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This paper reviews the use and structure of the Slepian model and the regression method and shows how they can be used to obtain good numerical approximations to various crossing variables. It gives a detailed account of the regression method for Gaussian processes with auxilliary variables chosen in a recursive way. It also presents a package of computer programs for the numerical calculations, and gives

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CENTER FOR STOCHASTIC PROCESSES

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SLEPIAN MODELS AND REGRESSION APPROXIMATIONS

IN CROSSING AND EXTREME VALUE THEORY

by

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and

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SLEPIAN MODELS AND REGRESSION APPROXIMATIONS IN CROSSING AND EXTREME VALUE THEORY

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Abstract

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This paper reviews the use and structure of the Slepian model and the regression method and shows how they can be used to obtain good numerical approximations to various crossing variables. It gives a detailed account of the regression method for Gaussian processes with auxilliary variables chosen in a recursive way. It also presents a package of computer programs for the numerical calculations, and gives numerical examples on excursion lengths as well as wavelength and amplitude distributions. Further examples deal with an engineering "jump–and–bump" problem, and excursions for a χ^2–process.

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Slepian models and regression approximations in crossing and extreme value theory

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1. INTRODUCTION

In the physical world, a random function is often described as a sequence of local maxima or minima, constituting a series of random waves. In fact, not only the visual impression of the process, but also many technologically important implications in such fields as metal fatigue caused by random vibrations, failure caused by excess load on a construction, etc., depend on the character of the process as a random wave form. The basic objects in this theory are *level crossings* and *local extremes*.

A *Slepian model* is a random function representation of the conditional behaviour of a stochastic process after events defined by level or curve crossings. In general, a Slepian model contains one regression term with random coefficients which describe the dependence on initial conditions such as the slope at the crossing, the value of the process at a pre-determined point, etc, and one residual term, which describes the deviations from the path set out by the initial conditions.

In crossing theory there are many variables of great practical interest which are very difficult to analyse, such as the distance between level crossings, maximum height of an excursion between two level crossings, etc. The distribution of these quantities can usually be written only in the form of infinite–dimensional integrals and there is therefore a large need for good numerical approximations. A *regression approximation* of a crossing variable is defined as the corresponding variable in the regression term of the Slepian model. Its distribution can often be found explicitly, or expressed as a finite–dimensional integral, which can be evaluated explicitly.

The model was first introduced by Slepian (1963) to describe the behaviour of a stationary Gaussian process after a zero crossing. The term Slepian model was introduced by Lindgren (1977). Ditlevsen (1985a) gives a review of different engineering applications, and Ditlevsen (1985b), (1986), and (1988) contain further applications.

The regression approximation based on the Slepian model was first used by Lindgren & Rychlik (1982) in connection with the analysis of random waves. It has been used to find practically useful upper and lower bounds for random wave characteristics such as wave length and amplitude and related quantities often used in reliablility, ocean engineering, structural and mechanical engineering, and other fields of technology; see e.g. Rychlik (1988, 1989a).

A further important use of the Slepian model is that of asymptotic expansions. After a crossing of a very high level many processes take on an almost deterministic form, which can be found from the leading terms in a Taylor expansion; see Lindgren (1983, 1984b, 1984c), Ditlevsen & Lindgren (1988).

Most of the work on Slepian models and regression approximations in crossing theory has been directed towards Gaussian processes or to functions of such processes. However, the Slepian model is not limited to Gaussian processes even if its structure may not be as explicit for non–Gaussian processes. The χ^2–process is a simple example of a non–Gaussian process for which the Slepian model is analytically tractable; see Aronowich & Adler (1985), (1986), and Lindgren (1989).

In this paper we shall give a review of the use and structure of the Slepian model in crossing theory and how it can be used together with the regression approximation to find good approximations to notoriously difficult distributions such as wavelength and amplitude, waiting–times between crossings of constant and moving barriers and other quantities.

Section 2 of this paper contains the basic form of Slepian models after level crossings and local extremes and the simple regression approximations for the length of an excursion and for the wave length and amplitude after a local maximum. In Section 3 we give mathematical theorems about the interpretation of Slepian models as weak limits of conditional

distributions given an upcrossing in the Kac & Slepian *horizontal window* sense (see Kac & Slepian (1959)), and as empirical limits in the observed series of level crossings.

Section 4 describes the regression approximation for excursion length as well as for wave length and amplitude, first in a the simple case with fixed regression variables, and then in a more complicated situation where the regression variables are chosen in a recursive way. In Section 5, finally, we apply the theory to an engineering example, chosen from structural mechanics, and to the level crossing distances in the χ^2–process. Numerical examples, based on a library of FORTRAN subroutines, are also presented in this section.

2. INTRODUCTORY EXAMPLES

As first examples of Slepian models and regression approximations we shall describe the behaviour of a stationary Gaussian process after upcrossings of a fixed level and after local maxima.

Suppose $\xi(t)$ is a stationary Gaussian process with mean zero and continuously differentiable sample paths. When needed we shall assume also the second derivative to exist. Let the covariance function be $r(t) = E(\xi(s)\cdot\xi(s+t))$ with spectral density $R(t)$, i.e.

$$r(t) = \int_{-\infty}^{\infty} \exp(i\omega t)R(\omega)\, d\omega, \tag{1}$$

and write

$$\lambda_0 = V(\xi(t)) = r(0) = \int_{-\infty}^{\infty} R(\omega)\, d\omega,$$

$$\lambda_2 = V(\xi'(t)) = -r''(0) = \int_{-\infty}^{\infty} \omega^2 R(\omega)\, d\omega,$$

$$\lambda_4 = V(\xi''(t)) = r^{IV}(0) = \int_{-\infty}^{\infty} \omega^4 R(\omega)\, d\omega,$$

for the spectral moments, whenever they exist.

For any fixed level u we say that a u–upcrossing occurs at time t_0 if

$$\xi(t_0) = u,\ \xi'(t_0) > 0, \tag{2}$$

and we write $N(u; T)$ for the number of u–upcrossings in the interval $[0,T]$. Then the mean number of upcrossings is given by Rice's formula, (see Leadbetter et al. 1983), which states

$$E(N(u;T)) = T\cdot\int_0^\infty y f_{\xi(0),\xi'(0)}(u,y)\,dy =$$

$$= T\cdot f_{\xi(0)}(u)\cdot E((\xi'(0))^+ \mid \xi(0)=u),$$

where $z^+ = \max(0,z)$. In the normal case this specializes to

$$E(N(u;T)) = T\cdot(2\pi)^{-1}(\lambda_2/\lambda_0)^{1/2}\exp(-u^2/2\lambda_0),$$

since then $\xi(0)$ and $\xi'(0)$ are independent and normal with mean zero and variances λ_0 and λ_2, respectively.

We shall also need a formula for the expected number of marked upcrossings, i.e. crossings which satisfy some additional condition, specified by a set A. Let $\eta(t)$ be a random vector which is jointly stationary with $\xi(t)$, and write $N(u,A; T)$ for the number of u–upcrossings by $\xi(t)$, $0\le t\le T$, which satisfy $\eta(t+\cdot) \in A$. Then

$$E(N(u,A; T)) = T\cdot\int_0^\infty y f_{\xi(0),\xi'(0)}(u,y)\cdot P(\eta(\cdot)\in A \mid \xi(0)=u,\ \xi'(0)=y)\,dy$$

$$= T\cdot f_{\xi(0)}(u)\cdot E((\xi'(0))^+\cdot I\{\eta(\cdot)\in A\} \mid \xi(0)=u); \tag{3}$$

see Leadbetter et al. (1983).

We shall have use for some formulas for the conditional mean and covariances for $\xi(t)$ given value and derivatives at a specified point. Since $\mathrm{Cov}(\xi'(s),\xi(s+t)) = -r'(t)$, $\mathrm{Cov}(\xi''(s),\xi(s+t)) = r''(t)$, and $\mathrm{Cov}(\xi''(s),\xi'(s+t)) = r'''(t)$, one has

$$E(\xi(s+t)\mid\xi(s)=u,\ \xi'(s)=y) = ur(t)/\lambda_0 - yr'(t)/\lambda_2, \tag{4}$$

$$E(\xi(s+t)\mid\xi(s)=u,\ \xi'(s)=y,\ \xi''(s)=z) =$$

$$= uA(t) + yB(t) + zC(t), \tag{5}$$

$$\mathrm{Cov}(\xi(s+t_1),\xi(s+t_2)\mid\xi(s)=u,\ \xi'(s)=y) =$$

$$= r(t_2-t_1) - r(t_1)r(t_2)/\lambda_0 - r'(t_1)r'(t_2)/\lambda_2, \tag{6}$$

$$\mathrm{Cov}(\xi(s+t_1),\xi(s+t_2)\,|\,\xi(s)=u,\ \xi'(s)=y,\ \xi''(s)=z) =$$
$$= r(t_2-t_1) - r(t_1)r(t_2)/\lambda_0 - r'(t_1)r'(t_2)/\lambda_2 - b(t_1)b(t_2), \qquad (7)$$

where

$$b(t) = \frac{\mathrm{Cov}\,(\,\xi(s+t),\xi''(s)\,|\,\xi(s),\xi'(s))}{V(\xi''(s)\,|\,\xi(s),\xi'(s))^{1/2}} = \frac{r''(t)+(\lambda_2/\lambda_0)r(t)}{\sqrt{\lambda_4-\lambda_2^2/\lambda_0}}.$$

DEFINITION 1: (Standard process) In the sequel, we shall illustrate the regression approximation by means of a stationary Gaussian low frequency white noise process $\xi(t)$, with covariance function

$$r(t) = \frac{\sin\, t\sqrt{3}}{t\sqrt{3}}$$

and spectral density, which is constant over $(-\sqrt{3}, \sqrt{3})$,

$$R(\omega) = 1/2\sqrt{3} \quad \text{for} \quad |\omega| < \sqrt{3}.$$

We shall call such a process a "standard process". It has $V(\xi(t)) = V(\xi'(t)) = 1$, and $V(\xi''(t)) = 1.8$. □

EXAMPLE 1: (Slepian model after u–upcrossings.) Let the process $\xi(t)$, $t\geq 0$, have upcrossings of the level u at $t_1 < t_2 < \ldots$, and consider the process a fixed time t after any one of these upcrossings, t_k, say. For different t_k–points, $\xi(t_k+t)$ takes on different values and thus behaves like the realization of a random sequence. The value of $\xi(t_k+t)$ depends to some extent on $\xi(t_k)=u$ and on (the unobserved) $\xi'(t_k)$. This dependence can be expressed through the regression function, (cf. (4)),

$$E(\xi(t_k+t) \mid \xi(t_k)=u,\ \xi'(t_k)=y) = ur(t)/\lambda_0 - yr'(t)/\lambda_2,$$

while the rest of the variation is described by the residual process

$$\Delta(t\,|\,t_k) = \xi(t_k+t) - E(\xi(t_k+t) \mid \xi(t_k),\ \xi'(t_k)).$$

considering t as a parameter, $\Delta(t|t_k)$ becomes a realization of a random process, which is Gaussian with mean zero and a covariance function which, by (6), s equal to

$$C(t_1, t_2) = r(t_2-t_1) - r(t_1)r(t_2)/\lambda_0 - r'(t_1)r'(t_2)/\lambda_2. \tag{8}$$

Thus we can write

$$\xi(t_k+t) = E(\xi(t_k+t) \mid \xi(t_k), \xi'(t_k)) + \Delta(t|t_k) =$$

$$= \xi(t_k)\cdot r(t)/\lambda_0 - \xi'(t_k)\cdot r'(t)/\lambda_2 + \Delta(t|t_k). \tag{9}$$

In formula (9) we could have replaced $\xi(t_k)$ by u, but we have chosen to retain it in order to indicate the dependence on the starting point. The derivative $\xi'(t_k)$ has still to be specified, and this is the tricky part of the Slepian model, since $\xi'(t_1)$, $\xi'(t_2)$, ... do not form a stationary sequence. However, in the long run, the empirical distribution of $\xi'(t_k)$, taken over all t_k, $k=1,2,...$ converges to a Rayleigh distribution with density

$$f^{erg}_{\xi'(t_k)}(y) = \lambda_2^{-1} y \exp(-y^2/2\lambda_2),\ y > 0. \tag{10}$$

Here ^{erg} stands for ergodic, or long run, distribution, meaning that it is the limit of the empirical distribution of $\xi'(t_k)$, i.e. for an ergodic process,

$$\int_0^Y f^{erg}_{\xi'(t_k)}(y)\, dy = \lim_{T\to\infty} \frac{\#\{t_k \in [0,T];\ \xi'(t_k) \le Y\}}{\#\{t_k \in [0,T]\}}.$$

To describe the long run properties of $\xi(t_k+t)$ as t_k runs through the set of all u–upcrossings, we therefore only have to replace in (9), $\xi(t_k)$ by u, and the derivative $\xi'(t_k)$ by a Rayleigh–distributed random variable η with density given by (10), and $\Delta(t|t_k)$ by an independent non–stationary Gaussian process $\Delta(t)$ with mean zero and covariance function $C(s,t)$ given by (8), thereby obtaining a process

$$\xi_u(t) = ur(t)/\lambda_0 - \eta r'(t)/\lambda_2 + \Delta(t). \tag{11}$$

This is the Slepian model for $\xi(t)$ at time t after an upcrossing of the level u. One should think of $\xi_u(t)$ as a stochastic generator, which, by repeatedly drawing realizations of η and $\Delta(t)$, describes the long run behaviour of $\xi(t_k+t)$ for $k = 1, 2, ...$ For example, $\xi_u'(0) = ur'(0)/\lambda_0 - \eta r''(0)/\lambda_2 + \Delta'(0) = \eta$ is distributed as $\xi'(t_k)$, $k = 1, 2, ...$

□

EXAMPLE 2: (Slepian model after local maximum.) Suppose $\xi(t)$, $t \geq 0$ has twice continuously differentiable sample paths and let it have local maxima at $t_1 < t_2 < \ldots$. This is similar to Exampel 1, but now $\xi'(t)$ has its zero downcrossings at t_k, and we want to describe $\xi(t_k+t)$ for different t_k. One way of doing this is to let $\xi(t_k+t)$ depend on the height and curvature at the maximum at t_k through the regression

$$E(\xi(t_k+t) \mid \xi(t_k)=u,\ \xi'(t_k)=0,\ \xi''(t_k)=z) =$$

$$= u \cdot A(t) + 0 \cdot B(t) + z \cdot C(t).$$

Defining $\Delta_m(t|t_k) = \xi(t_k+t) - E(\xi(t_k+t)|\xi(t_k),\xi'(t_k),\xi''(t_k))$, we can write

$$\xi(t_k+t) = \xi(t_k) \cdot A(t) + \xi''(t_k) \cdot C(t) + \Delta_m(t|t_k)$$

in analogy with (9). Both $\xi(t_k)$ and $\xi''(t_k)$ at the maximum are random quantities with a long run distribution which turns out to have the density (see Lindgren, 1970),

$$f^{erg}_{\xi(t_k),\xi''(t_k)}(u,z) =$$

$$= \text{const} \cdot (-z) \exp\{-\frac{1}{2(\lambda_0\lambda_4-\lambda_2^2)}(\lambda_0 z^2+2\lambda_2 uz+\lambda_4 u^2)\},\ z < 0\ . \qquad (12)$$

Now, let ξ_m and ζ_m be random variables with joint density (12), and let $\Delta_m(t)$ be a non–stationary Gaussian process, independent of (ξ_m, ζ_m), with mean zero and with covariance function $C_m(s,t)$ given by (7). Thus, ξ_m, ζ_m, and $\Delta_m(\cdot)$ describe $\xi(t_k)$, $\xi''(t_k)$, and $\Delta_m(\cdot|t_k)$ as t_k runs through the set of all local maxima.

The Slepian model for $\xi(\cdot)$ near local maxima is then defined as

$$\xi_{max}(t) = \xi_m A(t) + \zeta_m C(t) + \Delta_m(t). \qquad (13)$$

□

EXAMPLE 1 (contd.): (Regression approximation for length of an excursion above u.) Let $u > 0$ be a fixed level, and use model (11), $\xi_u(t) = ur(t)/\lambda_0 - \eta r'(t)/\lambda_2 + \Delta(t)$, to describe the behaviour of $\xi(t_k+t)$ after u–upcrossings. Let $T > 0$ be the time of the first downcrossing of the level u for $\xi_u(t)$, i.e. the length of the excursion above u that

started at time 0. It can also be defined as the smallest $t > 0$ for which $\xi_u(t) = u$. This equation can be solved for η, giving

$$\eta = -\frac{u(1-r(T)/\lambda_0)-\Delta(T)}{r'(T)/\lambda_2},$$

provided $r'(T) \neq 0$.

However, this equation is of no use when we want to find the distribution of T, since $\Delta(t)$ is a random process. Now, consider only the regression term in $\xi_u(t)$, which we shall denote by $\xi_u^r(t)$,

$$\xi_u^r(t) = ur(t)/\lambda_0 - \eta r'(t)/\lambda_2,$$

and define T^r to be the length of the excursion above u by $\xi_u^r(t)$, which means that, i.a.,

$$\xi_u^r(T^r) = ur(T^r)/\lambda_0 - \eta r'(T^r)/\lambda_2 = u. \tag{14}$$

Again, solving for η, we obtain η as a function of T^r only. Denoting this function by q,

$$q(t) = -\frac{u(1-r(t)/\lambda_0)}{r'(t)/\lambda_2},$$

we have

$$\eta = q(T^r) = -\frac{u(1-r(T^r)/\lambda_0)}{r'(T^r)/\lambda_2}.$$

(Here we need $u \neq 0$, since if $u=0$ the distribution of T^r is concentrated at the first zero of $r'(t)$.)

Now, there is a simple relation between the densities of η and T^r,

$$f_{T^r}(t) = f_\eta(q(t)) \cdot |J(t)| \cdot I(t), \tag{15}$$

where $I(t)$ is an indicator function,

$$I(t) = \begin{cases} 1 & \text{if } ur(s)/\lambda_0 - q(t)r'(s)/\lambda_2 > u \text{ for all } s,\ 0<s<t, \\ 0 & \text{otherwise,} \end{cases}$$

and $J(t)$ is the Jacobian of the transformation, i.e.

$$J(t) = \frac{dq(t)}{dt} = u \cdot \{\lambda_2/\lambda_0 + \lambda_2 \cdot \frac{(1-r(t)/\lambda_0)r''(t)}{r'(t)^2}\}.$$

Inserting the Rayleigh density (10), we obtain

$$f_{T^r}(t) = \lambda_2^{-1} q(t) \exp(-q(t)^2/2\lambda_2) \cdot |J(t)| \cdot I(t); \tag{16}$$

this is called the *regression approximation of zero order* for the density of T.

To facilitate generalizations we shall introduce the function

$$\xi^r_{ut}(s) = "\xi^r_u(s) \mid \xi^r_u(t)=u" = ur(s)/\lambda_0 - q(t)r'(s)/\lambda_2$$

obtained by replacing η by $q(t)$, to make $\xi^r_u(t) = u$. Writing (14) as

$$G(T^r)\eta = u - ur(T^r)/\lambda_0,$$

one can then write the Jacobian in the general form

$$J(t) = -\frac{\frac{d}{ds}\ \xi^r_{ut}(s)]_{s=t}}{\det\ G(t)}. \tag{17}$$

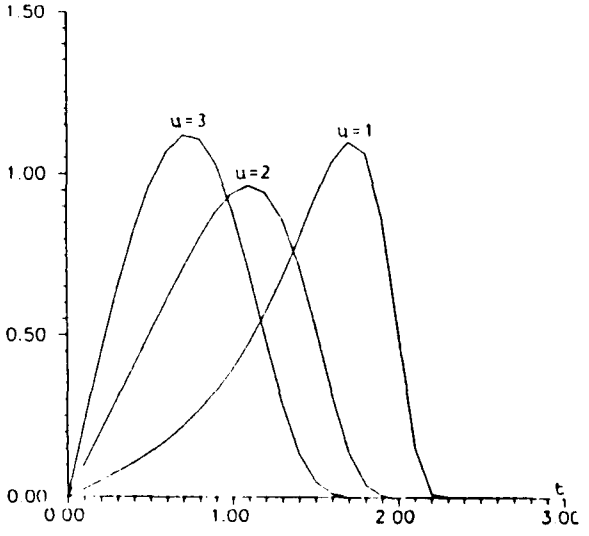

Figure 1: Density $f_{Tr}(t)$ of zero order regression approximation for length of an excursion above u for a standard process $\xi(t)$; u = 0, 1, 2, 3. □

EXAMPLE 2 (contd.): (Regression approximation for wavelength and amplitude.) We shall seek the distribution of the wavelength T and amplitude H after a local maximum, i.e. the horizontal and vertical distances between a maximum and the following minimum. Defined in terms of the model function $\xi_{max}(t) = \xi_m A(t) + \zeta_m C(t) + \Delta_m(t)$, the wave

length $T > 0$ is the time of the first zero upcrossing by $\xi'_{max}(t)$, while $H = \xi_{max}(0) - \xi_{max}(T)$.

The regression approximation of T, H is defined as the corresponding quantities T^r, H^r in the regression term $\xi^r_{max}(t) = \xi_m A(t) + \zeta_m C(t)$ of $\xi_{max}(t)$. Using $A(0)=1$, $C(0)=0$, one has

$$\xi_m A'(T^r) + \zeta_m C'(T^r) = 0, \tag{18}$$

$$H^r = \{\xi_m A(0) + \zeta_m C(0)\} - \{\xi_m A(T^r) + \zeta_m C(T^r)\} =$$

$$= \xi_m \cdot (1-A(T^r)) - \zeta_m \cdot C(T^r), \tag{19}$$

which can be solved for ξ_m, ζ_m. To write the solution in a form that can be generalized to more complicated problems, define

$$G(t) = \begin{bmatrix} 1-A(t) & C(t) \\ A'(t) & C'(t) \end{bmatrix}$$

and write equations (18) and (19) as

$$G(T^r)\,(\xi_m \;\; \zeta_m)^T = (H^r \;\; 0)^T.$$

If $\det G(T^r) \neq 0$, this has the general solution

$$(\xi_m \;\; \zeta_m)^T = G(T^r)^{-1}\,(H^r \;\; 0)^T \tag{20}$$

Written explicitly, this states that

$$\xi_m = H^r \cdot p(T^r) q(T^r),$$

$$\zeta_m = H^r \cdot q(T^r),$$

where

$$p(t) = -C'(t)/A'(t),$$

$$q(t) = -A'(t)/\{(1-A(t))C'(t) - A'(t)C(t)\}.$$

(The only reason for writing the functions in this form is that the density of T^r, H^r then can be written in a form which is familiar in engineering literature.)

The Jacobian of the transformation (20) can be expressed by means of the function

$$\xi^r_{th}(s)' = "\xi^r_{max}(s)' \mid \xi^r_{max}(t)'=0,\ \xi^r_{max}(0)-\xi^r_{max}(t)=h" =$$

$$= hp(t)q(t)A'(s) + hq(t)C'(s) = h\{p(t)q(t)A'(s) + q(t)C'(s)\}$$

obtained by replacing ξ_m, ζ_m by the solution (20), to make $T^r=t$, $H^r=h$. Then, in analogy with (17),

$$J(t) = -\frac{\frac{d}{ds}\ \xi^r_{th}(s)']_{s=t}}{\det\ G(t)} = hp'(t)q(t)^2.$$

With density of ξ_m, ζ_m given by (12), this finally gives the following density for T^r, H^r, which shall be called the "zero order approximation" of $f_{T,H}$:

$$f_{T^r,H^r}(t,h) = f_{\xi_m,\zeta_m}(hp(t)q(t),\ hq(t))\cdot|J(t,h)|\cdot I(t,h) =$$

$$= \text{const}\cdot I(t,h)\cdot h^2|q(t)^3p'(t)|\cdot$$

$$\cdot\exp\left\{-\frac{1}{2\epsilon^2}h^2q(t)^2\,(T_m/\pi)^4\,\{((\pi/T_m)^2p(t)+1)^2+\frac{\epsilon^2}{1-\epsilon^2}\}\right\}, \qquad (21)$$

where $T_m = \pi(\lambda_2/\lambda_4)^{1/2}$ is the mean wavelength, and $\epsilon = (1-\lambda_2^2/\lambda_0\lambda_4)^{1/2}$ is a measure of the spectral width of the process. Further, $I(t,h)$ is the indicator function which is equal to one if $\xi^r_{th}(s)' < 0$ for all s, $0 < s < t$, and zero otherwise. Note that if $h < 0$, then $\xi^r_{th}(s)' > 0$ for small $s > 0$, which implies that $I(t,h) = 0$ for $h < 0$.

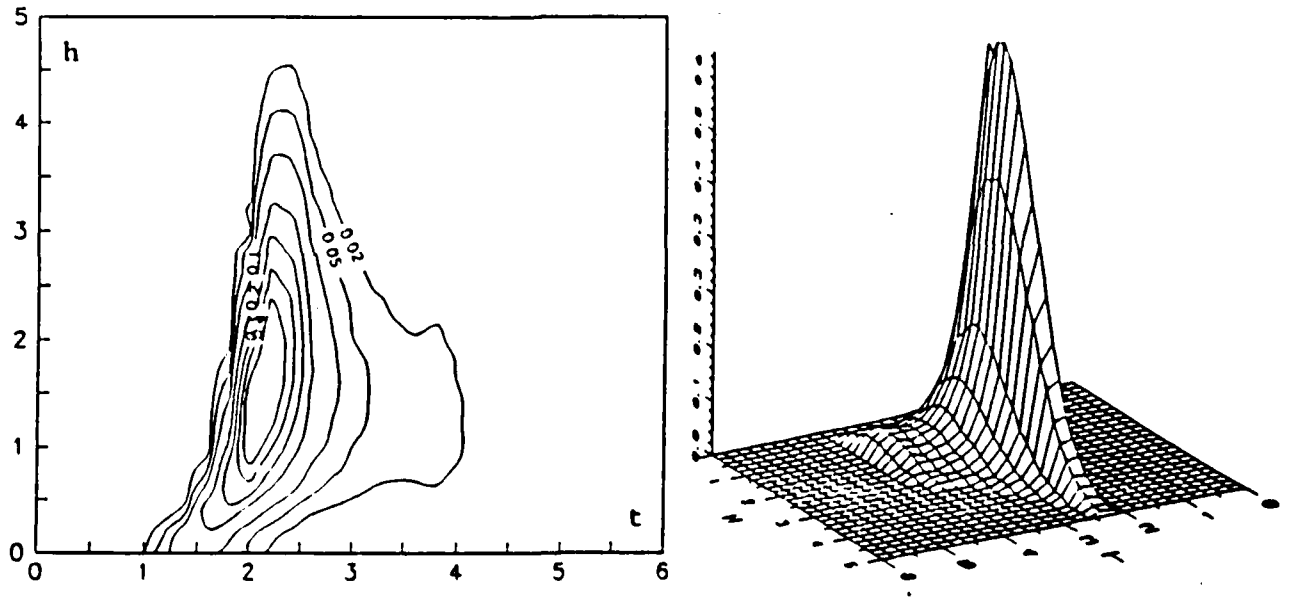

Figure 2: Level curves and perspectiv view of the density $f_{T^r,H^r}(t,h)$ of zero order regression approximation of wavelength and amplitude for standard process $\xi(t)$. □

The regression approximations described in Examples 1 and 2 are not very accurate, as approximations to the true excursion and wave distributions. However, they can be improved considerably by including some extra information into the regression term, like the value of the process or the derivative at some predefined time after the original crossing. This is illustrated in the following example.

EXAMPLE 3: (Regression approximation of wavelength and amplitude with supplementary information.) Consider the model $\xi_{max}(t) = \xi_m A(t) + \zeta_m C(t) + \Delta_m(t)$ after a local maximum, presented in Example 2. Let s_1 be a fixed timepoint, write $X_1 = \Delta_m'(s_1)$, and define the new residual

$$\Delta_1(t) = \Delta_m(t) - E(\Delta_m(t) \mid X_1) = \Delta_m(t) - X_1 b_1(t),$$

where $b_1(t)$ is a deterministic function. Thus,

$$\xi_{max}(t) = \xi_m A(t) + \zeta_m C(t) + X_1 b_1(t) + \Delta_1(t), \tag{22}$$

where X_1 is normal with mean zero and some variance $\sigma_1^2 = V(\Delta_m'(s_1))$, and $\Delta_1(t)$ is a non-stationary Gaussian process, the random variables (ξ_m,ζ_m), X_1, and the process $\Delta_1(\cdot)$ being independent.

We shall approximate the distribution of wavelength T and amplitude H by simple functions of (ξ_m,ζ_m) and X_1. To do this, write $H = M_0 - M_1$, where

$$M_0 = \xi_{max}(0), \quad M_1 = \xi_{max}(T).$$

The regression part of (22) is

$$\xi_{max}^r(t) = \xi_m A(t) + \zeta_m C(t) + X_1 b_1(t).$$

Define the *first order regression approximation* T_1^r as the wavelength of $\xi_{max}^r(t)$, i.e. take T_1^r to be the first t such that

$$\xi_m A'(T_1^r) + \zeta_m C'(T_1^r) + X_1 b_1'(T_1^r) = 0,$$

and define $M_0^r = \xi_{max}^r(0)$, $M_1^r = \xi_{max}^r(T_1^r)$, respectively. (Note that $A(0)=1$, $C(0)=0$, $b_1(0)=0$, which implies $M_0^r = \xi_m$.) Then, $H^r = M_0^r - M_1^r$.

Thus, we have replaced the three variables T, M_0, M_1 by the approximating T^r, M_0^r, M_1^r, which are connected to ξ_m, ζ_m, ξ_1 through the following system of equations,

$$M_0^r = \xi_m ,$$

$$M_1^r = \xi_m A(T_1^r) + \zeta_m C(T_1^r) + X_1 b_1(T_1^r),$$

$$0 = \xi_m A'(T_1^r) + \zeta_m C'(T_1^r) + X_1 b_1'(T_1^r).$$

Writing

$$G(t) = \begin{bmatrix} 1 & 0 & 0 \\ A(t) & C(t) & b_1(t) \\ A'(t) & C'(t) & b_1'(t) \end{bmatrix},$$

$$M = (M_0^r \;\; M_1^r \;\; 0)^T, \quad X = (\xi_m \;\; \zeta_m \;\; X_1)^T,$$

we get the relations

$$M = G(T_1^r)\cdot X, \quad X = G(T_1^r)^{-1}\cdot M,$$

and finally

$$f_{M_0^r,M_1^r,T_1^r}(m_0,m_1,t) = f_{\xi_m,\zeta_m,X_1}(u,z,x)\cdot |J(t,m_0,m_1)|\cdot I(t,h,x), \tag{23}$$

where $(u,z,x)^T = G(t)^{-1}\cdot(m_0,m_1,0)^T$ are functions of (t,m_0,m_1) and $h = m_0 - m_1$. In (23),

$$f_{\xi_m,\zeta_m,X_1}(u,z,x) = f_{\xi_m,\zeta_m}(u,z)\cdot f_{X_1}(x),$$

where the first density is given by (12) and $f_{X_1}(x)$ is a normal density. The Jacobian $J(t,m_0,m_1)$ is given by

$$J(t,m_0,m_1) = -\frac{\frac{d}{ds}\,\xi_{thx}^r(s)']_{s=t}}{\det\, G(t)} = \det\, G(t)^{-1}\cdot(uA''(t)+zC''(t)+xb_1''(t)),$$

where as before $\xi_{thx}^r(s)'$ denotes the process $\xi_{max}^r(s)'$ with u,z,x inserted to make $T^r=t$, $M_0^r=m_0$, $M_1^r=m_1$. The indicator function $I(t,h,x)$ in (23) is equal to one if $\xi_{thx}^r(s)' < 0$ for all s, $0 < s < t$, and zero otherwise.

Finally, the marginal density of T^r,H^r can be obtained from (23) by numerical integration of x over the set $\{x;\, I(t,h,x) = 1\}$. The accuracy in the approximation

depends on the choice of s_1, and there are many possible suggestions where to put it; see Section 4 for further details, and numerical examples of approximations of higher order.

□

3. SLEPIAN MODELS AS WEAK LIMITS

In Section 2 we have presented two basic Slepian models for a stationary Gaussian process $\xi(t)$ after level crossings and local maxima, respectively,

$$\xi_u(t) = ur(t)/\lambda_0 - \eta r'(t)/\lambda_2 + \Delta(t), \tag{11}$$

$$\xi_{max}(t) = \xi_m A(t) + \zeta_m C(t) + \Delta_m(t). \tag{13}$$

We have also given some intuitive interpretations of how they describe the behaviour of $\xi(t_k+t)$ when t_k is an upcrossing of a fixed level u, or a local maximum, respectively.

These intuitive interpretations can be made precise in two different directions. One is by the *horizontal window conditioning* principle, the other is the ergodic *frequency interpretation.* With the horizontal window conditioning, $\xi_u(t)$ is defined as the limit (in distribution) as $h \to 0$ of $\xi(s+t)$ conditioned on the event that $\xi(s)$ has a u–upcrossing somewhere in the horizontal window $[t_k–h, t_k]$. It was introduced by Kac & Slepian (1959), who also showed the connection with the frequency interpretation.

Both the horizontal window and the ergodic interpretation can be made in the sense of distributional convergence in a space of sufficiently smooth functions. This means that also distributions of variables such as excursion length, height of exceedances, wave length and amplitude, obtained from the Slepian model, can be interpreted in the horizontal window conditioning and in the ergodic frequency sense, provided the topology is chosen in an appropriate way. Such problems have been dealt with by i.a. Lamperti (1965), Lindgren (1977), Wilson (1983, 1986, 1988), Rusakov & Seleznjev (1988), and Seleznjev (1989a, b). See also Leadbetter et al. (1983, Ch. 10) for an introduction.

Suppose $\xi(t)$ has sample paths which are continuous functions (with probability one). Even if not absolutely needed for most of our results, we shall also require the sample

paths to be continuously differentiable. A sufficient condition for this is that the process is separable and the covariance function $-r''(t)$ of the derivative has an expansion

$$-r''(t) = \lambda_2 + o(|\log|t||^{-\alpha}) \text{ as } t \to 0,$$

for some constant $\alpha > 1$; see Cramér & Leadbetter (1967). (In Remark 3.1 we comment on what happens when the derivative is assumed to exist only in quadratic mean.)

Define $(C^1, \mathcal{B}^1)$ as the space of continuously differentiable functions on the real line, with metric

$$d(x, y) = |x(0)-y(0)| + \sum_{r=1}^{\infty} 2^{-r} \cdot \frac{\sup_{|t|\le r}|x'(t)-y'(t)|}{1 + \sup_{|t|\le r}|x'(t)-y'(t)|},$$

i.e. $x_n \to y$ as $n \to \infty$ means that $x_n(0) \to y(0)$ and $x_n'(t) \to y'(t)$ uniformly on all bounded intervals. Further, let $(C_u^1, \mathcal{B}_u^1)$ denote the subspace of $(C^1, \mathcal{B}^1)$, consisting of functions such that $x(0)=u$, and write, for $\tau>0$, $B_\tau = [0,\tau]$, and $B_{-\tau} = [-\tau,0]$. We consider $\xi(t)$ as a process defined on the space $(C^1, \mathcal{B}^1)$, with $\xi(t; x) = x(t)$; e.g. $\xi(t_k+\cdot)$ is, with probability one, and element of C_u^1. We also use the same notation for functionals defined on $(C^1, \mathcal{B}^1)$ and the corresponding random variables.

Let, for any interval B,

$$N(u; B) = \#\{t_k \in B\},$$

be the number of u–upcrossings, that fall in B. Similarly, for any set $A \in \mathcal{B}_u^1$, let

$$N(u,A; B) = \#\{t_k \in B;\ \xi(t_k+\cdot) \in A\},$$

be the number of u–upcrossings in B for which the sample function starting at t_k satisfies the condition given by the set A. These definitions generalize the definitions of $N(u; T)$ and $N(u,A; T)$ in Section 2.

We shall give some examples of relevant sets A. The first example describes the length of an excursion. Let τ_k be the length of the excursion above u starting at t_k, i.e.

$$\tau_k = \inf\{t > 0;\ \xi(t_k+t) = u\},$$

and, for $x \in C_u^1$, let

$$\tau = \inf\{t > 0;\, x(t) = u\}.$$

Then, for $A_T = \{x \in C_u^1;\, \tau < T\}$,

$$N(u,A_T;\, B) = \#\{t_k \in B;\, \tau_k < T\}$$

is the number of excursions, starting in B, with length less than T.

As a second example we consider the height of an excursion above u. With $M_m = \{x \in C_u^1;\, \sup_{0 \le t \le \tau} x(t) \le m\}$,

$$N(u,M_m;\, B) = \#\{t_k \in B;\, \sup_{0 \le t \le \tau_k} \xi(t_k+t) \le m\}$$

is the number of excursions in B which do not exceed m.

We can now define two different distributions on the space $(C_u^1, \mathscr{C}_u^1)$. First, let P_u be the *Palm distribution* for $\xi(\cdot)$ near u–upcrossings, defined by

$$P_u(A) = E(N(u,A;\, B_1))/E(N(u;\, B_1)).$$

(Note that here, by stationarity, B_1 can be replaced by any interval of length one.) Second, define $P_{u,T}$ to be the *empirical distribution* of $\xi(\cdot)$ near u–upcrossings, i.e. (for T large enough to make $N(u;\, B_T) > 0$),

$$P_{u,T}(A) = N(u,A;\, B_T)/N(u;\, B_T),$$

i.e. $P_{u,T}$ gives equal weight $1/n$ to each of the $n = N(u;\, B_T)$ functions $\xi(t_k+\cdot)$ starting at $t_1, .., t_n$.

THEOREM 3.1: The Slepian model $\xi_u(t) = ur(t)/\lambda_0 - \eta r'(t)/\lambda_2 + \Delta(t)$ has distribution given by P_u considered as an element in $(C_u^1, \mathscr{C}_u^1)$.

Proof: The finite–dimensional distributions of $\xi_u(t)$ are given by those of P_u. This follows from the general formula (3) for the expected number of marked crossings. One has only to take $\eta(t) = (\xi(t+s_1),..., \xi(t+s_n))$. Since the finite–dimensional distributions determine P_u, and ξ_u has continuously differentiable sample paths, the theorem follows. □

THEOREM 3.2: If ξ is an ergodic process, then, with probability one, $P_{u,T}$ converges weakly to P_u in $(C_u^1, \mathscr{C}_u^1)$ as $T \to \infty$.

Proof: The theorem states that, with probability one, the empirical probability measure over the realizations of $\xi(t_k+t)$, $k=1,2,...$, in $(C_u^1, \mathscr{C}_u^1)$ converges weakly towards P_u as $T \to \infty$. It follows from the the ergodic theorem, (see Cramér & Leadbetter, 1967), that, for each $A \in \mathscr{C}_u^1$, with probability one,

$$T^{-1} \cdot N(u,A; B_T) \to E(N(u,A; B_1)),$$

$$T^{-1} \cdot N(u; B_T) \to E(N(u; B_1)),$$

which shows that $P_{u,T}(A) \to P_u(A)$, with probability one. Since $(C_u^1, \mathscr{C}_u^1)$ is separable with the metric d, it follows from Billingsley (1968), that, with probability one, $P_{u,T}(A) \to P_u(A)$ for every $A \in \mathscr{C}_u^1$. □

THEOREM 3.3: The conditional distribution of $\{\xi(\cdot) \mid N(u; B_{-h})>0\}$ converges weakly to P_u in $(C_u^1, \mathscr{C}_u^1)$ as $h \downarrow 0$.

Proof: The finite dimensional conditional distributions of $\{\xi(s_1),...,\xi(s_m) \mid N(u;B_{-h}) > 0\}$ converge to P_u as $h \downarrow 0$, i.e.

$$P(\xi(s_j) < y_j,\ j=1,...,m \mid N(u; B_{-h}) > 0\} =$$

$$= \frac{P(N(u;B_{-h})>0 \quad \text{and} \quad \xi(s_j)<y_j,\ j=1,...,m\}}{P(N(u;B_{-h})>0)} \to$$

$$\to \frac{E(N(u, A_y;\ B_1))}{E(N(u;\ B_1))} = P(A_y), \tag{24}$$

where

$$A_y = \{x \in C_u^1;\ x(s_j) < y_j,\ j=1,...,m\}.$$

To prove (24), let t_0 be the last u–upcrossing of $\xi(t)$ before 0, so that $t_0 > -h$ if and only if $N(u;B_{-h}) > 0$. Then replace the event

$$\{N(u;B_{-h}) > 0 \text{ and } \xi(s_j) < y_j,\ j=1,\ldots,m\}$$

by

$$\{t_0 > -h \text{ and } \xi(t_0+s_j) < y_j,\ j=1,\ldots,m\}.$$

The error will be of small order. Then (24) is a consequence of the general result, for stationary point processes ν, that $h^{-1}P(\nu(B_h) > 0) \to E(\nu(B_1))$ as $h \to 0$, applied to the point processes of u–upcrossings $\{t_k\}$ and to the thinned point processes of u–upcrossings, which satisfy the extra condition $\xi(t_k+s_j) < y_j$, $j=1,\ldots,m$.

To show the full weak convergence on $(C_u^1, \mathcal{C}_u^1)$ we need the tightness of the conditional distributions of $\xi(0)$ and of the continuity modulus of ξ',

$$\omega_B(\xi', \delta) = \sup_{\substack{|s-t|<\delta \\ s,t\in B}} |\xi'(s)-\xi'(t)|$$

given that $N(u;B_{-h}) > 0$. In fact, $P(|\xi(0)| > K \mid N(u;B_{-h}) > 0) \to 0$ as $h \to 0$ for all $K > u$, so that $\xi(0)$ is tight.

Further we have to show the conditional tightness of $\omega_{B_r}(\xi',\delta)$, i.e. that, for each $\varepsilon > 0$, $r = 1, 2, \ldots$,

$$\lim_{\delta\to 0} \lim_{h\to 0} P\{\omega_{B_r}(\xi',\delta) > \varepsilon \mid N(u;B_{-h}) > 0\} = 0.$$

One can use the same technique as above, considering the point process N'_δ of u-upcrossings t_k such that

$$\omega_{B_{r+1}(t_k)}(\xi',\delta) \geq \varepsilon,$$

where $B_{r+1}(t_k)$ is the sphere of radius $r+1$ and center t_k. Then

$$P\{\omega_{B_r}(\xi',\delta) > \varepsilon \mid N(u;B_{-h}) > 0\} \leq P\{N'_\delta(B_{-h}) \geq 0 \mid N(u;B_{-h}) > 0\}$$

which tends to $E(N'_\delta(B_1))/E(N(u;B_1))$ as $h \to 0$. Here, $N'_\delta(B_1) \leq N(u;B_1)$ and $N'_\delta(B_1) \to 0$, with probability one, as $\delta \to 0$, which proves the tightness. □

Remark 3.1: If ξ has continuous sample paths with mean square derivatives, one can define the model ξ_u as an element in the space $(C_u, \mathscr{C}_u)$ of continuous functions (with $x(0) = u$), with metric for uniform convergence on bounded intervals and the corresponding statements to Theorems 3.1 and 3.2 hold. However, variables like excursion length and height after a u–upcrossing are examples of variables which are continuous on the space $(C_u^1, \mathscr{C}_u^1)$ but not on the simpler space $(C_u, \mathscr{C}_u)$. □

4. REGRESSION APPROXIMATIONS

4.1 Regression decomposition

Assume $\xi(\cdot)$, $\eta(\cdot)$ is a pair of smooth processes defined on the same probability space, and let T be the first zero crossing of ξ. Suppose that we are interested in the joint density of the crossing time T for ξ and the value of η at T, i.e.

$$(T,H) = (T(\xi), \eta(T)), \tag{25}$$

which are sometimes called *marked exit time* and *mark*, respectively. In this section, we shall illustrate the general regression approximation method, by constructing a sequence of approximations to the joint density of T and H.

The regression method is based on a decomposition of the processes $\xi(t)$, $\eta(t)$ into linear regression terms and independent residual processes. More precisely, let $\zeta = (\zeta_1,\ldots,\zeta_n)$, $n>1$, be a vector of regressors from the linear space spanned by $\xi(\cdot)$, $\eta(\cdot)$ and let the joint density of $\zeta_1,\ldots,\zeta_n$ be a smooth function. Suppose that the processes ξ and η can be decomposed as follows,

$$\begin{aligned} \xi(s) &= b_0(s) + \sum_{i=1}^{n} \zeta_i b_i(s) + \Delta_1(s), \\ \eta(s) &= c_0(s) + \sum_{i=1}^{n} \zeta_i c_i(s) + \Delta_2(s), \end{aligned} \tag{26}$$

where the functions b_i and c_i are defined explicitly, and Δ_1, Δ_2 are zero mean Gaussian residual processes, independent of $(\zeta_1,\ldots,\zeta_n)$. Furthermore, we shall assume that

$\mathrm{Var}(\zeta_i) = 1$, for all i, which only a matter of scaling. The conditional expectations of the $(\xi(s),\eta(s))$-processes, given $\zeta = (\zeta_1,\ldots,\zeta_n)$, are given by

$$\begin{aligned} E(\xi(s)\,|\,\zeta) &= b_0(s) + \sum_{i=1}^{n} \zeta_i b_i(s) = \xi^r(s), \\ E(\eta(s)\,|\,\zeta) &= c_0(s) + \sum_{i=1}^{n} \zeta_i c_i(s) = \eta^r(s). \end{aligned} \tag{27}$$

In Example 1 in Section 2 we had only one process,

$$\xi_u(t) - u = ur(t)/\lambda_0 - u - \eta r'(t)/\lambda_2 + \Delta(t),$$

which is of the form (26) with $n = 1$, $\zeta_1 = \eta/V(\eta)^{1/2}$ and

$$b_0(t) = ur(t)/\lambda_0 - u,$$

$$b_1(t) = -r'(t)/\lambda_2 \cdot V(\eta)^{1/2},$$

$$\Delta_1(t) = \Delta(t).$$

Example 2 involves two processes,

$$\xi_u'(t) = \xi_m A'(t) + \zeta_m C'(t) + \Delta'(t),$$

$$\xi_u(t) = \xi_m A(t) + \zeta_m C(t) + \Delta(t),$$

and is of the form (26) with $n = 2$, $\zeta_1 = \xi_m/V(\xi_m)^{1/2}$, $\zeta_2 = \zeta_m/V(\zeta_m)^{1/2}$, and

$$b_0(t) = c_0(t) = 0,$$

$$b_1(t) = A'(t)\cdot V(\xi_m)^{1/2}, \qquad b_2(t) = C'(t)\cdot V(\zeta_m)^{1/2},$$

$$c_1(t) = A(t)\cdot V(\xi_m)^{1/2}, \qquad c_2(t) = C(t)\cdot V(\zeta_m)^{1/2},$$

$$\Delta_1(t) = \Delta'(t), \qquad \Delta_2(t) = \Delta(t).$$

We shall now study the general model (26) in more detail in order to approximate the distribution of wave–length and amplitude, i.e. T, H. The regression approximation of (T,H) is defined as the corresponding variables in the regression functions (ξ^r,η^r), i.e. they are obtained by replacing in (26) the processes (ξ,η) by the regression curves. We shall denote the regression approximation of (T,H) by (T^r,H^r), i.e.

$$(T^r, H^r) = (T(\xi^r),\ \eta^r(T^r)). \tag{28}$$

As has been shown in the introductory examples, the general structure of Slepian models for the Gaussian processes is the decomposition (26). Although most of our applications of the regression method is in connection with Slepian models, it is not limited to the Slepian models and Gaussian residuals. However, since we require that all finite-dimensional distributions of the residual processes Δ_1, Δ_2 are given in explicit form, the residual processes are usually asumed to be Gaussian or functions of Gaussian vector processes; see Rychlik (1987d, 1989b). In addition, in Section 5.2, we shall present Slepian models for the χ^2-process. In that case, Slepian models will be defined by more complicated decompositions (functions of the linear decomposition (26)). The regression method can still be used for approximation of the (T,H)–distributions and it is defined as the corresponding variables in the nonlinear regressions obtained by replacing the residual processes by zero.

Finally, the results presented in this section can be easily extended to more complicated marked crossing problems, when ξ and η in (26) are vector valued processes, i.e. $T = (T_1,...,T_k)$ is a vector of the first k zero crossings of ξ; see Rychlik (1987b, 1988).

4.2 Simple regression approximation

We now turn to the evaluation of the density of (T^r, H^r). Obviously, T^r and H^r are functions of $(\zeta_1,...,\zeta_n)$ alone. However, since we are interested in the density of (T^r, H^r), we need an inverse mapping that expresses (ζ_1, ζ_2) as a function of $(T^r, H^r, \zeta_3,...,\zeta_n)$, leaving $(\zeta_3,...,\zeta_n)$ unaffected. This can be done, using the implicit definition of T^r, i.e.

$$b_0(T^r) + \sum_{i=1}^{n} \zeta_i b_i(T^r) = 0. \tag{29}$$

The required variable transformation can be written as follows,

$$\zeta_1 b_1(T^r) + \zeta_2 b_2(T^r) = -b_0(T^r) - \sum_{i=3}^{n} \zeta_i b_i(T^r),$$
$$\zeta_1 c_1(T^r) + \zeta_2 c_2(T^r) = H^r - c_0(T^r) - \sum_{i=3}^{n} \zeta_i c_i(T^r). \tag{30}$$

The linear equation system (30) has a unique solution (ζ_1, ζ_2) for $T^r=t$ if the determinant of the following matrix is not equal to zero,

$$G(t) = \begin{bmatrix} b_1(t) & b_2(t) \\ c_1(t) & c_2(t) \end{bmatrix}. \tag{31}$$

As in Examples 1 and 2, we can then solve (30) and obtain (ζ_1, ζ_2) as a function of $(T^r, H^r, \zeta_3,...,\zeta_n)$. Denoting this function by (q_1, q_2), i.e.

$$\begin{bmatrix} q_1(t,h,x_3,...,x_n) \\ q_2(t,h,x_3,...,x_n) \end{bmatrix} = G(t)^{-1} \cdot \begin{bmatrix} -b_0(t)-x_3 b_3(t)-...-x_n b_n(t) \\ h-c_0(t)-x_3 c_3(t)-...-x_n c_n(t) \end{bmatrix}, \tag{32}$$

the conditional density of $(T^r, H^r \mid \zeta_3,...,\zeta_n)$ can be obtained by a simple variable transformation, given in Lemma 4.1 below.

Before we state the lemma, we shall introduce two important functions, $\xi^r_{thx}(s)$ and $\eta^r_{thx}(s)$, defined by "$\xi^r(s) \mid \xi^r(t)=0,\ \eta^r(t)=h,\ \zeta_3,...,\zeta_n=x$" and "$\eta^r(s) \mid \xi^r(t)=0,\ \eta^r(t)=h,\ \zeta_3,...,\zeta_n=x$", respectively, i.e. for a fixed $(t,h,x) = (t,h,x_3,...,x_n)$,

$$\xi^r_{thx}(s) = b_0(s) + q_1(t,h,x)b_1(s) + q_2(t,h,x)b_2(s) + \sum_{i=3}^{n} x_i b_i(s)$$
$$\eta^r_{thx}(s) = c_0(s) + q_1(t,h,x)c_1(s) + q_2(t,h,x)c_2(s) + \sum_{i=3}^{n} x_i c_i(s). \tag{33}$$

For any continuously differentiable function f, let $I(t;f)$ be an indicator function, defined equal to one if $f(s)$ has no zero crossings in the interval $(0,t)$ and zero otherwise, i.e.

$$I(t;f) = \begin{cases} 1 & \text{if } f(s) > 0 \text{ or } f(s) < 0 \text{ for all } s,\ 0<s<t, \\ 0 & \text{otherwise}. \end{cases} \tag{34}$$

LEMMA 4.1: Let (T^r, H^r) be the regression approximation of (T, H), defined by (28). If the determinant $\det G(t) \neq 0$, the conditional density of $T^r,H^r \mid \zeta_3,\ldots,\zeta_n$ at $(t,h \mid x) = (t,h \mid x_3,\ldots,x_n)$ is given by

$$f(t,h \mid x) = I(t;\xi^r_{thx}) \cdot |J(t,h,x)| \, f_{\zeta_1,\zeta_2 \mid \zeta_3 \ldots \zeta_n}(q_1(t,h,x), q_2(t,h,x) \mid x), \tag{35}$$

where $f_{\zeta_1,\zeta_2 \mid \zeta_3 \ldots \zeta_n}$ is the conditional density of $\zeta_1,\zeta_2 \mid \zeta_3,\ldots,\zeta_n$ and the functions q_1, q_2 are defined by (32). The Jacobian is given by

$$J(t,h,x) = -\frac{\frac{d}{ds}\xi^r_{thx}(s)]_{s=t}}{\det G(t)}, \tag{36}$$

where $\xi^r_{thx}(s)$ is defined by (33) and the indicator function $I(t;\xi^r_{thx})$ is given by (34).

□

The marginal density of (T^r, H^r) can be obtained by multiplying the conditional density (35) by the density of $\zeta_3,\ldots,\zeta_n$ and integrating the x–variables, i.e. by computing $E(f(t,h \mid \zeta_3,\ldots,\zeta_n))$.

The Jacobian $J(t,h,x)$ is written in the form (36) to indicate the explicit dependence of the T^r,H^r-density on the derivative $\xi^r_{thx}(t)'$, i.e on the derivative of the regression ξ^r at the zero crossing at time t, when the mark η^r takes the value h.

The density of T^r,H^r is related to the marked zero crossing intensity through the following alternative formula:

$$\begin{aligned} f_{T^r,H^r}(t,h) &= E(f(t,h \mid \zeta_3 \ldots \zeta_n)) = \\ &= E(I(t;\xi^r) \mid \xi^r(t)' \mid \mid \xi^r(t)=0,\ \eta^r(t)=h) f_{\xi^r(t),\eta^r(t)}(0,h), \end{aligned} \tag{37}$$

where $I(t;\xi^r)$ is defined by (34). The proof of formula (37) is based on the observation that the conditional density $(\xi^r(t),\eta^r(t) \mid \zeta_3 \ldots \zeta_n)$ is given by

$$f_{\xi^r(t),\eta^r(t) \mid \zeta_3 \ldots \zeta_n}(0,h \mid x) = |\det G(t)|^{-1} f_{\zeta_1,\zeta_2 \mid \zeta_3 \ldots \zeta_n}(q_1(t,h,x),q_2(t,h,x) \mid x).$$

4.3 An explicit formula for density of (T,H)

It is instructive to write the true density of T,H in a form similar to formula (35), based on the regression approximation. To obtain this form, just introduce the residual processes Δ_1 and Δ_2 into the functions q_1, q_2 and the Jacobian J, (36), defined in Lemma 4.1. More precisely, define new q–functions $\bar{q}$, and a Jacobian $\bar{J}$ as follows,

$$\begin{bmatrix} \bar{q}_1(t,h,x_3,\ldots,x_n,\Delta) \\ \bar{q}_2(t,h,x_3,\ldots,x_n,\Delta) \end{bmatrix} = G(t)^{-1} \cdot \begin{bmatrix} -b_0(t)-x_3b_3(t)-\ldots-x_nb_n(t)-\Delta_1(t) \\ h-c_0(t)-x_3c_3(t)-\ldots-x_nc_n(t)-\Delta_2(t) \end{bmatrix} \tag{38}$$

$$\bar{J}(t,h,x,\Delta_1) = -\frac{\frac{d}{ds}\xi_{thx}(s)]_{s=t}}{\det\ G(t)}, \tag{39}$$

where $\Delta(s,t) = (\Delta_1(s), \Delta_2(t))$ and the function $\xi_{thx}(s)$ is defined by

$$\xi_{thx}(s) = b_0(s) + \bar{q}_1(t,h,x,\Delta)b_1(s)+\bar{q}_2(t,h,x,\Delta)b_2(s)$$

$$+ \sum_{i=3}^{n} x_i b_i(s) + \Delta_1(s). \tag{40}$$

Then the true conditional density of $(T,H \mid \zeta_3,\ldots,\zeta_n)$ can be written as an expectation of the density (35) over all sample paths of $\Delta = (\Delta_1,\Delta_2)$, with the modified $\bar{q}$– and $\bar{J}$–functions, i.e.

$$f_{T,H|\zeta_3,\ldots,\zeta_n}(t,h,x) = E_{\Delta_1,\Delta_2}\Big[I(t;\xi_{thx})\cdot|\bar{J}(t,h,x,\Delta_1)|\cdot$$

$$\cdot f_{\zeta_1\zeta_2|\zeta_3\ldots\zeta_n}(\bar{q}_1(t,h,x,\Delta), \bar{q}_2(t,h,x,\Delta)|x)\Big], \tag{41}$$

where $I(t;\cdot)$ is defined by (34).

The density of (T,H) can be written in the same form as (37) of (T^r,H^r), by replacing the regressions ξ^r, η^r by the original processes ξ, η, i.e.

$$f_{T,H}(t,h) = E(I(t;\xi)|\xi'(t)|\,|\,\xi(t)=0,\ \eta(t)=h)f_{\xi(t),\eta(t)}(0,h). \tag{42}$$

Observe that formula (42) holds for a broader class of processes, which contains more general nonlinear decompositions; see Durbin (1987), Rychlik (1987c). However, for the purpose

of approximation and numerical calculation, the explicit formula (41), based on the decomposition (26), is more useful than formula (42).

The expectation (41) is difficult to evaluate exactly, since the indicator $I(t;\xi_{thx})$ depends on the whole realization of ξ_{thx}, i.e. on $\Delta_1(\cdot)$. Nevertheless, the formula is the basis for improvement of the accuracy of the regression approximations of the (T,H)-density, by approximation of the residual processes Δ_1, Δ_2, by simpler processes $\hat{\Delta}_1$, $\hat{\Delta}_2$, which are functions of a finite number of random variables $Y = (Y_1,...,Y_k)$, say, defined on the same probability space as Δ_1, Δ_2. The $(\hat{\Delta}_1,\hat{\Delta}_2)$-processes can be chosen in many different ways, e.g. as cosine polynomials, splines etc. In the next subsection we shall use regression curves.

4.4 More complicated approximations

Let $Y = (Y_1,...,Y_k)$ be a random vector defined on the residual processes Δ_1, Δ_2. The general formula for the regression approximation (T_k^r,H_k^r) of the (T, H)-density is obtained by replacing, in (41), the Δ_1- and Δ_2-residual processes by their conditional expectations, given Y, i.e. by $\hat{\Delta} = (\hat{\Delta}_1,\hat{\Delta}_2)$,

$$\hat{\Delta}_1(Y)(s) = E(\Delta_1(s)|Y), \qquad \hat{\Delta}_2(Y)(s) = E(\Delta_2(s)|Y). \tag{43}$$

More precisely, the regressions $\hat{\Delta}_1,\hat{\Delta}_2$ in (43) define a new decomposition of the (ξ,η)-processes by splitting the residuals Δ_1,Δ_2, i.e.

$$\xi(s) = \xi^r(s) + \hat{\Delta}_1(Y)(s) + \Delta_1(Y)(s),$$

$$\eta(s) = \eta^r(s) + \hat{\Delta}_2(Y)(s) + \Delta_2(Y)(s),$$

where, for i=1,2,

$$\Delta_i(Y_1...Y_k)(s) = \Delta_i(s) - E(\Delta_i(s)|Y_1...Y_k). \tag{44}$$

The regression approximations T_k^r, H_k^r are defined as

$$(T_k^r, H_k^r) = \{T(\xi^r(\cdot) + \hat{\Delta}_1(Y)(\cdot)),\ \eta^r(T_k^r) + \hat{\Delta}_1(Y)(T_k^r)\}.$$

The conditional density $f^k(t,h|x,y)$ of $(T_k^r,H_k^r|\zeta,Y)$ can be obtained in a similar way as in Lemma 4.1. The explicit formula can be written as follows. With

$$\begin{bmatrix} q_1(t,h,x,y) \\ q_2(t,h,x,y) \end{bmatrix} = G(t)^{-1} \cdot \begin{bmatrix} -b_0(t)-x_3b_3(t)-\ldots-x_nb_n(t)-\dot{\Delta}_1(y)(t) \\ h-c_0(t)-x_3c_3(t)-\ldots-x_nc_n(t)-\dot{\Delta}_2(y)(t) \end{bmatrix} \tag{45}$$

$$J(t,h,x,y) = -\frac{\frac{d}{ds}\xi^r_{thxy}(s)]_{s=t}}{\det G(t)}, \tag{46}$$

and $\xi^r_{thxy}(s)$ and $\eta^r_{thxy}(s)$ defined by

$$\xi^r_{thxy}(s) = b_0(s) + q_1(t,h,x,y)b_1(s)+q_2(t,h,x,y)b_2(s) \tag{47a}$$

$$+ \sum_{i=3}^{n} x_ib_i(s) + \dot{\Delta}_1(y)(s),$$

$$\eta^r_{thxy}(s) = c_0(s) + q_1(t,h,x,y)c_1(s)+q_2(t,h,x,y)c_2(s) \tag{47b}$$

$$+ \sum_{i=3}^{n} x_ic_i(s) + \dot{\Delta}_2(y)(s),$$

the conditional density $f^k(t,h|x,y)$ of $(T_k^r,H_k^r|\zeta,Y)$ is equal to

$$f^k(t,h|x,y) = I(t;\xi^r_{thxy}) \cdot |J(t,h,x,y)|$$

$$f_{\zeta_1\zeta_2|\zeta_3\ldots\zeta_n}(q_1(t,h,x,y),\, q_2(t,h,x,y)|x). \tag{48}$$

(Observe that the Y–variables are independent of ζ.)

The density of the simplest regression approximation, formula (35), corresponds to the choice $k=0$, i.e. there are no extra regressors at all besides ζ. Thus the approximating variables will be denoted as T_0^r,H_0^r and the conditional density $f(t,h|x)$, formula (35), will be denoted by $f^0(t,h|x)$.

The regressors $Y_1,\ldots,Y_k$ can be choosen with great freedom to contain simple but efficient information about (T,H). One of the main features of the structure of approximation (48) is that it can easily be improved by just adding more regressors in a recursive way. In the following subsection we present a recursive procedure of choosing the regressors $Y_1,\ldots,Y_k$, suitable for inplementation in a recursive programming language.

4.5 A recursive procedure for choosing the regressors $Y_1,...,Y_k$

The regressors $Y_1,...,Y_k$ should be chosen in such a way that the conditional density of $(T_k^r,H_k^r \mid \zeta)$ is a good approximation to the exact formula for the density (41) of (T, H), with as few regressors as possible. Consequently, strategies of choosing the Y_i-variables are usually quite complicated and can depend both on the properties of the ξ_{thx}^r-, and η_{thx}^r-functions and on the residual processes $\Delta = (\Delta_1,\Delta_2)$. We shall now present a class of regression approximations (T_k^r, H_k^r) for the (T,H)-distribution, based on recursive selection of regressors Y_i.

Assume that we have designed a strategy to select a unique variable Y_1, defined as a function of the residuals, $\Delta(s,t) = (\Delta_1(s),\Delta_2(t))$. More precisely, assume that we have been given a zero-mean Gaussian process Δ, with a covariance $r_\Delta = (r_{\Delta_1},r_{\Delta_2},r_{\Delta_1,\Delta_2})$, and deterministic functions $f = (f_1,f_2)$, corresponding to the functions ξ_{thx}^r, η_{thx}^r, and assume further, that we have defined a functional, Proc, on the sample paths of $\Delta = \omega$, with f as the parameter, such that the regressor Y_1 is defined by $Y_1(\omega) = Proc(\omega ; f)$.

The same procedure, Proc, can then be used to define further regressors $Y_2, ...,Y_k$, provided it has the following structure:

(a) Y_1=Proc(Δ,f) is a Gaussian random variable,

(b) the regression curves $\hat{\Delta}(Y_1)(s,t)$ in (43) can be evaluated explicitly,

(c) for any real y_1, the residual processes $\Delta_1(y_1)(s)$, $\Delta_2(y_1)(s)$ in (44) are zero-mean jointly Gaussian processes, with explicitly defined covariance function $r_{\Delta \mid Y_1=y_1}$.

Then, the regressors $Y_1, ...,Y_k$ can be defined recursively:

(i) $Y_1 = Proc(\Delta; \xi_{thx}^r, \eta_{thx}^r)$,

(ii) for any value $y = (y_1,...,y_{k-1})$ of the regressors $Y_1,...,Y_{k-1}$, $Y_k = Proc(\Delta(y); \xi_{thxy}^r, \eta_{thxy}^r)$,

where $\Delta(y)(s) = (\Delta_1(s)-E(\Delta_1(s) \mid Y_1...Y_{k-1}), \Delta_2(s)-E(\Delta_2(s) \mid Y_1...Y_{k-1}))$ are the residual processes (44) and ξ_{thxy}^r, η_{thxy}^r are defined by (17).

We turn now to the problem of choosing the procedure $\mathrm{Proc}(\cdot)$. Many different algorithms can be proposed and we shall classify them into two main categories. The first one, which will be called *deterministic procedures*, selects Y_1 from the linear space spanned by Δ, independently of the properties of ξ^r_{thx}, η^r_{thx}. Consequently, the conditional covariances $r_{\Delta|Y_1\ldots Y_k=y}$ do not depend on y. As examples, we give the simple procedure

$\mathrm{Proc}_1(\Delta;\ \xi^r_{thx},\eta^r_{thx})$:

Choose $Y_1=\Delta_1(s_1)$, where s_1, $0<s_1<t$, is a fixed point such that $\mathrm{Var}(\Delta_1(s_1))=\sup_{0<s<t}\mathrm{Var}(\Delta_1(s))$,

and the more complicated algorithm

$\mathrm{Proc}_2(\Delta;\ \xi^r_{thx},\eta^r_{thx})$:

Choose $Y_1=\Delta_1(s_1)$, where s_1, $0<s_1<t$, is a fixed point such that $\sup_{0<s<t}\mathrm{Var}(\Delta_1(s)|\Delta_1(s_1))$ is minimized.

The second class of procedures for choosing the first regressor Y_1, will be called *stochastic procedures*, and here $r_{\Delta|Y_1\ldots Y_k=y}$ depends explicitly on y. This happens when one allows Y_1 to depend on ξ^r_{thx}, η^r_{thx}. An example of a stochastic procedure, used in Rychlik (1987d, 1989b), is the following algorithm:

$\mathrm{Proc}_3(\Delta;\ \xi^r_{thx},\eta^r_{thx})$:

Choose $Y_1=\Delta_1(s_1)$, where s_1, $0<s_1<t$, is a fixed point such that, if $\xi^r_{thx}(0)>0$, $P(\xi^r_{thx}(s_1)>0)$ is minimized, otherwise, i.e. if $\xi^r_{thx}(0)<0$, $P(\xi^r_{thx}(s_1)<0)$ is minimized.

The difference between a stochastic and a determinstic procedure is that in the stochastic procedure the regressors $Y_1,\ldots,Y_k$, depend on the sample path of the residual processes Δ_1, Δ_2, leading to more accurate approximations.

Finally, we give a recursive formula for the conditional density $(T^r_k,H^r_k \mid \zeta_3,\ldots,\zeta_n)$ defined by recursive selection of the regressors $Y=(Y_1,\ldots,Y_k)$. Obviously, using the

conditional density $f^k(t,h|x,y)$, given by (48), of $(T_k^r,H_k^r \mid \zeta,Y)$, we obtain the $(T_k^r,H_k^r|\zeta)$-density in the form of a k–dimensional integral,

$$f^k(t,h|x) = E_Y(f^k(t,h|x,Y)), \tag{49}$$

which has to be evaluated numerically. Since the regressors Y are choosen in a recursive way, the value of the conditional density $f^k(t,h|x)$ depends only on the functional Proc, and the starting values Δ, ξ_{thx}^r, η_{thx}^r. We shall express this dependence explicitly by the notation $f^k_{Proc(\Delta;\xi_{thx}^r,\eta_{thx}^r)}(t,h|x)$. The recursive formula for $f^k(t,h|x)$ is then obtained by writing the integration in (49) in iterated form, i.e.

$$f^k_{Proc(\Delta;\xi_{thx}^r,\eta_{thx}^r)}(t,h|x) = \int E_{Y_2\ldots Y_k}(f^k(t,h|x,y,Y_2\ldots Y_k))\, f_{Y_1}(y)\, dy$$

$$= \int f^{k-1}_{Proc(\Delta|Y_1=y;\xi_{thxy}^r,\eta_{thxy}^r)}(t,h|x)\, f_{Y_1}(y)\, dy.$$

The main advantage of the recursive representation of the conditional density of $(T_k^r,H_k^r|\zeta)$ is that it can be easily implemented in a computer program, in particular when the programming language allows recursive functions.

4.6 Bounds for the T,H–density

In some applications, one needs exact bounds of the approximations error. In the following we shall briefly present upper and lower bounds for the (T,H)–density, based on regression approximations.

Let (t,h,x) be a fixed vector for which bounds for the conditional density of $T,H|\zeta_3,\ldots,\zeta_n$ are required. Suppose that $\Delta_1(t) = \Delta_2(t) = \Delta_1'(t) = 0$, a.s. This seems at first to be a strong assumption, but can easily be attained by inclusion of $\Delta_1(t)$, $\Delta_2(t)$, $\Delta_1'(t)$ in the vector $(\zeta_1,\ldots,\zeta_n)$. (Observe that this was not required when the regression approximations of T,H were defined.) Consequently, formula (41) for the density of $T,H|\zeta_3,\ldots,\zeta_n$ simplifies to

$$f_{T,H|\zeta_3\dots\zeta_n}(t,h|x) = E_{\Delta_1}\left[I(t;\xi_{thx})\right] \cdot |J(t,h,x)| \cdot$$

$$\cdot f_{\zeta_1,\zeta_2|\zeta_3\dots\zeta_n}(q_1(t,h,x),\, q_2(t,h,x)|x). \qquad (50)$$

Assume that from knowledge of the values (t,h,x), we can determine whether the first crossing of ξ_{thx}, defined by (40), is an up- or a down-crossing, i.e we assume that $(\xi(0),\xi'(0))$ is included in $\zeta_1,\dots,\zeta_n$. Suppose that the first zerocrossing of ξ_{thx} is an upcrossing, and hence the expectation in (50) is the following probability,

$$E_{\Delta_1}\{I(t;\xi_{thx})\} = P(\xi_{thx}(s) < 0 \text{ for all } s,\ 0<s<t). \qquad (51)$$

Observe that $\xi_{thx}(s)$ is a continuously differentiable Gaussian process. In Rychlik (1987d, 1989b) we have proposed a recursive procedure, similar to the regression approximation, to bound probabilities of type (51).

4.6 Numerical examples

In this section we shall illustrate higher order approximations to the excursion length distribution in Example 1 and to the wavelength and amplitude density of Examples 2 and 3. Computer programs for the algorithms will be presented in the next section, where also details about the approximations can be found.

First consider the excursion length distribution for a standard process. Figure 3,a–d, shows excursion length distributions after upcrossings of levels $u = 0, 1, 2, 3$, calculated with $k = 1, 2, 3$ extra regressors. For comparison, the figures also show the simple approximations with $k = 0$. As seen, it is necessary to use extra information in order to catch the bimodal distribution after a zero upcrossing. For higher levels even the simple approximation gives reasonable results, and in fact for $u = 2, 3$ the curves for $k = 1, 2, 3$ coincide.

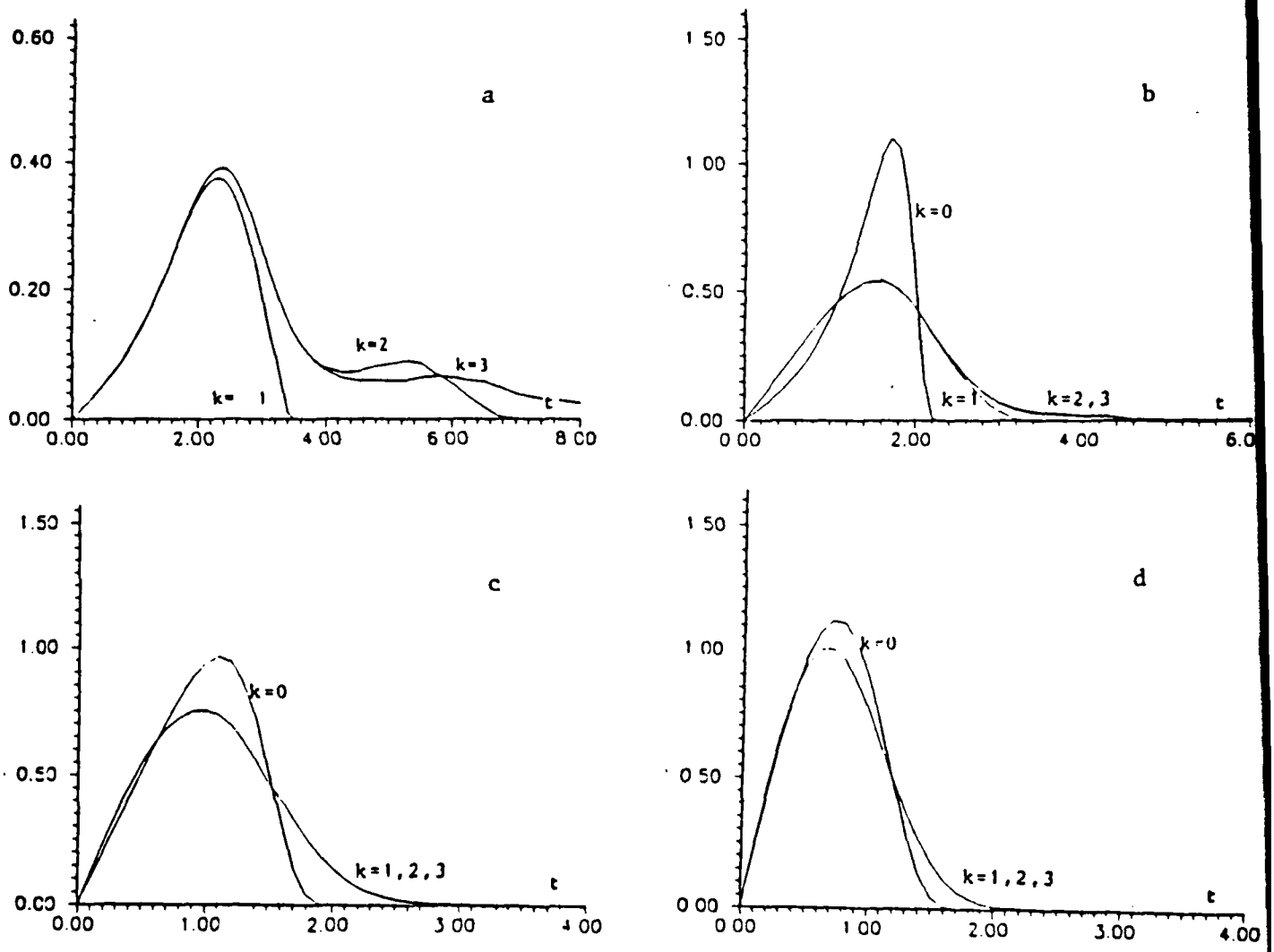

Figure 3: Regression approximations according to formula (19) to excursion length density after upcrossing of a level u for a standard process; (a) u=0, (b) u=1, (c) u=2, (d) u=3. The order of approximation is k = 0, 1, 2, 3.

As a second example we show the higher order approximations to the wavelength and amplitude distributions considered in Examples 2 and 3. As can be seen from Figure 4, (a)–(c) higher order approximations are smoother, in particular at the left, smaller end of the distribution. The zero order approximation, shown in Figure 2, has a cut off point at $t \approx 4$, to the left of which it is zero.

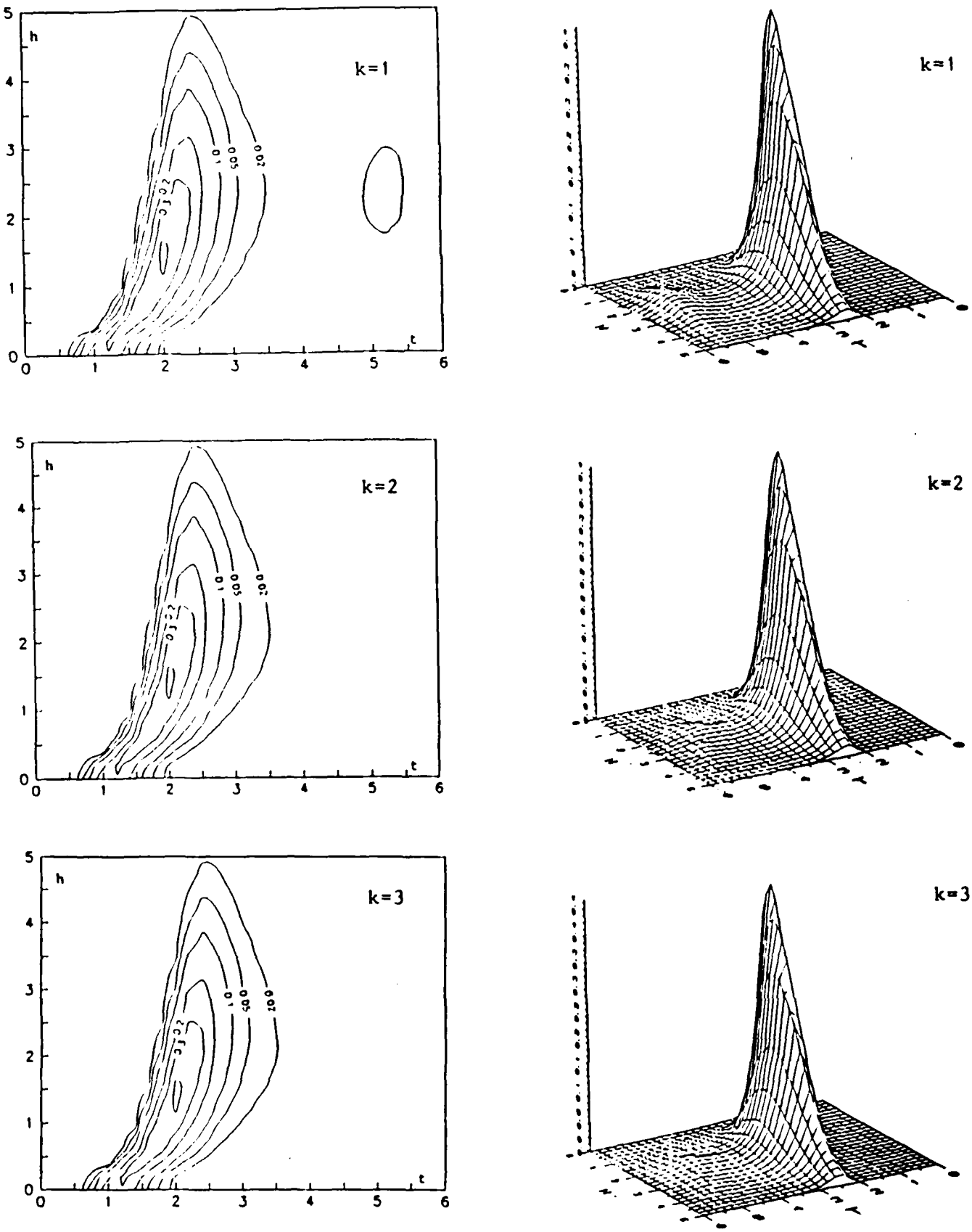

Figure 4: Regression approximation according to formula (49) for wavelength and amplitude density for a standard process; k = 1, 2, 3.

4.7 Computer programs

The numerical examples presented in this paper have been evaluated by means of a library of computer programs, CROSSREG, which contains the regression approximations for the following densities:

EXCREG: excursion time above the level u for the Slepian model $\xi_u(t)$; Example 1;

WAMPREG: wavelength and amplitude after local maximum; Example 2;

JUMPREG: length of jump for a car travelling on a random road; Example 5.1;

CHIREG: excursion time above the level u^2 for the Slepian model for the χ^2–process; Section 5.2, formula (62), $n = 2$.

The programs are implemented in FORTRAN 77 and have been run on a PC/AT and also on a MicroVAXII.

Using the library one can evaluate both the simple regression approximations, without any additional regressors, Section 4.2, and the more complicated approximations discussed in Section 4.4. The additional regressors $Y = (Y_1,...,Y_k)$ are selected recursively, using a sligthly modified stochastic procedure of the type Proc_3; see Section 4.5. The main modification lies in the introduction of a stopping criterion, which allows us to use different numbers of regressors for different sample paths of the residual process. The stopping criterion tests whether, for given values of t and $y = (y_1,...,y_k)$, the conditional density of the regression approximation, $f_{T_k^r|Y}(t|y)$ is sufficiently close to the conditional density $f_{T|Y}(t|y)$ of the true T. This stopping rule reduces the necessary amount of numerical computations drastically.

The programs can be used on a PC/AT, with a reasonable computation time, for as many as four additional regressors. As can be seen in the examples in this paper, this is enough to compute very accurate approximations for a large class of processes of practical interest. In addition, in order to speed up computations, we have designed a numerical integration algorithm, specially well suited for the regression approximation. A more complete description of the programs will be given elsewhere.

5. APPLICATIONS

5.1 Random vibration of mechanical structures

Stochastic processes and linear filtering theory is often used to describe the behaviour of mechanical structures, such as bridges, high towers, aeroplane wings, car suspensions, etc, subjected to external random forces. Loads created by the environment, such as wind and wave forces, are described as stochastic processes, while the structure itself is described in terms of linear or non–linear transfer functions. Processes of interest are forces and displacements within the structure, and the crossing and extremal properties of these are of great importance for the reliability of the structure. Examples of important quantities are high peak values, length of excursions above high levels, and period and wave analysis.

Most models used to describe random vibration are high dimensional, and the external forces even form a continuous random field, as for example the wind over the whole extension of a long bridge. We shall here present a simplified example with only one forcing process and a transfer function with a simple non–linearity, which contains many of the characteristic features of a random mechanical system.

Example 5.1 ("Jumps and bumps"; Lindgren (1981)) Consider the movements of a car travelling with constant speed on a rough road, considered as a locally stationary stochastic process. The suspension system of the car should guarantee safe roadholding and high ride comfort. The roadholding depends on the forces between wheel and road and the ride comfort mainly on the acceleration forces on the driver and passengers. An undesirable event occurs when the force between a wheel and ground is less than a required level. In an extreme case, a wheel can even leave the ground for a short while – it jumps and bumps. We shall describe the jump–and–bump situation by means of a Slepian model and use the regression technique to derive the distribution of the length of the flight starting at the jump.

To simplify the analysis we shall consider a one–wheeled car with mass m which drives with constant speed v on a randomly profiled road. The suspension system consists

of a linear spring with length P_0, stiffness k, and damping coefficient c. (A four-wheeled can be treated similarly as a multivariate system; see Jogréus (1986).) Let $Y(t)$ denote the height of the road above a zero reference level at time t, and let $X(t)$ be the extension of the spring from its unloaded position. (Then $Y(t) = Y_o(vt)$ where $Y_o(s)$ denotes the road elevation at distance s from the starting point.)

If the wheel has no mass, the following differential equation governs the movements of the car as long as no jumps occur,

$$m(X''(t) + Y''(t)) + cX'(t) + kX(t) = -gm \tag{52}$$

where $g = 9.81$ m/s^2 is the acceleration of gravity. This implies that $E(X(t)) = -gm/k$.

As long as no jumps occur, the vertical acceleration of the car is

$$X''(t) + Y''(t) = -g - m^{-1}(cX'(t) + kX(t)),$$

but if the wheel is not permanently attached to the ground, this acceleration can never be less than $-g$. Therefore a jump occurs every time the normal force

$$N(t) = cX'(t) + kX(t) \tag{53}$$

(which has mean $-gm$) has an upcrossing of the zero level. Then an excursion starts, during which the movements of the car and spring are purely deterministic.

If t_k denotes the time of a jump, the spring expands exponentially during the jump according to the equation

$$X(t_k+t) = X(t_k) \exp(-kt/c), \tag{54}$$

while the car follows the parabolic curve

$$P_0 + X(t_k) + Y(t_k) + t(X'(t_k)+Y'(t_k)) - gt^2/2. \tag{55}$$

The road level $Y(t_k+t)$ is of course not affected by the jump, but the fact that a jump did occur at t_k affects its distribution, so we need the conditional distribution of $Y(t_k+t)$ given that a jump has occurred.

We now assume, as is often done in technical literature, that the road profile $Y(t)$ is a Gaussian process, which is stationary at least locally. This means that, if no jumps were allowed to occur and the wheel was in permanent contact with the ground, also the

normal force $N(t)$ and the spring extension $X(t)$ would be Gaussian processes. One could then use the theory in previous chapters to describe the crossings and extremal properties of $N(t)$ and $X(t)$. However, if jumps may occur, the Gaussian character of the process is destroyed, but if jumps are rare, one could think of the process as being renormalized after each jump, and use the Slepian model based on normal theory to describe the local behaviour near jumps. The assumptions that jumps occur unfrequently is a realistic assumption for real cars.

In this example, we shall assume that jumps do not occur at all and that $X(t)$ is always defined by equation (52), and we nall consider equations (54) and (55) as fictitious excursions, to be described by Slepian models.

We assume that $Y(t) = Y_o(vt)$ is a stationary Gaussian process with mean zero and (speed dependent) covariance function r_Y possessing spectral density

$$R_Y(\omega) = v^{-1} R_{Y_o}(\omega/v),$$

where R_{Y_o} is the spectral density for the road. We let $X(t)$ be the stationary (Gaussian) solution of equation (52), rewritten as

$$mX''(t) + cX'(t) + k(X(t)+gm/k) = -mY''(t), \tag{52'}$$

and write $N(t) = cX'(t)+kX(t)$. All covariance properties of the involved processes can then be derived from $R_Y(\omega)$ and the transfer function $H(\omega)$ of the system (52'),

$$H(\omega) = \frac{m\omega^2}{k + ic\omega - m\omega^2}. \tag{56}$$

We need the following variance and covariances for the involved processes,

$$r_Y(t) = \int e^{i\omega t} R_Y(\omega)\, d\omega,$$

$$r_{N,Y}(t) = \mathrm{Cov}(N(0), Y(t)) = \int e^{-i\omega t} m\omega^2 \{H(\omega)+1\} R_Y(\omega)\, d\omega,$$

$$r_{N',Y}(t) = \mathrm{Cov}(N'(0), Y(t)) = \int e^{-i\omega t} mi\omega^3 \{H(\omega)+1\} R_Y(\omega)\, d\omega,$$

and

$$\sigma_N^2 = \int m^2\omega^4 \, |H(\omega)+1|^2 \, R_Y(\omega) \, d\omega,$$

$$\sigma_{N'}^2 = \int m^2\omega^6 \, |H(\omega)+1|^2 \, R_Y(\omega) \, d\omega.$$

Now, let t_k, $k = 1, 2, \ldots$ be the times of zero upcrossings for the process $N(t)$, i.e. the potential jump times, and consider the movements of spring and car during the fictitious flights, as defined by (54) and (55). The Slepian model for $Y(t_k+t)$ after a zero upcrossing by $N(t_k)$ consists of a regression term on $N(t_k)=0$, and $N'(t_k)$ which, being a derivative at a level upcrossing in a stationary Gaussian process, has a Rayleigh distribution with density

$$z\sigma_{N'}^{-2} \exp(-z^2/2\sigma_{N'}^2), \quad z > 0. \tag{57}$$

Let ζ_0 be a Rayleigh variable with density (57). The direct model for $Y(t_k+t)$ is then

$$Y^*(t) = b^*(t) + \zeta_0 b_0^*(t) + \Delta^*(t), \tag{58}$$

where, with $E(N(0)) = -gm$,

$$b^*(t) + z_0 b_0^*(t) = E(Y(t) \mid N(0)=0, N'(0)=z_0).$$

Further, the Gaussian residual process $\Delta^*(t)$ is independent of ζ_0 and has mean zero and covariance function given by the conditional covariance function of $Y(t)$ given $N(0)$ and $N'(0)$. However, if we want to describe the extension of the flight starting at t_k we need also the joint distribution of the derivative $N'(t_k)$, the starting values in (54) and (55) and the residual process in (58).

We first introduce some notation. Define the random vectors $\nu = (N(0), N'(0))^T$, $\pi = (X(0)+Y(0), X'(0)+Y'(0), X(0))^T$ and note that (ν, π) has a 5–dimensional normal distribution with mean

$$\begin{bmatrix} \mu_\nu \\ \mu_\pi \end{bmatrix} = (-gm, \ 0, \ -gm/k, \ 0, \ -gm/k)^T$$

and covariance matrix

$$\begin{bmatrix} \Sigma_{\nu\nu} & \Sigma_{\nu\pi} \\ \Sigma_{\pi\nu} & \Sigma_{\pi\pi} \end{bmatrix} = \begin{bmatrix} \sigma_n^2 & 0 & \sigma_{n,x+y} & \sigma_{n,x'+y'} & \sigma_{n,x} \\ 0 & \sigma_{n'}^2 & \sigma_{n',x+y} & \sigma_{n',x'+y'} & \sigma_{n',x} \\ \sigma_{n,x+y} & \sigma_{n',x+y} & \sigma_{x+y}^2 & 0 & \sigma_{x+y,x} \\ \sigma_{n,x'+y'} & \sigma_{n',x'+y'} & 0 & \sigma_{x'+y'}^2 & \sigma_{x'+y',x} \\ \sigma_{n,x} & \sigma_{n',x} & \sigma_{x+y,x} & \sigma_{x'+y',x} & \sigma_{x,x} \end{bmatrix}.$$

The conditional distribution of $\pi|\nu=(0,z)$ is therefore tri-variate normal with mean

$$\begin{bmatrix} -gm/k + gm\sigma_n^{-2}\sigma_{n,x+y} \\ 0 \quad + gm\sigma_n^{-2}\sigma_{n,x'+y'} \\ -gm/k + gm\sigma_n^{-2}\sigma_{n,x} \end{bmatrix} + z\cdot\begin{bmatrix} \sigma_{n'}^{-2}\sigma_{n',x+y} \\ \sigma_{n'}^{-2}\sigma_{n',x'+y'} \\ \sigma_{n'}^{-2}\sigma_{n',x} \end{bmatrix} = A + z\cdot B, \text{ say,}$$

and covariance matrix

$$E = \Sigma_{\pi\pi} - \Sigma_{\pi\nu}\Sigma_{\nu\nu}^{-1}\Sigma_{\nu\pi}.$$

To get starting values for X, Y, X', Y' in (54) and (55), let ζ_0 be a Rayleigh variable, with density (57), and let $\zeta = (\zeta_1, \zeta_2, \zeta_3)^T$ be a 3–dimensional normal variable, independent of ζ_0, with mean zero and covariance matrix E. Then the starting values $(X(t_k)+Y(t_k), X'(t_k)+Y'(t_k), X(t_k))^T$ are distributed as

$$A + \zeta_0 B + \zeta. \tag{59}$$

The complete Slepian model for $Y(t_k+t)$ is now

$$\text{Road}(t) = b(t) + \sum_{j=0}^{3} \zeta_j b_j(t) + \Delta(t), \tag{60}$$

where the regression is defined as

$$b(t) + \sum_{j=0}^{3} \zeta_j b_j(t) = E\{Y(t) \mid \nu = (0,\zeta_0)^T, \pi = A+\zeta_0 B+\zeta\},$$

and the residual $\Delta(t)$ is independent of $(\zeta_0, \zeta_1, \zeta_2, \zeta_3)$ and has the covariance function

$$r_\Delta(s, t) = \text{Cov}(Y(s),Y(t) \mid \nu, \pi).$$

Equations (54) and (55) then give the following expressions for the length of the spring, height of the car, and height of the wheel during a flight,

$$\text{Spring}(t) = (a_3+\zeta_0 b_3+\zeta_3)\cdot\exp(-kt/c),$$

$$\text{Car}(t) = P_0 + (a_1+\zeta_0 b_1+\zeta_1) + t(a_2+\zeta_0 b_2+\zeta_2) - gt^2/2,$$

$$\text{Wheel}(t) = \text{Car}(t) - P_0 - \text{Spring}(t).$$

Combining these with the road model (60), we get the height of the wheel above road as

$$\begin{aligned}\text{Height}(t) &= \text{Wheel}(t) - \text{Road}(t) = \\ &= (a_1+\zeta_0 b_1+\zeta_1) + t(a_2+\zeta_0 b_2+\zeta_2) - gt^2/2 - (a_3+\zeta_0 b_3+\zeta_3)\exp(-kt/c) - \\ &\quad - b(t) - \sum_{j=0}^{3} \zeta_j b_j(t) - \Delta(t) = \\ &= \{a_1+a_2 t - gt^2/2 - a_3\cdot\exp(-kt/c) - b(t)\} + \zeta_0\{b_1+b_2 t - b_3\cdot\exp(-kt/c) - b_0(t)\} + \\ &\quad + \zeta_1\{1-b_1(t)\} + \zeta_2\{t-b_2(t)\} - \zeta_3\{\exp(-kt/c)+b_3(t)\} - \Delta(t).\end{aligned}$$

This model process is of the form (26), containing one deterministic function, four random coefficient functions, and one Gaussian residual.

We now consider the first time the wheel hits the ground after a jump, i.e. the first zero (down)crossing of Height(t), and we want to find its distribution using the method of Section 4. One then has to observe that Height(t) already contains four random coefficients, which have to be conditioned on. Further, to get good accuracy in the approximation one may have to condition on some values of the residual $\Delta(t)$, thus increasing the number of regressors to a point where the computation time becomes prohibitive.

It is therefore more efficient to slightly reformulate the model, including the three normal coefficients ζ_1,ζ_2,ζ_3 into the residual, and write the model as the sum of one determinstic function, one Rayleigh term and one Gaussian residual,

$$\begin{aligned}\text{Height}(t) &= \{a_1+a_2 t - gt^2/2 - a_3\cdot\exp(-kt/c) - b(t)\} + \\ &\quad + \zeta_0\{b_1+b_2 t - b_3\exp(-kt/c) - b_0(t)\} + \Delta^0(t),\end{aligned} \tag{61}$$

where now $\Delta^0(t)$ is nonstationary Gaussian with a covariance function given by r_Δ and

the covariances between ζ_1, ζ_2, ζ_3. This will permit the computer algorithm to pick as regressors those values of $\Delta^0(t)$ which contribute the most to the distribution of the first zero crossing. Note that (61) is of the same simple form as the original Slepian model (11).

The following numerical example shows the regression approximation for the time of first zero of Height(t) when the Gaussian road process $Y_0(s)$ is of the standard form with spectral density

$$R_{Y_0}(\omega) = \sigma_Y^2/2\sqrt{3} \quad \text{for} \quad |\omega| < \sqrt{3}.$$

(This is rather different from real road spectra (see Lindgren (1981) for references) but is chosen here as a standard process.) Further $k = 0.64$ N/m, $c = 0.08$ Ns/m, and $m = 0.01$ kg. We have chosen two different speed values, $v = 2$ m/s, 4 m/s. With $\sigma_Y^2 = 0.5$, the jump intensity, i.e. the mean number of zero upcrossings per time unit for N(t), is 0.035 and 0.85, respectively.

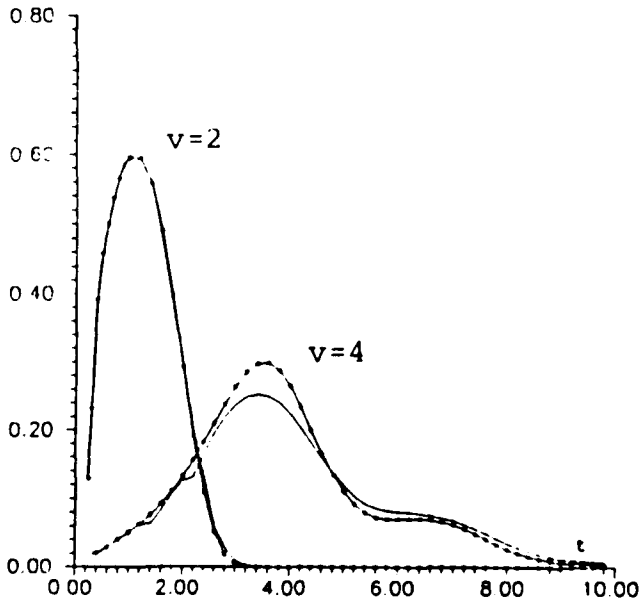

Figure 5: Regression approximations to jump length density for v=2 and v=4; (——— k = 0, * — * — * k = 2)

With a model of the form (61), the regression approximation of the time for the first zero are shown in Figure 5 for the two v-values. For $v = 2$ m/s it suffices to use one extra regressor, while for $v = 4$ m/s one needs two in order to find the characteristic details of

the distribution. As seen, the speed is important; for $v = 4$ m/s the distribution of jump length has two tops, while it is unimodal for low speed. This can be explained by the periodicity of the road profile. For $v = 4$ m/s, the wheel can jump over the first hill and hit the second.

5.2 Regression approximations for crossings in the χ^2–processes

A χ^2–process is the sum of squares of Gaussian processes,

$$\chi^2(t) = \sum_{i=1}^{n} Z_i^2(t),$$

where $Z_1(t), \ldots, Z_n(t)$ are independent, with mean zero and variance one. In the general χ^2–process, $Z_i(s)$ and $Z_k(t)$ need not be independent for $i \neq k$, $s \neq t$, but in order to keep complexity down, we shall here consider only χ^2–processes with independent components with common covariance function $r(t)$. Let $\lambda_2 = V(Z_i'(t)) = -r''(0)$. (The envelope of a Gaussian process is a simple example of a χ^2–process with dependent components; see Ditlevsen & Lindgren (1988), and Lindgren (1989) for details on the Slepian model.)

The Slepian model for $\chi^2(\cdot)$ after upcrossings of a level u^2 was studied by Aronowich & Adler (1986). In the following simple form, given by Lindgren (1989),

$$Y_u^+(t) = (b_0(t) - \zeta_1 b_1(t) + \Delta_1(t))^2 + \sum_{i=2}^{n} (\zeta_i b_1(t) + \Delta_i(t))^2,$$

ζ_1 is a standard Rayleigh variable, $\zeta_2, \ldots, \zeta_n$ are independent standard Gaussian variables and the residual processes Δ_i are independent zero mean Gaussian with covariance function given by (8). Note that here are ony two types of b–functions, given by

$$b_0(t) = ur(t),$$

$$b_1(t) = r'(t)/\lambda_2^{1/2}.$$

Suppose we are interested in the length T of an excursion above u^2 by the χ^2-process $\chi^2(t)$, i.e. of the first T for which $Y_u^+(T) = u^2$. In order to put the problem in the general setting of Section 4, we introduce the process

$$\xi(t) = Y_u^+(t) - u^2 =$$

$$= (b_0(t) - \zeta_1 b_1(t) + \Delta_1(t))^2 + \sum_{i=2}^{n} (\zeta_i b_1(t) + \Delta_i(t))^2 - u^2. \qquad (62)$$

As in Section 4, the regression method is based on a decomposition of $\xi(t)$ into a regression term and independent residual processes, but here the decomposition is not linear as in equation (26) but quadratic, which leads to mixed terms of the form $\zeta_i \Delta_j(s)$.

However, the most important common property of (26) and (62) is that in both cases, the ξ–process is defined by a mapping of $\zeta = (\zeta_1, ..., \zeta_n)$ and $\Delta(t) = (\Delta_1(t), ..., \Delta_n(t))$, i.e. $\xi(t) = F(t, \zeta, \Delta(t))$. Under some additional assumptions on F, one can generalise the results of Section 4, to cover this more complicated situation; see Rychlik (1987b, 1989b).

Like in Section 4, the regression approximation of T is defined as the corresponding variables in the regression functions ξ_k^r, obtained by replacing in (62), the residual processes Δ_i by the regressions $\hat{\Delta}_i$ on some additional explanatory vectors $Y^i = (Y_1^i, ..., Y_k^i)$, i.e

$$\xi_k^r(t) = (b_0(t) - \zeta_1 b_1(t) + \hat{\Delta}_1(t))^2 + \sum_{i=2}^{n} (\zeta_i b_1(t) + \hat{\Delta}_i(t))^2 - u^2, \qquad (63)$$

where

$$\hat{\Delta}_i(t) = E(\Delta_i(t) | Y^i). \qquad (64)$$

In general, the number of regressors in the vector Y^i may depend on i, i.e. $k=k_i$. However, in our examples the residual processes Δ_i are independent and identically distributed, and we choose the same function to define the regressors for each residual, i.e. $Y^i = Y(\Delta_i(\cdot))$. This choice leads to independent and identically distributed regressions (64). In addition, since the first squared process in (62) is exactly the same as the Slepian model (11), we can use the same procedure to select $Y^1 = Y(\Delta_1(\cdot))$ as we used in Section 4.6 for the distribution of excursion length distribution.

We shall denote the regression approximation of T by T_k^r, i.e. $T_k^r = T(\xi_k^r)$. Since Δ_i are Gaussian processes, the regressors Y^i can be choosen using the same methods as in Section 4.5.

We begin with the true density of T. As before, using the implicit definition of T, we find a transformation which represents one of ζ_i's as a function of T, the remaining ζ_j's and the residual processes, Δ_j, i.e.

$$(b_0(T)-\zeta_1 b_1(T)+\Delta_1(T))^2 + \sum_{i=2}^{n} (\zeta_i b_1(T)+\Delta_i(T))^2 - u^2 = 0. \tag{65}$$

Since ζ_1 has a larger variance than the other ζ_i–variables, we solve (65) for ζ_1. Assuming $b_1(t) \neq 0$, then, for $T = t$, $(\zeta_2 \ldots \zeta_n) = x$ and $\Delta = \omega$, we have

$$\zeta_1 = (-\{u^2 - \sum_{i=2}^{n} (x_i b_1(t)+\omega_i(t))^2\}^{1/2} - b_0(t) - \omega_1(t))/b_1(t) = q_2(t,x,\omega) \tag{66}$$

or

$$\zeta_1 = (\{u^2 - \sum_{i=2}^{n} (x_i b_1(t)+\omega_i(t))^2\}^{1/2} - b_0(t) - \omega_1(t))/b_1(t) = q_1(t,x,\omega). \tag{67}$$

Similarily to (39), the Jacobians J_i, i=1,2, of the transformations q_i, (66, 67), are obtained by the implicit function theorem for evaluation of $\partial\zeta_1/\partial T$, viz.

$$J_i(t,x,\omega) = -\frac{\frac{d}{ds}\xi^i_{tx\omega}(s)\,|_{s=t}}{(-1)^i\{u^2-\Sigma_{i=2}^{n}(x_i b_1(t)+\omega_i(t))^2\}^{1/2} b_1(t)}, \tag{68}$$

where the function $\xi^i_{tx\omega}(s)$ is defined by "$\xi(s)\,|\,\xi(t)=0,\ \zeta_2,\ldots,\zeta_n=x,\ \Delta=\omega$", i.e.

$$\xi^i_{tx\omega}(s) = (b_0(s)-q_i(t,x,\omega)b_1(s)+\omega_1(s))^2 + \sum_{i=2}^{n} (x_i b_1(s)+\omega_i(s))^2 - u^2. \tag{69}$$

Then the true conditional density of $(T\,|\,\zeta_2,\ldots,\zeta_n)$ can be written as an expectation over all sample paths of $\Delta = (\Delta_1,\ldots,\Delta_n)$, of the density $(T\,|\,\zeta_2,\ldots,\zeta_n,\Delta)$, which is defined by the variable transformation (66, 67), i.e.

$$f_{T\,|\,\zeta_2\ldots\zeta_n}(t,x) = \sum_{i=1}^{2} E_\Delta\left[I(t;\xi^i_{tx\omega})\cdot|J_i(t,x,\Delta)|\,q_i(t,x,\Delta)\exp\{-0.5q_i(t,x,\Delta)^2\}\right], \tag{70}$$

since ζ_1 is a Rayleigh variable independent of $\zeta_2, \ldots, \zeta_n$. The indicator function $I(t;\cdot)$ is defined by (34), i.e.

$$I(t;f) = \begin{cases} 1 & \text{if } f(s) > 0 \text{ for all } s,\ 0<s<t, \\ 0 & \text{otherwise}. \end{cases} \tag{71}$$

Once again, the density of T can be written in the "marked crossing" form, see (42), i.e.

$$f_T(t) = E(I(t;\xi)|\xi'(t)| \mid \xi(t)=0)\cdot f_{\xi(t)}(0).$$

The expectation in (70) is difficult to evaluate exactly, since the indicator $I(t;\xi_{tx\omega})$ depends on the whole realization of $\xi^i_{tx\omega}$, i.e. on Δ.

Finally, the conditional density of the regression approximation $T^r_k|\zeta_2,\ldots,\zeta_n$ is obtained by replacing in (70) the residual processes $\Delta = (\Delta_1,\ldots,\Delta_n)$ by their regressions $\hat{\Delta} = (\hat{\Delta}_1,\ldots,\hat{\Delta}_n)$ as in (63). In the simplest case, k=0, the residuals Δ_i are replaced by constant functions equal to zero. Then the expectation in (70) disappears, leading to an explicit density, viz.

$$f_{T^r_0|\zeta_2\ldots\zeta_n}(t|x) = \sum_{i=1}^{2} I(t;\xi^i_{tx0})\cdot|J_i(t,x,0)|q_i(t,x,0)\exp\{-0.5q_i(t,x,0)^2\}\Big]. \qquad (72)$$

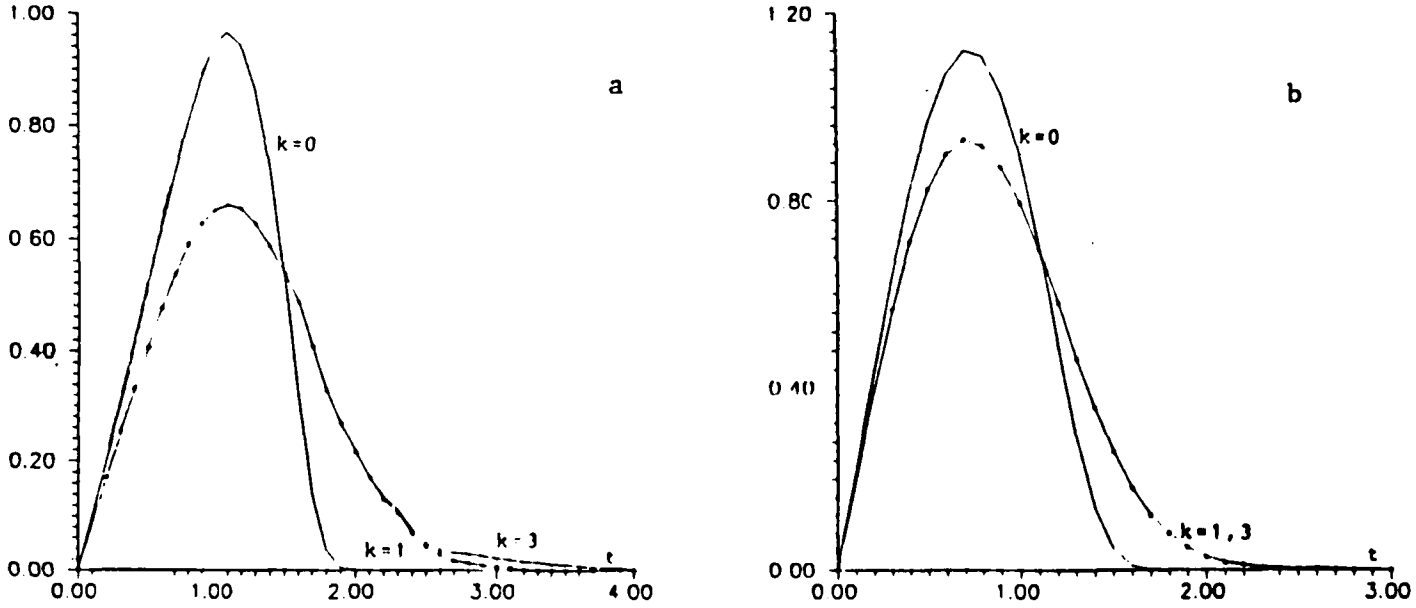

Figure 6: Regression approximation of excursion length density after a u–upcrossing for χ^2–process, the sum of squares of two standard normal processes; (a) u = 2, (b) u = 3; k = 0 (———), k = 1 (* —— *), k = 3 (———).

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