# FORMULA SCORING

# BASIC THEORY AND APPLICATIONS

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Formula scoring is a systematic study of measurement statistics expressed as sums of products of item scores. The theory is currently being used to compute non-parametric estimates of ability distributions, item response functions, and option response functions. The theory has been used to design algorithms for estimating item response functions from adaptive test data (on-line calibration), monitoring and correcting drift in observed score distributions for adaptive tests (on-line equating), computing optimal tests for cheating, and combining appropriateness measurement information from several subtests. In this paper a portion of the theory is developed from a few principles. Applications are considered to the problems of deciding whether ability has the same distribution in two demographic groups, to finding latent class models that are equivalent to item response models, and to controlling drift in adaptive testing programs. Keywords:

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## ABSTRACT


Formula scoring is the systematic study of measurement statistics expressed as linear combinations of products of item scores. The theory is currently being used to compute non-parametric estimates of ability distributions, item response functions, and option response functions. The theory has been used to design algorithms for estimating item response functions from adaptive test data (on-line calibration), monitoring and correcting drift in observed score distributions for adaptive tests (on-line equating), computing optimal tests for cheating, and combining appropriateness measurement information from several subtests. In this paper a portion of the theory is developed from a few principles. Applications are considered to the problems of deciding whether ability has the same distribution in two demographic groups, to finding latent class models that are equivalent to item response models, and to controlling drift in adaptive testing programs.

FORMULA SCORING

BASIC THEORY AND APPLICATIONS

## Preface

For several years, Bruce Williams and I have been presenting applications of a new approach to measurement, which we call *formula scoring*. Our presentations to the annual ONR Contractor's Conferences have been punctuated with the phrase, "It can be shown ... ." This technical report begins a series of papers providing proofs of these claims. An attempt will be made to derive formula score theory from a few basic principles.

This version of the report is being used to introduce graduate students to the work in our laboratory. Very explicit, computational proofs are provided for some basic results. A shorter version is being prepared for publication.

Thanks to Bruce Williams and Fritz Drasgow there are many data-based applications[1] of formula scoring, which are now starting to appear in print[2]. The data-based applications are not suitable for motivating this paper because Bruce's programs use concepts that are developed in later papers. Therefore an alternative way to motivate the report had to be found.

Three examples of results that can be obtained with the theory have been selected to motivate the theory. I don't think the results would have been discovered without the theory. Each seems surprising - at least to me - and somewhat contrary to conventional psychometric wisdom. Each result can be easily proven with the theory. And each result seems hard to prove without reproducing the reasoning in the theory.

## Some Examples to Motivate the Theory

Formula score theory can be used to derive some unexpected, hopefully useful, consequences of the assumptions of item response theory. Three examples follow.

The examples are valid for parametric and non-parametric item response models. Except where noted, the results hold for all "continuous, one-dimensional, probabilistic item response models for bounded abilities." Thus, item response functions are permitted to have any shape, provided they are continuous functions of one variable with values strictly between zero and one. The cumulative distribution of ability also is permitted to have any shape, provided there is some - possibly very large - interval such that the distribution is zero or one outside this interval.

### Example One: Checking for ability distribution differences

A quick way to recognize ability distribution differences is to check average tests scores. Thus, if girls on the average have higher test scores than boys on an unbiased test it is safe to conclude that ability is distributed differently among girls and boys. The converse obviously is not true because very different distributions may have same mean.

Using observed scores to check for group ability differences is believed to be uniquely uncomplicated for the Rasch model. Since the number right score is a sufficient statistic for estimating ability it might be expected that it is possible to determine the presence or absence of group ability differences by comparing distributions of number right score. This (incorrect) assertion can also be expressed as follows:

> There is a set of statistics $X_0, X_1, \ldots X_n$ such that the group ability distributions are different if and only if at least one of the statistics has different expected values among girls and boys.

Here $n$ is the number of items on the test, and $X_j$ is the statistic which is one if exactly $j$ items were answered correctly and zero otherwise.

The theory shows that the Rasch model is *not* unique in having a small number of diagnostic statistics. The theory also shows what can *and cannot* be concluded when corresponding pairs of expectations are equal.

For *any* item response model, Rasch model or other, there is a set of statistics $X_1, X_2, \ldots X_J$ such that if at least one pair of corresponding expected values differ, then the group ability distributions are different. But if corresponding expected values are equal, then the distributions still may be different. However, it can be shown that no statistical test (using only the answers to the $n$ items for data) exists that can demonstrate the difference! In particular, for a test satisfying the Rasch model if boys and girls have equal expected $X_j$'s , then ability may be distributed differently in the two populations, but no analysis of test data can be used to demonstrate the difference. Details follow the proof of Theorem One.

Recall that for the Rasch model each item response function $P_i$

$$P_i(t) = \text{Prob}(\text{correct answer for item } i \mid \text{ability} = t)$$

can be written in the form $P_i(t) = [1+e^{-(t-b_i)}]^{-1}$ for some constant $b_i$ . To avoid mathematical digressions irrelevant to the main points of this paper, it will generally be assumed that for $i \neq j$ , $b_i \neq b_j$ . Thus no two Rasch model items have exactly the same item response function.

As an example of another model having a small set of diagnostic statistics, consider the generalization of the Rasch model having item response functions given by the following equation

$$P_i(t) = c_i + (1-c_i)[\ 1 + e^{-a(t-b_i)}]^{-1} .$$

As with the Rasch model, it will generally be assumed that different items have different difficulties. Thus if $i \neq j$ , $b_i \neq b_j$ . For this model $J$ is less than or equal to the number of items, and $X_j$ can be taken to be the score that is one if item $j$ is answered correctly and zero otherwise. (If for some $i \neq j$ , $b_i = b_j$ , then a somewhat more complicated set of $X_j$ must be used, but $J$ is still small.)

Incidentally, these results are related to the identifiability of ability distributions. Since different distributions can give the same vector of expected $X_j$'s , the ability distribution is not identifiable, even when the item response functions are completely specified.

Example Two: How to turn an item response model for an ability continuum into an isomorphic latent class model with finitely many classes

Suppose we are given an item response model with continuous item response functions $\neq 0,1$ and a continuous ability density $f$ . Using the theoretical results in this paper it can be shown that it is possible to select abilities $t_0 < t_1 < \ldots t_J$ and numbers $p(t_0), p(t_1), \ldots p(t_J)$ such that for each item response pattern $u^*$ , the "manifest probability"

Prob(Sampling an examinee with item response patttern $u^*$) ,

which is ordinarily computed by i tegrating the likelihood function,

$$\int_c^d \text{lik}(u^* | \text{ ability} = t)\, f(t)\, dt$$

can be computed by evaluating the sum

$$\sum_{j=0}^{J} \text{lik}(u^* | \text{ ability} = t_j)\, p(t_j) \ .$$

For the item response functions given by the formulas in Example One, $J$ can by set equal to the number of items.

Since the manifest probabilities sum to one, $\Sigma\, p(t_j) = 1$ . Thus if $p(t_j) \geq 0$ for $j \leq J$ , we have a latent class model with $J+1$ classes that is isomorphic to the continuous latent trait model.

I haven't found a simple proof based only on the results in this paper of the existence $t_j$ with $p(t_j) \geq 0$ . However the result also is true and is proven in next paper in this series. In any event, even when some of the $p(t_j)$ are negative the result seems able to greatly reduce computation times in some applications noted below.

Example Three: On-line equating or Simulation results without simulation

Consider two subtests, say, word knowledge (WK) and arithmetic reasoning (AR), of a computer administered adaptive test such as the adaptive version of the Armed Services Vocational Aptitude Battery (ASVAB). Suppose the item pool for WK has just been changed by introducing some new items that haven't been ministered often enough to highly motivated examinees to have well estimated item response functions. To analyze and control the effect of the new items on the distribution of an observed score $\hat{\theta}_{WK}$ we wish to calculate three functions, usually computed by simulation:

$$F_1(t) = \text{Expectation} \{ \hat{\theta}_{WK} \mid \theta_{WK} = t \}$$
$$F_2(t) = \text{Variance} \{ \hat{\theta}_{WK} \mid \theta_{WK} = t \}$$
$$P(x|t) = \text{Prob} \{ \hat{\theta}_{WK} \leq x \mid \theta_{WK} = t \} .$$

$F_1$ and $F_2$ show how the first two conditional moments of the observed score are affected by the new items and can be used to make corrections. For example, if $F_2(-1)$ is observed to increase very much when the new items replace easy old items then countermeasures such as adding more easy items can be tried. $P(x|t)$ provides the remaining moments. It can be used to predict how the marginal distribution of $\hat{\theta}_{WK}$ will be affected by future changes in the ability distribution.

Since the item response functions for the new items are not known, simulation is not possible. (When the score $\hat{\theta}_{WK}$ is a Bayes mode or maximum likelihood ability estimate, then item parameter estimates derived from small samples of not highly motivated examinees may be used to compute the score, but such estimates are not suitable for including in a simulation.) Thus, the following result is of interest.

It is generally possible to use the item response functions for the *old* WK items to compute functions $c_0(t), c_1(t), \ldots c_K(t)$ and to sort examinees into groups using only an AR score $\hat{\theta}_{AR}$ . According to the theory, the conditional expectation of $\hat{\theta}_{WK}$ (computed from item scores for both old and new items) can be calculated with the formula

$$\text{Expectation} ( \hat{\theta}_{WK} \mid \theta_{WK}=t ) =$$

$$\sum_{k=0}^{K} c_k(t) \text{ Expectation} ( \hat{\theta}_{WK} \mid \hat{\theta}_{AR} \text{ is in the } k\text{th score group}) .$$

In words, we use $\hat{\theta}_{AR}$ to group examinees and then compute the conditional expected WK score as a linear combination $\hat{\theta}_{WK}$ group averages. The $\hat{\theta}_{WK}$ score is computed using item scores for both old WK items and new WK items. However, only the well estimated old WK item response functions are used to compute the coefficients of the linear combination. In this way the effect of introducing new items on an observed score *at each ability level* can be calculated from actual data. Since the method does not use item parameter estimates for the new items, it is not adversely affected by item parameter estimation error on the new items.

A similar formula gives the conditional variance since for the same $c_j$ and groups

$$\text{Expectation} ( \hat{\theta}^2_{WK} \mid \theta_{WK}=t )$$

$$= \sum_{k=0}^{K} c_k(t) \text{ Expectation}( \hat{\theta}^2_{WK} \mid \hat{\theta}_{AR} \text{ is score group } k ) .$$

Finally, for the random variable defined by

$$X = \begin{cases} 1 \text{ if } \hat{\theta}_{WK} \leq x \\ 0 \text{ otherwise} \end{cases}$$

the conditional distribution of $\hat{\theta}_{WK}$ is given by

$$\text{Prob}( \hat{\theta}_{WK} \leq x \mid \theta_{WK}=t) = \text{Expectation}( X \mid \theta_{WK}=t)$$

$$= \sum_{k=0}^{K} c_k(t) \text{ Expectation}( X \mid \hat{\theta}_{AR} \text{ is in group } k ) \quad .$$

The calculation of these three conditional expected values illustrates a more general result described in the discussion of "quasidensities" (Section Two, below).

NOTES

1. Formula score theory currently is being used to compute non-parametric maximum likelihood estimates of ability distributions, item response functions, and option response functions. The theory has been used to design algorithms for estimating item response functions from adaptive test data without interrupting testing (on-line calibration), to compute optimal tests for cheating, and to combine appropriateness measurement information from several subtests. The theory yields measures of item bias and test dimensionality. The theory seems to lead to a tractible, non-parametric, multidimensional item response theory, which is currently being developed. The theory is also being applied to what might be called "online equating," i.e., monitoring and correcting changes in the distribution of observed scores for an adaptive test as the test's item pool is replenished.

2. Drasgow, F., Levine, M.V., Williams, B., McLaughlin, M.E., and Candell, G.L. Modelling incorrect responses with multilinear formula score theory. Applied Psychological Measurement, In press, 1989; Drasgow, F., Levine, M.V., and McLaughlin, M.E. Multitest extensions of appropriateness indices. Applied Psychological Measurement, accepted for publication, 1989.

## Section One

## Formula Score Theory and Equivalent Distributions

Formula score theory systematically studies measurement statistics expressed as linear combinations of products of item scores. The theory begins with an equivalence relation on ability distributions.

We consider a fixed test of $n$ items. A pair of distributions $F$ and $G$ are defined to be *equivalent* relative to the test if every statistic computed from the test's item scores has the same distribution under the hypothesis

$$H_0: \text{ Ability has cumulative distribution } F$$

as under the alternative hypothesis

$$H_1: \text{ Ability has cumulative distribution } G .$$

Notice that there is no way whatsoever to use item responses on the test being analyzed to distinguish between a pair of equivalent distributions. For if $F$ is equivalent to $C$ and if the statistic $X$ is used for hypothesis testing, then decisions based on $X$ will be no more valid than decisions based on the flip of a coin or other irrelevant random process.

Notice also that equivalence is defined relative to a fixed test of specified items. Thus a pair of distributions may be equivalent relative to the test, but distinguishable if one more item is added to the test. In fact, if one of the items is replaced by a slightly different item, the equivalence relation may be changed. This is a significant limitation of the present algebraic version of the theory. Later papers on applications use metric concepts to get around this problem.

The main result of this section is a characterization of equivalent distributions in terms of the expected values of finitely many statistics. Comments on implications and applications of this result are at the end of this section.

## Item Response Theory and Formula Score Theory

To make the paper more nearly self-contained and to make explicit just what assumptions of item response theory are used to prove the new results, we begin with some definitions from item response theory.

An item response model provides a probability measure for set $\{a\}$, which is interpreted as a set of possible or actual examinees. There are two types of random variables in item response theory: observed *item scores* $u_1(a)$, $u_2(a)$, ... $u_n(a)$ and unobserved *abilities* $\theta(a)$. Item scores are either one or zero. "$u_i(a)=1$" is interpreted as "examinee a successfully answered item i ."

In this paper, the abilities $\theta(a)$ are numbers. However, after some routine changes, all of the results in this paper and their proofs generalize to multidimensional abilities, i.e., vector-valued $\theta(a)$'s .

Item response theory relates item scores to abilities with functions $P_i$ called *item response functions*

$$P_i(t) = \text{Prob}(\ u_i=1|\theta=t\ ) .$$

$P_i(t)$ is interpreted as the probability of observing $u_i(a) = 1$ , when examinee a is sampled from all those with ability t .

In this paper, details about the item response functions are generally left unspecified. Only continuity and a weak condition, $0 < P_i(t) < 1$ , are assumed. These conditions are also implied by the parametric formulas of most item response models.

Formula scoring differs from much of item response theory on the domain of definition of the item response functions. In item response models $P_i(t)$ is usually defined for all numbers $t$, despite the fact that the models predict essentially the same behavior from examinees with ability 20 and 20,000 and despite the fact that applications of the parametric models usually proceed as if abilities were bounded.

In this section the domain of definition of the item response functions can be bounded or unbounded. However, in the following sections $P_i(t)$ is defined only for $t$ in an interval of finite length. Some discussion of this point is at the end of this section.

The main assumption of item response theory is *local independence*. It asserts that item responses are conditionally independent, i.e., for any sequence of zeros and ones

$$u_1^*, u_2^*, \ldots u_n^*$$

and any ability $t$

$$\text{Prob}\{ u_1=u_1^* \ \& \ u_2=u_2^* \ \ldots \ u_n=u_n^* \mid \theta = t\} = \Pi_i \ \text{Prob}\{ u_i=u_i^* \mid \theta=t\} .$$

In item response theory analyses of data, the item responses are recorded and inferences are made about $\theta$. Only the item responses are observed. Thus if the word "statistic" is to be reserved for random variables that are functions of the observables, only functions of the $u_i$ are statistics. Since the range of each $u_i$ is finite, every function of the $u_i$ is a random variable. *Thus* X *is a statistic if and only if* X *is a function of item scores.*

The set of all statistics for a test is obviously a vector space since a linear combination of functions of item scores is a function of item scores. Since the $u_i$ take on only finitely many values, every statistic can be written as a polynomial in the item scores. In fact, since $u_i^2 = u_i$

every statistic is a linear combination of the following statistics, which are called *elementary formula scores*,

$$1$$

$$u_1,\ u_2,\ \ldots\ u_n$$

$$u_1u_2,\ u_1u_3,\ \ldots\ u_{n-1}u_n$$

$$\cdot\ \cdot\ \cdot$$

$$\prod_{i=1}^{n} u_i\ .$$

Thus the elementary formula scores, or some subset of the these scores, form a basis for the vector space of all statistics. Since there are finitely many ( $2^n$ ) elementary formula scores, *the set of all statistics is a finite dimensional vector space*.

The *regression function* $R_X(\cdot)$ or conditional expectation function of a statistic $X$

$$R_X(t) = E(X|\theta=t)$$

expresses the conditional expected value of the statistic as a function of ability. Since every statistic is a linear combination of the elementary formula scores, local independence implies that each regression function can be written in at least one way as a linear combination of the following functions

$$1$$

$$P_1(t),\ \ldots\ P_n(t)$$

$$P_1(t)P_2(t),\ P_1(t)P_3(t),\ \ldots\ P_{n-1}(t)P_n(t)$$

$$\cdot\ \cdot\ \cdot$$

$$\prod_{i=1}^{n} P_i(t)\ .$$

The central concept of formula score theory is the canonical space. The *canonical space* (CS) of a test is the vector space of regression functions of statistics. Obviously it is the vector space spanned by the *square-free monomials*, i.e. the products of item response functions without repeated factors, listed above. Thus, *the canonical space is a finite dimensional vector space of continuous, real-valued functions.*

### An Alternative Characterization of Equivalent Distributions

Using the canonical space it is possible to derive a simpler test for equivalent distributions. The definition would have us check the *distribution* of *every* statistic. It will be shown that only *finitely* many statistics need to be considered and that all that needs to be known about each statistic is its expected value. First, some notation.

$F$ will be used in all sections of this paper to denote the (generally unknown) ability distribution. For any statistic $X$ and number $x$, the distribution function of $X$ evaluated at $x$ can be written

$$\text{Prob}(X \leq x) = \int \text{Prob}(X \leq x \mid \theta = t)\, dF(t) \ .$$

If $G$ is $F$ or any other distribution, then the distribution of $X$ relative to $G$ evaluated at $x$ will be denoted by $P(x;X,G)$. Thus

$$P(x;X,G) = \int P(X \leq x \mid \theta = t)\, dG(t) \ .$$

Similarly, the expected value of $X$ and the expected value of $X$ relative to distribution $G$ are denoted by

$$E(X) = \int E(X \mid \theta = t)\, dF(t)$$

$$E(X;G) = \int E(X \mid \theta = t)\, dG(t) \ .$$

Using this notation the definition of equivalent distributions given earlier can be succinctly expressed: *Two distributions* $F_1$ *and* $F_2$ *are* *equivalent if for all statistics* $X$ *and real* $x$

$$P(x;X,F_1) = P(x;X,F_2) \ .$$

Theorem One is an alternative characterization of equivalent distributions.

Theorem One: Let $J+1$ be the dimension of the canonical space. Then there are $J$ statistics $X_1, X_2, \ldots X_J$ such that $F_1$ is equivalent to $F_2$ if and only if

$$E(X_j;F_1) = E(X_j;F_2) \quad \text{for } j=1, \ldots , J \ .$$

Furthermore, if $Y_0, Y_1, \ldots Y_J$ are any statistics with linearly independent regression functions, then $F_1$ is equivalent to $F_2$ if and only if $E(Y_j;F_1) = E(Y_j;F_2)$ for $j=0, 1, \ldots J$ .

Proof: Let $h_0, \ldots h_J$ be a basis for the canonical space. Since the constant function is in the CS, $h_0$ can be taken to be the constant function, $h_0(t) = 1$ . Since the $h_j$ are in the CS, there are statistics $X_j$ such that $h_j(t) = E(X_j|\theta=t)$ for $0 \le j \le J$ . For any statistic $X$ and real $x$ , the regression function of the indicator random variable, $\chi$

$$\chi = \begin{cases} 1, \text{ if } X(u_1, \ldots , u_n) \le x \\ 0, \text{ if } X(u_1, \ldots , u_n) > x \end{cases}$$

is in the canonical space and consequently can be written

$$E(\chi|\theta=t) = \sum_{j=0}^{J} \alpha_j h_j(t) \ .$$

Therefore for $i=1,2$

$$\begin{aligned} P(x;X,F_i) &= \int \Sigma_j \alpha_j h_j(t) dF_i(t) \\ &= \Sigma_j \alpha_j E(X_j;F_i) \ . \end{aligned}$$

Since $E(X_0;F_1) = \int 1 \ dF_1(t) = 1 = E(X_0;F_2)$ ,

$$E(X_j;F_1) = E(X_j;F_2) \quad \text{for} \quad j=1, \ldots, J$$

implies that $F_1$ and $F_2$ are equivalent. Conversely, each $X_j$ can be written as a sum of products of the binary item scores,

$$X_j = \sum_{\nu=1}^{2^n} a_\nu v_\nu$$

where $v_1, v_2, \ldots v_\nu, \ldots v_{2^n}$ is an enumeration of the $2^n$ elementary formula scores. Since $v_\nu$ is either zero or one, for $i=1$ or $2$

$$E(v_\nu;F_i) = 1-P(0;v_\nu,F_i) \; .$$

Therefore "$F_1$ is equivalent to $F_2$" implies

$$\begin{aligned} E(X_j;F_1) &= \sum_\nu a_\nu E(x_\nu;F_1) \\ &= \sum_\nu a_\nu [1-P(0;v_\nu,F_1)] \\ &= E(X_j;F_2) \; . \end{aligned}$$

Finally, if $J+1$ statistics $Y_j$ have linearly independent regression functions $g_j$ then for some non-singular $(J+1)\times(J+1)$ matrix $A=(a_{ij})$, $g_j(\cdot)=\sum_k a_{jk}h_k(\cdot)$ . The remainder of the proof follows routinely from $E(Y_j;F_i) = \sum_k a_{jk}E(X_k;F_i)$ for $j=0, 1, \ldots J$ and $i=1,2$ .

## Implications and Applications

The theorem has negative implications for distribution estimation. We have observed that when $J$ is small, two distributions with clearly different shapes can be equivalent. As noted in Example Two a discrete distribution on a few points may turn out to be indistinguishable from a distribution with a continuous density. Thus, even when item response functions are known, it is not possible to consistently estimate the ability distribution without additional assumptions.

Note that for some applications it is valuable to know that ability distributions are equivalent. Returning to Example One of the Prefa -, if the ability distributions for boys and girls are equivalent relative to the test, then any selection procedure based on test results is as likely to select a boy as a girl.

The theorem shows, as was asserted in Example One, that by checking finitely many pairs of expected values, a difference between the ability distributions can be demonstrated. In Section 3 it is shown that $J$ can be small. For the Rasch model and its generalization, $J$ can be taken equal to the number of test items and $X_j$ can be taken to be the *jth* item score. Thus a necessary and sufficient condition for there to be a *demonstrable* difference between distributions is that there be at least one item on which the proportion of boys passing the item is different from the proportion of girls.

For other models $J$ can be large and the $X_j$ may be complicated. Models with large $J$ are discussed in Section 4. The task of computing $J$ and $X_j$ is also discussed in Section 4.

Example Two illustrates a second situation in which distribution equivalence may have practical importance. In Example Two we considered replacing an ability distribution having a continuous density with a step function having finitely many steps. The goal in doing so was to reduce integrals to sums. (In Section 3 a procedure for calculating the location and size of the steps is described.) In optimal appropriateness measurement[1] it is necessary to integrate over ability to obtain a uniformly most powerful test for cheating and other forms of aberrance. Even for unidimensional tests a great deal of computing is required to compute the theoretical manifest probabilities in Example Two. For multidimensional tests and "multi-unidimensional" test batteries such as ASVAB considerably more computation is required.

So far we have successfully avoided computing multiple integrals in our analyses of test batteries in which each subtest measures a different ability[2] by using approximations. The results in this section indicate an alternative, more general way[3] to calculate probabilities. Since an integral must be evaluated for each of thousands of examinees and since multivariate quadrature requires a lot of computation, replacing a continuous multivariate with an equivalent discrete distribution on a small number of points is very desireable.

This section is concluded with comments on the issue of bounded and unbounded ability continua, which is raised by Theorem One.

## Why Bounded Abilities

Sometimes whatever is being measured by a test is intrinsically bounded. Adding extremely hard items to a test generally changes what is being measured and may cause a test to fail to be unidimensional. Thus a calculus item is not a very hard arithmetic item but an item measuring an ability or achievement other than what is being measured by a grade school subtraction test. At the other extreme, a child totally ignorant of subtraction occupies a lower end point on the measurement scale.

Theorem One raises questions about the domain of definition of the $P_i$ and also motivates considering bounded continua. Suppose that on a particular test no examinee has an ability outside the interval $[-5,5]$. Then there can be a pair of *inequivalent* distributions $F_1$ and $F_2$ such that $F_1(t) = F_2(t)$ for $|t| \leq 5$, even though no empirical study can distinguish between $F_1$ and $F_2$. This awkward situation can be kept from occuring by defining the item response functions as functions of abilities in [-5,5] . If the $P_i$ are defined only for $|t| \leq 5$ , then the CS becomes a set of functions defined on an interval. Distributions that agree on the interval will then be equivalent in the sense of Theorem One as well as in the intuitive sense. Thus by treating the $P_i$ as functions of a *bounded* variable the intuitive and technical meanings of "equivalent" can be brought closer together. Alternatively, attention can be limited to ability distributions that are zero or one outside this interval. Both options are developed in the next section.

The assumption of boundedness turns out to be very weak. In any practical measurement situation, it can be trivially satisfied by considering a very large interval, an interval so large that the probability of sampling an examinee outside the interval for all practical purposes is zero. For theoretical work, boundedness can be imposed on a test model by transforming abilities without affecting the only assumptions being made about item response functions: continuity and $0 < P_i(t) < 1$ .

NOTES

1. Levine, M.V. and Drasgow, F., Optimal Appropriateness Measurement. Psychometrika, 1989.

2. Drasgow, F., Levine, M.V., and McLaughlin, M.E. Multitest extensions of appropriateness indices. Applied Psychological Measurement, accepted for publication, 1989.

3. The method can be thought of as a quadrature technique developed for evaluating the integrals that occur in psychometric applications. The selection of the quadrature points and weights is discussed in Section 3. Each quadrature formula is exact for some set of integrands. The new method is exact for integrating functions in the CS.

## Section Two

## An Inner Product and Quasidensities

When abilities are bounded, the CS has an inner product with a simple statistical interpretation. And each distribution function can be treated as if it had a continuous derivative. This "derivative," the quasidensity, is the subject of this section.

In the remainder of this paper it will be assumed that there are numbers $c<d$ such that $\text{Prob}\ (c \le \theta \le d) = 1$ . Item response functions will be treated as functions defined on $[c,d]$ , and the canonical space will be a set of functions defined on $[c,d]$ . After these changes are made the function $\langle\cdot,\cdot\rangle$ defined on pairs of functions $f,g$ in the CS by

$$\langle f,g\rangle = \int_c^d f(t)g(t)dt$$

becomes an inner product.

Note that when the ability distribution has a density and this density is in the CS, then the inner product has a statistical interpretation. For if $R(t) = E(X|\theta=t)$ is the regression function of a statistic $X$ and if the ability distribution has a density $f$ also in the CS, then $\langle R,f\rangle$ is the expectation of $X$ . The major result of this section is to generalize this property to situations in which the ability density is not in the CS and to situations in which the ability distribution is not differentiable. It will be shown that there is a *unique* continuous function $g$ in the CS such that for all statistics $X$

$$\begin{aligned} E(X) &= \int_c^d E(X|\theta=t)\ dF(t) \\ &= \int_c^d E(X|\theta=t)\ g(t)dt \\ &= \langle R_X,g\rangle\ . \end{aligned}$$

<u>Theorem Two</u>: <u>If</u> $P(c\le\theta\le d) = 1$ , <u>then there is a unique continuous function</u>

g <u>in the CS such that for every statistic</u> X

$$E(X) = \int_c^d E(X|\theta=t)\ g(t)dt\ .$$

<u>Proof</u>: Let $h_0, h_1, \ldots h_J$ be an orthonormal basis for the CS relative to its inner product $<\cdot,\cdot>$ . Thus $<h_i,h_j> = 1$ or zero according to whether $i =, \neq j$ . For each $j \le J$ a statistic $X_j$ can be found such that $E(X_j|\theta=t) = h_j(t)$ because every function in the CS is the regression function of at least one statistic. Let X be any statistic and $R_X$ its regression function. Since the $h_j$ form a basis for the CS, $R_X$ can be written

$$R_X(\cdot) = \Sigma_j\ b_j h_j(\cdot)$$

for some constants $b_j$ . Since the $h_j$ are orthonormal $<R_X,h_j> = b_j$ and

$$R_X(\cdot) = \Sigma_j\ <R_X,h_j>h_j(\cdot)\ .$$

Consequently

$$\begin{aligned}
E(X) &= \int_c^d R_X(t)\ dF(t) \\
&= \int_c^d \Sigma_j\ <R_X,h_j>h_j(t)\ dF(t) \\
&= \Sigma_j\ <R_X,h_j> \int_c^d h_j(t)dF(t) \\
&= \Sigma_j\ <R_X,h_j>E(X_j) \\
&= \Sigma_j \int_c^d R_X(t)h_j(t)dt\ E(X_j) \\
&= \int_c^d R_X(t)\ \Sigma_j\ E(X_j)h_j(t)dt \\
&= \int_c^d E(X|\theta=t)g(t)dt
\end{aligned}$$

for $g = \Sigma\ E(X_j)h_j(\cdot)$ in the CS.

To prove uniqueness, suppose that for some h in the CS

$$E(X) = \int_c^d R_X(t)h(t)dt$$

for all statistics X . Since the $h_j$ form a basis, $h(\cdot) = \Sigma\ \alpha_j h_j(\cdot)$ for some constants $\alpha_j$ . Since the $h_j$ are orthonormal, for $X = X_j$

$$E(X_j) = \int_c^d R_{X_j}(t)h(t)dt$$

$$= \int_c^d h_j(t) \sum_k \alpha_k h_k(t)dt$$

$$= \sum_k \alpha_k \langle h_j, h_k \rangle$$

$$= \alpha_j \ .$$

Thus $h=g$ , as was to be proven.

If $G=F$ or any other distribution function, then $G$ will be called a *distribution on [c,d]* if for $t<c$ , $G(d) - G(t) = 1$ . If $G$ is $F$ or any other distribution on $[c,d]$ then a function $g$ in the canonical space is called the *quasidensity*[1] *for G* if for all statistics $X$

$$E(X;G) = \int_c^d E(X|\theta=t)\ g(t)dt\ .$$

Note that Theorem Two implies that every distribution on $[c,d]$ has a unique quasidensity. Furthermore the proof shows that the quasidensity for $G$ can be written as

$$g(\cdot) = \sum_{j=0}^{J} E(X_j;G)h_j(\cdot)$$

where $\{h_j\}_{j=0}^{J}$ is any orthonormal basis for the CS and each $X_j$ satisfies $R_{X_j} = h_j$ . Since the quasidensity is unique, the choice of the orthonormal basis and statistics $X_j$ used in the formula is inconsequential.

At the end of this section some facts about quasidensity densities are listed and proven. The quasidensity for the unit step at -1 is shown to have the simple form $g(t) = \sum_{j\leq J} h_j(-1)h_j(t)$ where $\{h_j\}_{j=0}^{J}$ is any orthonormal basis for the CS. This formula was used to compute an approximation to the quasidensity for the unit step at -1 . The first 19

$h_j$'s for 100 three parameter logistic items by the methods in Section 4. Figure One shows the graph of $q(t) = \sum_{j \leq 18} h_j(-1)h_j(t)$ . If $q(t)$ is multiplied times any of the 100 logistic functions and integrated, the result should be very close to $P_i(-1)$ . $|P_i(-1) - \int_c^d P_i(t)q(t)\,dt|$ was found to be generally small, as shown in Table One.

For shorter tests, the quasidensity of the unit step function can be computed without approximation. The graph shown in Figure One is typical.

The precision of the approximation shown in Table One serves to illustrate a point developed in Section Four: For some purposes, high dimensional canonical spaces can be approximated by much lower dimensional spaces.

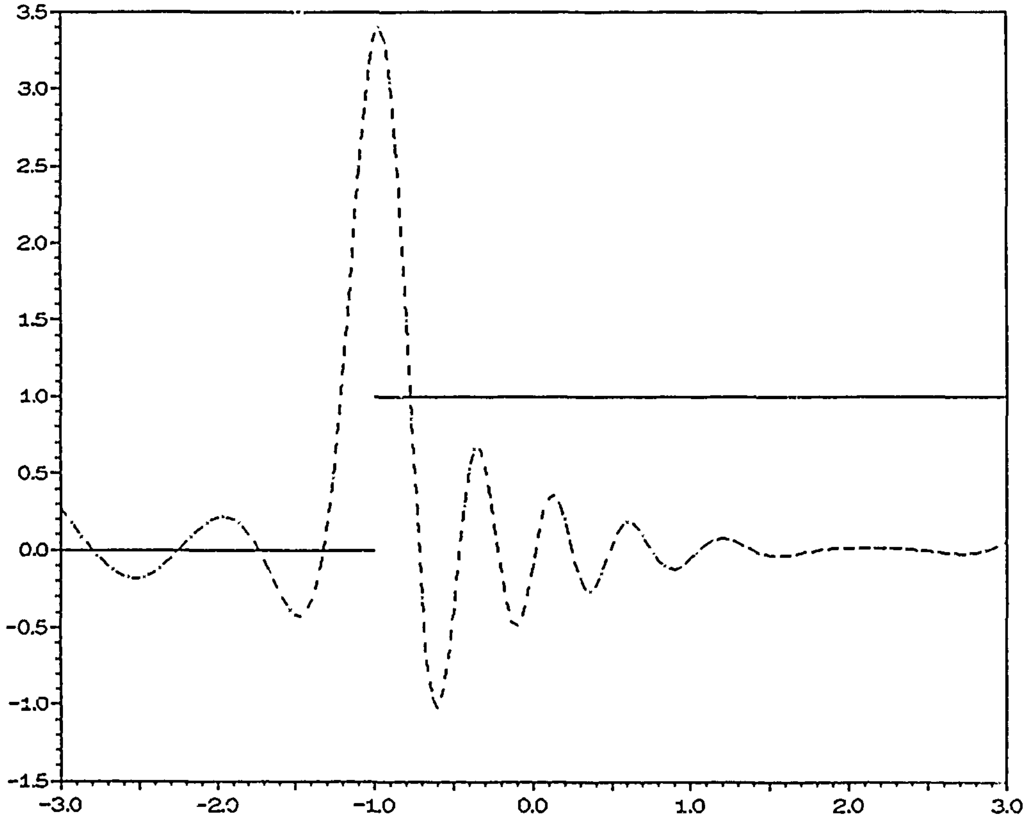


Figure One: Cumulative distribution function for the unit step function at $\theta = -1$ and its quasidensity

Table One: $P_i(-1)$ and an Approximation

| item | $P_i(-1)$ | $\int P_i q$ | diff | item | $P_i(-1)$ | $\int P_i q$ | diff |
|---|---|---|---|---|---|---|---|
| 1. | .1223 | .1223 | .0000 | 2. | .0601 | .0601 | .0000 |
| 3. | .0852 | .0852 | .0000 | 4. | .1639 | .1639 | .0000 |
| 5. | .1449 | .1449 | .0000 | 6. | .1878 | .1878 | .0000 |
| 7. | .2958 | .2958 | .0000 | 8. | .2058 | .2058 | .0000 |
| 9. | .2601 | .2601 | .0000 | 10. | .3345 | .3345 | .0000 |
| 11. | .2380 | .2380 | .0000 | 12. | .2093 | .2093 | .0000 |
| 13. | .3024 | .3023 | .0001 | 14. | .2965 | .2965 | .0000 |
| 15. | .3385 | .3385 | .0000 | 16. | .4869 | .4869 | .0000 |
| 17. | .2798 | .2795 | .0003 | 18. | .7576 | .7575 | .0001 |
| 19. | .4482 | .4483 | .0000 | 20. | .8665 | .8665 | .0000 |
| 21. | .7634 | .7634 | .0000 | 22. | .9014 | .9012 | .0002 |
| 23. | .7804 | .7804 | .0000 | 24. | .9054 | .9054 | .0000 |
| 25. | .8695 | .8696 | .0000 | 26. | .1391 | .1391 | .0000 |
| 27. | .2832 | .2832 | .0000 | 28. | .2334 | .2334 | .0000 |
| 29. | .1463 | .1463 | .0000 | 30. | .1504 | .1504 | .0000 |
| 31. | .1396 | .1396 | .0000 | 32. | .1374 | .1374 | .0000 |
| 33. | .2578 | .2578 | .0000 | 34. | .2314 | .2313 | .0001 |
| 35. | .2262 | .2262 | .0000 | 36. | .1881 | .1880 | .0000 |
| 37. | .2521 | .2521 | .0000 | 38. | .2788 | .2788 | .0001 |
| 39. | .3256 | .3256 | .0000 | 40. | .2676 | .2673 | .0003 |
| 41. | .3734 | .3734 | .0000 | 42. | .5322 | .5322 | .0000 |
| 43. | .6150 | .6149 | .0001 | 44. | .6617 | .6614 | .0003 |
| 45. | .7948 | .7948 | .0001 | 46. | .7852 | .7851 | .0001 |
| 47. | .7835 | .7835 | .0000 | 48. | .8159 | .8159 | .0000 |
| 49. | .8228 | .8227 | .0001 | 50. | .9064 | .9062 | .0001 |
| 51. | .1133 | .1133 | .0000 | 52. | .0662 | .0662 | .0000 |
| 53. | .0605 | .0605 | .0000 | 54. | .2013 | .2013 | .0000 |
| 55. | .2024 | .2024 | .0000 | 56. | .2697 | .2697 | .0000 |
| 57. | .3809 | .3809 | .0000 | 58. | .1809 | .1809 | .0000 |
| 59. | .3495 | .3495 | .0000 | 60. | .3370 | .3370 | .0000 |
| 61. | .1521 | .1521 | .0000 | 62. | .2812 | .2812 | .0000 |
| 63. | .2931 | .2931 | .0000 | 64. | .2673 | .2673 | .0000 |
| 65. | .2569 | .2569 | .0000 | 66. | .3876 | .3876 | .0000 |
| 67. | .4459 | .4459 | .0000 | 68. | .6903 | .6903 | .0000 |
| 69. | .6179 | .6179 | .0000 | 70. | .8457 | .8454 | .0003 |
| 71. | .7718 | .7718 | .0000 | 72. | .7427 | .7427 | .0000 |
| 73. | .8167 | .8167 | .0000 | 74. | .8800 | .8800 | .0000 |
| 75. | .8775 | .8774 | .0000 | 76. | .1406 | .1406 | .0000 |
| 77. | .2074 | .2074 | .0000 | 78. | .2022 | .2022 | .0000 |
| 79. | .0660 | .0660 | .0000 | 80. | .2454 | .2454 | .0000 |
| 81. | .2858 | .2858 | .0000 | 82. | .0996 | .0996 | .0000 |
| 83. | .1365 | .1365 | .0000 | 84. | .1368 | .1368 | .0001 |
| 85. | .2095 | .2095 | .0000 | 86. | .1741 | .1740 | .0000 |
| 87. | .2888 | .2888 | .0000 | 88. | .2685 | .2684 | .0001 |
| 89. | .3565 | .3565 | .0000 | 90. | .4457 | .4457 | .0000 |
| 91. | .3742 | .3742 | .0000 | 92. | .3632 | .3632 | .0000 |
| 93. | .7894 | .7894 | .0000 | 94. | .4970 | .4970 | .0000 |
| 95. | .7856 | .7856 | .0000 | 96. | .7681 | .7681 | .0000 |
| 97. | .8536 | .8532 | .0004 | 98. | .7984 | .7984 | .0000 |
| 99. | .8159 | .8159 | .0000 | 100. | .9671 | .9674 | .0003 |

Averages: .414162 .414137 .000025

## An Application of Quasidensities

As an illustrative application[2], we return to Example Three of the Preface. Let $X$ be a statistic such as $\hat{\theta}_{WK}$ for which we desire $E(X|\theta=t)$ . Let $M_1, M_2, \ldots M_K$ be binary random variables indicating group membership. For example in Example Three, $K$ numbers $x_k$ in the range of $\hat{\theta}_{AR}$ can be used to define variables of the form

$$M_k = 1 \text{ if } |\hat{\theta}_{AR} - x_k| \le .5 \text{ , else zero}$$

dividing examinees into $K$ not necessarily disjoint groups. Let $q_1, \ldots q_K$ be the quasidensities for the (conditional) distributions

$$F_k(t) = \text{Prob} ( \theta \le t \mid M_k = 1 ) .$$

Suppose $K$ is large enough and the $F_k$ different enough so that some subset of the $q_k$ forms a basis for the CS. Let $q(\cdot;s)$ be the quasidensity of the unit step at $s$ in $[c,d]$ . Then there must be numbers $c_k=c_k(s)$ such that

$$q(t;s) = \sum_{k\le K} c_k(s)q_k(t) \text{ , } c\le t\le d .$$

From the definition of $q(\cdot;s)$ we have

$$E(X|\theta=s) = \int_c^d E(X|\theta=t)q(s;s) \, dt .$$

Thus

$$\begin{aligned} E(X|\theta=s) &= \int_c^d E(X|\theta=t) \sum_{k\le K} c_k(s)q_k(t) \, dt \\ &= \sum_{k\le K} c_k(s) \int_c^d E(X|\theta=t)q_k(t) \, dt \\ &= \sum_{k\le K} c_k(s) \, E(X|M_k=1) . \end{aligned}$$

Thus the regression function on the left - expressing a conditioning on an *unobserved* ability - equals a linear combination expected values of observed scores for the *objectively defined* groups.

To apply this result $K$ is taken to be large, $q(\cdot;s)$ is computed with the identity (derived at the end of this section)

$$q(\cdot;s) = \sum_j h_j(s)h_j(t) .$$

The $q_k$ are estimated by maximum likelihood. The $c_k(\cdot)$ are computed for each $s$ by minimizing a quadratic objective function such as

$$Q(c_1, \ldots c_K) = \int_c^d [q_s(t) - \Sigma c_k(s)q_k(t)]^2 dt .$$

In this way a conditional expected value of a statistic given ability can be computed when simulation is not possible or practical.

In addition to the three examples in Example Three, there is the interesting special case of $X=u_{n+1}$ , the item score for a new item, and

$$E(X|\theta=t) = P_{n+1}(t) ,$$

its item response function. Thus the formula at the bottom of page 24 expresses an unknown item response function as a linear combination of the expected values of statistics.

## Summary of Properties of Quasidensities

Throughout this summary, let $\{h_j\}_{j=0}^{J}$ be an orthonormal basis for the CS and $\{X_j\}_{j=0}^{J}$ be statistics satisfying $E(X_j|\theta=t) = h_j(t)$ for $c \le t \le d$ .

Properties One, Two, and Three are useful for guessing the shape of the quasidensity when F has a density in the CS or is closely approximated by a distribution on [c,d] with a density in the CS. Property Four can be used even if no close approximation of F has a dens·ty in the CS. Property Five underscores the identifiability of the quasidensity by exhibiting a strongly consistent (albeit, inefficient) estimate for the quasidensity.

<u>Defining Property of Quasidensities</u>: <u>A function</u> g <u>in the CS is the quasidensity for</u> G <u>if for all statistics</u> X

$$\int_c^d E(X|\theta=t)\ dG(t) = \int_c^d E(X|\theta=t)g(t)\ dt$$

<u>Formula for Quasidensities</u>: $g(t) = \Sigma_j\ E[X_j;G]h_j(t)$

<u>Quasidensity for Step Functions</u>: <u>Let</u> $G_s$ <u>be the unit step at</u> s <u>and</u> $q(\cdot;s)$ <u>its quasidensity</u>. <u>Then</u>

$$q(t;s) = \Sigma_j\ h_j(s)h_j(t)$$

<u>Proof</u>: $E[X_j;G_s] = \int_c^d h_j(t)\ dG_s(t) = h_j(s)$

<u>Property One</u>: <u>If</u> G <u>has a continuous density</u> G′ <u>and</u> G′ <u>is in the canonical space then</u> G′ <u>is the quasidensity of</u> G .

<u>Proof</u>: $\langle R_X, G' \rangle = E(X;G)$ for all statistics X .

<u>Property Two</u>: <u>If</u> G <u>has a (not necessarily continuous) density</u> G′ <u>then the quasidensity of</u> G <u>is the projection of</u> G′ <u>into the canonical space in the sense that the quasidensity</u> g <u>is the unique minimizer in</u>

the CS of

$$\int_c^d [G'(t) - g(t)]^2 dt .$$

Proof: The general function in the CS can be written $h(t,d) = \Sigma_j [E(X_j;G) - d_j] h_j(t)$ for some vector of constants $d$. Since $E(X_j;G) = \int_c^d h_j(t)G'(t)dt$ and since the $h_j$ are linearly independent it suffices to show that $h(t,0)$ is a minimizer. This follows from the identity

$$\int_c^d [G'(t) - h(t,d)]^2 dt = \int_c^d G'^2 - \Sigma E(X_j;G)^2 + \Sigma d_j^2 .$$

Property Three: If distributions are close, then their quasidensities are close in the following sense:

If $F_1$ and $F_2$ be distributions on $[c,d]$ with quasidensities $q_1$ and $q_2$ and $\int_c^d [F_1(t) - F_2(t)]^2 dt \le \epsilon$, then $\int_c^d [q_1(t) - q_2(t)]^2 dt \le \epsilon$

Proof: For $i=1,2$ $F_i$ can be written $F_i = q_i + (F_i - q_i) = q_i + r_i$. For any orthonormal basis $\{h_j\}$, $\langle r_i, h_j \rangle = 0$ for each $j$. Thus for any $h$ in the CS, $\langle r_i, h \rangle = 0$. Consequently

$$\int_c^d [F_1(t) - F_2(t)]^2 dt = \int_c^d [q_1(t) - q_2(t)]^2 dt$$

$$+ 0$$

$$+ \int_c^d [r_1(t) - r_2(t)]^2 dt$$

$$\ge \int_c^d [q_1(t) - q_2(t)]^2 dt .$$

Property Four: The quasidensity of the limit of a convergent sequence of distributions on $[c,d]$ is the limit of the corresponding sequence of quasidensities. More precisely,

If $\{G_n\}$ is a sequence of distribution functions on $[c,d]$ weakly convergent to a distribution $G$ on $[c,d]$, then the sequence of

<u>quasidensities of the</u> $G_n$ <u>converges uniformly to the quasidensity of</u> $G$ .

<u>Proof</u>: Let $X$ be any statistic. Since the regression function for $X$ is continuous, by Helly's second theorem $\lim E(X,G_n) = \lim \int_a^b E(X|\theta=t)\, dG_n(t) = E(X;G)$ . Uniformity follows from the continuity of quasidensities.

The ability distribution clearly isn't determined by item response data. This is obvious from Theorem One. When $J$ is small, markedly different distributions can be equivalent. The quasidensity, on the other hand, can be recovered from item response data. The formula for the quasidensity shows that all one needs to estimate the quasidensity from data is the expected values of finitely many statistics.

<u>Property Five</u>: <u>The quasidensity is determined by item response data in the sense that there is a strongly consistent quasidensity estimation procedure</u>.

<u>Proof</u>: The variance of each $X_j$ must be finite because there are only finitely many possible values for $X_j$ , one for each of $2^n$ possible response patterns. Consequently $X_{j,N}$ , the sample average for $N$ randomly sampled examinees, tends to $E(X_j)$ with probability one as sample size is increased. In fact, the multivariate strong law of large numbers implies that the vector of sample means $\langle X_{0,N}, \ldots X_{J,N}\rangle$ almost surely converges to the vector of expected values $\langle E(X_0), \ldots, E(X_J)\rangle$ . Since the quasidensity $g$ for the ability distribution $F$ satisfies

$$g(t) = \sum_{j=0}^{J} E(X_j)h_j(t)$$

the random function defined by

$$g_N(t) = \sum_{j=0}^{J} X_{j,N} h_j(t) , \quad c \le t \le d$$

almost surely converges to the quasidensity. Furthermore, the convergence must be uniform in $t$ because the $h_j$ are continuous on $[c,d]$ .

NOTES

1. The term seems apt because the prefix "quasi" means "to some degree, in some manner." Although $g(t)$ may be negative,

$$\int_c^d h(t)\, dG(t) = \int_c^d h(t)g(t)\, dt$$

at least for every function $h$ in the CS.

2. There is a technical problem beyond the scope of this paper that arises in applications of this type. When the CS has been computed from only a subset of the test items then $R_X(t) = E[X|\theta=t]$ may not be in the CS. In this case the analysis yields an estimate of the projection of $R_X$ into a subspace of the CS computed from all the test items. We have observed that when only a small number of items have not been included the projection and $R_X(t)$ agree to several decimals, provided the not included items are not extremely easy, extremely hard or otherwise atypical.

## Section Three

### The Canonical Space
### Logistic Models and the Examples

This section contains proofs and additional details for assertions made earlier about the examples. We begin the study of computing the dimensionality of the CS and selecting basis functions $h_j$ and statistics $X_j$ for some simple models.

### The Rasch Model and its Generalization

In Examples One and Two it was asserted that the generalization of the Rasch Model has $J$ less than or equal to the number of items and that the item response functions or some subset of them form a basis.

If $P_i(t) = c_i + (1-c_i)[\ 1 + e^{-a(t-b_i)}]^{-1}$ then we can solve for $e^{at}$ and obtain

$$e^{at} = e^{ab_i} \frac{P_i(t)-c_i}{1-P_i(t)}$$

Thus for $i \neq j$

$$e^{ab_i}[\ 1-P_j(t)\ ][\ P_i(t)-c_i\ ] = e^{ab_j}[\ 1-P_i(t)\ ][\ P_j(t)-c_j\ ]\ .$$

If $b_i \neq b_j$ , then this equation can be simplified to obtain an expression of form

$$P_i(t)P_j(t) = a + bP_i(t) + cP_j(t)$$

where a, b, and c are independent of $t$ . Thus any product of two item response functions can be rewritten as a linear combination of the item response functions plus a constant. Using this fact it's easy to prove the assertions concerning these models in Example One.

<u>If item response functions satisfy the formula for the Rasch model or its generalization with</u> $b_i \neq b_j$ <u>for</u> $i \neq j$ , <u>then</u>

1. <u>The dimensionality of the canonical space is less than or equal to one plus the number of items</u>
2. <u>The constant function and the item response functions or some subset of these functions form a basis for the CS</u>
3. <u>The item scores satisfy the condition on the</u> $X_i$ <u>in Theorem One and Example One</u>.

<u>Proof</u>: Since the square-free monomials span the canonical space, it is sufficient to show that every square-free monomial can be expressed as a linear combination of the $P_i$ plus the constant function $h_0(t)=1$ . Any square-free monomial containing two or more of the item response functions can be written in form $RP_iP_j$ for $i \neq j$ for $R$ equal to a square-free monomial not divisible by $P_i$ or $P_j$ . Thus $RP_iP_j = aR + bRP_i + cRP_j$ can be rewritten as the linear combination of three square-free monomials, each of which has fewer factors than the original monomial. By iterating this process one eventually obtains a linear combination of square-free monomials depending on one of the $P_i$ or none of the $P_i$ (i.e. $h_0$). Thus $h_0$ and the $P_i$ span the CS, which proves 1. and 2. The remaining assertion follows from $E(u_i | \theta = t) = P_i(t)$ .

## Selecting Points for Example Two

In Example Two we considered changing integrals to sums. It was asserted that there were numbers $t_0, t_1, \ldots t_J$ and $p(t_0), p(t_1), \ldots p(t_J)$ such that for any vector of zeros and ones $u^*$, the manifest or pattern probability

$$\int \text{lik}(u^*|t)\ dF(t)$$

could be written

$$\Sigma_k \text{ lik}(u^*|t_k)p(t_k)\ .$$

This is an example of a more general result, proven in this subsection: For any statistic $X$ (including the statistic that is one if the observed item response pattern equals $u^*$ and zero otherwise)

$$\int E[X|\theta=t]\ dF(t) = \Sigma_k E[X|\theta=t_k]p(t_k)\ .$$

The choice of the $t_k$ and computation of the $p(t_k)$ is also discussed. We use the notation $q(\cdot;t_k)$ for the quasidensity of the unit step function at $t_k$ and the fact that $q(\cdot;t_k) = \Sigma_j h_j(t_k)h_j(\cdot)$ for any orthonormal basis for the CS.

The result need only be proven for bounded ability continua since any item response model with continuous $P_i \neq 0,1$ can be transformed by an invertible transformation to a bounded model. The proof is split into two parts: The existence of a basis consisting of quasidensities and interpretation of the $p(t_k)$ .

The results indicate the following procedure for selecting points and computing p's for a model with CS having basis $\{h_j\}_{j=0}^J$ :

1. Choose $t_0, t_1, \ldots t_J$ such that the matrix

$$\begin{bmatrix} h_0(t_0) & h_1(t_0) & \cdots & h_J(t_0) \\ h_0(t_1) & h_1(t_1) & \cdots & h_J(t_1) \\ \vdots & \vdots & & \vdots \\ h_0(t_J) & h_1(t_J) & \cdots & h_J(t_J) \end{bmatrix}$$

is nonsingular

2. Compute $p(t_0), p(t_1), \ldots p(t_J)$ by solving the linear equations $g(t_j) = \Sigma_k\, p(t_k) q(t_j; t_k)$ for $j=0,1, \ldots J$ where $g$ is the quasidensity of $F$ .

For the generalization of the Rasch model, the procedure can be simplified; the $p(t_k)$ can be found by solving the system of linear equations

$$E(u_i) = \Sigma_k\, p(t_k) P_i(t_k) \quad i=1, \ldots n$$
$$1 = \Sigma_k\, p(t_k)$$

Generalizations and proofs follow.

<u>If a test has continuous item response functions</u> $\neq 0,1$ <u>defined on an interval</u> $[c,d]$ <u>then the CS has a basis consisting of quasidensities of unit step distributions</u>.

<u>Proof</u>: Let $\{h_j\}_{j=0}^{J}$ be an orthonormal basis for the CS and let $h(t)$ denote the column vector

$$h(t) = \langle h_0(t), h_1(t), \ldots h_J(t)\rangle^T .$$

Since the $h_j$ are linearly independent there must be $J+1$ values of $t$ such that the vectors $h(t_0), h(t_1), \ldots h(t_J)$ are linearly independent. It follows that the partitioned matrix $[h(t_0), h(t_1), \ldots h(t_J)]$ has an inverse, say $A=(a_{ij})$ . Consequently, using Kronecker's delta notation each $h_j$ can be written as a linear combination of the quasidensities $q(\cdot; t_i)$

$$\begin{aligned} h_j(t) &= \Sigma_k\, h_k(t)\, \delta_{kj} \\ &= \Sigma_k\, h_k(t) \left[\Sigma_m h_k(t_m) a_{mj}\right] \\ &= \Sigma_m\, a_{mj}\, \Sigma_k h_k(t) h_k(t_m) \end{aligned}$$

$$= \Sigma_m a_{mj} q(t;t_m) .$$

Thus the quasidensities form a basis for the CS.

As a corollary, we have

<u>The quasidensities of unit step distributions at</u> $t_0, t_1, \ldots t_J$ <u>span the CS if and only if</u> $[h(t_0), h(t_1), \ldots h(t_J)]$ <u>is non-singular</u>.

In practice on this type of problem we compute the $t_k$ recursively. After having chosen $t_0, t_1, \ldots t_k$ we choose $t_{k+1}$ such that $h(t_{k+1})$ makes a relatively large angle with its projection into the linear space spanned by $h(t_0), h(t_1), \ldots h(t_k)$ .

After the $t_k$ are selected the calculation of the $p(t_k)$ is straight forward. Since the quasidensities for the $t_k$ form a basis for the CS, the ability distribution's quasidensity is a linear combination of the $q(\cdot;t_k)$ and the coefficients of the combination are unique. The $p(t_k)$ are simply the coefficients of the linear combination.

<u>Let</u> $\{q(\cdot;t_k)\}_{k=0}^{J}$ <u>be a basis for the CS and the quasidensity for the ability distribution be</u> $\Sigma_k p(t_k)q(\cdot;t_k)$ . <u>Then for any statistic</u> $X$ , $E(X) = \Sigma_k E(X|\theta=t_k)p(t_k)$ . <u>In particular for any vector of zeros and ones</u> $u^*$ , $Prob(u=u^*) = \Sigma_k Prob(u=u^*|\theta=t_k)p(t_k)$ .

<u>Proof</u>: Let $X$ be any statistic. Then from the defining property of quasidensities

$$E(X) = \int_c^d E(X|\theta=t) \Sigma_k p(t_k)q(t;t_k) dt$$

$$= \Sigma_k p(t_k) \int_c^d E(X|\theta=t)q(t;t_k) dt$$

$$= \Sigma_k p(t_k) E(X|\theta=t_k)$$

In particular for any vector of zeros and ones $u^*$ if $X$ is the random variable that is one if $u=u^*$ and zero otherwise, $Prob(u=u^*) = E(X)$

$= \Sigma_k \text{Prob}(u=u^* | \theta=t_k)p(t_k)$ .

## Models with Very Large J

If J is small, as is the case with the Rasch model and its generalization, then standard techniques can be used for computing an orthonormal basis. ‿wever, if the dimensionality of the CS is as large as the number of square-free monomials ( $2^n$ ) then computing an orthonormal basis is problematical. To conclude this section it is shown that for the most commonly used item response models, the three parameter logistic models, J+1 typically is equal to its upper bound.

Item response functions are three parameter logistic (3PL) if

$$P_i(t) = c_i + (1-c_i)[1 + e^{-a_i(t-b_i)}]^{-1}$$

for some item parameters $a_i>0$, $b_i$, and $c_i$ in $[0,1)$ . It is natural to consider the item parameters random variables because in most applications they are estimated from data. Suppose the sampling distribution of the estimated parameters has a continuous density. Then the following result is of interest.

<u>If the joint distribution of the</u> n <u>item parameter vectors</u> $\langle a_i, b_i, c_i \rangle$ <u>has a continuous density, then with probability one the CS of the 3PL item response model defined with sampled item parameters will have dimension</u> $2^n$

Thus, for example, if one begins with the any published set of estimated item parameters for an application of the 3PL model and adds an independent normally distributed "error" with zero mean and very small variance, say $10^{-10}$, to each of the 3n parameters, then with probability one either one of the a's or c's will be moved outside its allowed range or a 3PL model with J as large as it possible can be will be obtained.

Proof: With probability one, the functions

$$e^{a_1 t}, e^{a_2 t}, \ldots e^{a_n t}$$

will be algebraicly independent over the reals, i.e. will not satisfy any nontrivial polynomial with real coefficients. But if $J+1<2^n$ then one of the square-free monomials can be expressed as a linear combination of the remaining monomials. On multiplying both sides of the equation giving one monomial as a linear combination of the others by positive $\Pi_i[e^{a_i t} + e^{a_i b_i}]$ one obtains a polynomial in the $e^{a_i t}$ and a contradiction to the hypothesis $J+1<2^n$ .

## Section Four

## Large Canonical Spaces

Consider Example Three for a test with large CS for an application currently in progress. In a large scale simulation we are attempting to monitor and control the changes in a Bayes modal ability estimate as new items are introduced into a 100 item adaptive test item pool. The item response function estimates for the new items are not expected to be very accurate because of motivation, test format, and ability distribution differences between the item response function estimation sample and the examinees in the application. The methods to be reviewed in this section permit us to compute as many as we need of the roughly $2^{100}$ orthornormal $h_j$ for the test consisting of old items.

The trick is to compute the $h_j$ one-at-a-time in such a way that the $h_j$ needed to complete the application are computed first. Thus the CS is treated as the union of nested vector spaces $CS_K$

$$CS_K = \text{Span}(h_0, h_1, \ldots h_K)$$

where functions in only a dozen or so spaces can be and need be accurately computed. Some details follow.

We wish to approximate $E(\hat{\theta}_{WK}|\theta=t) = R(t)$ , where $\hat{\theta}_{WK}$ is the Bayes mode adaptive test score. It turns out that although $J$ is very large, the projection $\hat{R}$ of $R$ into the twelfth space

$$\hat{R}(t) = \Sigma_{j\le 12} <R,h_j>h_j(t)$$

is very close to $R(t)$ . Now if $\hat{q}(\cdot;s) = \Sigma_{j\le 12} h_j(s)h_j(t)$ is the projection $q(\cdot;s)$ into the twelfth space then $\int_c^d E(\hat{\theta}_{WK}|\theta=t)\hat{q}(t;s)\, dt = \hat{R}(t)$ . Thus if we can write $\hat{q}(\cdot;s)$ as

$$\hat{q}(\cdot;s) = \Sigma_{k\le K} c_k(s)q_k(\cdot)$$

a linear combination of quasidensities for the K AR score groups $q_k(\cdot)$ , then

$$\hat{R}(t) = \Sigma_{k \le K} c_k(s) E[\hat{\theta}_{WK} | \hat{\theta}_{AR} \text{ is in } M_k] .$$

The point is that if an application can be completed using $h_0, h_1, \ldots h_K$ only then it may be possible to proceed *as if* J=12 .

This section describes a general technique used by our laboratory for calculating the $h_j$ one-at-a-time in such a way that functions that are likely to be needed for an application are well approximated by a function in $CS_K$ for small K .

## The General Method

The first step of our approach to large spaces is to select a set of functions $\{f_\nu\}_{\nu=1}^{N}$ that span the CS *and* are such that the function of two variables $\Sigma_\nu f_\nu(s) f_\nu(t)$ can be easily evaluated. For example if $f_1, f_2, \ldots f_\nu, \ldots f_{2^n}$ is any enumeration of the square-free monomials then the $f_\nu$ span the CS. Furthermore for any s and t

$$\sum_{\nu=1}^{2^n} f_\nu(s) f_\nu(t) = \prod_{1}^{n} [1 + P_i(s) P_i(t)]$$

can be evaluated with 2n-1 multiplications and n additions. (This identity can be verified by induction on test length n .) Other examples of tractible spanning sets and additional criteria for spanning sets are discussed below.

There are two important points to be emphasized here. Although there are generally billions of $f_\nu$ to enter into the sum $H(s,t) = \Sigma_\nu f_\nu(s) f_\nu(t)$ , the multiplicative formula for H(s,t) requires only n additions and 2n-1 multiplications. Second, the ordering of the $f_\nu$ is inconsequential. Whereas the outcome of a Gram-Schmidt orthogonalization applied to the square-free monomials or any other large set of functions $f_\nu$ would be very order dependent, the calculation of H is not.

The next step in computing the $h_j$ can be carried out with commercial software or can be converted to a eigenvalue/eigenvector problem: Compute positive numbers $\lambda$ and functions h not identically equal to zero such that each h is in the CS and satisfies

$$\lambda h(\cdot) = \int_c^d H(\cdot,t)h(t)\,dt$$

where $H(s,t) = \Sigma_\nu\ f_\nu(s)f_\nu(t)$ . There will be only finitely many different values of $\lambda$ such that there is some $h\neq 0$ in the CS satisfying the equation. Since the h's are in the CS there can be only finitely many linearly independent solutions h for any $\lambda$ . Thus any maximal set of linearly independent solutions can be subscripted and arranged in order of their subscript so that $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_K > 0$ for some $K \leq J$ and $\lambda_j h_j(\cdot) = \int_c^d H(\cdot,t)h_j(t)\,dt$ .

Without loss of generality we can set $\langle h_j,h_j\rangle = 1$ since $h_j(t)$ is a solution for $\lambda_j$ if and only if $h_j(\cdot)/\langle h_j,h_j\rangle$ is. Since the set of all h's corresponding to any $\lambda$ form a vector space, they can be selected to be orthonormal. Since it can be shown h's with different $\lambda$'s are orthogonal, the $h_j$ will form an orthonormal set of vectors In fact it is easy to show that when the $f_\nu$ span the CS, K=J and the $h_j$ computed in this way form an orthonormal basis for the CS. If an application suggests a set of $f_\nu$ that don't span the CS, then K<J and the $h_j$ will be a basis for whatever subspace the $f_\nu$ span.

Note that except in the unusual case that more than one h correpsonds to one $\lambda$ , the h's are fully ordered by their $\lambda$'s . Even if for some j , $\lambda_j = \lambda_{j+1}$ , the h's corresponding to different $\lambda$'s will be ordered and we can still speak of $h_j$ occuring early or late in the sequence of h's .

The ordering is important because for various reasons (cumulative numerical errors and the fact that $\lambda_j$ is very close to zero for large $j$ ) the $h_j$ that occur early in the sequence are relatively easy to compute (although the remaining $h_j$ can be very hard to compute).

There are two related advantages in arranging tne computation of basis functions as described above. The $h_j$ with large $\lambda_j$ , which are easy to compute, can be computed without computing the $h_j$ with small $\lambda$ , which can be very hard to compute. This is important because $\lambda_j$ generally measures the relative importance $h_j$ in representing functions in several senses. For example, if $f_\nu$ is approximated by its projection into $CS_K = \text{span}\{h_0, \ldots h_K\}$ , which turns out to be $\hat{f}_\nu(\cdot) = \sum_{j\le K} \langle f_\nu, h_j\rangle h_j$ for $K<J$ , then the total error

$$\sum_\nu \int [f_\nu(t) - \hat{f}_\nu(t)]^2 \, dt$$

is simply $\sum_{j>K} \lambda_j$ . (This sum can be evaluated even if $J$ is very large because

$$\int_c^d H(t,t)\, dt - \sum_{j\le K} \lambda_j = \sum_{j>K} \lambda_j \text{ .)}$$

As a bonus, the method also delivers a set of statistics $X_j$ needed for Example One and Theorem One (i.e., statistics such that $h_j(t) = E[X_j|\theta=t]$ for all $t$ in $[c,d]$ ). Details are given in the final subsection.

## Some Examples of Spanning Sets

In addition to the square-free monomials we use the $2^n$ likelihood functions for short tests. Here

$$f_\nu(t) = \prod_{i=1}^{n} P_i(t)^{u^*_{i,\nu}} [1-P_i(t)]^{(1-u^*_{i,\nu})}$$

where $u^*_1, \ldots u^*_\nu, \ldots u^*_{2^n}$ is any enumeration of the $2^n$ item response patterns. For these functions

$$H(s,t) = \Sigma_\nu\, f_\nu(s) f_\nu(t)$$
$$= \prod_1^n \{P_i(s)P_i(t)+[1-P_i(s)][1-P_i(t)]\} ,$$

which can be easily evaluated. (This also can be proven by induction on test length $n$ after noting that each likelihood function can be written as $f_\nu(t) = \prod_{i=1}^{n} \{u^*_{i,\nu}P_i(t) + (1-u^*_{i,\nu})[1-P_i(t)]\}$ .) These functions certainly span the CS because any square-free monomial can be written as a linear combination of likelihood functions. (To prove this, simply write the general monomial $\prod_{j\le r} P_{i_j}$ as the sum of the likelihoods for patterns $u^*$ with $u^*_{i_1}=u^*_{i_2}= \ldots u^*_{i_r}=1$ .)

For adaptive tests and long tests satisfying (exactly or approximately) an algebraic property described below, we use likelihood functions for selected subtests. For example to study a fixed length adaptive test of 15 items with a 100 item pool it is natural to consider the $\binom{100}{15} \ll 2^{100}$ likelihood functions with fifteen factors since every statistic computed from an examinee's score depends on only 15 item scores.

The discussion of the Rasch model introduces a second rationale for forming the $f_\nu$ from the likelihood functions for short subtests. Recall that for the Rasch model every polynomial in the CS could be rewritten as a "polynomial" in the CS, no monomial of which contained 2 or more factors. This property is remarkably general. For the 3PL model (and most of its generalizations) every polynomial in the CS can be rewritten as a linear combination of monomials with five or fewer factors, at least to a surprisingly high degree of approximation[1].

When every function in the CS can be expressed as a linear combination of square-free monomials with five or fewer factors, then the CS is spanned by the likelihood functions from subtests with five factors. There are still an enormous number of likelihood functions $f_\nu$ that can be formed from from all five item subtests. Nonetheless $H(s,t) = \Sigma f_\nu(s)f_\nu(t)$ can be computed efficiently for these functions as follows:

Let $F_i(s,t)$ abbreviate $P_i(s)P_i(t) + [1-P_i(s)][1-P_i(t)]$ .
Let $H_i^m(s,t)$ denote the sum of the likelihood functions for all $i$ item subtests formed from the first $m$ items.

To initialize set

$$H_1^1(s,t) = F_1(s,t)$$

$$H_i^1(s,t) = 0 \quad \text{for} \quad i=2,3,\ldots 5 .$$

To update, compute

$$H_i^{m+1} = F_{m+1}H_{i-1}^m \quad \text{for} \quad i=2,\ldots 5$$

$$H_1^{m+1} = F_{m+1} + H_1^m$$

If in the update step $H_5^{m+1}$ is computed first, followed by $H_4^{m+1}$ , etc., then $H_j^{m+1}$ can be written over $H_j^m$ and the amount of storage required by the algorithm can be kept small.

Most of our current applications to one dimensional ability tests use this algorithm. Although some of the CS may be left out, the algorithm in practice works very well. It is the only algorithm that has consistently produced useful results with long tests.

## Reduction of Proofs to Matrix Algebra

A number of assertions were made without proof concerning the solutions for the functional equation

$$\psi(h) = \lambda h$$

where $\psi(h)(\cdot) = \int_c^d H(\cdot,t)h(t)\,dt$ for $H(s,t) = \Sigma_\nu f_\nu(s)f_\nu(t)$ . By taking advantage of the finite dimensionality of the CS these proofs can be obtained with matrix algebra. In this section the reduction to matrix algebra is indicated after a few of the assertions are proven directly.

First $\psi$ is a transformation of the CS to itself because the $f_\nu$ are in the CS and $\psi(h) = \Sigma_\nu \langle f_\nu,h\rangle f_\nu$ is a linear combination of the $f_\nu$ . $\psi$ is thus a linear mapping of a finite dimensional vector space into itself.

To show that the eigenfunctions of $\psi$ span the CS it is neccesary to show that $\psi$ maps the CS onto the CS. Equivalently, since the CS is finite dimensional, one may show $\psi(h)=0$ implies $h=0$ . To show this one can write $f_\nu(\cdot) = \sum_{j=0}^{J} a_{\nu j}g_j(\cdot)$ for some orthonormal basis $\{g_j\}_{j=0}^{J}$ . The matrix $A=(a_{\nu j})$ must have rank $J+1$ since the $f_\nu$ span the CS. If $\psi(h)=0$ , then $0 = \langle g_j,\psi(h)\rangle = e_j^T A^T A\langle g,h\rangle$ , $j=0,\ldots J$ where $e_j$ is the $j$*th* unit vector and $\langle g,h\rangle$ is the column vector of $\langle g_j,h\rangle$'s . Thus $A^T A\langle g,h\rangle=0$ . Since $A^T A$ has rank $J+1$ , $\langle g,h\rangle=0$ , i.e., $h$ is orthogonal to each $g_j$ . Thus $h=0$ .

The existence of eigenfunctions in the CS and the fact that the eigenfunctions span the CS can be shown with matrix algebra. To introduce matrix notation, for each $t$ in $[c,d]$ let $f(t)$ be the column vector with $\nu$*th* coordinate $f_\nu(t)$ . Then $H(s,t)$ is the scalar product of $f(s)$ and $f(t)$ . Let $Q$ denote the matrix of definite integrals

$Q = \int_c^d f(t)f^T(t)\,dt$ , i.e. $Q$ is the matrix with typical entry $q_{\nu\nu'} = \langle f_\nu, f_{\nu'} \rangle$ .

Q must be positive definite or positive semidefinite since for any vector $a$ , $a^TQa = \int_c^d [a\cdot f(t)]^2\,dt \geq 0$ . Therefore for some $K$ , $Q$ can be written $Q = [a^0, a^1, \ldots a^K]^T D [a^0, a^1, \ldots a^K]$ for $K+1$ orthonormal vectors $a^j$ and a diagonal matrix $D$ having positive diagonal entries $d_j > 0$ . For $0 \leq j \leq K$ let $h_j$ be defined by

$$h_j(t) = d_j^{-1/2}\, a^j \cdot f(t) \ .$$

Since each $h_j$ is a linear combination of functions in the CS, each must be in the CS. The $h_j$ are orthonormal since

$$\begin{aligned}\langle h_j, h_k \rangle &= d_j^{-1/2} d_k^{-1/2}\, a^{jT} \int_c^d f(t)f^T(t)\,dt\; a^k \\ &= d_j^{-1/2} d_k^{-1/2}\, a^{jT} Q a^k \ . \\ &= \begin{cases} 0 \ , & \text{if } j \neq k \\ 1 \ , & \text{if } j = k \end{cases}\end{aligned}$$

In fact the $h_j$ must be eigenfunctions of $\psi$ because

$$\begin{aligned}\psi(h_j) &= \int_c^d f^T(t) f(\cdot)\, d_j^{-1/2}\, a^j \cdot f(t)\,dt \\ &= d_j^{-1/2}\, a^{jT} \int_c^d f(t)f^T(t)\,dt\; f(\cdot) \\ &= d_j^{-1/2}\, a^{jT} Q f(\cdot) \\ &= d_j^{1/2}\, a^{jT} f(\cdot) \\ &= d_j h_j \ .\end{aligned}$$

$K$ must equal $J$ because otherwise $\psi$ would not map the CS onto the CS. Thus the eigenfunctions form an orthonormal basis for the CS.

## The Statistics $X_j$

In Example One and Theorem One statistics with regression functions equal to $h_j$ were needed. Of course such statistics exist because every function in the CS, by definition, is the regression function of at least one statistic. Finding a statistic matching a function fortunately turns out to be easy for bases formed from eigenfunctions.

When the $h_j$ are obtained as eigenfunctions, these statistics are calculated in two steps. First, the examinee's data is transformed into a continuous function $X(t)$ . Then a statistic is obtained by computing $\langle X,h_j\rangle/\lambda_j$ .

For concreteness consider the second example of the general method in which each $f_\nu$ is a likelihood function. The general technique applied to this example gives $X(t)$ equal to the familiar likelihood function as the random function

$$X(t) = \prod_{i=1}^{n} [u_i P_i(t) + (1-u_i)Q_i(t)]$$

and $X_j = \int_c^d X(t)h_j(t)\, dt/\lambda_j$ .

To verify that the regression function for this statistic is $h_j$ , we compute as follows. The regression function for $X_j$ evaluated at $\theta=s$ is

$$\begin{aligned} E[X_j|\theta=s] &= \lambda_j^{-1}E[\int X(t)h_j(t)\, dt|\theta=s] \\ &= \lambda_j^{-1} \int \prod_{i=1}^{n}{}' [P_i(s)P_i(t) + Q_i(s)Q_i(t)]h_j(t)\, dt \\ &= \lambda_j^{-1} \int H(s,t)h_j(t)\, dt \\ &= h_j(s) . \end{aligned}$$

The general rule for obtaining a random function $X(t)$ for arbitrary $f_\nu$ is to make the replacements

$P_1(s) \to u_1$

$P_2(s) \to u_2$

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$P_n(s) \to u_n$

in $f_\nu(s)$ to obtain a random variable $Y_\nu(u)$ from $f_\nu(s)$. A random function $X$ is defined by

$$X(t) = \Sigma_\nu \, Y_\nu(u) f_\nu(t) \; .$$

Finally, a random variable having regression function equal to the $j$*th* basis function is obtained as $\int_c^d X(t) h_j(t) \, dt/\lambda_j$. To summarize

Let $H(s,t) = \Sigma f_\nu(s) f_\nu(t)$ for functions in the CS $f_\nu$ not necessarily spanning the CS. Let $h$ satisfy $\int_c^d H(\cdot,t)h(t)\,dt = \lambda h(\cdot)$ for positive $\lambda$. For each $t$ in $[c,d]$ let $X(t)$ be the random variable obtained by replacing each $P_i(s)$ by $u_i$ in the formula defining $H(s,t)$. If $X_j = \langle X, h_j \rangle / \lambda_j$, then $E[X_j | \theta = t] = h_j(t)$ for $c \le t \le d$.

Note, the transformation $f_\nu(s) \to Y_\nu$ generally cannot be defined on the CS because if two items have the same item response function, then we can have $f_\nu(\cdot) = f_{\nu'}(\cdot)$ as functions in the CS but $Y_\nu \neq Y_{\nu'}$. The problem can be avoided by regarding $f_\nu(s)$ as a polynomial with real coefficients in algebraicly independent variables $P_1(s), P_2(s), \ldots P_n(s)$.

Proof: $E[X(t) | \theta = s] = H(s,t)$.

NOTES

1. Levine, M. and Williams, B. Latent trait theory as fundamental measurement, Paper presented at Society for Mathematical Psychology Annual Conference, Irvine, California, 1989.

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