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ABILITY DISTRIBUTIONS

PATTERN PROBABILITIES AND QUASIDENSITIES

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December 1989

Prepared under Contract No. N00014-83K-0397, NR 150-518
and No. N00014-86K-0482, NR 4421546.

Sponsored by the Cognitive Science Program
Office of Naval Research.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a REPORT SECURITY CLASSIFICATION: Unclassified

1b. RESTRICTIVE MARKINGS:

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE:

3 DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release: distribution unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S): Measurement Series 89-2

5. MONITORING ORGANIZATION REPORT NUMBER(S):

6a. NAME OF PERFORMING ORGANIZATION: Michael V. Levine, Model Based Measurement Lab.

6b. OFFICE SYMBOL *(If applicable)*:

6c. ADDRESS *(City, State, and ZIP Code)*: University of Illinois, 210 Education Building, 1310 S. Sixth St., Champaign, IL 61820

7a. NAME OF MONITORING ORGANIZATION: Cognitive Science Program, Office of Naval Research

7b. ADDRESS *(City, State, and ZIP Code)*: Code 1142PT, 800 North Quincy St., Arlington, VA 22217-5000

8a. NAME OF FUNDING/SPONSORING ORGANIZATION:

8b. OFFICE SYMBOL *(If applicable)*:

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: N00014-83-K-0397, N00014-86-K-0482

8c. ADDRESS *(City, State, and ZIP Code)*:

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
61153N	RR04204	RR04204-01	NR 150-518 NR 4421546

11. TITLE *(Include Security Classification)*: Ability Distributions, Pattern Probabilities, and Quasidensities

12. PERSONAL AUTHOR(S): Levine, Michael V.

13a TYPE OF REPORT: Final Report

13b TIME COVERED: FROM ________ TO ______

14. DATE OF REPORT *(Year, Month, Day)*: 1989, December

15. PAGE COUNT: 40

16 SUPPLEMENTARY NOTATION:

17 COSATI CODES

FIELD	GROUP	SUB-GROUP
05	09	

18 SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*: Item response theory, latent trait theory, latent class theory, multiple choice tests, density estimation, distribution estimation, ability distributions, (continued)

19 ABSTRACT *(Continue on reverse if necessary and identify by block number)*

The quasidensity is a useful surrogate for the ability density in item response theory analyses of multiple choice tests. Like the ability density, it can be used to calculate the probability of sampling an examinee with a specified pattern of responses. Sometimes it is preferable to the density because it is continuous (densities need not be continuous), it is unique (two very different densities can give exactly the same pattern probabilities and all other expected values of random variables that are functions of item responses), it always exists (a discontinuous ability distributions has a quasidensity, but it does not have a density). Some large sample results are proven for quasidensity estimation. It is shown that the maximum likelihood quasidensity estimate (mle) is strongly consistent. The asymptotic distribution of the mle is derived. Some new results on the relation between latent class and latent trait models are also presented. It is shown that every item response model with a smooth density and smooth item response→(continued)

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION: Unclassified

22a NAME OF RESPONSIBLE INDIVIDUAL: Dr. Charles Davis

22b TELEPHONE *(Include Area Code)*: 202-696-4046

22c OFFICE SYMBOL: ONR 1142CS

DD Form 1473, JUN 86 — *Previous editions are obsolete.*

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

18. continued ...
nonparametric density estimation, foundations of measurement, formula score theory, Rasch model, logistic models, consistent estimation, asymptotic distribution, quasidensities, large sample theory, consistency, asymptotic normality.

19. continued ...
functions is isomorphic to the latent class model obtained with the same item response functions and a discrete ability distribution. An upper bound for the number of points is derived.

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Unannounced	☐
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Preface

This paper was originally intended as a report on some large sample results for quasidensity and pattern probability estimation, results that provide a foundation for our methods for ability distribution estimation and item response function estimation. While attempting to determine the generality of one of the results, I happened upon a promising new line of research. The new methods and results (Section Three) admittedly haven't been integrated into this report very well. The main result, relating models with a continuum of abilities to finite models, seems at least as important as the large sample results. (It asserts that every item response model with a smooth ability distribution and smooth item response functions is isomorphic to a latent class model obtained by replacing the ability distribution with a discrete distribution.) The reader primarily interested in the generality of latent class models may wish to skim Sections One and Two for notation and then read Section Three. Two separate papers eventually will be prepared for publication.

ABILITY DISTRIBUTIONS, PATTERN PROBABILITIES, AND QUASIDENSITIES

Introduction

This paper solves a problem closely related to ability distribution estimation, which is arguably the central problem of item response theory. The problem, quasidensity estimation, comes up when one needs to know the probability of sampling an examinee with a specified item response pattern from a very large pool of examinees. The situation in which item response functions are specified but nothing is known about the distribution of ability is considered in this paper.

If the ability distribution has a density, then the density can be used to calculate the probability of sampling an examinee with a specified pattern. A pattern's probability is simply the integral of the product of the pattern's likelihood function times the ability density.

It is shown that even if the ability distribution is a step function or some other distribution that doesn't have a density there is an essentially unique, continuous function that can be used in place of a density to compute pattern probabilities. The integral of the product of this function (the quasidensity) and any pattern's likelihood function is exactly equal to the pattern's probability.

When the ability distribution is unknown, estimates must be used. Quasidensity estimation is easier than ability distribution estimation because the quasidensity is identifiable, but the ability distribution is not. Many different ability distributions will fit large samples equally well (Levine, 1989). By contrast, the maximum likelihood quasidensity estimate is unique (Section II.2, below).

Quasidensity estimation is related to ability distribution estimation in two ways. First, under general conditions the indefinite integral of the quasidensity equals or closely approximates the cumulative distribution of ability. Second, the methods of quasidensity estimation have been generalized to obtain a nonparametric theory for ability density estimation.

As others (Lord, 1970; Samejima, 1981) have observed, item response function estimation is closely related to ability distribution estimation. The results in this paper are central to our current work on nonparametric item response function and option response function estimation.

A quasidensity estimation theory is developed in this paper. The quasidensity is represented as a linear combination of orthogonal functions. The set of linear combination coefficients is shown to be convex and compact. It is shown that the maximum likelihood estimate of the coefficients is strongly consistent. The asymptotic distribution of the coefficients is derived.

Some general results on ability distributions are also proven. For example, it is shown that for the most commonly used item response models, every ability distribution is equivalent to a distribution with only finitely many points of increase. An upper bound for the number of points of increase is obtained.

Section One

Quasidensities, Parameterizations, and Approximations

In many applications it is necessary to compute the probability of sampling an examinee with a specified item response function. For example, pattern probabilities are needed in situations in which it is important to decide whether because of cheating, language problems or an ill-advised test-taking strategy, an individual's test-taking behavior is so unlike other examinees' that his/her test score is virtually uninterpretable. With pattern probabilities a uniformly most powerful statistical test for faulty answer sheets can be computed (Levine and Drasgow, 1988).

In this section a general strategy for computing pattern probabilities is derived and discussed. We begin with some notation and a brief review of basic item response theory.

Section I.1: Terminology, Notation and Assumptions

Let $u^* = \langle u_1^*, u_2^*, \ldots, u_n^* \rangle$ denote a vector of ones and zeros indicating right and wrong answers to n test items. Sampled vectors are *locally independent* relative to a random variable θ if the conditional probability of sampling pattern u^* given any value of θ can be factored and written

$$\text{Prob}(u = u^* | \theta = t) = \prod_{i=1}^{n} \text{Prob}(u_i = u_i^* | \theta = t)$$

where u denotes the sampled vector, u_i is its *ith* component and t is one of the possible values of θ .

Usually in item response theory the *item response functions* $P_i(t) = \text{Prob}(u_i = 1 | \theta = t)$ are assumed to have some specific (generally logistic) functional form with values strictly between zero and one. The results in this paper (except for the last part of Section III) assume only that the item response functions are continuous functions with values that are strictly between zero and one.

Usually item response functions are defined over an unbounded range and applied over a finite range, typically the interval $[-3,3]$. The results in this paper only use item response function values over a finite range of abilities. Thus, throughout this paper the P_i are continuous functions with values strictly between zero and one that are defined on some finite closed interval $[c,d]$. This results in no loss of generality because the interval can be very large and because an unbounded ability continuum can be transformed into an interval. In applications, we make the interval $[c,d]$ big enough so that the assumption that all abilities are in $[c,d]$ is plausible.

This paper is concerned with *distributions on* $[c,d]$, i.e. distributions of random variables that are between c and d with probability one. The condition that a distribution function G is a distribution on $[c,d]$ can be expressed without explicitly referring to random variables as follows: for $t < c$, $G(t) = 1-G(d)$.

To summarize, the results to follow assume

1. local independence relative to a unidimensional random variable θ,
2. continuous item response functions defined on an interval $[c,d]$ and taking values strictly between zero and one, and
3. $\text{Prob}(c \le \theta \le d) = 1$.

Section I.2: The Canonical Space and its Quasidensities

The assumption that the P_i are strictly between zero and one implies that for every pattern u^*, the *pattern likelihood function* $\ell(u^*,\cdot)$ given by

$$\ell(u^*,t) = \mathrm{Prob}(u=u^*|\theta=t) = \prod_1^n P_i(t)^{u_i^*} [1 - P_i(t)]^{1-u_i^*}$$

will also be a continuous, positive function defined on $[c,d]$. By forming linear combinations of likelihood functions we obtain a finite dimensional real vector space which is called the test's *canonical space* (CS). Thus f is in the CS if and only if for some real constants a_ν

$$f(\cdot) = \sum_{\nu=1}^{2^n} a_\nu \ell(u_\nu^*,\cdot)$$

where $u_1^*, u_2^*, \ldots u_\nu^* \ldots$ is any enumeration of the 2^n possible item response patterns. Since the functions in the CS are continuous, an inner product for the CS is defined by $\langle f,g\rangle = \int_c^d f(t)g(t)\,dt$.

Note that since the likelihood functions may be linearly dependent, it may be possible to write a function as a linear combination of likelihood functions in several ways. The uniqueness referred to in the following result applies to functions, not vectors of coefficients (a_ν) .

I.1 <u>There is a unique function</u> g <u>in the CS such that for all response patterns</u> u^*

$$\mathrm{Prob}(u=u^*) = \int_c^d \ell(u^*,t)g(t)\,dt \ .$$

<u>Equivalently, for any distribution</u> G <u>on</u> $[c,d]$ <u>there is a unique</u> g <u>in the CS such that</u>

$$\int_c^d \ell(u^*,t)\,dG(t) = \int_c^d \ell(u^*,t)g(t)\,dt$$

<u>Proof</u>: The formula

$$h = \sum_{\nu} a_{\nu} \ell_{\nu} \rightarrow E[h(\theta)] = \sum_{\nu} a_{\nu} \mathrm{Prob}(u=u_{\nu}^{*})$$

defines a linear mapping on the CS since expectation is linear. Since every linear functional defined on an inner product space that is isomorphic to a Euclidean space can be written as an inner product, there is some g in the CS such that the mapping can be written

$$\sum_{\nu} a_{\nu} \ell_{\nu}(\cdot) \rightarrow \left< \sum_{\nu} a_{\nu} \ell_{\nu}(\cdot), g \right> .$$

In particular, if all the a's are zero except one, then for every u_{ν}^{*}

$$\mathrm{Prob}(u=u_{\nu}^{*}) = \left< \ell(u_{\nu}^{*}, \cdot), g \right> .$$

If $\bar{g}$ also satisfies these conditions then for all ν

$$\begin{aligned} 0 &= \left< \ell_{\nu}, g \right> - \left< \ell_{\nu}, \bar{g} \right> \\ &= \left< \ell_{\nu}, g - \bar{g} \right> . \end{aligned}$$

Thus $g-\bar{g} = 0$ because no nonzero element of a vector space can be orthogonal to all of the vector space's generators. //

The function g is called the *quasidensity* of G because it functions like a density in the calculation of pattern probabilities. In fact, it can be used in place of a density to calculate the expected value of any statistic that is a function of item responses (Levine, 1989, Section 2). Although the quasidensity integrates to one, it is generally not non-negative. A discussion of their properties can be found in (Levine, 1989).

When an orthonormal basis for the CS is available the quasidensity has a simple formula which is often used.

I.2: <u>If</u> θ <u>has distribution</u> G <u>where</u> G <u>is a distribution on</u> [c,d] <u>and</u>

$$h_0, h_1, \ldots h_J$$

<u>is an orthonormal basis for the canonical space</u>, <u>then the quasidensity for</u> G <u>is</u>

$$g(\cdot) = \sum_{J=0}^{J} E[h_j(\theta)]h_j(\cdot) \quad .$$

<u>Proof</u>: Since $\{h_j\}_{j=0}^{J}$ is an orthonormal basis, each pattern likelihood function satisfies $\ell(u_\nu^*, t) = \Sigma_j <\ell_\nu, h_j> h_j(t)$. Consequently

$$\begin{aligned} P(u=u_\nu^*) &= \int_c^d \Sigma_j <\ell_\nu, h_j> h_j(t)\, dG(t) \\ &= \Sigma_j <\ell_\nu, h_j> E[h_j(\theta)] \\ &= \Sigma_j \int_c^d \ell_\nu(t) h_j(t) dt\, E[h_j(\theta)] \\ &= \int_c^d \ell(u_\nu^*, t)\, \Sigma_j E[h_j(\theta)] h_j(t) dt. \end{aligned}$$

From uniqueness proven in I.1 it follows that $g(\cdot) = \Sigma_j E[h_j(\theta)]h_j(\cdot)$. //

The value of the quasidensity in studying pattern probabilities derives from the following obvious but very useful fact:

I.3: <u>If</u> $\{h_j\}_{j=0}^{J}$ <u>is a basis for the CS and the quasidensit</u>[illegible] <u>distribution of</u> θ <u>satisfies</u>

$$g(\cdot) = \sum_{0}^{J} \pi_j h_j(\cdot)$$

<u>then</u>

$$\text{Prob}(u = u_\nu^*) = \sum_{0}^{J} \pi_j <h_j, \ell_\nu> \ .$$

Thus, each pattern probability can also be represented as an inner product of a known vector depending only on the pattern and a vector that must be estimated.

Section I.3: Approximations and Bases

For applications, the size of J is important. For the Rasch model $(P_i(t) = [1+e^{-(t-b_i)}]^{-1})$ J cannot be larger than the number of items, n (Ievine, 1989, Section 2). On the other hand for the three parameter logistic model $(P_i(t) = c_i + (1-c_i)[1+e^{-a_i(t-b_i)}]^{-1})$, J may be as large as 2^n. Fortunately by careful choice of the functions h_j, approximations can be obtained that have very few terms but are still accurate in one sense or another. Two examples follow.

For a first example suppose it is desirable to keep the total squared error

$$\sum_{\nu} [\mathrm{Prob}(u = u_\nu^*) - \text{approximated } \mathrm{Prob}(u = u_\nu^*)]^2$$

small. A basis can be obtained by analyzing the function of two variables

$$H(s,t) = \prod_{i=1}^{n} \{P_i(s)P_i(t) + [1-P_i(s)][1-P_i(t)]\}$$

defined for $c \le s,\ t \le d$. By solving the functional equation

$$\int_c^d H(s,t)h(s)ds = \lambda h(t)$$

one obtains a maximal set of orthonormal functions h_j in the CS and positive constants $\lambda_0 \ge \lambda_1 \ge \ldots \lambda_J > 0$ such that

$$H(s,t) = \sum_{j=0}^{J} \lambda_j h_j(s) h_j(t) .$$

These functions can be shown to form an orthonormal basis for the CS (Levine 1989). From this fact and the identity

$$H(s,t) = \sum_{v=1}^{2^n} \ell(u_\nu^*,s)\ell(u_\nu^*,t)$$

it follows that the total squared error with a K<J term ap imation satisfies

$$\sum_{\nu} [\mathrm{Prob}(u = u_{\nu}^{*}) - \sum_{j<K} \pi_j <\ell_{\nu},h_j>]^2 = \sum_{j\geq K} \lambda_j <g,h_j>^2$$
$$< \lambda_K \int_c^d g^2(t)dt$$

when $\pi_j = <g,h_j>$.

This relation is important because for all tests we have analyzed, the λ_j very rapidly decrease to zero. Typically, K=15 provides very accurate least squares pattern probability approximation.

To introduce a second and final example concerning the choice of a CS basis, suppose it is important to control the maximum absolute error. In addition suppose an approximation of the distribution function G is available. Then it is often possible to select an orthonormal basis h_j such that $E[h_j(\theta)]$ will be small for large j . Of course, small $E[h_j(\theta)]$ guarantees that the finite sum $\sum_{j\geq K} |E[h_j(\theta))|$ is also small. This is important because for $\pi_j = <g,h_j> = E[h_j(\theta)]$ the K term approximation satisfies

$$|\mathrm{Prob}(u - u_{\nu}^{*}) - \sum_{j<K} \pi_j <\ell_{\nu},h_j>| = \sum_{j\geq K} |\pi_j <\ell_{\nu},h_j>|$$
$$\leq (\sum_{j\geq K} |\pi_j|)^{1/2} (\sum_{j\geq K} |\pi_j| <\ell_{\nu},h_j>^2)^{1/2}$$
$$\leq (\sum_{j\geq K} |\pi_j|)^{1/2} \max_{j\geq K} |\pi_j|^{1/2} (\sum_{j\geq K} <\ell_{\nu},h_j>^2)^{1/2}$$
$$\leq (\sum_{j\geq K} |\pi_j|)^{1/2} \max_{j\geq K}|\pi_j|^{1/2} (\int_c^d \ell_{\nu}^2(t)dt)^{1/2}$$
$$\leq \max_{j\geq K}|\pi_j|^{1/2} (\sum_{j\geq K} |\pi_j|)^{1/2} (d-c)^{1/2} \max_{c\leq t\leq d} \ell_{\nu}(t)$$

This paper is concerned with using maximum likelihood estimation of the π_j, from sampled patterns to obtain approximations of pattern probabilities. It will be shown that the maximum likelihood estimate for the vector π is strongly consistent and asymptotically efficient. In addition some results about the set of vectors π corresponding to distributions are proven.

Section Two

Uniqueness and Consistency of the Maximum Likelihood Estimate

In the remainder of this paper F is used to denote the unknown distribution of the ability random variable θ , $\{h_j\}_{j=0}^{J}$ is a fixed orthonormal basis for the CS, and π is the vector of expectations with jth coordinate $\pi_j = E[h_j(\theta)]$. Thus the pattern probability for u^* can be written

$$P_\pi(u^*) = \Sigma <\ell(u^*,\cdot),h_j> \pi_j$$

or with the abbreviation $\beta_j(u^*) = <\ell(u^*,\cdot),h_j>$, $P_\pi(u^*) = \beta(u^*)\cdot\pi$. If G is the distribution of θ or any other distribution on $[c,d]$ and $g_{\pi'}(\cdot) = \Sigma\pi_j' h_j(\cdot)$ is its quasidensity then the pattern probabilities obtained by using G in place of F are given by

$$P_{\pi'}(u^*) = \int_c^d \ell(u^*,t)\ dG(t)$$

$$= \beta(u^*)\cdot\pi' .$$

This section begins the task of estimating F's quasidensity, or equivalently the vector π , from a sample of patterns u^* .

Section II.1: Distributions Viewed as Points in a Convex, Compact Subset of Euclidean Space

Procedures for recovering the unknown π_j from samples of observed patterns have been developed. Each requires the maximization of some continuous function defined over a set A of vectors corresponding to quasidensities

$$A = \{\pi' \text{ in } E^{J+1} : \Sigma \pi_j' h_j(\cdot) \text{ is the quasidensity of at least one distribution on } [c,d]\} .$$

The set A has three properties that greatly simplify maximization:

i. Convexity: the line segment connecting any two points in

A is also in A .

ii. Boundedness: There is a constant κ such that for all π' in A $|\pi'-\pi| = \langle\pi'-\pi,\pi'-\pi\rangle^{1/2} \le \kappa$.

iii. Closure: Every continuous, bounded function defined on A has a maximizer in A .

The proof requires a simple result that is repeatedly used elsewhere.

II.1: <u>The vectors</u> $\{\beta(u_\nu^*)\}_{\nu=1}^{2^n}$ <u>span a</u> J+1 <u>dimensional vector space</u>. <u>Equivalently</u>, <u>for any choice of positive constants</u> w_ν <u>the matrix</u>

$$\sum_{\nu=1}^{2^N} \beta(u_\nu^*)\beta(u_\nu^*)^T w_\nu$$

<u>is positive definite</u>.

<u>Proof</u>: Let $\ell_{\nu_1},\ell_{\nu_2} \ldots \ell_{\nu_{J+1}}$ be likelihood functions forming a basis for the CS. Since $\ell_{\nu_i}(\cdot) = \Sigma_j\ \beta_j(u_{\nu_i}^*)\ h_j(\cdot)$, linear independence of the ℓ_{ν_i} implies linear independence of the $\beta(u_{\nu_i}^*)$. Thus the set $\{\beta(u_\nu^*)\}_{\nu=1}^{2^n}$ of J+1 vectors contains J+1 linearly independent vectors. //

II.2: <u>The set</u> A <u>of coordinates of distributions on</u> [c,d] <u>is convex</u>, <u>closed and bounded</u>.

<u>Proof</u>: (i) Convexity: For π^1 and π^2 in A and $0 \le \epsilon \le 1$ let G_1 and G_2 be distributions on [c,d] with quasidensities g_{π^1} and g_{π^2} respectively. Since for any positive ϵ , $G_3 = \epsilon G_1 + (1-\epsilon)G_2$ is also a distribution on [c,d] and since for any h in the CS $\int_c^d h(t)\ d[\epsilon G_1(t)+(1-\epsilon)G_2(t)] = \epsilon \int_c^d h(t)dG_1(t) + (1-\epsilon) \int_c^d h(t)dG_2(t)$, it follows from I.1 that $g_{\epsilon\pi^1+(1-\epsilon)\pi^2}$ is the quasidensity of G_3 . Thus a convex combination of points in A is in A .

(ii) Boundedness: Let π' be any vector in A other than π . Let, $\bar{\pi}'$ be the intersection of the unit sphere about π and the ray from π passing through π' . Thus $|\bar{\pi}' - \pi| = 1$ and $\bar{\pi}' = [\pi' - \pi]/k + \pi$, (or equivalently, $\pi' = \pi + k(\bar{\pi}' - \pi)$) for $k=|\pi-\pi'|>0$. Since $\pi'\cdot\beta(u^*) = P_{\pi'}(u^*)$ and $\pi\cdot\beta(u^*) = P_\pi(u^*)$ are probabilities,

$$0 \le P_{\pi'}(u^*) = [\pi + k(\bar{\pi}' - \pi)]\cdot\beta(u^*) \le 1$$

and

$$-1 \le -P_\pi(u^*) \le k(\bar{\pi}' - \pi)^T\beta(u^*) \le 1-P_\pi(u^*) \le 1 .$$

After squaring and summing over all 2^n patterns we obtain

$$0 \le k^2(\bar{\pi}' - \pi)^T \sum_{u^*} \beta(u^*)\beta(u^*)^T (\bar{\pi}' - \pi) \le 2^n .$$

But in II.1, $\sum_{u^*} \beta(u^*)\beta(u^*)^T$ was shown to be positive definite. Since $|\bar{\pi}' - \pi| = 1$, the expression multiplying k^2 must be at least as large as the smallest eigenvalue of this matrix. Thus for $\kappa^2 = 2^n/[\text{smallest eigenvalue}]$, $|\bar{\pi}' - \pi| \le \kappa$, and A is bounded.

(iii) Closure: Let $\{\pi^n\}$ be a Cauchy sequence in A . Since A is bounded, the sequence converges to some π' in E^{J+1} . To show π' is in A , let $\{G_n\}$ be any sequence of distributions on $[c,d]$ such that g_{π^n} is the quasidensity of G_n . By Helly's theorem, $\{G_n\}$ contains a subsequence $\{G_{m(n)}\}$ such that $G_{m(n)}$ converges to some distribution G on $[c,d]$ at every point of continuity of G . Since each h_j is continuous, $\int_c^d h_j(t)\, dG_{m(n)}(t) \to \int_c^d h_j(t)\, dG(t)$. Thus, by I.2 $\pi_j^{m(n)} \to \int_c^d h_j(t)\, dG(t)$. Since $\{\pi^{m(n)}\}$ is a subsequence of $\{\pi^n\}$, $\pi_j^{m(n)} \to \int_c^d h_j(t)\, dG(t) = \pi_j'$, and $g_{\pi'}$ is the quasidensity of G . //

Note that since the CS is also a metric space with distance $(g,h) =$

$\langle g-h,g-h\rangle^{1/2}$ and since $\pi' \to g_{\pi'}$ is an isometry, it follows that the subset of the CS corresponding to quasidensities of distributions on $[c,d]$ is also convex and compact.

Section II.2: Uniqueness of the Maximum Likelihood Estimate

Consider now drawing a random sample of N patterns

$$u_1^*,\ u_2^*,\ \ldots\ u_a^*,\ \ldots\ u_N^*$$

and attempting to recover the quasidensity g_π by maximizing the sample likelihood function $\prod_{a=1}^{N} P_{\pi'}(u_a^*)$ defined for vectors π' in A or its logarithm

$$L_N(\pi') = \sum_{a=1}^{N} \log \pi' \cdot \beta(u_a^*) \ .$$

It will be shown that if the sample is large enough that L_N almost surely has a unique maximum $\hat{\pi}_N$.

II.3: <u>With probability one</u> $L_N(\cdot)$ <u>eventually has a unique maximizer in</u> A .

<u>Proof</u>: For each pattern u^* and vector π' in A , $P_{\pi'}(u^*) = \pi' \cdot \beta(u^*)$ is positive. Therefore, $L_N(\cdot)$ is defined and continuous on A . Since A is compact, $L_N(\cdot)$ has at least one maximizer in A . Suppose $\pi' \neq \pi''$ both maximize A . Since the line segment connecting two points of A is entirely in A , a function of one variable is defined by

$$p(\epsilon) = L_N[\pi' + \epsilon(\pi' - \pi'')]$$

for $0 \leq \epsilon \leq 1$. From the formula for L_N and the fact the π' and π'' are maximizers, p has 2 continuous derivatives, and $p(0) = p(1)$. Since the second derivative of p is the negative of a sum of squares

$$p''(\epsilon) = -\sum_{a=1}^{N} w_a(\epsilon)[(\pi' - \pi'') \cdot \beta(u_a^*)]^2$$

for $w_a(\epsilon) = \{P_{\pi'}(u_a^*) + \epsilon[P_{\pi''}(u_a^*) - P_{\pi'}(u_a^*)]\}^{-2} > 0$ and $\max_\epsilon\{p(\epsilon)\} = p(0) = p(1)$, p must be constant. Thus, the second derivative of p evaluated at, say, $\epsilon=.5$ must be zero. However,

$$0 = -\sum_{a=1}^{N} w_a(.5)\ (\pi'-\pi'')^T \beta(u_a^*)\beta(u_a^*)^T(\pi'-\pi'')$$

implies that the N vectors $\beta(u_a^*)$ span a subspace of dimension less than $J+1$. Since each of the 2^n response patterns has positive probability of being sampled and since by II.1 the full set spans a space of dimension $J+1$, with probability one eventually a linearly independent set of $J+1$ patterns will be sampled, and the maximizer will henceforth be unique. //

Section II.3: Strong Consistency of the Maximum Likelihood Estimate

In order to study the asymptotic behavior of $\hat{\pi}_N$, the maximum likelihood estimate, it is convenient to have an open set containing A on which any L_N can be extended to a differentiable function. To this end we choose a positive number d such that if a vector x is within distance d of at least one point of A , then $x \cdot \beta(u^*) \geq 0$ for all patterns u^* .

II.4: <u>Let</u> $d = \inf \operatorname{UNION}_{u^*} \{|x-\pi'| : \pi' \text{ is in } A \text{ and } x\cdot\beta(u^*)=0\}$ <u>and</u>

$$A^+ = \{x : \text{ for some } \pi' \text{ in } A ,\ |x-\pi'|<d\} .$$

<u>Then</u> A^+ <u>is an open set containing</u> A <u>on which the formula</u>

$$\sum_{a=1}^{N} \log x\cdot\beta(u_a^*)$$

<u>extends</u> L_N <u>to a differentiable function defined on</u> A^+ .

<u>Proof</u>: It remains only to show d is positive. Since A is compact, for each u^* the set $\{|x-\pi'| : x\cdot\beta(u^*) = 0 \text{ and } \pi' \text{ is in } A\}$ has a positive minimum. Thus d is the minimum of 2^n positive numbers. //

If there is some π' in **A** such that $P_{\pi'}(\cdot)$ assigns exactly the same probabilities to patterns as $P_{\pi}(\cdot)$ then it will not be possible to prove $\hat{\pi}_N$ converges to π . Thus the following intrinsically important result is needed.

II.5: <u>If</u> $\pi' \neq \pi''$ <u>and both are in</u> A , <u>then for at least one pattern</u> u^* , $P_{\pi'}(u^*) \neq P_{\pi''}(u^*)$.

<u>Proof</u>: If $P_{\pi'}(\cdot) = P_{\pi''}(\cdot)$, then for all u^*

$$0 = \beta(u^*) \cdot (\pi' - \pi'') \ .$$

Since from II.1 the $\beta(u^*)$ span a J+1 dimensional space, the J+1 vector π' must equal π'' . //

Finally, strong consistency for the maximum likelihood estimate can be proven by an argument Wald used to prove the consistency in a different context (Wald, 1949).

II.6: (Strong consistency of the mle) <u>With probability</u> 1 , $\hat{\pi}_N$ <u>converges to</u> π .

<u>Proof</u>: Using the inequality $\log x < x-1$ for $x \neq 1$ and II.5 it follows that for $\pi' \neq \pi$, $E \log P_{\pi'}(u) < E \log P_{\pi}(u)$. For if the set $D = \{ u^* : P_{\pi'}(u^*) \neq P_{\pi}(u^*) \}$ is not empty then

$$\begin{aligned}
E\,[\log P_{\pi'}(u)] - E[\log P_{\pi}(u)] & \\
&= E \log[P_{\pi'}(u) \,/\, P_{\pi}(u)] \\
&= \sum_{u^* \text{ in } D} P_{\pi}(u^*) \log\,[P_{\pi'}(u^*)/P_{\pi}(u^*)] \\
&< \sum_{u^* \text{ in } D} P_{\pi}(u^*)\,[P_{\pi'}(u^*)/P_{\pi'}(u^*) - 1] \\
&= \sum_{D} P_{\pi'}(u^*) - \sum_{D} P_{\pi}(u^*)
\end{aligned}$$

$$= \sum_{D} P_{\pi'}(u^*) - [1 - \sum_{u^* \text{ not in } D} P_{\pi}(u^*)]$$

$$= 0$$

To obtain a finite open covering of some subsets of A, note that $x \cdot \beta(u^*)$ is positive for x in A^+. For $\pi' \neq \pi$ let $s(u^*,\pi',\rho) = \sup\{\pi'' \cdot \beta(u^*) : \pi''$ is in A^+ and $|\pi'-\pi''| < \rho\}$. Since for π'' in A, π'' in A^+ and $|\pi'-\pi''| < \rho$

$$\begin{aligned} \pi'' \cdot \beta(u^*) &= [\pi' + (\pi''-\pi')] \cdot \beta(u^*) \\ &\leq P_{\pi'}(u^*) + |\pi''-\pi'||\beta(u^*)| \\ &< P_{\pi'}(u^*) + \rho|\beta(u^*)| , \end{aligned}$$

$\log s(u^*,\pi',\rho) \leq \log P_{\pi'}(u^*) + \rho|\beta(u^*)| / P_{\pi'}(u^*)$. Consequently

$$\begin{aligned} E \log s(u,\pi',\rho) &\leq E \log P_{\pi'}(u) + \rho E[|\beta(u)| / P_{\pi'}(u)] , \\ &= E \log P_{\pi}(u) \\ &\quad - [E \log P_{\pi}(u) - E \log P_{\pi'}(u)] \\ &\quad + \rho E[|\beta(u)| / P_{\pi'}(u)] \end{aligned}$$

and for each $\pi' \neq \pi$ in A, a positive $\rho(\pi')$ less than d can be selected so that

$$E \log s(u,\pi',\rho(\pi')) < E \log P_{\pi}(u) .$$

Let B be a closed subset of A not containing π. To show that with probability one

$$\sup_{\pi'' \text{ in } B} \prod_{a=1}^{N} P_{\pi''}(u_a^*)/P_{\pi}(u_a^*)$$

tends to zero as N tends to infinity, consider the open covering of A formed by the sets

$$B(\pi') = \{\pi'' \text{ in } A^+ : |\pi''-\pi'| < \rho(\pi')\} .$$

Since B is also compact $\pi'_1, \pi'_2, \ldots \pi'_m$ can be selected such that $B(\pi'_1)$, $B(\pi'_2), \ldots B(\pi'_m)$ covers B . By the strong law of large numbers,

$$\frac{1}{N} \sum_{a=1}^{N} \{\log s(u_a^*, \pi'_i, \rho(\pi'_i)) - \log P_\pi(u_a^*)\}$$

almost surely tends to $E \log s(u, \pi'_i, \rho(\pi'_i)) - E \log P_\pi(u) < 0$.

Consequently with probability one,

$$\sum_{a=1}^{N} \log s(u_a^*, \pi'_i, \rho(\pi'_i)) / P_\pi(u_a^*) \to -\infty$$

and $\lim_N \sup_{\pi'' \text{ in } B_i} \prod_{a=1}^{N} \pi'' \cdot \beta(u_a^*)/P_\pi(u_a^*) = 0$. Since

$$0 < \sup_{\pi'' \text{ in } B} \prod_{a=1}^{N} P_{\pi''}(u_a^*)$$

$$\leq \sum_{i=1}^{m} \sup_{\pi'' \text{ in } B_i} \prod_{a=1}^{N} \pi'' \cdot \beta(u_a^*) ,$$

with probability one

$$\lim_N \sup_{\pi'' \text{ in } B} \prod_{a=1}^{N} P_{\pi''}(u_a^*) / \prod_{a=1}^{N} P_\pi(u_a^*) = 0.$$

Finally, to show $\hat{\pi}_N$ converges almost surely to π , it is shown that for any positive ϵ , with probability one, $|\hat{\pi}_N - \pi|$ is eventually less than ϵ .

Let B be the closed set excluding π ,

$$B = \{\pi' \text{ in } A : |\pi' - \pi| \geq \epsilon \} .$$

With probability one

$$\sup_{\pi' \text{ in } B} \prod_{a=1}^{N} P_{\pi'}(u_a^*) / \prod_{a=1}^{N} P_\pi(u_a^*)$$

is less than one for sufficiently large N . Thus with probability one $\hat{\pi}_N$ is eventually in the complement of B , i.e., $|\hat{\pi}_N - \pi|$ is eventually less than ϵ . //

Section III
A Dichotomy for Ability Distributions

A common starting point for studying the asymptotic distribution of maximum likelihood estimates is the family of likelihood equations

$$0 = \frac{\partial}{\partial \pi_j} L_N(\hat{\pi}_N) \qquad j=0,1, \ldots J .$$

It turns out that these equations are false for our current formulation of the estimation problem.

The equations are valid if $\hat{\pi}_N$ is the maximum of L_N over an open $J+1$ dimensional subset of **A** . But **A** has no $J+1$ dimensional subsets because, as shown below, it is a J dimensional subset of a $J+1$ dimensional space.

In this section **A** is reparameterized as a J dimensional set, i.e., the points of **A** are expressed as functions of J numbers. The reparameterization suggests a dichotomy of the distributions on $[c,d]$. We distinguish "regular" distributions that correspond to points in the interior of the new set of parameters for **A** and "irregular" distributions that correspond to boundary points. The distinction is important because in this paper the asymptotic distribution of the mle is worked out in detail only for regular distributions.

In the process of attempting to show that the irregular distributions were pathological and safe to ignore a surprising result was obtained. It was found that for the most popular item response models, *every* distribution is equivalent to some discrete probability distribution on at most $J+1$ points.

Section III.1: Reparameterization

Since $1 = \Sigma_{u^*} P_{\pi'}(u^*) = \pi'\cdot\Sigma_{u^*} \beta(u^*)$ every vector π' in A satisfies the equation $\pi'\cdot\bar{\beta} = 1$ for $\bar{\beta} = \Sigma_{u^*} \beta(u^*)$. Thus A is contained in a J dimensional subset of E^{J+1} and no point of A has an open neighborhood contained in A .

To reparameterize A let $\{\bar{\beta}/|\bar{\beta}|, z^1, z^2, \ldots z^J\}$ be an orthonormal basis for E^{J+1} and Z be the $(J+1)\times J$ matrix

$$Z = [z^1, z^2, \ldots z^J] .$$

Thus $x \to Z^T x$ maps the set of vectors orthogonal to $\bar{\beta}$ one-to-one onto E^J . Since $x \to Z^T x$ is also linear, it follows that for any fixed π^0 in A

$$\pi' \to Z^T(\pi' - \pi^0)$$

maps A one-to-one, onto some convex, compact subset of E^J and that each π' in A can be expressed as

$$\pi' = \pi^0 + Zs$$

for exactly one J vector s in a convex, compact subset of E^J .

Let $B = \{Z^T(\pi' - \pi^0) : \pi' \epsilon A\}$ be the set of possible J vectors s . It will be shown that B is J dimensional so that distributions on [c,d] can be classified into two non-empty sets: regular distributions with s in the interior of B and irregular distributions corresponding to vectors on the boundary of B . Before resuming the study of the distribution of the maximum likelihood estimate some facts about regular and irregular distributions will be proven.

Section III.2: Regular and Irregular Distributions

A distribution on $[c,d]$, its quasidensity $g_{\pi'}$, and its coordinate vector π' will be called *regular* if for some $\rho>0$ the J-dimensional open ball centered at π'

$$\{\pi'' \text{ in } E^{J+1} : \pi''\cdot\bar{\beta} = 1 \text{ and } |\pi'' - \pi'|<\rho\}$$

is a subset of A . If a distribution is not regular, then it (and its quasidensity and coordinate vector) will be called *irregular.* Equivalently, a distribution with quasidensity $g_{\pi'}$ is regular if and only if $Z^T(\pi'-\pi^0)$ is an interior point of B .

The uniform distribution and most distributions expected in applications are regular. However, if the pattern likelihood function $\ell(u^*,t)$ is unimodal with maximum at t_0 in $[c,d]$, then the unit step function at t_0 is an irregular distribution on $[c,d]$. The first result shows that some distributions are regular, i.e., that B is J dimensional.

III.1 <u>If</u> G <u>is a distribution on</u> $[c,d]$ <u>with a positive quasidensity and if for</u> $c<t<d$ <u>the quasidensity of</u> G <u>evaluated at</u> t <u>equals the derivative of</u> G <u>at</u> t <u>then</u> G <u>is regular</u>. <u>In particular</u>, <u>the uniform distribution is regular</u>.

<u>Proof</u>: If $\frac{d}{dt}G(t) = \sum_0^J \pi'_j h_j(t) = g_{\pi'}(t) > 0$ for $c<t<d$, and g_π is also positive at c and d , then $\min_t g_{\pi'}(t) > 0$. If $|\pi''-\pi'|_\infty = \max_j |\pi''_j - \pi'_j|$ is sufficiently small, then $|g_{\pi''-\pi'}(t)| \le 1/2 \min_t g_{\pi'}(t)$. Consequently for sufficiently small $|\pi''-\pi'|_\infty$

$$\begin{aligned} g_{\pi''}(t) &= g_{\pi'}(t) + g_{\pi''-\pi'}(t) \\ &\ge \min_t g_{\pi'}(t) - \max_t |g_{\pi'-\pi''}(t)| \end{aligned}$$

>0 .

Since $|\ |_2$ and $|\ |_\infty$ determine the same topology on E^{J+1} , for some $\rho_0>0$, $|\pi''-\pi'|_2 < \rho_0$ implies $g_{\pi''}(t)>0$ for t in $[c,d]$.
To show $|\pi''-\pi'|<\rho_0$ and $\pi''\cdot\bar{\beta}=1$ imply π'' is in A it suffices to show that these assumptions imply that the function $\bar{G}$

$$\bar{G}(x) = \begin{cases} 0 & \text{if } c < x \\ \int_c^x g_{\pi''}(t)dt & \text{if } c \le x \le d \\ 1 & \text{if } d < x \end{cases}$$

is a distribution on $[c,d]$ and $g_{\pi''}$ is its quasidensity. The only nontrivial step in verifying that $\bar{G}$ is a distribution on $[c,d]$ is showing $\bar{G}(d)=1$. Since for all t , $\Sigma_{u^*} \ell(u^*,t) = 1$, $\bar{G}(d) = \int_c^d g_{\pi''}(t)dt = \int_c^d \Sigma_{u^*} \ell(u^*,t)g_{\pi''}(t)dt = \Sigma_j <\Sigma_{u^*} \ell(u^*,\cdot),h_j>\pi''_j = \bar{\beta}\cdot\pi'' = 1$. Consequently, $\bar{G}$ is a distribution on $[c,d]$, $g_{\pi''}$ is the restriction of its probability density to $[c,d]$, $g_{\pi''}$ is its quasidensity (by the uniqueness of quasidensities in I.1), and π'' is in A . In particular, since $1 = \Sigma_{u^*} \ell(u^*,t)$ is in the CS, the uniform density $g(t) = [d-c]^{-1}$ is in the CS, and the uniform distribution is regular. //

The next result gives examples of regular distributions that do not have continuous densities. It shows how to approximate any distribution on $[c,d]$ with a regular distribution.

III.2 <u>If $t_0<t_1\ldots <t_J$ are in $[c,d]$ and the vectors $\pi^i = \left(h_j(t_i)\right)$ are linearly independent then for any vector α in E^{J+1} if</u>

$$\alpha_j > 0 \quad j = 0, \ldots J$$

$$\Sigma_j\alpha_j = 1$$

<u>then the discrete distribution function</u>

$$G(t) = \sum_{(i:\ t_i\le t)} \alpha_i$$

<u>is regular</u>. <u>More generally if</u> $G_0, G_1, \ldots G_i \ldots G_J$ <u>are distributions on</u> $[c,d]$ <u>with quasidensities</u> g_{π^i} <u>and the</u> π^i <u>are linearly independent</u>, <u>then</u> $\alpha>0$ <u>and</u> $\Sigma\, \alpha_i=1$ <u>imply that</u> $\Sigma_i \alpha_i G_i$ <u>is a regular distribution on</u> $[c,d]$.

<u>Proof</u>: Clearly each for vector $\alpha>0$ such that $\Sigma\, \alpha_i=1$ the convex combination $\Sigma_i\alpha_i\, G_i(\cdot)$ of distributions G_i on $[c,d]$ is a distribution on $[c,d]$. Let $\alpha^0>0$ be the vector of coefficients of any such combination. A value of ρ will be computed to show $\Sigma_i\, \alpha_i^0\, \pi^i$ is regular. For any α

$$|\Sigma_i\, \alpha_i^0\, \pi^i - \Sigma_i\, \alpha_i\, \pi^i\, |^2 = \sum_{ij} (\alpha_i^0 - \alpha_i)\, \pi^i \pi^{j^T} (\alpha_j^0 - \alpha_j)$$

$$= (\alpha^0 - \alpha)^T\, Q(\alpha^0 - \alpha)$$

for positive definite or positive semidefinite Q . Since the π^i are linearly independent, Q is definite. Consequently $|\Sigma_i\, \alpha_i^0\, \pi^i - \Sigma_i\, \alpha_i\, \pi^i| \geq |\alpha^0 - \gamma|\epsilon^{1/2}$, where $\epsilon>0$ is the smallest eigenvalue of Q . Since the $J+1$ linearly independent π^i form a basis for E^{J+1}, and for any π' in E^{J+1}, there is a unique α' in E^{J+1} such that $\pi' = \Sigma\, \alpha_i'\pi^i$. If

$$|\Sigma_i\, \alpha_i^0\, \pi^0 - \pi'| < \frac{1}{2}\, \epsilon^{1/2} \min_i(\alpha_i^0)$$

then

$$|\alpha^0 - \alpha'| < \frac{1}{2} \min(\alpha_i^0) ,$$

$$\max|\alpha_i^0 - \alpha_i'| < \frac{1}{2} \min(\alpha_i^0) ,$$

and

$$0 < \alpha_i' \quad \text{for} \quad i = 0, 1, \ldots J .$$

Furthermore, if $\pi'\cdot\bar{\beta} = 1$, then

$$1 = (\sum_i \alpha_i' \pi^i)\cdot\bar{\beta}$$

$$= \sum_i \alpha_i'(\pi'\cdot\bar{\beta})$$

$$= \sum \alpha_i' .$$

Thus for $\rho = \frac{1}{2}\epsilon^{1/2}\min(\alpha_i^0)$,

$$|\sum \alpha_i^0 \pi^i - \pi'| < \rho \text{ and } \pi'\cdot\bar{\beta} = 1$$

imply $g_{\pi'}$ is the quasidensity of some distribution on $[c,d]$, i.e. π' is in A and $\sum \alpha_i^0 G_i$ is regular. //

The following result helps to visualize irregular distributions.

III.3 <u>If</u> π' <u>is regular and</u> π'' <u>is in</u> A <u>then for sufficiently small positive</u> ϵ , $\pi'' + (1+\epsilon)(\pi'-\pi'')$ <u>is also in</u> A .

Thus to obtain an irregular point one starts with any two points of A , π^0 and $\pi'\neq\pi^0$. Since A is convex, closed and bounded one can move through A along the ray from π^0 through π' to a point $\pi^0 + k(\pi'-\pi^0)$ in A such that $\pi^0 + k'(\pi'-\pi^0)$ is not in A for any $k'>k$. Thus $\pi^0 + k(\pi'-\pi^0)$ can not be regular.

<u>Proof</u>: $\bar{\beta}\cdot[\pi'' + (1+\epsilon)(\pi'-\pi'')] = 1 + (1+\epsilon)(1-1) = 1$ and, for sufficiently small positive ϵ , $|\pi' - (\pi''+(1+\epsilon)(\pi'-\pi''))| = \epsilon|\pi'-\pi''| < \rho$. //

Irregular distributions can be obtained from the many unimodal functions (e.g. most likelihood functions) in the CS too.

III.4 <u>If</u> f <u>is a function in the CS with a unique maximizer</u> t_0 , <u>i.e., if</u> $f(t) \geq f(t_0)$ <u>implies</u> $t = t_0$ <u>for all</u> t <u>in</u> $[c,d]$, <u>then the unit step function at</u> t_0 <u>is irregular</u>.

<u>Proof</u>: Since f is a linear combination of likelihood functions for any distribution function on $[c,d]$, $\int_c^d f(t)\,dG(t) = \langle f,g\rangle$ where g is the

quasidensity of G . In particular for $g_{\pi(t_0)}$ = the quasidensity of the unit step at t_0 , $\langle f, g_{\pi(t_0)}\rangle = f(t_0)$. Note that if G is the distribution of a random variable ω then $\langle f,g\rangle = E[f(\omega)]$. Thus for every quasidensity π' in A , $\langle f, g_{\pi'}\rangle \le \max_t f(t) = f(t_0)$. In particular for $g_{\pi(t_1)}$ the quasidensity of the unit step at $t_1 \ne t_0$,

$$\langle f, g_{\pi(t_1)}\rangle = f(t_1) < f(t_0) = \langle f, g_{\pi(t_0)}\rangle .$$

Consequently for any $\epsilon>0$, for $\pi(\epsilon) = \pi(t_1) + (1+\epsilon)[\pi(t_0)-\pi(t_1)]$

$$\langle f, g_{\pi(\epsilon)}\rangle = \langle f, g_{\pi(t_0)}\rangle + \epsilon[\langle f, g_{\pi(t_0)}\rangle - \langle f, g_{\pi(t_1)}\rangle]$$

$$= f(t_0) + \epsilon[f(t_0)-f(t_1)] > f(t_0) .$$

Thus $g_{\pi(\epsilon)}$ cannot be the quasidensity of any distribution on $[c,d]$ and from III.3, the unit step at t_0 is not regular. //

In fact a stronger result can be proven. It can be shown that the unit step at a maximizer corresponds to a point π' in A situated like a vertex of a polyhedron or a boundary point of an ellipsoid: If $\epsilon>0$ and π'' is also in A , then $\pi' + (1+\epsilon)(\pi''-\pi')$ is not in A . In other words, π' is not an interior point of the intersection of A and any line through π' .

III.5 <u>If</u> f <u>is a function in the CS with unique maximizer</u> t_0 , <u>then</u> $\pi(t_0)$ <u>is not an interior point of the intersection of</u> A <u>and any line through</u> π' .

Proof: Let G be any distribution on $[c,d]$ other than the unit step at t_0 . Let $[c',d']$ be any closed subinterval of $[c,d]$ not containing t_0 such that $\int_{c'}^{d'} dG(t) > 0$. Then for $g_{\pi'}$ equal to the quasidensity of G ,

$$\langle f, g_{\pi'}\rangle = \int_{c'}^{d'} f(t)\, dG(t) + \int_{t \text{ not in } [c',d']} f(t)\, dG(t)$$

$$\leq \max_{c'\leq t\leq d'} f(t) \int_{c'}^{d'} dG(t) + f(t_0) \left[1 - \int_c^d dG(t)\right]$$

$$< f(t_0) \int_{c'}^{d'} dG(t) + f(t_0) \left[1 - \int_c^d dG(t) \right]$$

$$= f(t_0) = \langle f, g_{\pi(t_0)}\rangle \ .$$

From the argument in III.4 applied to $\pi(\epsilon) = \pi' + (1+\epsilon)[\pi(t_0)-\pi']$ it follows that for all $\pi' \neq \pi(t_0)$ in **A** for $\epsilon>0$, $\pi(\epsilon)$ is not in **A** . //

Not every irregular distribution is mapped to one of these points that "stick out" from **A** like a vertex or a point of positive Gaussian curvature on a boundary. By taking linear combinations of unimodal functions in the CS one can construct bimodal and multimodal functions with modes of equal height. The reasoning used in III.4 and III.5 can then be used to show that **A** has edges and faces too.

III.6 <u>Let</u> f <u>be a function in the CS having modes of equal height at</u> $t_0 < t_1 < \ldots t_m$ <u>so that</u> $f(t_0)=f(t_1)= \ldots f(t_m)$ <u>and for</u> t <u>in</u> $[c,d]\backslash\{t_0, t_1, \ldots t_m\}$, $f(t) < f(t_0)$. <u>Let</u> $G_0, G_1, \ldots G_m$ <u>denote the unit step functions at</u> $t_0, t_1, \ldots t_m$. <u>Then for any positive numbers</u> α_i <u>such that</u> $\sum_0^m \alpha_i = 1$, <u>the distribution</u> $\Sigma\, \alpha_i G_i$ <u>is irregular</u>.

Proof: Let $g_{\pi(s)}$ be the quasidensity of the unit step function at s in $[c,d]\backslash\{t_0,t_1, \ldots t_m\}$. Then $\langle f,g_{\pi(s)}\rangle = f(s) < f(t_0) = \sum_0^m \alpha_i f(t_i)$. Thus for $g_{\pi(t_i)}$ equal to the quasidensity of the unit step at t_i and

$$\pi(\epsilon) = \pi(s) + (1+\epsilon)[\Sigma\, \alpha_i\pi(t_i) - \pi(s)]$$

$\epsilon>0$ implies $\langle f,g_{\pi(\epsilon)}\rangle = f(t_0) + \epsilon[f(t_0)-f(s)] > f(t_0)$. Thus $\Sigma\, \alpha_i G_i$ is irregular. //

The following corollary reconciles the apparent contradiction between

III.2 and III.6 .

III.7 <u>If for</u> f <u>in the CS there are more than</u> J <u>numbers</u> t <u>satisfying</u> $f(t) \geq f(s)$ <u>for all</u> s <u>in</u> $[c,d]$ <u>then the quasidensities of the unit step distributions at these numbers are linearly dependent</u>.

Two ability distributions are called *equivalent* if the probability distribution of any function of the item scores does not depend on which of the two ability distributions is used to compute the distribution. Thus, if F and G are equivalent, then data cannot be used to determine which of the two distributions is correct. A necessary and sufficient condition for two distributions on $[c,d]$ to be equivalent is that they have the same quasidensity (Levine, 1989). We will show that for logistic models all distributions are equivalent to discrete distributions. The result is valid for models with item response functions P_i such that for each t there is a power series for P_i that converges absolutely in some neighborhood of t .

III.8 <u>If the constant functions are the only functions in the CS that are constant on some nonempty open subset of</u> $[c,d]$, <u>then every distribution on</u> $[c,d]$ <u>is equivalent to a distribution with at most</u> $J+1$ <u>points of increase</u>.

<u>Proof</u>: It is sufficient to prove that there are finitely many points of increase because A has been shown to be J dimensional. It is sufficient to limit attention to irregular distributions because if π' is not on the boundary ∂A of A then compactness of A implies that for any π^1 in ∂A we can choose $t>1$ such that $\pi^2=\pi^1+t(\pi^1-\pi')$ is also on ∂A . The equation

$$\pi' = \frac{t-1}{t}\pi' + \frac{1}{t}\pi^2$$

shows that π' corresponds to a probability mixture of distributions mapped to the boundary of A .

Let G be a distribution on $[c,d]$ with quasidensity $g_{\pi'}$ for π' on ∂A . Let $H=\{x$ in E^{J+1} : $n \cdot x=c\}$ be a hyperplane in E^{J+1} containing π' and no points of the interior of A . Without loss of generality it can be assumed that $\mathbf{x}$ in $\mathbf{A}$ implies $n \cdot x \leq c$ because $n \cdot \mathbf{x}=c$ implies $(-n) \cdot \mathbf{x}=-c$. It follows that among distributions on $[c,d]$ G is a maximizer of $\int_c^d \Sigma_j n_j h_j(t)\, dG(t) = n \cdot \pi'$. Either $g_n(\cdot) = \Sigma\, n_j h_j(\cdot)$ is constant on some subinterval of $[c,d]$ (and therefore constant) or there are only finitely many numbers t such that $g_n(t) \geq g_n(s)$ for all s in $[c,d]$. g_n cannot be constant for otherwise $n \cdot \pi'' = \int_c^d g_n(t) \cdot g_{\pi''}(t)\, dt = c$ would be independent of π'' and A would have no interior points. Thus for finitely many numbers $c \leq t_1 < \ldots t_K \leq d$ $g_n(t_k) = \max_{[c,d]} g_n(t)$. Thus a distribution $\bar{G}$ (such as G) maximizes $\int_c^d g_n(t)\, d\bar{G}(t)$ if and only if it is equivalent to

$$\sum_{k=1}^{K} \alpha_k F_k(\cdot)$$

where F_k is the unit step at t_k for some positive numbers α_k such that $\Sigma\, \alpha_k = 1$. //

Section Four
Asymptotic Normality for Regular Distributions

In this section it will be shown that if the ability distribution is regular then $\hat{\pi}_N$, the maximum likelihood estimate of π , is asymptotically normal. A formula is derived for the asymptotic dispersion matrix.

Throughout this section the distribution for θ is assumed to be regular. Throughout this section let ρ be a fixed positive number chosen so that the intersection of the open ball with radius ρ centered at π

$$\{\pi' \text{ in } E^{J+1} : |\pi'-\pi| < \rho\}$$

and the hyperplane

$$\{\pi' \text{ in } E^{J+1} : \pi'\cdot\bar{\beta} = 1\}$$

is a subset of A . As in Section III.1, let $z^1, z^2, \ldots z^J$ be any orthonormal basis for the annihilator of $\bar{\beta}$ and let $Z = [z^1, z^2, \ldots z^J]$ be the $(J+1)\times J$ matrix formed from the z's so that $\pi' \to ZZ^T\pi'$ is the orthogonal projection onto the annihilator of $\bar{\beta}$.

To describe the asymptotic behavior of $\hat{\pi}_N$ we will need the information matrix, i.e. the matrix I of expected second derivatives with typical entry I_{ij}

$$\begin{aligned} I_{ij} &= -E \frac{\partial^2}{\partial\pi'_i\partial\pi'_j} \log P_{\pi'}(u)\Big|_{\pi'=\pi} \\ &= -\sum_{u^*} P_\pi(u^*) \frac{\partial^2}{\partial\pi'_i\partial\pi'_j} \log \pi'\cdot\beta(u^*)\Big|_{\pi'=\pi} \\ &= \sum_{u^*} P_\pi(u^*) \frac{\beta_i(u^*)\beta_j(u^*)}{[\pi\cdot\beta(u^*)]^2} \quad . \end{aligned}$$

Thus I can be written in the form $\sum_{u^*} w(u^*)\, \beta(u^*)\beta(u^*)^T$ and II.1 can be used to show it is non-singular. Since the columns of Z are independent, II.1 also implies that for any positive weights $w_\nu = w(u^*_\nu)$ the matrix

$\sum_{\nu=1}^{2^n} w_\nu \, Z^T\beta(u_\nu^*) \, \beta^T(u_\nu^*)Z$ is non-singular. In particular, Z^TIZ is non-singular.

Since $\hat{\pi}_N$ almost surely converges to π, with probability one $\hat{\pi}_N$ will eventually be within ρ of π. For $\hat{s}_N = Z^T(\hat{\pi}_N - \pi)$ the mle can be written

$$\hat{\pi}_N = \pi + Z\hat{s}_N \ .$$

Since $|\hat{\pi}_N - \pi| = |\hat{s}_N|$, almost surely $|\hat{s}_N| < \rho$ for sufficiently large N. Defining M_N on open $\{ s \text{ in } E^J : |s| < \rho \}$ by $M_N(s) = L_N[\pi + Zs]$ it follows that $\hat{s}_N$ maximizes M_N and consequently must satisfy the equations

$$0 = \frac{\partial}{\partial s_j} M_N(s) \qquad j = 1,2, \ldots J \ .$$

In fact, $\hat{s}_N$ almost surely eventually is the only solution of the equations with length less than ρ because with probability one, for sufficiently large N $J+1$ patterns with linearly independent β's will be sampled, and this implies that the Hessian matrix evaluated at $|s| < \rho$

$$\partial^2 M_N(s) = -Z^T[\sum_{a=1}^{N} \beta(u_a^*)\beta^T(u_a^*) \, P^{-2}_{\pi + Zs}(u_a^*) \,]Z$$

is definite .

The asymptotic distribution of $\hat{\pi}_N$ is obtained with Taylor's formula applied to the gradient of M_N .

IV.2 (Asymptotic distribution of the maximum likelihood estimate). Let $z^1, \ldots z^J$ be any orthonormal basis for $\mathrm{Nul}(\bar{\beta})$ and $Z = [z^1, \ldots z^J]$. If the ability distribution is regular then $\hat{\pi}_N$ converges in distribution to

$$\pi + N^{-1/2}Zn$$

<u>where n is multinormal with zero mean and covariance matrix $(Z^T IZ)^{-1}$</u>.

<u>Proof</u>: Since $\hat{s}_N = Z^T(\hat{\pi}_N - \pi)$ implies that $Z\hat{s}_N = \hat{\pi}_N - \pi$, the theorem can be proven by showing that $N^{1/2}\hat{s}_N$ converges in distribution to multinormal n. This will be done by showing that for any non-zero J-vector t, the random variable $N^{1/2}t\cdot\hat{s}_N$ is asymptotically normal with mean zero and variance $t^T(Z^T IZ)^{-1}t$.

Each component of the gradient of M_N

$$\frac{\partial}{\partial s_k} M_N(s) = \sum_{a=1}^{N} [P_{\pi+Zs}(u_a^*)]^{-1} \beta(u_a^*)\cdot z^k \qquad k = 1, \dots J$$

is defined and has continuous partial derivatives of order two for $|s| < \rho$. Thus when $|\hat{s}_N|$ is less than ρ there will be some $\epsilon_{N,k}$, $0<\epsilon_{N,k}<1$, such that for each $k\leq J$

$$0 = \frac{\partial}{\partial s_k} M_N(0) + \sum_i \frac{\partial}{\partial s_i}\frac{\partial}{\partial s_k} M_N(0)\hat{s}_{N,i} + \frac{1}{2}\sum_i [\sum_j \frac{\partial^2}{\partial s_i \partial s_j}\frac{\partial}{\partial s_k} M_N(\epsilon_{N,k}\hat{s}_N)\hat{s}_{N,j}]\hat{s}_{N,i}$$

$$= \frac{\partial}{\partial s_k} M_N(0) + \sum_i [\frac{\partial^2}{\partial s_i \partial s_k} M_N(0) + \frac{1}{2}\sum_j \frac{\partial^3}{\partial s_i \partial s_j \partial s_k} M_N(\epsilon_{N,k}\hat{s}_N)\hat{s}_{N,j}]\hat{s}_{N,i} .$$

In matrix notation

$$0 = \partial M_N(0) + [\partial^2 M_N(0) + \frac{1}{2} C_N]\hat{s}_N$$

and

$$[-\frac{1}{N}\partial^2 M_N(0) - \frac{1}{N}\frac{1}{2} C_N]\hat{s}_N = \frac{1}{N}\partial M_N(0) .$$

The right hand side of the last equation, being a mean of independent, identically distributed random vectors with zero expectation and covariance matrix $Z^T IZ$, is asymptotically normal with expectation 0 and covariance matrix $Z^T IZ/N$. Thus $N^{-1/2}\partial M_N(0)$ is asymptotically normal with mean zero and covariance matrix $Z^T IZ$. Since the summands in

$$-\frac{1}{N}\,\partial^2 M_N(0) = \frac{1}{N}\,\{\sum_{a=1}^{N}[P_\pi(u_a^*)]^{-2}Z^T\beta(u_a^*)\beta(u_a^*)^T Z\}$$

are independent and identically distributed, $-\frac{1}{N}\,\partial^2 M_N(0)$ converges almost surely to the non-singular matrix of expected values

$$E[P_\pi(u)^{-2}Z^T\beta(u)\beta(u)^T Z] = \sum_{\nu=1}^{2^n} P_\pi(u_\nu^*)[P_\pi(u_\nu^*)]^{-2}Z^T\beta(u_\nu^*)\beta(u_\nu^*)^T Z$$
$$= Z^T IZ\ .$$

The k*th* row and i*th* column of C_N is

$$\sum_j \frac{\partial^3}{\partial s_i \partial s_j \partial s_k}\ M_N(\epsilon_{N,k}\hat{s}_N)\ \hat{s}_{N,j}$$
$$= \sum_{a=1}^{N}\ [P^{-3}_{\pi+\epsilon_{N,k}Z\hat{s}_N}(u_a^*)]\ \left(\beta(u_a^*)\cdot z^k\right)\left(\beta(u_a^*)\cdot z^i\right)\left(\beta(u_a^*)\cdot Z\hat{s}_N\right)\ .$$

Since $\hat{s}_N$ converges almost surely to zero and the probabilities are bounded away from zero, the matrix $N^{-1}C_N$ converges almost surely to a matrix of zeros. Consequently the matrix D_N

$$D_N = \{-\frac{1}{N}\,\partial^2 M_N(0) - \frac{1}{N}\,\frac{1}{2}\,C_N\}$$

converges almost surely to non-singular $Z^T IZ$. Thus with robability one D_N is eventually non-singular and eventually both

$$\hat{s}_N = D_N^{-1}[N^{-1}\partial M_N(0)]$$

and

$$N^{1/2}t^T\hat{s}_N = t^T D_N^{-1}[N^{-1/2}\partial M_N(0)]\ .$$

Let $Y_N^T = \begin{cases} t^T D_N^{-1} & \text{, if } D_N \text{ is non-singular} \\ 0^T & \text{, otherwise} \end{cases}$

and $X_N = N^{-1/2}\ \partial M_N(0)$ so that $Y_N^T X_N$ almost surely eventually equals $N^{1/2}t\cdot\hat{s}_N$. If it can be shown that $Y_N^T X_N$ converges in distribution to $t^T(Z^T IZ)^{-1}X$ where X is multivariate normal with covariance matrix $Z^T IZ$ then it follows that $Y_N^T X_N$ and $N^{1/2}t^T\hat{s}_N$ are asymptotically normal with variance $t^T(Z^T IZ)^{-1}(Z^T IZ)[t^T(Z^T IZ)^{-1}]^T = t^T(Z^T IZ)^{-1}t$, and the proof will be complete. $Y_N^T - t^T(Z^T IZ)^{-1} \xrightarrow{wp1} 0^T$ so $Y_N^T - t^T(Z^T IZ)^{-1} \xrightarrow{p} 0^T$. Since

X_N converges in distribution, $[Y_N^T - t^T(Z^TIZ)^{-1}]X_N \xrightarrow{d} 0$ and $\xrightarrow{p} 0$.
Finally, since $t^T(Z^TIZ)^{-1}X_N \xrightarrow{d} t^T(Z^TIZ)^{-1}X$,

$$Y_N^TX_N = [Y_N^T - t^T(Z^TIZ)^{-1}]X_N + t^T(Z^TIZ)^{-1}X_N \xrightarrow{d} t^T(Z^TIZ)^{-1}X \ . \ //$$

Acknowledgements

This research was supported by the Cognitive Science Program of the Office of Naval Research contract N00014-83K-0397, NR 150-518 and N00014-86K-0482, NR 4421546. I am indebted to Bruce Williams, Tim Davey, Charles Davis, Fritz Drasgow, Brian Junker, and Gary Thomasson for comments on earlier versions of this work. Conversations with J.O. Ramsay were also useful.

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