $0 < S_1 < t$, to minimize $P_1^+(t; y_{u(t)}(\cdot|z^{(0)}))$, and, by (23), given the values $z^{(0)} = (\mathbf{r}, \mathbf{r}_1)$, choose the time s_1 to minimize $P(v(s_1) < y_{u(t)}(s_1|z^{(0)}) < u(s_1))$.*

Step n: given the time points $S_1, \ldots, S_{n-1}$ choose the time S_n, $0 < S_n < t$, to minimize $P_1^+(\cdot\ y_{u(t)}(\cdot|z^{(n-1)}))$, i.e. given $\mathbf{z}^{(n-1)}$, choose the time s_n to minimize $P(v(s_n) < y_{u(t)}(s_n|z^{(n-1)}) < u(s_n))$.

Since, for any $z^{(n)} \in R^{(n+2)k}$, $n \geq 0$, the procedure P defines $P_1^+(t; y_{u(t)}(\cdot|z^{(n)}))$, $P_1^-(t; y_{u(t)}(\cdot|z^{(n)}))$. then, using (24), we can recursively evaluate the bounds $P_n^+(t; y_{u(t)}(\cdot|z^{(0)}))$, $P_n^-(t; y_{u(t)}(\cdot|z^{(0)}))$. Hence, the upper and lower bounds $f_u^+(t;n)$, $f_u^-(t;n)$ are defined by

$$f_u^+(t;n) = \int P_n^+(t; y_{u(t)}(\cdot|z^{(0)}))f(z^{(0)})f_{\zeta^{(0)}}(z^{(0)})\,dz^{(0)}, \tag{25}$$

$$f_u^-(t;n) = \int P_n^-(t; y_{u(t)}(\cdot|z^{(0)}))f(z^{(0)})f_{\zeta^{(0)}}(z^{(0)})\,dz^{(0)},$$

where $f(z^{(0)}) = \big(y_u'(t|z^{(0)}) - u'(t)\big)^+ f(u(t)|\mathbf{r})$. In the following subsection we present a program BOUND, which evaluates the bounds $P_n^+(t; y_{u(t)}(\cdot|z^{(0)}))$, $P_n^-(t; y_{u(t)}(\cdot|z^{(0)}))$ in (25), for the special case when $y_{u(t)}(\cdot|z^{(0)})$ is a zero-mean Gaussian process.

3.4 Program BOUND. The procedure BOUND evaluates the upper and lower bound for the probability

$$Prob = P\big(u(s) > \Delta(s) > v(s) \text{ for all } s, 0 < s < t\big), \tag{26}$$

where Δ is a continuously differentiable zero-mean Gaussian process. We begin with a simple lemma.

LEMMA 6. *Let X be a Gaussian variable with mean m and variance σ^2, then*

$$E[X^+] = \sigma \cdot \Psi\left(\frac{m}{\sigma}\right),$$

where $\Psi(x) = \phi(x) + x\Phi(x)$, ϕ and Φ being the standardized normal density and distribution functions.

In all calculations, we have approximated the Φ-distribution by Hermite polynomials, and hence the Ψ-function is very accurately approximated by an explicit function.

We turn now to the description of the procedure BOUND

Input variables:

1: n - number of iterations, $n \geq 0$,

2: t - fixed time point, $t > 0$,

3: $u(s), v(s)$ - continuously differentiable barriers, $u(s) > v(s)$, $0 < s < t$,

4: $u'(s), v'(s)$ - derivatives of the barriers,

5: $r(s_1, s_2)$ - covariance function of a zero-mean Gaussian process Δ, $0 < s_1, s_2 < t$,

6: $r_1(s_1, s_2)$ - covariance function $Cov(\Delta'(s_1), \Delta(s_2)) = \frac{\partial}{\partial s_1} r(s_1, s_2)$, $0 < s_1, s_2 < t$,

7: $\sigma_1^2(s)$ - variance of the derivative Δ', i.e. $\sigma_1^2(s) = Var(\Delta'(s))$, $0 < s < t$,

Output variables:

P_n^+ - upper bound for the probability *Prob* (26),

P_n^- - lower bound for the probability *Prob* (26).

Algorithm:

Since the bounds P_n^+, P_n^- are functions of $u(\cdot), v(\cdot)$ and the covariance function of the process Δ, r_Δ say, we express it in notation by introducing $P_n^+(u, v, r_\Delta)$ and $P_n^-(u, v, r_\Delta)$.

If $n = 0$, then by (22)

$$P_0^+ = 1,$$

$$P_0^- = 1 - \int_0^t E\big[(\Delta'(s) - u'(s))^+ \big| \Delta(s) = u(s)\big] f_{\Delta(s)}(u(s))\, ds$$
$$- \int_0^t E\big[(\Delta'(s) - v'(s))^- \big| \Delta(s) = v(s)\big] f_{\Delta(s)}(v(s))\, ds.$$

Using Lemma 6, the lower bound P_0^- can be written in more explicit way, i.e.

$$P_0^- = 1 - \int_0^t \frac{\sigma_2(s)}{\sigma(s)} \left(\Psi\Big(\frac{u(s)b(s) - u'(s)}{\sigma_2(s)}\Big)\phi\Big(\frac{u(s)}{\sigma(s)}\Big) + \Psi\Big(\frac{v'(s) - v(s)b(s)}{\sigma_2(s)}\Big)\phi\Big(\frac{v(s)}{\sigma(s)}\Big) \right) ds,$$

where the Ψ-function is defined in Lemma 6 and $b(s) = r_1(s,s)/r(s,s)$,

$$E[\Delta'(s)|\Delta(s) = u(s)] = -\,u(s) \cdot b(s),$$

$$E[\Delta'(s)|\Delta(s) = v(s)] = -\,v(s) \cdot b(s),$$

$$\sigma_2^2(s) = Var(\Delta'(s)|\Delta(s)) = \sigma_1^2(s) - \frac{r_1(s,s)^2}{r(s,s)}.$$

Observe that the integral must be evaluated numerically.

If $n > 1$, then, by the procedure P, we choose s_1, $0 < s_1 < t$, a fixed time point for which the probability $P(u(s_1) > \Delta(s_1) > v(s_1))$ is minimized, i.e.

$$P(u(s_1) > \Delta(s_1) > v(s_1)) = \min_{0<s<t} \Big(\Phi\Big(\frac{u(s)}{\sigma(s)}\Big) - \Phi\Big(\frac{v(s)}{\sigma(s)}\Big)\Big),$$

and by recursion (24) we have

$$P_n^+(u, v, r_\Delta) = \int_{v(s_1)}^{u(s_1)} P_{n-1}^+\big(u - xb_1(s), v - xb_1(s), r_{\Delta|\Delta(s_1)}\big) f_{\Delta(s_1)}(x)\, dx \tag{27}$$
$$\approx \sum_{i=1}^{N} P_{n-1}^+\big(u - x_i b_1(s), v - x_i b_1(s), r_{\Delta|\Delta(s_1)}\big) f_{\Delta(s_1)}(x_i) h_i,$$

where x_i, h_i are suitable nodes and weights, respectively, $u(s_1) > x_i > v(s_1)$, $i = 1, \ldots, N$ and

$$xb_1(s) = E[\Delta(s)|\Delta(s_1) = x] = x\frac{r(s, s_1)}{r(s_1, s_1)}. \tag{28}$$

The same recursion can be given for the lower bound P_n^-, by replacing in (27) "+" by "−".

Finally, for each x_i, we evaluate the bounds $P_{n-1}^+(u - x_i b_1(s), v - x_i b_1(s), r_{\Delta|\Delta(s_1)})$, $P_{n-1}^-(u - x_i b_1(s), v - x_i b_1(s), r_{\Delta|\Delta(s_1)})$, using the procedure BOUND with the following input variables;

1: $n-1$,

2: t, unchanged,

3: $u(s) - x_i b_1(s), v(s) - x_i b_1(s)$, where $b_1(s)$ is given by (28), $0 < s < t$,

4: $u'(s) - x_i b_1'(s), v'(s) - x_i b_1'(s)$, where $b_1'(s)$ is given by (29), $0 < s < t$,

5: covariance function (30),

6: covariance function (31),

7: variance (32),

$$b_1'(s) = \frac{r_1(s_1, s)}{r(s_1, s_1)}, \tag{29}$$

$$r_{\Delta|\Delta(s_1)}(s,t) = r(s,t) - \frac{r(s,s_1)r(s_1,t)}{r(s_1,s_1)}, \tag{30}$$

$$Cov(\Delta'(s), \Delta(t)|\Delta(s_1)) = r_1(s,t) - \frac{r_1(s,s_1)r(s_1,t)}{r(s_1,s_1)}, \tag{31}$$

$$Var(\Delta'(s)|\Delta(s_1)) = \sigma_1^2(s) - \frac{r_1^2(s_1,s)}{r(s_1,s_1)}. \tag{32}$$

The output variables P_n^+ and P_n^- are now defined by

$$P_n^+ = \sum_{i=1}^{N} P_{n-1}^+\Big(u - x_i b_1(s), v - x_i b_1(s), r_{\Delta|\Delta(s_1)}\Big) f_{\Delta(s_1)}(x_i) h_i, \tag{33}$$

$$P_n^- = \sum_{i=1}^{N} P_{n-1}^-\Big(u - x_i b_1(s), v - x_i b_1(s), r_{\Delta|\Delta(s_1)}\Big) f_{\Delta(s_1)}(x_i) h_i.$$

Here, we assume that the procedure BOUND is programed in a computer language which allows recursive functions, e.g. APL.

Finally, an obvious question is whether one should use in (19) fixed times $s_1, \ldots, s_n$ instead of random $S_1, \ldots, S_n$ is would drasticly reduces the number of times one have to evaluate

equations (28-32). However, tests disclosed that such a procedure is usually much slower. The reason is that the probability $P(u(s_1) > \Delta(s_1) > v(s_1)) \geq P(u(S_1) > \Delta(S_1) > v(S_1))$, and hence we have to increase N, number of nodes x_i, to compute (33).

3.5 Concluding remarks. In this subsection we discuss the convergence of the bounds (19), in the case when S_i are deterministic and dense points in the interval $[0,t]$. This does not prove that the bounds obtained using the procedure P converges to the density of T_u. However, the procedure evaluates simultaneously the upper and lower bounds for the density of T_u, so the accuracy and the convergence of the bounds can be easily checked. In addition, in examples presented in the next section, the bounds based on the random times S_i are more accurate then the bounds based on deterministic points s_i and the algorithm is much faster.

REMARK 7. *Under assumptions of Theorem 4, if $\{s_n\}_{n=1}^{+\infty}$ is a dense subset of the interval $[0,t]$, then as n tends to infinity*

$$I(y;s_1,\ldots,s_n) \downarrow I_{(0,t)}(y), \text{ a.s.},$$

$$E\big[(y'(t)-u'(t))^+(y'(s)-u'(s))^+ I(y;s_1,\ldots,s_n)\big|y(t)=u(t),y(s)=u(s)\big] \downarrow 0,$$

$$E\big[(y'(t)-u'(t))^+(y'(s)-v'(s))^- I(y;s_1,\ldots,s_n)\big|y(t)=u(t),y(s)=v(s)\big] \downarrow 0.$$

Consequently, the upper and lower bounds $f_u^+(t;n)$, $f_u^-(t;n)$, (19), with $S_i = s_i$, $i = 1,2,\ldots$, converge monotonically to the density of T_u as n goes to infinity. (The proof is similar to the proof of Theorem 11 in [13].)

Observe that Theorems 3 and 4 can be used to construct many different types of upper and lower bounds. For example, another type of lower bound are obtained by replacing the indicator $I(y;S)$ in (16), by 1, and overestimating the indicator $I_{(0,s)}(y)$, by $I(y;\hat{S}_1,\ldots,\hat{S}_n)$, where $\hat{S}_1,\ldots,\hat{S}_n$ can be chosen by a procedure similar to P, see [13]. However, an important property of (19), which distinguishes it from the other approaches, is that the same points $\boldsymbol{S}$ are used in both upper and lower bounds leading to a more efficient algorithm.

4. Applications

4.1 Slepian model process. Let $y(t)$, $t \geq 0$, be a stationary zero-mean ergodic Gaussian process with covariance function r, and assume that its sample paths are a.s. continuously differentiable. A sufficient condition [1] for this is that the process is separable and that

$$r''(s) = \lambda - o(|log|s||^{-\alpha}),$$

as $s \to 0$, for some $\alpha > 1$. Assume that the process y is normalized so that $\lambda_0 = \mathrm{Var}(y(0)) = \lambda_2 = \mathrm{Var}(y'(0)) = 1$, which is only a matter of scaling.

In following subsections, we are interested in the "long run" properties of the process y after downcrossings of the level u; consequently we are introducing the Slepian model process ξ_u for y after a downcrossing of the level u. This is the stochastic process $\xi_u(\cdot)$ which is distributed as the long run distribution of $y(\omega, t_k + \cdot)$, when t_k runs over all u-downcrossings of $y(\omega, \cdot)$. Mathematical details about Slepian processes and long run probabilities can be found in [5], Ch. 10, and [6, 7]. We now give a simple representation of the Slepian model process ξ_u.

Consider a zero-mean Gaussian process Δ, with covariance function

$$\mathrm{Cov}(\Delta(s), \Delta(t)) = \mathrm{Cov}(y(s), y(t) | y(0), y'(0)) = r(s-t) - r(t)^2 - r'(t)^2, \tag{34}$$

since $\mathrm{Var}(y(0)) = \mathrm{Var}(y'(0)) = 1$, and let R be a standard Rayleigh variable, with mean $\sqrt{\frac{\pi}{2}}$, independent of Δ. Then the Slepian model process ξ_u is given by

$$\xi_u(s) = ur(s) + Rr'(s) + \Delta(s). \tag{35}$$

Obviously, the process ξ_u satisfies the assumptions of Theorem 4, and hence we can use the bounds (19) developed in Section 3.3.

In following numerical examples, we shall use the process y with covariance functions given by

$$r(s) = \frac{\sin \sqrt{3}\, s}{\sqrt{3}\, s}. \tag{36}$$

4.2 The distribution of wave-length and amplitude. Assume $y(t)$ is a zero-mean Gaussian process, which described water elevation at a fixed point. A question that arises in oceanography is that of the "empirical" or "long run" distribution of the zerocrossing wave-length and amplitude T, H. By this we mean the difference between the time of the zero downcrossing and the following zero upcrossing and the lowest value of y in this interval, see Figure 1. Since the "long run" properties of y after zero downcrossings are described by Slepian model process ξ_0 (35), we have

$$T, H \stackrel{D}{=} T(\xi_0), H(\xi_0).$$

For notational convenience, we drop the subscript "0" in ξ_0 and in the following the Slepian model process after zero downcrossing is denoted by ξ.

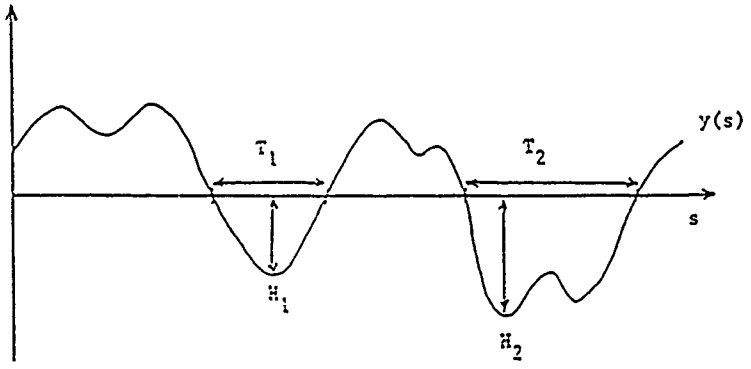

Figure 1. Definition of zerocrossing wavelength and amplitude T, H.

Obviously, $H(\xi) > h$ if and only if $\xi(s)$, $s > 0$, reaches the level $-h$ before it crosses the level 0 again. (Observe that $\xi(0) = 0$, $\xi'(0) < 0$.) Thus the distribution of T, H can be expressed using the densities of the absorptions times of ξ, T_u, T_v, with $u(s) = 0, v(s) = -h$, respectively, i.e.

$$P(T \leq t, H \leq h) = \int_0^t f_{T_u}(s)\, ds, \tag{37}$$

$$P(H \leq h) = \int_0^{+\infty} f_{T_u}(s)\, ds = 1 - \int_0^{+\infty} f_{T_v}(s)\, ds. \tag{38}$$

In (38) we are using a fact that, for Gaussian processes, the probability that ξ stays for ever between the finite barriers is zero, i.e.

$$\int_0^{+\infty} [f_{T_v}(s) + f_{T_u}(s)]\, ds = 1.$$

Note that the distribution of T can be obtained from (37) by choosing the lower barrier $v(s) = -\infty$.

In order to use (38) for bounding the distribution F_H, we have to approximate the infinite region of integration by some bounded interval. For many processes of practical interest, there exists a positive constant T_0, such that $f_{T_u}(s) \approx 0$, in (37), for all $s > T_0$. Now, using the lower bounds (19) for the density of T_u, $u(s) = 0, v(s) = -\infty$, respectively, we can find T_0 as the first time when

$$\int_0^{T_0} f_u^-(s; n)\, ds \geq 1 - \epsilon,$$

for some small ϵ, i.e. $P(T > T_0) \leq \epsilon$.

We turn now to the presentation of the bounds $f_u^+(s; n), f_u^-(s; n)$ for the density of $T_u(\xi)$ obtained using the procedure P of Section 3.3. The bounds for the $T_v(\xi)$ density can be derived in the similar way.

For a fixed value h, the formula (1) for the density of T_u, with $u(s) = 0, v(s) = -h$, is given by

$$f_{T_u}(t) = E\big[I_{(0,t)}(\xi)\xi'(t)^+ \big| \xi(t) = 0\big] f_{\xi(t)}(0). \tag{39}$$

The formula (39) can also be expressed in terms of y, i.e.

$$\begin{aligned} f_{T_u}(t) &= c \cdot E\big[I_{(0,t)}(y) y'(0)^- y'(t)^+ \big| y(0) = 0, y(t) = 0\big] f_{y(0),y(t)}(0,0) \\ &= c \cdot \int_{-\infty}^{0} \int_0^{+\infty} P\big(0 > y(s) > -h \text{ for all } s, 0 < s < t \big| y'(0) = z, y'(t) = z_1, \\ &\qquad y(0) = 0, y(t) = 0\big) f(z, z_1)\, dz_1\, dz, \end{aligned}$$

where c^{-1} is the average number of zero downcrossings per unit interval

$$c = E[y'(0)^- | y(0) = 0] f_{y(0)}(0) = (2\pi)^{-1} \sqrt{\frac{\lambda_2}{\lambda_0}}, \tag{40}$$

by the celebrated Rice formula [9], and

$$f(z,z_1) = |z|z_1 f_{y'(0),y'(t),y(0),y(t)}(z,z_1,0,0), \qquad z \le 0 \text{ and } z_1 \ge 0. \tag{41}$$

In the following we assume that t, h, z, z_1 are fixed values. Many of formulas will depend on t, h, z, z_1, however for notational convenience we shall not always write this dependence explicitly.

Let $m(s)$ be the following conditional expectation

$$m(s) = E\big[y(s)\big|y'(0) = z, y'(t) = z_1, y(0) = 0, y(t) = 0\big],$$

and let Δ be a zero-mean Gaussian process with a covariance function $\hat{r}(s_1,s_2)$ given by

$$\hat{r}(s_1,s_2) = \mathrm{Cov}\big(y(s_1), y(s_2)\big|y'(0), y'(t), y(0), y(t)\big).$$

It means that $m(\cdot) + \Delta(\cdot) \stackrel{D}{=} y(\cdot)\big|y'(0) = z, y'(t) = z_1, y(0) = 0, y(t) = 0$.

As before, let r be the covariance of the y process. Then with

$$\mathbf{C}(t) = \begin{pmatrix} 1 & -r''(t) & 0 & -r'(t) \\ -r''(t) & 1 & r'(t) & 0 \\ 0 & r'(t) & 1 & r(t) \\ -r'(t) & 0 & r(t) & 1 \end{pmatrix},$$

and

$$\mathbf{c}(s) = \big(-r'(s), -r'(s-t), r(s), r(s-t)\big),$$

the mean $m(s)$ and a covariance function $\hat{r}(s_1,s_2)$ are given by

$$m(s) = \mathbf{c}(s)\mathbf{C}(t)^{-1}\begin{pmatrix} z \\ z_1 \\ 0 \\ 0 \end{pmatrix},$$

$$\hat{r}(s_1,s_2) = r(s_1 - s_2) - \mathbf{c}(s_1)\mathbf{C}(t)^{-1}\mathbf{c}(s_2)^T.$$

Using the process Δ, the formula for the density of T_u can be written as

$$f_{T_u}(t) = c \cdot \int_{-\infty}^{0}\int_{0}^{+\infty} P\big(-m(s) > \Delta(s) > -h - m(s) \text{ for all } s, 0 < s < t\big) f(z,z_1)\, dz_1\, dz, \tag{42}$$

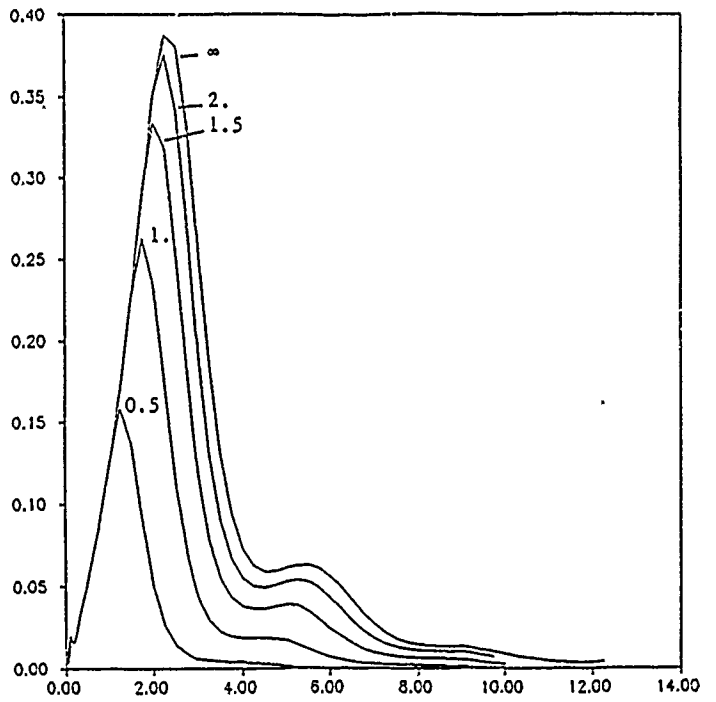

Figure 2. Densities of T_u, $h = 0.5, 1, 1.5, 2, +\infty$, for the covariance (36).

where $f(z, z_1)$ is given by (41).

Now, upper and lower bounds (19) $f_u^+(t,n)$, $f_u^-(t,n)$, respectively, are obtained, using the procedure BOUND of Section 3.4, by over and under estimating the probability in (42).

We turn now to presentation of the numerical bounds for the "long run" wave-length and amplitude distribution for Gaussian process y with covariance function r (36).

Table 1 shows bounds $f_u^+(t;0), \ldots, f_u^+(t;4)$, $f_u^-(t;4), \ldots, f_u^-(t;1)$ for the zerocrossing wavelength density, i.e. $u(s) = 0$, $v(s) = -\infty$, for the process y with covariance (36). We can see, that the upper and lower bounds are almost identical. In addition, the integral of the lower bound $f_u^-(t;4)$, over the interval $[0, 12.5]$, is 0.999, indicating that only 0.1% of all waves are longer than 12.5.

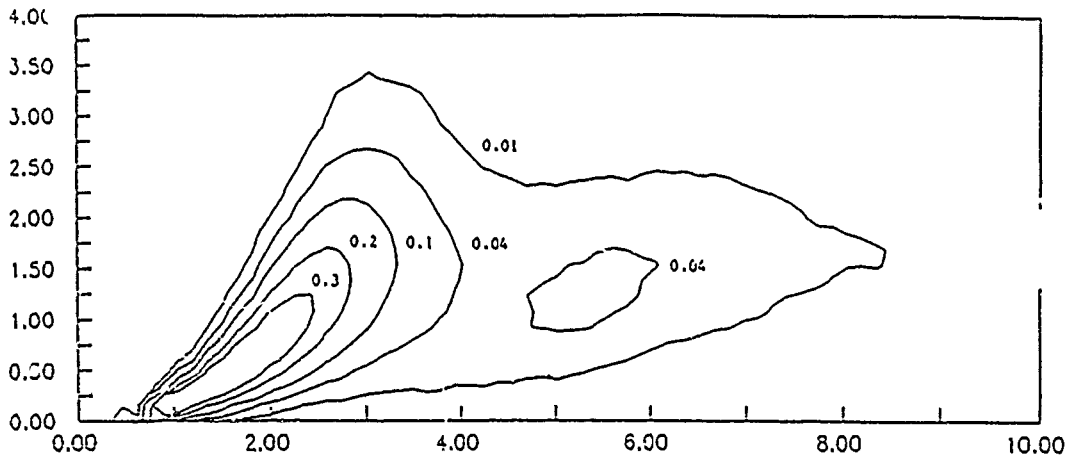

Figure 3. Isolines of joint density of wavelength and amplitude T, H, for the covariance (36).

TABLE 1. *Bounds* $f_u^+(t;0)$, $f_u^+(t;1)$, $f_u^+(t;2)$ $f_u^+(t;3)$, $f_u^+(t;4)$, $f_u^-(t;4)$, $f_u^-(t;3)$, $f_u^-(t;2)$, $f_u^-(t;1)$, *for the zerocrossing wave-length density, i.e.* $u(s) = 0, v(s) = -\infty$, *covariance function* r *(36).*

t	$f_u^+(t;0)$	·	·	·	$f_u^+(t;4)$	$f_u^-(t;4)$	·	·	$f_u^-(t;1)$
1	0.123	·	·	·	·	·	·	·	0.123
2	0.352	·	·	·	·	·	·	·	0.352
3	0.257	·	·	·	·	·	·	·	0.257
4	0.078	0.073	·	·	·	·	·	0.07	0.072
5	0.114	0.063	0.062	·	·	·	0.062	0.06	0.057
6	0.231	0.071	0.062	0.056	·	·	0.056	0.056	0.056
7	0.173	0.039	0.033	0.027	0.026	·	0.026	0.026	0.025
8	0.109	0.032	0.021	0.015	0.014	·	0.014	0.014	0.013
9	0.158	0.059	0.023	0.017	0.013	·	0.013	0.008	0.000
10	0.205	0.061	0.019	0.013	0.009	·	0.009	0.005	0.000
11	0.146	0.040	0.012	0.009	0.006	0.004	0.004	0.001	0.000
12	0.128	0.043	0.018	0.008	0.005	0.002	0.000	0.000	0.000

In order to bound the distribution of zerocrossing amplitude we have to bound density of T_u for the barriers $u(s) = 0$ and $v(s) = -h$, $h > 0$. For a fixed t, the density of T_u is increasing function of h. This is shown on Figure 2, where we present the density of T_u for $h = 0.5, 1, 1.5, 2, +\infty$. Finally, the joint density of wavelength and amplitude T, H is obtained by numerical differentiation, on h, of the density of T_u, see Figure 3.

4.3 The distribution of Rainflow cycle amplitude. When a piece of metal is subjected to a periodically varying load small microscopic inhomogeneities can develop into open cracks, leading to fatigue failure after a random amount of time. The distribution of fatigue life length depends on the amplitudes of the applied "load cycles". One then needs a rule to combine the damages caused by the different cycles. The most commonly used damage rule is due to Palmgren & Miner, and postulates that the total damage caused by a stress history $\{S_k\}$ of load cycles is

$$D(t) = \sum_{k=1}^{k(t)} \frac{1}{N_{S_k}},$$

where the sum is extended over all cycles completed at time t and N_s is the median cycle life obtained from tests with constant amplitude s. The median life is predicted to be the time t which makes $D(t)$ greater than or equal to one. In most situations, the median cycle life N_s is large, between 10^4 and 10^7, and therefore, by ergodicity of the load process, the fatigue life is predicted as

$$T = \frac{1}{c \cdot \int \frac{1}{N_s} f_S(s)\, ds}, \tag{43}$$

where f_S is the density of the ergodic (long run) distribution of the cycle amplitude S_k and c is a mean number of a cycles counted in the unit interval $[0, 1]$.

Dowling [2] has studied the accuracy of the predictors of the fatigue life T based on eight most commonly used counting methods, and finds that only the rainflow cycle (RFC) counting method leads to prediction agreeing with actual lifes.

Due to the great importance of the RFC-counting method, many different algorithms have been proposed in the literature. However, most of them have a complicated "sequential" structure which makes them difficult to apply when their statistical properties are studied. The following definition of RFC-cycle, given in [12], is more convenient for statistical analysis of long run properties of the RFC-cycles.

DEFINITION 8:. *Let $y(\tau)$, $-T \le \tau \le T$, be a load function, and let $\{t_k\}$, with $\cdots < t_{-1} < 0 \le t_0 < t_1 < \ldots$, be the times of the local maxima of $y(\cdot)$. For a local maximum at time t_i, let t_i^+ be the time for the first upcrossing after t_i of the level $y(t_i)$ (or $t_i^+ = T$ if no such upcrossing exists for $t_i \le \tau \le T$), and let t_i^- be the time for the last downcrossing of $y(t_i)$ before t_i (or $t_i^- = -T$ if no such downcrossing exists for $-T \le \tau \le t$). Let the lowest minima in the intervals (t_i^-, t_i) and (t_i, t_i^+) occur at t_l, t_r, respectively, and let t_i^* be the time when the higher of the minima $y(t_l)$, $y(t_r)$ occur, i.e.*

$$t_i^* = \begin{cases} t_r & \text{if } y(t_l) \le y(t_r), \\ t_l & \text{otherwise.} \end{cases}$$

The RFC-count attaches to a maximum at time t_i a Rainflow cycle originating at t_i, defined as a pair of the maximum $y(t_i)$ and minimum $y(t_i^*)$, the amplitude of the cycle is given by

$$S_i = y(t_i) - y(t_i^*),$$

see Figure 4. Furthermore, the empirical bivariate distribution of the local maximum $M = y(t_i)$ and the corresponding RFC-minimum $m = y(t_i^*)$ is defined as

$$F_{M,m}(u,v;y,T) = \frac{\#\{t_i \in [-T,T]; y(t_i) \le u, y(t_i^*) \le v\}}{\#\{t_i \in [-T,T]\}}. \tag{44}$$

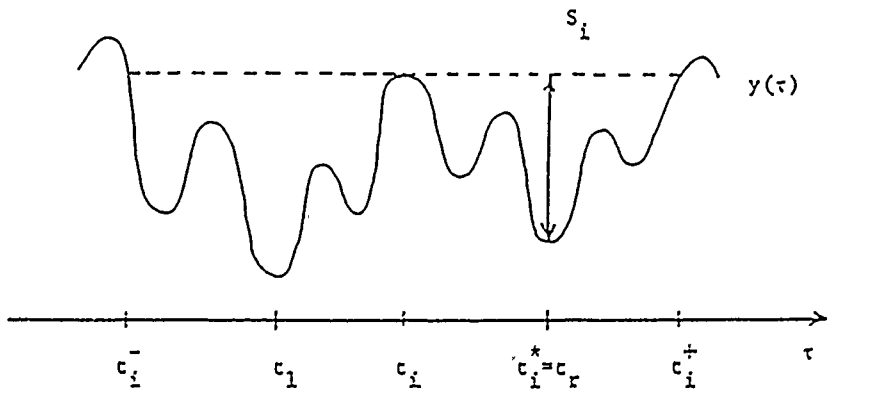

Figure 4. Definition of Rainflow cycle.

For some functions $y(\cdot)$, the empirical distribution $F_{M,m}(u,v;y,T)$ (44) diverges as $T \to +\infty$. However, when y is a sample path of an ergodic process a limit of $F_{M,m}(u,v;y,T)$, as $T \to +\infty$, exists almost surely and defines a bivariate distribution function $F^{erg}_{M,m}$, say. Obviously, since the RFC-amplitude $S \stackrel{D}{=} M - m$, once, knowing the ergodic distribution $F^{erg}_{M,m}$, we can evaluate the predictor of fatigue life (43). In the following we shall present an approximation of the ergodic RFC-distribution, based on the bounds for the absorptions times T_u, T_v, $u(s) = u, v(s) = v$, presented in previous sections.

Observe, that the marginal distribution of m is the same as the ergodic distributions of the height of local minima and for Gaussian processes can be given in an explicit formula. Hence, the evaluation of the ergodic distribution $F^{erg}_{M,m}$ is equivalent to calculation of $P^{erg}(M > u, m \leq v)$, i.e.

$$F^{erg}_{M,m}(u,v) = F^{erg}_{m}(v) - P^{erg}(M > u, m \leq v). \tag{45}$$

Now, using ergodic properties of marked point processes, see Leadbetter et al. [5], Chapter 10, the probability $P^{erg}(M > u, m \leq v)$ is given by

$$P^{erg}(M > u, m \leq v) = \frac{E[\#\{t_i \in [0,1]; y(t_i) > u, y(t_i^*) \leq v\}]}{E[\#\{t_i \in [0,1]\}]},$$

where t_i are the times of local maxima, see Definition 8. In [14], we have proved that the mean $E[\#\{t_i \in [0,1]; y(t_i) > u, y(t_i^*) \leq v\}]$ is equal to the mean number of u-downcrossings, by y, in the interval $[0,1]$, which are followed by a downcrossing of the level v without crossing the barrier u in between. More exactly, for a fixed u, v, let $\{s_i\}$, $s_i \geq 0$, be a sequence of downcrossings of the level u, then

$$\begin{aligned} P^{erg}(M > u, m \leq v) &= c \cdot \frac{E[\#\{s_i \in [0,1]; y(s_i + t) \text{ crosses the level } v \text{ before } u, t > 0\}]}{E[\#\{s_i \in [0,1]\}]} \\ &= c \cdot P(\xi_u(t) \text{ crosses the level } v \text{ before } u, t > 0), \end{aligned}$$

where ξ_u is the Slepian model process for y after u-downcrossing (35) and c is given by

$$c = \frac{E[\#\{s_i \in [0,1]\}]}{E[\#\{t_i \in [0,1]\}]} = \sqrt{\frac{\lambda_2}{\lambda_4}} \exp^{-u^2/2},$$

where $\lambda_4 = Var(y''(0))$, see (40).

Finally, by the definition of the variable T_u, with $u(s) = u, v(s) = v$, we have

$$c^{-1} \cdot P^{erg}(M > u, m \leq v) = \int_0^{+\infty} f_{T_v}(s)\, ds = 1 - \int_0^{+\infty} f_{T_u}(s)\, ds, \tag{46}$$

since, for Gaussian processes, the probability that ξ_u stays for ever between the finite barriers is zero. As in the previous section, we have to approximate the infinite region of integration in (46), by some finite interval. If the level u is relatively small, e.g. $u < \sigma$, $\sigma^2 = \mathrm{Var}(y(0))$, or the levels u, v are close to each other, then one can usually find a constant T_0, such that

$$\int_0^{T_0} [f_v^-(s;n) + f_u^-(s;n)]\, ds \geq 1 - \epsilon, \tag{47}$$

for some small positive ϵ, where f_u^-, f_v^- are lower bounds (19) for the densities of T_u, T_v, respectively. However, in the case of the high positive u and low negative v, the tails of densities of T_u and T_v become very long, consequently T_0 is also large. Hence, in order to find suitable T_0, we have to use bounds $f_u^-(s;n), f_v^-(s;n)$ with high values of n, which causes numerical difficulties. Consequently, in the following, we present an approximative method to evaluate the probability $P^{erg}(M > u, m \leq v)$ in the case when levels u, v are high and low, respectively.

As in the previous section, we assume that y is a zero-mean Gaussian ergodic process. For a positive constant T_0, denote by $P_u(T_0)$ and $P_v(T_0)$ the following truncated integrals

$$\begin{aligned} P_u(T_0) &= \int_0^{T_0} f_{T_u}(s)\, ds, \\ P_v(T_0) &= \int_0^{T_0} f_{T_v}(s)\, ds. \end{aligned} \tag{48}$$

By the definition of T_v, T_u variables the integral in (46) can be written as follows

$$\begin{aligned} \int_0^{+\infty} f_{T_v}(s)\, ds = P_v(T_0) + (1 - P_v(T_0) - P_u(T_0)) \cdot P(\xi_u(t) \text{ crosses the level } v \\ \text{before } u, t > T_0 \,|\, \xi_u(t) \text{ stays between } v, u \text{ for all } t, 0 < t \leq T_0). \end{aligned} \tag{49}$$

Obviously, if (47) is satisfied, i.e. $(1 - P_v(T_0) - P_u(T_0)) < \epsilon$, the second term in (49) is less than ϵ and can be disregarded.

It is well known that for Gaussian processes, see Leadbetter et. al. [5] for suitable conditions, the point processes of downcrossings of levels u and v converges to independent Poisson processes, as $u \to +\infty$ and $v \to -\infty$. Furthermore, by (34), if the covariance function $r(t)$ and its derivative $r'(t)$, of the process y, converge to zero as t goes to infinity, then for large t, we have $\xi_u(t) \overset{D}{\approx} y(t)$, where $\overset{D}{\approx}$ denotes approximative equality in distribution. Consequently, we propose to approximate the conditional probability in (49) by the corresponding probability evaluated for independent Poisson processes with the same crossing intensities as ξ_u, i.e.

$$P^{app}(T_0) = \int_{T_0}^{+\infty} \lambda_v^-(t) \cdot \exp^{-\int_{T_0}^{t} \lambda_v^-(s)+\lambda_u^+(s)\,ds}\, dt, \tag{50}$$

where the intensities λ_v^-, λ_u^+ are given by Rice formula, i.e.

$$\lambda_v^-(t) = E[\xi_u'(t)^- | \xi_u(t) = v] f_{\xi_u(t)}(v)$$

$$\lambda_u^+(t) = E[\xi_u'(t)^+ | \xi_u(t) = u] f_{\xi_u(t)}(u).$$

Now, for fixed T_0, by replacing the conditional probability in (49) by $P^{app}(T_0)$, we obtain an approximation $P(u, v; T_0)$, say, for the probability $P^{erg}(M > u, m \leq v)$, viz.

$$P(u, v; T_0) = P_v(T_0) + (1 - P_v(T_0) - P_u(T_0)) \cdot P^{app}(T_0). \tag{51}$$

Finally, by combining (45) and (51) we obtain an approximation of the joint distribution of maximum M and the RFC minimum m

$$F_{M,m}(u, v; T_0) = F_m^{erg}(v) - P(u, v; T_0). \tag{52}$$

We turn now to the numerical example. Let the covariance function of the process y be given by (36). Figure 5 shows the approximation $P(u, v; T_0)$ as a function of T_0, for $u = 2$ and $v = 0.5$, 0., -0.5, -1., -1.5, -2., -2.5, -3.. We can see that $P(u, v; T_0)$ (51) stabilizes very quickly, indicating that the constant T_0 can be chosen as low as 5, what substantially reduces the numerical effort to evaluate the probabilities $P_v(T_0), P_u(T_0)$ (48). Figure 6 shows the level curves of the approximation (52), $T_0 = 10$, of the distribution of (M, m) covariance (36).

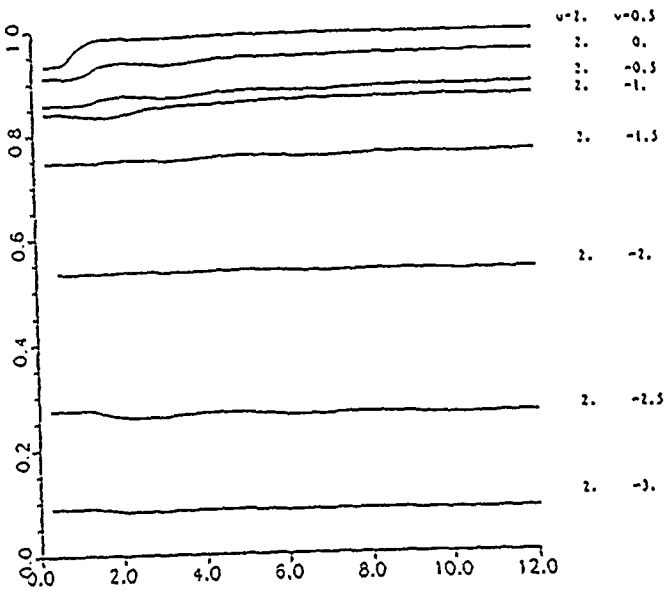

Figure 5. Approximations $P(u, v; T_0)$ (51), for $u = 2$ and $v = 0.5, 0., \ldots, -3.$, as a function of T_0, for covariance function (36).

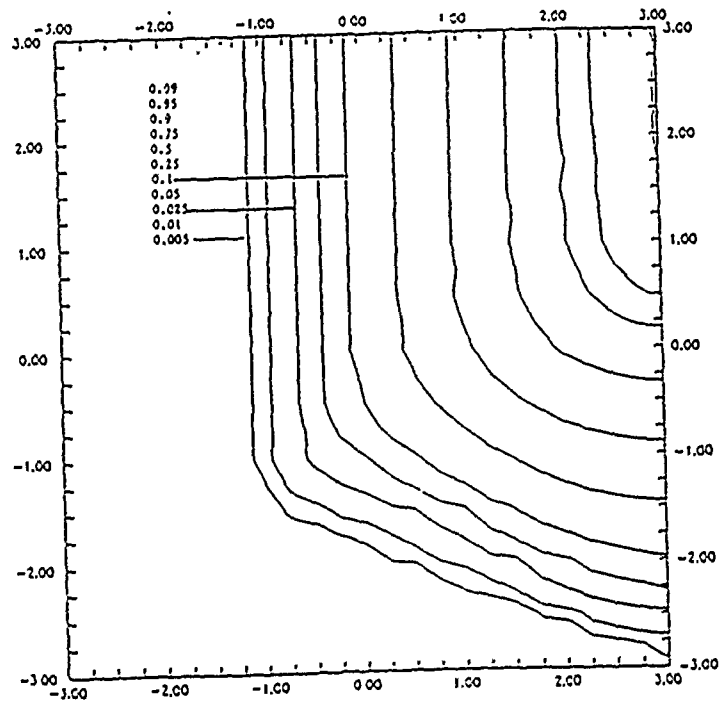

Figure 6. Isolines of approximative distribution (52) $F_{M,m}(u, v; T_0)$, $T_0 = 10$, of maximum M and the RFC minimum m, for covariance function (36).

Recently, Ford [4] and Nielsen [7] have proposed approximations of the joint distribution of

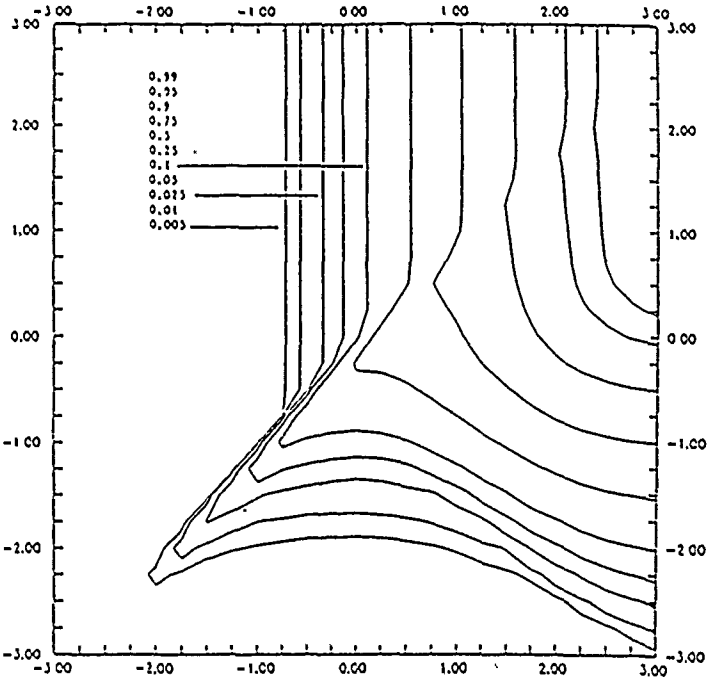

Figure 7. Isolines of approximative distribution (52) $F_{M,m}(u,v;T_0)$, $T_0 = 0$, of maximum M and the RFC minimum m, for covariance function (36).

(M,m) equivalent to $F_{M,m}(u,v;0)$. Since $F_{M,m}(u,v;0)$ approximation is based on the assumption that u-upcrossings and v-downcrossings of Slepian process ξ_u are independent Poisson point processes, this approximation can be accurate only for high positive u and low negative v, see Figure 7.

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