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A Basic Theorem of Complementarity for  
the Generalized Variational-like Inequality Problem

by  
Jen-Chih Yao

TECHNICAL REPORT SOL 89-16

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# A Basic Theorem of Complementarity for The Generalized Variational-like Inequality Problem

by  
Jen-Chih Yao

## Abstract

In this report, a basic theorem of complementarity is established for the generalized variational-like inequality problem introduced by Parida and Sen. Some existence results for both generalized variational inequality and complementarity problems are established by employing this basic theorem of complementarity. In particular, some sets of conditions that are normally satisfied by a nonsolvable generalized complementarity problem are investigated.

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# A Basic Theorem of Complementarity for The Generalized Variational-like Inequality Problem

by Jen-Chih Yao

## 0. Notation

Let  $\mathbf{R}^n$  denote  $n$ -dimensional Euclidean space with the usual inner product and norm  $\langle x, x \rangle = \|x\|^2$ . Let  $\mathbf{R}_+^n$  denote the nonnegative orthant of  $\mathbf{R}^n$ . For  $K, B \subset \mathbf{R}^n$ , let  $\text{int}_K(B)$  denote the relative interior of  $B$  in  $K$  and  $K \setminus B$  denote the set of points of  $K$  which are not in  $B$ .

## 1. Introduction

Given a subset  $K$  of  $\mathbf{R}^n$  and a function  $f$  from  $\mathbf{R}^n$  into itself, the classical *variational inequality problem*, denoted by  $VIP(f, K)$  is to find a vector  $\bar{x} \in K$  such that

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0, \forall x \in K.$$

This original problem has been extensively studied in the past years. For example, see Eaves [2], Hartman and Stampacchia [4], Moré [9], and Pang [10]. Recently Parida and Sen [12] introduced the following generalized variational-like inequality problem for point to set mapping. Let  $K$  and  $C$  be subsets of  $\mathbf{R}^n$  and  $\mathbf{R}^m$  respectively. Given two maps  $\theta : K \times C \rightarrow \mathbf{R}^n$  and  $\tau : K \times K \rightarrow \mathbf{R}^n$ , and a point-to-set mapping  $F : K \rightarrow C$ , the *generalized variational-like inequality problem*, denoted by  $GVIP(F, \theta, \tau, K, C)$  is to find  $\bar{x} \in K, \bar{y} \in F(\bar{x})$  such that

$$\langle \theta(\bar{x}, \bar{y}), \tau(x, \bar{x}) \rangle \geq 0, \forall x \in K.$$

We use  $GVIP(F, \theta, K, C)$  to denote a special problem of the type  $GVIP(F, \theta, \tau, K, C)$  where  $\tau(x, y) = x - y$ . If we further assume that  $\theta(x, y) = y$  and  $C = \mathbf{R}^n$ , then  $GVIP(F, \theta, K, C)$  reduces to  $GVIP(F, K)$  which was introduced by Fang and Peterson [3].

Let  $f$  be a mapping of  $\mathbf{R}^n$  into itself. The classical *complementarity problem*, denoted by  $CP(f)$ , is to find a vector  $x \in \mathbf{R}^n$  such that

$$x \geq 0, f(x) \geq 0, \langle x, f(x) \rangle = 0.$$

The nonlinear  $CP$  was first studied by Cottle [1]. In [12], Parida and Sen also introduced the following generalized complementarity problem. Given  $K$  a closed convex cone of  $\mathbf{R}^n$ ,  $C$  a closed convex subset of  $\mathbf{R}^m$ ,  $\theta : K \rightarrow \mathbf{R}^n$  single-valued function,  $F : K \rightarrow C$  a point-to-set mapping, the

generalized complementarity problem, related to  $GVIP(F, \theta, K, C)$  and denoted by  $GCP(F, \theta, K, C)$  is to find a vector  $x \in K$  and a vector  $y \in F(x)$  such that

$$\theta(x, y) \in K^*, \langle \theta(x, y), x \rangle = 0$$

where  $K^* = \{y \in \mathbf{R}^n : \langle y, x \rangle \geq 0, \forall x \in K\}$  is the polar cone of  $K$ . When  $\theta(x, y) = y$  and  $C = \mathbf{R}^n$ ,  $GCP(F, \theta, K, C)$  reduces to  $GCP(F, K)$  which was introduced by Saigal [15].

In this report, our aim is to prove a basic theorem of complementarity for  $GVIP(F, \theta, \tau, K, C)$ . This main result is given in Section 2. Our work in this section is closely related to the work of Kojima [6]. In Section 3, we employ the main result to establish some existence results for the generalized complementarity problem. Finally, in Section 4 we investigate some sets of conditions that will normally be satisfied by a nonsolvable  $GCP(F, K)$ .

## 2. The Main Result

A subset  $K$  of  $\mathbf{R}^n$  is *contractible* if it can be continuously deformed into a point within the set itself. That is,  $K$  is contractible if there exists a continuous mapping  $g : K \times [0, 1] \rightarrow K$  such that  $g$  is the identity mapping on  $K \times \{1\}$  and  $g$  is a constant mapping on  $K \times \{0\}$ . Let  $K$  and  $C$  be subsets of  $\mathbf{R}^n$  and  $\mathbf{R}^m$  respectively. Let  $F$  be a point-to-set mapping from  $K$  into  $C$ . The mapping  $F$  is said to be *upper continuous* at  $x \in K$  if and only if a sequence  $\{x_n\}$  converging to  $x$ , and a sequence  $\{y_n\}$  with  $y_n \in F(x_n)$  converging to  $y$ , imply  $y \in F(x)$ .  $F$  is said to be *uniformly compact near  $x$*  if there exists a neighborhood  $V$  of  $x$  such that  $F(V) = \bigcup_{u \in V} F(u)$  is bounded. If  $F$  is both upper continuous and uniformly compact on  $K$ , then  $F(B)$  is compact for any compact subset  $B$  of  $K$  [16, Lemma 2.6]. In order to obtain our main result, we need the following lemma which is a slightly different version of a theorem of Mas-Colell [7, Theorem 3].

**Lemma 2.1.** *Let  $B \subset \mathbf{R}^n$  be compact and convex. For  $t \geq 0$ , let  $F : B \times [0, t] \rightarrow B$  be upper continuous and uniformly compact such that  $F(x, t)$  is bounded and contractible for each  $(x, t)$ . Then there is a connected set  $T \subset B \times [0, t]$  which intersects both  $B \times \{0\}$  and  $B \times \{t\}$  such that  $x \in F(x, t), \forall (x, t) \in T$ .*

**Proof.** This follows from [8, Theorem 2.2] and [7, Theorem 3].  $\square$

A point-to-set mapping  $D : \mathbf{R}_+ \rightarrow \mathbf{R}^n$  is said to be *isotone* if  $D(r) \subset D(s)$  whenever  $r \leq s$ . We now can state our main result.

**Theorem 2.2.** (The basic theorem of complementarity) *Let  $K$  and  $C$  be closed and convex subsets of  $\mathbf{R}^n$  and  $\mathbf{R}^m$  respectively. Let  $F : K \rightarrow C$  be an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in K$ . Let  $D : \mathbf{R}_+ \rightarrow \mathbf{R}^n$  be a continuous and isotone point-to-set mapping such that  $D(t) \subset K$  is nonempty bounded and convex  $\forall t \in \mathbf{R}_+$ ,  $D(0) = \{0\}$  and  $\bigcup\{D(t) : t \in \mathbf{R}_+\}$  is closed. Let  $\theta : K \times C \rightarrow \mathbf{R}^n$  and  $\tau : K \times K \rightarrow \mathbf{R}^m$  be continuous. Suppose that*

(i)  $\langle \theta(x, y), \tau(x, x) \rangle \geq 0, \forall (x, y) \in K \times C,$

(ii) *for each fixed  $(x, y, t) \in K \times C \times \mathbf{R}_+$ , the set*

$$H(x, y, t) = \{u \in D(t) : \langle \theta(x, y), \tau(u, x) \rangle = \min_{s \in D(t)} \langle \theta(x, y), \tau(s, x) \rangle\}$$

is contractible.

Then there is a closed connected set  $S$  in  $K$  such that

(iii) each  $x \in S$  solves  $GVIP(F, \theta, \tau, D(t), C)$  for some  $t \in \mathbf{R}_+$ ,

(iv) for each  $t \in \mathbf{R}_+$ , there is an  $x \in S$  which solves  $GVIP(F, \theta, \tau, D(t), C)$ .

Furthermore,  $S$  can be chosen so that it is maximal or minimal.

**Proof.** For each  $t \in \mathbf{R}_+$ , let  $S(t) = \{x \in D(t) : x \text{ solves } GVIP(F, \theta, \tau, D(t), C)\}$ . By [16, Corollary 3.1.4],  $S(t) \neq \emptyset, \forall t \in \mathbf{R}_+$ . Let  $X = \bigcup_{t \in \mathbf{R}_+} S(t)$ . Then  $X$  is closed. To see this, let  $\{x_n\}$  be a convergent sequence in  $X$  with limit  $x$ . Then there exist  $t_n \geq 0$  such that  $x_n \in S(t_n), \forall n$ . This implies that  $x_n \in \bigcup_{t \in \mathbf{R}_+} S(t), \forall n$ . Since  $\bigcup_{t \in \mathbf{R}_+} S(t)$  is closed,  $x \in \bigcup_{t \in \mathbf{R}_+} S(t)$ . So  $x \in D(t)$  for some  $t \geq 0$ . Let  $\bar{t} = \inf\{t \in \mathbf{R}_+ : x \in D(t)\}$ . Then since  $D$  is continuous,  $x \in D(\bar{t})$ . Also if  $\{t_n\}$  has a limit point  $r$  such that  $r < \bar{t}$ , then the continuity of  $D$  will force  $x \in D(r)$  which contradicts the choice of  $\bar{t}$ . Let  $v_n = \min(t_n, \bar{t}), \forall n$ . Then  $v_n \rightarrow \bar{t}$  as  $n \rightarrow \infty$ . Since  $\{x_n\}$  converges to  $x$ , the set  $A = \{x_n\} \cup \{x\}$  is compact. Therefore  $F(A)$  is also compact. Since  $x_n$  solves  $GVIP(F, \theta, \tau, D(t_n), C)$ , there is a  $y_n \in F(x_n)$  such that

$$\langle \theta(x_n, y_n), \tau(u, x_n) \rangle \geq 0, \forall u \in D(t_n).$$

Then since  $F(A)$  is compact,  $\{y_n\}$  has a converging subsequence  $\{y_{n'}\}$  such that  $y_{n'} \rightarrow y \in F(A)$ . Because  $F$  is upper continuous, we have  $y \in F(x)$ . Now for each  $z \in D(\bar{t})$ , from the continuity of  $D$ , there is a sequence  $\{z_n\}$  such that  $z_n \in D(v_n) \subset D(t_n)$  and  $z_n$  converges to  $z$ . By considering the subsequence  $\{z_{n'}\}$  corresponding to  $\{x_{n'}\}$ , we have

$$\langle \theta(x_{n'}, y_{n'}), \tau(z_{n'}, x_{n'}) \rangle \geq 0, \forall n'.$$

Since  $\theta, \tau$  are continuous, we have as  $n' \rightarrow \infty$

$$\langle \theta(x, y), \tau(z, x) \rangle \geq 0.$$

Hence  $\langle \theta(x, y), \tau(z, x) \rangle \geq 0, \forall z \in D(\bar{t})$ . Since  $x \in D(\bar{t})$  and  $y \in F(x)$ , we conclude that  $x \in S(\bar{t}) \subset X$ . Therefore  $\bigcup_{t \in \mathbf{R}_+} S(t)$  is closed. Let  $S$  be the maximal connected component of  $\bigcup_{t \in \mathbf{R}_+} S(t)$  containing the origin. Since  $\bigcup_{t \in \mathbf{R}_+} S(t)$  is closed,  $S$  is also closed. If  $x \in S$ , then  $x \in \bigcup_{t \in \mathbf{R}_+} S(t)$ . Thus  $x \in S(t)$  for some  $t \in \mathbf{R}_+$ . Hence (iii) is satisfied.

To prove (iv), let  $t \in \mathbf{R}_+$  be given. Since  $D(t)$  is compact, so is  $F(D(t))$ . Let  $H(t)$  be the convex hull of  $F(D(t))$ . It is easy to see that  $\Pi$  is upper continuous on  $K \times C \times \mathbf{R}_+$ . Now define

$$G : D(t) \times H(t) \times [0, t] \rightarrow D(t) \times H(t)$$

by

$$G(x, y, t) = (\Pi(x, y, t), F(x)).$$

Suppose  $(x_n, y_n, t_n) \rightarrow (x, y, k)$  and  $(w_n, v_n) \in G(x_n, y_n, t_n)$  such that

$$(w_n, v_n) \rightarrow (w, v) \text{ as } n \rightarrow \infty.$$

Then since  $F$  is upper continuous,  $v \in F(x)$ . For each  $z \in D(k)$ , since  $D$  is continuous, there exists a sequence  $\{z_n\}$  converging to  $z$  such that  $z_n \in D(t_n) \forall n$ . Since  $w_n \in \Pi(x_n, y_n, t_n), \forall n$ ,

$$\langle \theta(x_n, y_n), \tau(w_n, x_n) \rangle \leq \langle \theta(x_n, y_n), \tau(z_n, x_n) \rangle, \forall n.$$

Since  $\theta$  and  $\tau$  are continuous, by letting  $n \rightarrow \infty$ , we have

$$\langle \theta(x, y), \tau(w, x) \rangle \leq \langle \theta(x, y), \tau(z, x) \rangle, \forall z \in D(k).$$

Hence  $w \in D(k)$  and thus  $(w, v) \in G(x, y, k)$ . Consequently,  $G$  is upper continuous. Also since  $\Pi(x, y, k)$  and  $F(x)$  are compact and contractible,  $G(x, y, k)$  is compact and contractible for each  $(x, y, k)$ . By Lemma 2.1, there is a connected set  $T \subset D(t) \times H(t) \times [0, t]$  such that

$$T \cap D(t) \times H(t) \times \{0\} \neq \emptyset, T \cap D(t) \times H(t) \times \{t\} \neq \emptyset \text{ and } (x, y) \in G(x, y, k), \forall (x, y, k) \in T.$$

Let  $g$  be the projection mapping from  $D(t) \times H(t) \times [0, t]$  onto  $D(t)$  and  $E = g(T)$ . Since  $g$  is continuous and  $T$  is connected,  $E$  is also connected. Note that  $(x, y) \in G(x, y, k)$  if and only if  $(x, y)$  solves  $GVIP(F, \theta, \tau, D(k), C)$ . Therefore  $E \subset \bigcup_{t \in \mathbf{R}_+} S(t)$ . Since  $T \cap D(t) \times H(t) \times \{0\} \neq \emptyset$  and  $T \cap D(t) \times H(t) \times \{t\} \neq \emptyset$ ,  $0 \in E$  and there is an  $x \in E \cap S(t)$ . By the definition of  $S$ ,  $E \subset S$ . Thus  $x \in S$  solves  $GVIP(F, \theta, \tau, D(t), C)$ . Hence (iv) is also satisfied.

It remains to show the existence of a minimal closed connected set in  $K$  satisfying (iii) and (iv). Let  $\Gamma$  be the family of all closed connected sets in  $K$  satisfying (iii) and (iv). By what we have shown,  $\Gamma \neq \emptyset$ . We introduce set inclusion relation  $\subset$  as a partial ordering on  $\Gamma$ . Let  $\Lambda$  be a chain (linearly ordered set) in  $\Gamma$ ; that is, for any  $S, T \in \Lambda$ , either  $S \subset T$  or  $T \subset S$ . We want to show that  $\Lambda$  has a lower bound so that we can apply Zorn's Lemma to ensure a minimal set in  $\Gamma$ . Let  $\bar{S} = \bigcap \{S : S \in \Lambda\}$ . Clearly  $\bar{S}$  satisfies (iii). For each  $t \in \mathbf{R}_+$ , consider the family  $\Omega = \{S \cap S(t) : S \in \Lambda\}$  of closed subsets of  $S(t)$ . Let  $\{S(t_i) \cap S(t) : i = 1, \dots, m\}$  be any finite subfamily of  $\Omega$ . Since  $\Lambda$  is a chain,  $\bigcap_{i=1}^m (S(t_i) \cap S(t)) = (\bigcap_{i=1}^m S(t_i)) \cap S(t) = S(t_\ell) \cap S(t) \neq \emptyset$ , for some  $\ell, 1 \leq \ell \leq m$ . So  $\Omega$  has the finite intersection property. Since  $S(t)$  is compact, we conclude that  $\bar{S} \cap S(t) \neq \emptyset$  (see e.g. Willard[67, 17.4]). Therefore  $\bar{S} \in \Gamma$  and  $\bar{S}$  is a lower bound of  $\Lambda$ . Hence by Zorn's Lemma,  $\Gamma$  has a minimal element. That is, there is a minimal connected closed set  $S \in K$  satisfying (iii) and (iv).  $\square$

#### Remarks.

- (i) The convexity assumption in Lemma 2.1 and Theorem 2.2 is not essential. As remarked by Mas-Colell [7], it can be replaced by *contractible absolute neighborhood retract*.
- (ii) The uniform compactness of the point-to-set mapping  $F$  is essential in Theorem 2.2 as the following example illustrates. Let  $K = [0, \infty)$  and  $C = \mathbf{R}$ . Let  $\theta(x, y) = y$ ,  $\tau(x, y) = x - y$  and  $F$  be defined by

$$F(x) = \begin{cases} \{1/x\} & \text{if } x > 0 \\ \{-1\} & \text{if } x = 0 \end{cases}$$

Finally, let  $D(t) = [0, t]$  for  $t \in \mathbf{R}_+$ . Then all the conditions of Theorem 2.2 except that  $F$  is uniformly compact are satisfied. But it can be seen that  $GVIP(F, \theta, D(t), C)$  has no solution for all  $t > 0$ .

The following two theorems are immediate.

**Theorem 2.3.** *Let  $K$  and  $C$  be nonempty closed and convex subsets of  $\mathbf{R}^n$  and  $\mathbf{R}^m$  respectively. Let  $F : K \rightarrow C$  be an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in K$ . Let  $D : \mathbf{R}_+ \rightarrow \mathbf{R}^n$  be a continuous and isotone point-to-set mapping such that  $D(t) \subset K$  is nonempty, bounded, convex  $\forall t \in \mathbf{R}_+$ ,  $D(0) = \{0\}$  and  $\bigcup\{D(t) : t \in \mathbf{R}_+\}$  is closed. Let  $\theta : K \times C \rightarrow \mathbf{R}^n$  be continuous. Then there is a closed connected set  $S$  in  $K$  such that*

- (i) each  $x \in S$  solves  $GVIP(F, \theta, D(t), C)$  for some  $t \in \mathbf{R}_+$ ,
- (ii) for each  $t \in \mathbf{R}_+$ , there is an  $x \in S$  which solves  $GVIP(F, \theta, D(t), C)$ .

Furthermore,  $S$  can be chosen so that it is maximal or minimal.  $\square$

**Theorem 2.4.** *Let  $K$  be a nonempty closed and convex subset of  $\mathbf{R}^n$ . Let  $F : K \rightarrow \mathbf{R}^n$  be an upper continuous point to set map such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in K$ . Let  $D : \mathbf{R}_+ \rightarrow \mathbf{R}^n$  be a continuous and isotone point-to-set mapping such that  $D(t) \subset K$  is nonempty, bounded, convex  $\forall t \in \mathbf{R}_+$ ,  $D(0) = \{0\}$  and  $\bigcup\{D(t) : t \in \mathbf{R}_+\}$  is closed. Then there is a closed connected set  $S$  in  $K$  such that*

- (i) each  $x \in S$  solves  $GVIP(F, D(t))$  for some  $t \in \mathbf{R}_+$ ,
- (ii) for each  $t \in \mathbf{R}_+$ , there is an  $x \in S$  which solves  $GVIP(F, D(t))$ .

Furthermore,  $S$  can be chosen so that it is maximal or minimal.  $\square$

Let  $K$  be a convex set in  $\mathbf{R}^n$  and  $f : K \rightarrow [-\infty, +\infty]$  be a convex function. A vector  $x^*$  is said to be a *subgradient* (see, e.g. Rockafaller [14]) of  $f$  at a point  $x$  if

$$f(z) \geq f(x) + \langle x^*, z - x \rangle, \forall z \in K.$$

The set of all subgradients of  $f$  at  $x$  is called the *subdifferential* of  $f$  at  $x$  and is denoted by  $\partial f(x)$ . When  $f(x) = \delta(x|K)$ , that is,  $f$  is the *indicator function* of  $K$ , then  $x^* \in \partial \delta(x|K)$  if and only if  $x \in K$  and  $\langle x^*, z - x \rangle \leq 0$  for all  $z \in K$ . Thus  $\partial \delta(x|K)$  is the *normal cone* to  $K$  at  $x$  (empty if  $x \notin K$ ).

**Remark.** Suppose that  $K$  is a closed convex cone. Let  $D : \mathbf{R}_+ \rightarrow K$  be defined as  $D(t) = \{x \in K : \|x\| \leq t\}$ . Then  $D(t)$  is compact and convex for all  $t \in \mathbf{R}_+$ . By Theorem 2.3, there is a closed connected subset  $S$  in  $K$  such that  $S \cap S(t) \neq \emptyset, \forall t \in \mathbf{R}_+$ , where  $S(t)$  is the nonempty solution set for  $GVIP(F, \theta, D(t), C)$ . Thus for each  $t$ , there exist  $x_t \in K$  and  $y_t \in F(x_t)$  such that

$$\langle \theta(x_t, y_t), x - x_t \rangle \geq 0, \forall x \in D(t).$$

Equivalently, for each  $t$ ,  $x_t$  solves the following ordinary convex program:

$$\text{minimize } \langle \theta(x_t, y_t), x \rangle + \delta(x|K) \text{ subject to } \|x\|^2 \leq t^2.$$



By the generalized Kuhn-Tucker conditions (see e.g. Rockafellar [14, Theorem 20.2, 28.3]), there exist  $\lambda_t \geq 0, x_t^0 \in \partial\delta(x_t|K)$  such that

$$\begin{aligned}\lambda_t(\|x_t\|^2 - t^2) &= 0, \\ \theta(x_t, y_t) + 2\lambda_t x_t + x_t^0 &= 0.\end{aligned}$$

If  $\lambda_t = 0$  for some  $t$ , then since  $x_t^0 \in \partial\delta(x_t|K)$ ,

$$\langle \theta(x_t, y_t), x - x_t \rangle \geq 0, \forall x \in K.$$

For each  $x \in K$ , since  $K$  is a convex cone,  $x + x_t \in K$ . So we have

$$\langle \theta(x_t, y_t), x \rangle \geq 0, \forall x \in K.$$

Hence  $\theta(x_t, y_t) \in K^*$ . Also since  $\langle \theta(x_t, y_t), x_t \rangle = \langle -x_t^0, x_t \rangle = 0$ ,  $x_t$  solves  $GCP(F, \theta, K, C)$ . Therefore, if for some positive  $t$ ,  $GVIP(F, \theta, D(t), C)$  has a solution which is an interior point of  $D(t)$ , then  $GCP(F, \theta, K, C)$  has a solution. This observation justifies the title of this report.

Since continuous single-valued functions are clearly upper continuous, uniformly compact and contractible, the next two corollaries follow from Theorem 2.4 directly.

**Corollary 2.5.** (Kojima [6, Theorem 2.1]) *Suppose that  $D$  is a continuous and isotone point-to-set mapping from  $\mathbf{R}_+$  into the class of all nonempty compact convex subsets of  $\mathbf{R}^n$  such that  $D(0) = \{0\}$  and that  $\bigcup\{D(t) : t \in \mathbf{R}_+\}$  is closed, and let  $f$  be a continuous function from  $\mathbf{R}^n$  into itself. Then there is a closed connected subset  $S$  of  $\mathbf{R}^n$  such that*

- (i) *for each  $x \in S$  there is a  $t \in \mathbf{R}_+$  such that  $x$  is a stationary point of the pair  $(f, D(t))$ ,*
- (ii) *for each  $t \in \mathbf{R}_+$  there is a stationary point  $x \in S$  of the pair  $(f, D(t))$ .*

Furthermore,  $S$  can be chosen so that it is maximal or minimal.  $\square$

**Corollary 2.6.** (Eaves [2, Theorem 3]) *Let  $D(t) = \{x \in \mathbf{R}_+^n : \langle x, d \rangle \leq t\}$  for each  $t \in \mathbf{R}_+$ , where  $d$  is a positive vector and let  $f$  be a continuous function from  $\mathbf{R}^n$  into itself. Then there is a closed connected subset  $S$  of  $\mathbf{R}^n$  such that*

- (i) *for each  $x \in S$ , there is a  $t \in \mathbf{R}_+$  such that  $x$  is a stationary point of the pair  $(f, D(t))$ ,*
- (ii) *for each  $t \in \mathbf{R}_+$ , there is a stationary point  $x \in S$  of the pair  $(f, D(t))$ .*

Furthermore,  $S$  can be chosen so that it is maximal or minimal.  $\square$

Let  $K$  be a closed convex pointed cone in  $\mathbf{R}^n$  and  $K^*$  be its polar cone. Let  $d \in \text{int}(K^*)$  (since  $K$  is pointed,  $\text{int}(K^*)$  is not empty). Let  $D(t) = \{x \in K : \langle x, d \rangle \leq t, \forall t \in \mathbf{R}_+\}$ . Then  $D(t)$  is compact and convex for all  $t \in \mathbf{R}_+$ . For if we let  $\alpha_x$  be the angle between  $d$  and  $x$ , then since  $d \in \text{int}(K^*)$ , there is an  $\alpha_0$  such that  $0 < \alpha_0 < \pi/2$  and  $\alpha_0 \leq \alpha_x, \forall x \in K$ . Thus if  $x_n \in D(t)$  such that  $\|x_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ , for some  $t \geq 0$ , then  $t \geq \langle d, x_n \rangle = \|d\|\|x_n\|\cos\alpha_x \geq \|d\|\|x_n\|\cos\alpha_0 \rightarrow \infty$

as  $n \rightarrow \infty$  which is a contradiction. Since  $GVIP(F, K)$  is a special case of  $GVIP(F, \theta, \tau, K, C)$ , the next corollary follows from Theorem 2.4 directly.

**Corollary 2.7.** *Let  $D_t = \{x \in K : \langle x, d \rangle \leq t\}$  for each  $t \in \mathbf{R}_+$ , where  $d \in \text{int}(K^*)$  and let  $F$  be an upper continuous mapping from  $K$  into  $\mathbf{R}^n$  such that  $F(x)$  is contractible and uniformly compact near  $x$  for each  $x \in K$ . Then there is a closed connected subset  $S$  of  $K$  such that*

- (i) *each  $x \in S$  is a stationary point of the pair  $(F, D_k)$  for  $k = \langle d, x \rangle$ .*
- (ii) *for each  $k \geq 0$  there is a stationary point  $x \in S$  of the pair  $(F, D_k)$ .*

□

We note that Corollary 2.7 is similar to an assertion due to Saigal [15, Theorem 4.1] which may not be true in general by considering the example from Remark (ii) following Theorem 2.2.

It is worth noting that we don't need to require  $K$  to be pointed, solid, or a cone in Theorem 2.2. So Theorems 2.2, 2.3 and 2.4 can be expected to have wider applications than most of the other versions of the basic theorem of complementarity.

Following Kojima's definition [6], any set  $S$  in Theorems 2.2, 2.3 or 2.4 will be called a *Browder set* of tuples  $(F, \theta, \tau, K, C, D)$ ,  $(F, \theta, K, C, D)$  or  $(F, K, D)$ , respectively.

### 3. Application of The Main Results to The GCP

In this section, we shall apply the basic theorems of complementarity established in Section 2 to generalized complementarity problems. The first result is that if some Browder set is bounded, then  $GVIP$  has a solution.

**Theorem 3.1.** *If  $\bigcup\{D(t) : t \in \mathbf{R}_+\} = K$ , then any bounded Browder set for  $(F, \theta, \tau, K, C, D)$ ,  $(F, \theta, K, C, D)$  or  $(F, K, D)$  contains a solution for  $GVIP(F, \theta, \tau, K, C)$ ,  $GVIP(F, \theta, K, C)$  and  $GVIP(F, K)$  respectively.*

**Proof.** Let  $S$  be a bounded Browder set for  $(F, \theta, \tau, K, C, D)$ . Let  $\{t_n\}$  be a nonnegative and increasing sequence such that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Since  $S \cap D(t) \neq \emptyset, \forall t \geq 0$ , there exists a sequence  $\{x_n\} \subset S$  such that  $x_n$  solves  $GVIP(F, \theta, \tau, D(t_n), C), \forall n$ . Now since  $S$  is closed and bounded, it is compact. So  $\{x_n\}$  has a subsequence  $\{x_{n'}\}$  such that  $x_{n'} \rightarrow x \in S$ . Also there exist  $y_{n'} \in F(x_{n'})$  such that

$$\langle \theta(x_{n'}, y_{n'}), u - x_{n'} \rangle \geq 0, \forall u \in D(t_{n'}).$$

Since  $F(x)$  is bounded,  $F(x)$  is compact for all  $x \in K$ . Let  $A = \{x_{n'}\} \cup \{x\}$ . Then  $A$  is compact and thus  $F(A)$  is compact. Therefore there exists a subsequence  $\{y_{n''}\}$  of  $\{y_{n'}\}$  such that  $y_{n''} \rightarrow y$  as  $n'' \rightarrow \infty$ . Then since  $F$  is upper continuous,  $y \in F(x)$ . Now for each  $x \in K$ , since  $D(t)$  is monotone in  $t$ ,  $D(t_{n''}) \subset D(t_{(n+1)'})$ , and so there exists a positive integer  $m$  such that  $x \in D(t_{n''}), \forall n \geq m$ . Then

$$\langle \theta(x_{n''}, y_{n''}), x - x_{n''} \rangle \geq 0, \forall n \geq m.$$

Let  $n''$  go to  $\infty$ . We have  $\langle \theta(x, y), x - x \rangle \geq 0$ . Therefore we conclude that

$$\langle \theta(x, y), x - \bar{x} \rangle \geq 0, \forall x \in K.$$

Hence  $\bar{x} \in S$  solves  $GVIP(F, \theta, \tau, K, C)$ . Finally the other two statements follow directly by noting that  $GVIP(F, \theta, K, C)$  and  $GVIP(F, K)$  are special cases of  $GVIP(F, \theta, \tau, K, C)$ .  $\square$

The following corollaries are immediate.

**Corollary 3.2.** *Let  $K$  be a closed convex cone and  $\bigcup\{D(t) : t \in \mathbf{R}_+\} = K$ .*

- (a) *If there exists a bounded Browder set for  $(F, \theta, K, C, D)$ , then  $GCP(F, \theta, K, C)$  has a solution.*
- (b) *If there exists a bounded Browder set for  $(F, K, D)$ , then  $GCP(F, K)$  has a solution.*

**Proof.** This follows from Theorem 3.1 and [16, Corollary 4.2].  $\square$

**Corollary 3.3.** *Let  $K$  be a closed convex cone. If there exists a bounded Browder set for  $(f, D)$  such that  $\bigcup\{D(t) : t \in \mathbf{R}_+\} = \mathbf{R}_+^n$ , then there is a solution for  $CP(f)$ .  $\square$*

If we exploit the proof of Theorem 3.1 more deeply, it is easy to see that under the condition  $\bigcup\{D(t) : t \in \mathbf{R}_+\} = K$ , the existence of a converging sequence  $\{x_n\}$  such that  $x_n \in S \cap D(t_n)$ , for some Browder set  $S$  and a sequence  $\{t_n\}$  converging to  $\infty$  guarantees the existence of a solution to the corresponding generalized variational inequality problem. By essentially the same argument as in Theorem 3.1, we have the following theorem.

**Theorem 3.4.** *Let  $K$  be a nonempty closed convex subset of  $\mathbf{R}^n$ . Suppose that  $\bigcup_{t \in \mathbf{R}_+} D(t) = K$ , where  $D$  is a continuous isotone point-to-set mapping from  $\mathbf{R}_+$  into the family of nonempty bounded and convex subsets of  $K$ . If there exists a nonnegative sequence  $\{t_n\}$  converging to  $\infty$  and some Browder set  $S$  for  $(F, \theta, \tau, K, C, D)$ ,  $(F, \theta, K, C, D)$  or  $(F, K, D)$  respectively such that the sequence  $\{x_n\}$  with  $x_n \in S \cap D(t_n), \forall n$  is convergent, then there exists a solution for  $GVIP(F, \theta, \tau, K, C)$ ,  $GVIP(F, \theta, K, C)$  or  $GVIP(F, K)$ , respectively.  $\square$*

The following corollary is immediate.

**Corollary 3.5.** *Let  $K$  be a nonempty closed convex cone in  $\mathbf{R}^n$ . Suppose that  $\bigcup\{D(t) : t \in \mathbf{R}_+\} = K$ , where  $D$  is a continuous isotone point-to-set mapping from  $\mathbf{R}_+$  into the family of nonempty bounded and convex subsets of  $K$ . If there exists a nonnegative sequence  $\{t_n\}$  converging to  $\infty$  and some Browder set  $S$  for  $(F, \theta, K, C, D)$  or  $(F, K, D)$  respectively such that the sequence  $\{x_n\}$  with  $x_n \in S \cap D(t_n), \forall n$  is convergent, then there exists solution for  $GCP(F, \theta, K, C)$  or  $GCP(F, K)$ , respectively.*

**Proof.** This follows directly from Theorem 3.4 and [16, Corollary 4.2].  $\square$

Let  $K$  be a closed convex cone in  $\mathbf{R}^n$ . We say that a subset  $U \subset K \setminus D$  separates  $D$  from  $\infty$ , if each unbounded connected subset of  $K$  which intersects  $D$  also intersects  $U$ .

**Theorem 3.6.** *Let  $D$  be a continuous and isotone point-to-set mapping from  $\mathbf{R}_+$  into the class of nonempty bounded and convex subsets of a closed convex cone  $K$  such that  $\bigcup\{D(t) : t \in \mathbf{R}_+\} = K$ .*

and  $D(0) = \{0\}$ . Let  $U$  be a bounded subset of  $K$  which separates  $D(r)$  from  $\infty$  for some  $r \in \mathbf{R}_+$  and suppose that for each  $x \in U$ , there is a  $w \in \bigcap \{ \text{int}_K(D(t)) : x \in D(t) \}$  such that

$$\langle \theta(x, y), w - x \rangle \leq 0, \forall y \in F(x).$$

Then  $GCP(F, \theta, K, C)$  has a solution in each Browder set of  $(F, \theta, K, C, D)$ .

**Proof.** Let  $S$  be any Browder set of  $(F, \theta, K, C, D)$ . If  $S$  is bounded, then the result follows from Theorem 3.1. Suppose that  $S$  is unbounded. In this case there exists  $x \in S \cap U$ . So there exist  $t \in \mathbf{R}_+$  and  $\bar{y} \in F(\bar{x})$  such that  $\bar{x} \in D(t)$  and

$$\langle \theta(\bar{x}, \bar{y}), x - \bar{x} \rangle \geq 0, \forall x \in D(\bar{t}).$$

By hypothesis, there exists a  $w \in \text{int}_K(D(\bar{t}))$  such that

$$\langle \theta(\bar{x}, \bar{y}), w - \bar{x} \rangle \leq 0, \forall y \in F(\bar{x}).$$

In particular  $\langle \theta(\bar{x}, \bar{y}), w - \bar{x} \rangle \leq 0$ . Also since  $w \in D(\bar{t})$ ,  $\langle \theta(\bar{x}, \bar{y}), w - \bar{x} \rangle \geq 0$ . Thus  $\langle \theta(\bar{x}, \bar{y}), w - \bar{x} \rangle = 0$ . Now choose  $\lambda > 0$  sufficiently small such that  $w + \lambda \bar{x} \in D(\bar{t})$ . Then we have

$$\langle \theta(\bar{x}, \bar{y}), w + \lambda \bar{x} - \bar{x} \rangle = \lambda \langle \theta(\bar{x}, \bar{y}), \bar{x} \rangle \geq 0.$$

Thus  $\langle \theta(\bar{x}, \bar{y}), \bar{x} \rangle \geq 0$ . On the other hand, since  $0 \in D(\bar{t})$ ,  $\langle \theta(\bar{x}, \bar{y}), \bar{x} \rangle \leq 0$ . Hence  $\langle \theta(\bar{x}, \bar{y}), \bar{x} \rangle = 0$ . It remains to show that  $\theta(\bar{x}, \bar{y}) \in K^*$ . To this end, for each  $x \in K$ , let  $\lambda_x > 0$  be such that  $w + \lambda_x x \in D(\bar{t})$ . Then

$$\langle \theta(\bar{x}, \bar{y}), w + \lambda_x x - \bar{x} \rangle = \lambda_x \langle \theta(\bar{x}, \bar{y}), x \rangle \geq 0.$$

So  $\langle \theta(\bar{x}, \bar{y}), x \rangle \geq 0$ . Thus  $\theta(\bar{x}, \bar{y}) \in K^*$ . Therefore  $\bar{x}$  solves  $GCP(F, \theta, K, C)$ .  $\square$

The next corollary follows from Theorem 3.6 directly by letting  $\theta(x, y) = y$ .

**Corollary 3.7.** Let  $D$  be a continuous and isotone point-to-set mapping from  $\mathbf{R}_+$  into the class of nonempty bounded and convex subsets of a closed convex cone  $K$  such that  $\bigcup \{ D(t) : t \in \mathbf{R}_+ \} = K$  and  $D(0) = \{0\}$ . Let  $U$  be a bounded subset of  $K$  which separates  $D(r)$  from  $\infty$  for some  $r \in \mathbf{R}_+$  and suppose that for each  $x \in U$ , there is a  $w \in \bigcap \{ \text{int}_K(D(t)) : x \in D(t) \}$  such that

$$\langle y, w - x \rangle \leq 0, \forall y \in F(x).$$

Then  $GCP(F, K)$  has a solution in each Browder set of  $(F, K, D)$ .  $\square$

From Corollary 3.7, we get the following corollary directly by letting  $K = \mathbf{R}_+^n$ .

**Corollary 3.8.** (Kojima [6, Theorem 5.1]) Let  $D$  be a continuous and isotone point-to-set mapping from  $\mathbf{R}_+$  into the class of nonempty bounded and convex subsets of  $\mathbf{R}_+^n$  such that  $\bigcup \{ D(t) : t \in \mathbf{R}_+ \} = \mathbf{R}_+^n$  and  $D(0) = \{0\}$ . Suppose that for each  $x \in U$  there is a  $w \in \bigcap \{ \text{int}_{\mathbf{R}_+^n}(D(t)) : x \in D(t) \}$  such that  $\langle f(x), w - x \rangle \leq 0$ , where  $U$  is a bounded subset of  $\mathbf{R}_+^n \setminus \{0\}$  which separates the origin from  $\infty$ . Then  $CP(f)$  has a solution in each Browder set of the pair  $(f, D)$ .  $\square$

A point-to-set mapping  $F$  defined on a set  $K$  is said to be  $\beta$ -copositive on  $K$  if there exists an increasing function  $\beta : [0, \infty) \rightarrow [0, \infty)$  with  $\beta(0) = 0$  and  $\beta(r) \rightarrow \infty$  as  $r \rightarrow \infty$ , and there exists a  $y_0 \in F(0)$  such that for all  $x \in K$ , we have

$$\langle y - y_0, x \rangle \geq \|x - x_0\| \beta(\|x\|), \forall y \in F(x).$$

Let  $K$  and  $C$  be subsets of  $\mathbf{R}^n$  and  $\mathbf{R}^m$  respectively. Given a point-to-set mapping  $F$  from  $K$  into  $C$  and a continuous mapping  $\theta$  from  $K \times C$  into  $\mathbf{R}^n$ , let  $F_\theta$  be a point-to-set mapping defined by  $F_\theta(x) = \{\theta(x, y) : y \in F(x)\}$  on  $K$ . we have

**Theorem 3.9.** *Let  $K$  be a nonempty closed convex set containing 0 in  $\mathbf{R}^n$ . Let  $F : K \rightarrow \mathbf{R}^m$  be a contractible-valued point-to-set mapping which is upper continuous and uniformly compact. If  $F_\theta$  is  $\beta$ -copositive, then  $GVIP(F, \theta, K, C)$  and hence  $GCP(F, \theta, K, C)$  has a solution.*

**Proof.** Let  $D(t) = \{x \in K : \|x\| \leq t\}, \forall t \in \mathbf{R}_+$ . Let  $S$  be any Browder set of  $(F, \theta, K, D)$ . Suppose  $S$  is unbounded. Then there exists  $x_n \in S, \forall n$  such that  $\|x_n\| \rightarrow \infty$ , as  $n \rightarrow \infty$ . Thus for each  $n$ , there exist  $t_n \geq 0, x_n \in D(t_n)$  and  $y_n \in F(x_n)$  such that

$$\langle \theta(x_n, y_n), u - x_n \rangle \geq 0, \forall u \in D(t_n).$$

Since  $F_\theta$  is  $\beta$ -copositive on  $K$ , there exists an increasing function  $\beta : [0, \infty) \rightarrow [0, \infty)$  with  $\beta(0) = 0$  and  $\beta(r) \rightarrow \infty$  as  $r \rightarrow \infty$  and a  $z \in F(0)$  such that

$$\langle \theta(x, y) - \theta(0, z), x \rangle \geq \|x\| \beta(\|x\|), \forall x \in K, y \in F(x).$$

In particular we have  $\langle \theta(x_n, y_n) - \theta(0, z), x_n \rangle \geq \|x_n\| \beta(\|x_n\|), \forall n$ . Since  $\|x_n\| \rightarrow \infty$ , as  $n \rightarrow \infty$ , there is an  $m$  such that  $x_m \neq 0$  and  $\beta(\|x_m\|) - \|\theta(0, z)\| > 0$ . So

$$\begin{aligned} \langle \theta(x_m, y_m), x_m \rangle &\geq \langle \theta(0, z), x_m \rangle + \|x_m\| \beta(\|x_m\|) \\ &\geq \|x_m\| (\beta(\|x_m\|) - \|\theta(0, z)\|) \\ &> 0. \end{aligned}$$

On the other hand since  $x_m$  solves  $GVIP(F, \theta, D(t_m), C)$  and  $0 \in D(t_m)$ , we have  $\langle \theta(x_m, y_m), x_m \rangle \leq 0$ . This is a contradiction. Hence  $S$  is bounded. Therefore by Theorem 3.1  $GVIP(F, \theta, K, C)$  has a solution. The second assertion follows from [16, Corollary 4.2].  $\square$

The following corollaries are immediate.

**Corollary 3.10.** *Let  $K$  be a nonempty closed convex set containing 0 in  $\mathbf{R}^n$ . Let  $F : K \rightarrow \mathbf{R}^m$  be a contractible-valued point-to-set mapping which is upper continuous and uniformly compact. If  $F$  is  $\beta$ -monotone, strongly copositive or strongly monotone, then  $GVIP(F, K)$  has a solution.  $\square$*

**Corollary 3.11.** *Let  $K$  be a nonempty closed convex cone in  $\mathbf{R}^n$ . Let  $F : K \rightarrow \mathbf{R}^m$  be a contractible-valued point-to-set mapping which is upper continuous and uniformly compact. If  $F$  is  $\beta$ -copositive,  $\beta$ -monotone, strongly copositive or strongly monotone, then  $GCP(F, K)$  has a solution.  $\square$*

We note that Theorem 3.9 extends some results of Saigal ([15, Theorem 3.1]) and Karamardian ([5, Theorem 4.3]).

#### 4. The Necessary Conditions for Nonsolvable GCP

In this section, we aim to investigate some sets of conditions that are normally satisfied by a nonsolvable generalized complementarity problem  $GCP(F, K)$  via the basic theorem of complementarity. The impetus behind this investigation is that the existence results for generalized complementarity problems are normally established by considering some special mappings, e.g. strongly monotone point-to-set mappings. In general, the mappings under consideration do not possess this nice property. These new conditions will be helpful in analyzing the existence of solutions to generalized complementarity problems.

Let  $K$  be a convex subset in  $\mathbf{R}^n$  and  $F : K \rightarrow \mathbf{R}^n$  a point-to-set mapping. We say that  $F$  is *convex* on  $K$  if the set  $G(F) \cap (K \times \mathbf{R}^n)$  is convex, or equivalently, for all  $x, y \in K$  and  $0 \leq \lambda \leq 1$ ,

$$\lambda F(x) + (1 - \lambda)F(y) \subset F(\lambda x + (1 - \lambda)y).$$

**Theorem 4.1.** *Let  $K \subset \mathbf{R}^n$  be a closed convex cone and  $F : K \rightarrow \mathbf{R}^n$  an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in K$ . Let  $z^* \in F(0)$  and suppose  $F$  is convex. If  $GCP(F, K)$  has no solution, then there exist  $x^* \in K$ ,  $y^* \in F(x^*)$ ,  $x^0 \in \partial\delta(x^*|K)$ , such that*

$$x^0 = -(y^* - z^* - \langle x^*, y^* - z^* \rangle x^*), \quad \langle x^*, x^0 \rangle = 0, \quad x^* \neq 0.$$

**Proof.** Let  $D : \mathbf{R}_+ \rightarrow K$  be defined as  $D(t) = \{x \in K : \|x\| \leq t\}$ . Then  $D(t)$  is compact and convex for all  $t \in \mathbf{R}_+$ . By Theorem 2.4, there is a closed connected subset  $S$  in  $K$  such that  $S \cap S(t) \neq \emptyset, \forall t \in \mathbf{R}_+$ , where  $S(t)$  is the nonempty solution set for  $GVIP(F, D(t))$ . Thus for each  $t$ , there exist  $x_t \in K$  and  $y_t \in F(x_t)$  such that

$$\langle y_t, x - x_t \rangle \geq 0, \forall x \in D(t).$$

Equivalently, for each  $t$ ,  $x_t$  solves the following ordinary convex program:

$$\text{minimize } \langle y_t, x \rangle + \delta(x|K) \text{ subject to } \|x\|^2 \leq t^2.$$

By the generalized Kuhn-Tucker conditions, there exist  $\lambda_t \geq 0$ ,  $x_t^0 \in \partial\delta(x_t|K)$  such that

$$\lambda_t(\|x\|^2 - t^2) = 0, \tag{1}$$

$$y_t + 2\lambda_t x_t + x_t^0 = 0. \tag{2}$$

If  $\lambda_t = 0$  for some  $t$ , then since  $x_t^0 \in \partial\delta(x_t|K)$ ,

$$\langle y_t, x - x_t \rangle \geq 0, \forall x \in K.$$

For each  $x \in K$ , since  $K$  is a convex cone,  $x + x_t \in K$ . So we have

$$\langle y_t, x \rangle \geq 0, \forall x \in K.$$

Hence  $y_t \in K^*$ . Also since  $\langle y_t, x_t \rangle = \langle -x_t^0, x_t \rangle = 0$ ,  $x_t$  solves  $GCP(F, K)$  which is a contradiction. Therefore we conclude that  $\lambda_t > 0, \forall t \in \mathbf{R}_+$ . Consequently by (1),  $\|x_t\| = t, \forall t \in \mathbf{R}_+$ . Let  $x^t = x_t/t, y^t = y_t/t, \forall t$  and  $C = \{x : \|x\| = 1\}$ . Then  $x^t \in C, \forall t$ . Since  $C$  is compact,  $\{x^t\}$  has a convergent subsequence  $\{x^{t_n}\}$  with limit  $x^*$ . Clearly  $x^* \neq 0$ . Without loss of generality, we assume  $t_n \geq 1, \forall n$ . Then since  $F$  is convex, we have

$$\begin{aligned} y^{t_n} + (1 - t_n)z^* &\in 1/t_n F(x_{t_n}) + (1 - 1/t_n)F(0) \\ &\subset F(x^{t_n}). \end{aligned}$$

Therefore since  $F$  is upper continuous and uniformly compact,  $\{y^{t_n}\}$  has a convergent subsequence  $\{y^{v_n}\}$  with limit  $y^0$  such that  $y^* = y^0 + z^* \in F(x^*)$ . From (2), we have

$$\langle y_{v_n} + 2\lambda_{v_n} x_{v_n}, x_{v_n} \rangle = 0, \forall n.$$

Thus  $\lambda_{v_n} \rightarrow -\langle x^*, y^* - z^* \rangle / 2 \geq 0$  and again by (2),

$$x_{v_n}^0 / v_n \rightarrow x^0 = -(y^* - z^* - \langle x^*, y^* - z^* \rangle x^*).$$

For any  $x \in K$ , since  $tx \in K, \forall t \geq 0$  and  $x_t^0 \in \partial\delta(x_t|K)$ , we have

$$\langle x_{v_n}^0, v_n x - x_{v_n} \rangle \leq 0, \forall n.$$

By dividing both sides of the above inequality by  $v_n$  and passing to a limit, we have

$$\langle x^0, x - x^* \rangle \leq 0, \forall x \in K.$$

Thus  $x^0 \in \partial\delta(x^*|K)$  and  $\langle x^0, x^* \rangle = 0$ . Hence the result follows.  $\square$

We note that the  $z^*$  in Theorem 4.1 is arbitrarily chosen.

Let  $K$  be a closed convex cone in  $\mathbf{R}^n$  and  $F : K \rightarrow \mathbf{R}^n$  a point-to-set mapping. We say that  $F$  is *positively homogeneous* on  $K$ , if  $G(F) \cap (K \times \mathbf{R}^n)$  is a cone, or equivalently

$$F(\lambda x) = \lambda F(x), \forall \lambda > 0.$$

See Rockafaller [13] for more details on convex or positively homogeneous point-to-set mappings. We note that the convexity on Theorem 4.1 is used to ensure the existence of the limit  $y^*$  only. It is easy to see that the positive homogeneity of  $F$  is also sufficient for the existence of  $y^*$ . With this observation, the next theorem follows by essentially the same argument as in Theorem 4.1.

**Theorem 4.2.** *Let  $K \subset \mathbf{R}^n$  be a nonempty closed convex cone and  $F : K \rightarrow \mathbf{R}^n$  an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in K$ . Suppose  $F$  is positively homogeneous on  $K$ . If  $GCP(F, K)$  has no solution, then there exist  $x^* \in K, y^* \in F(x^*), x^0 \in \partial\delta(x^*|K)$ , such that*

$$x^0 = -(y^* - \langle x^*, y^* \rangle x^*), \langle x^*, x^0 \rangle = 0, x^* \neq 0. \square$$

The next corollary shows how Theorem 4.2 can be employed to obtain an existence result for  $GCP(F, K)$ .

**Corollary 4.3.** *Let  $K \subset \mathbf{R}^n$  be a nonempty closed convex cone and  $F : K \rightarrow \mathbf{R}^n$  an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in K$ . Suppose  $F$  is positively homogeneous on  $K$ . If  $\langle x, y \rangle \geq 0, \forall x \in K, y \in F(x)$ , then  $GCP(F, K)$  has a solution.*

**Proof.** Suppose that  $GCP(F, K)$  has no solution. Then by Theorem 4.2, there exist  $x^* \in K, y^* \in F(x^*), x^0 \in \partial\delta(x^*|K)$ , such that

$$x^0 = -(y^* - \langle x^*, y^* \rangle x^*), \langle x^*, x^0 \rangle = 0, x^* \neq 0.$$

Note that  $\langle x^*, y^* \rangle \leq 0$ . If  $\langle x^*, y^* \rangle = 0$ , then  $y^* = -x^0$ . Since  $x^0 \in \partial\delta(x^*|K), y^* \in K^*$ . Also since  $\langle x^*, y^* \rangle = \langle -x^0, x^* \rangle = 0, y^*$  solves  $GCP(F, K)$  which is a contradiction. Therefore  $GCP(F, K)$  has solutions.  $\square$

In the case that  $K = \mathbf{R}_+^n$ , the conclusion in Theorem 4.1 can be refined as the following theorem shows.

**Theorem 4.4.** *Let  $F : \mathbf{R}_+^n \rightarrow \mathbf{R}^n$  an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in \mathbf{R}_+^n$ . Let  $z^* \in F(0)$  and suppose  $F$  is convex. If  $GCP(F, \mathbf{R}_+^n)$  has no solution, then there exist  $x^*, \bar{x} \geq 0, y^* \in F(x^*), \bar{y} \in F(\bar{x}),$  scalars  $\bar{\lambda} > 0, \lambda^* \geq 0,$  such that*

$$v = y^* - z^* + \lambda^* x^* \geq 0, \langle x^*, v \rangle = 0, x^* \neq 0, \quad (3)$$

$$u = \bar{y} + \bar{\lambda} \bar{x} \geq 0, \langle \bar{x}, u \rangle = 0, \bar{x} \neq 0, \quad (4)$$

$$\langle x^*, u \rangle = \langle \bar{x}, v \rangle = 0. \quad (5)$$

**Proof.** Let  $D : \mathbf{R}_+ \rightarrow K$  be defined as  $D(t) = \{x \in \mathbf{R}_+^n : \|x\| \leq t\}$ . Then  $D(t)$  is compact and convex for all  $t \in \mathbf{R}_+$ . By Theorem 2.4, there is a closed connected subset  $S$  in  $K$  such that  $S \cap S(t) \neq \emptyset, \forall t \in \mathbf{R}_+$ , where  $S(t)$  is the nonempty solution set for  $GVIP(F, D(t))$ . Thus for each  $t$ , there exist  $x_t \in \mathbf{R}_+^n$  and  $y_t \in F(x_t)$  such that

$$\langle y_t, x - x_t \rangle \geq 0, \forall x \in D(t).$$

Equivalently, for each  $t, x_t$  solves the following ordinary convex program:

$$\text{minimize } \langle y_t, x \rangle \text{ subject to } \|x\|^2 \leq t^2, x \geq 0.$$

By the Kuhn-Tucker condition (see e.g. Rockafellar [14, Theorem 28.3]), there exist  $\lambda_t \geq 0$  such that

$$\lambda_t (\|x\|^2 - t^2) = 0, \quad (6)$$



$$y_t + 2\lambda_t x_t \geq 0, \quad (7)$$

$$\langle y_t + 2\lambda_t x_t, x_t \rangle = 0. \quad (8)$$

If  $\lambda_t = 0$  for some  $t$ , then

$$y_t \geq 0 \text{ and } \langle y_t, x_t \rangle = 0.$$

Thus  $x_t$  solves  $GCP(F, K)$  which is a contradiction. Therefore we conclude that  $\lambda_t > 0, \forall t \in \mathbf{R}_+$ . Consequently by (6),  $\|x_t\| = t, \forall t \in \mathbf{R}_+$ . Let  $x^t = x_t/t, y^t = y_t/t, \forall t$  and  $C = \{x : \|x\| = 1\}$ . Then  $x^t \in C, \forall t$ . Since  $C$  is compact,  $\{x^t\}$  has a convergent subsequence  $\{x^{t_n}\}$  with limit  $x^*$ . Clearly  $x^* \neq 0$ . Without loss of generality, we assume  $t_n \geq 1, \forall n$ . Then since  $F$  is convex, we have

$$\begin{aligned} y^{t_n} + (1 - t_n)z^* &\in 1/t_n F(x_{t_n}) + (1 - 1/t_n)F(0) \\ &\subset F(x^{t_n}). \end{aligned}$$

Therefore since  $F$  is upper continuous and uniformly compact,  $\{y^{t_n}\}$  has a convergent subsequence  $\{y^{v_n}\}$  with limit  $y^0$  such that  $y^* = y^0 + z^* \in F(x^*)$ . From (8), we have

$$\langle y_{v_n} + 2\lambda_{v_n} x_{v_n}, x_{v_n} \rangle = 0, \forall n.$$

Thus  $\lambda_{v_n} \rightarrow \lambda^* = -\langle x^*, y^* - z^* \rangle / 2 \geq 0$  and thus by (7),

$$(y_{v_n} + 2\lambda_{v_n} x_{v_n}) / v_n \rightarrow v = (y^* - z^* + \lambda^* x^*) \geq 0.$$

It is clear from (8) that  $\langle v, x^* \rangle = 0$ . Therefore (3) holds. Finally to show (4) and (5), let

$$I_1 = \{i : x_i^* > 0\}, I_2 = \{i : x_i^* = 0 \text{ and } x_i^{v_n} > 0 \text{ for all sufficiently large } v_n\} \text{ and } I = I_1 \cup I_2.$$

Choose  $v_n^0$  such that for all  $v_n \geq v_n^0, x_i^{v_n} > 0$  for  $i \in I$  and  $x_i^{v_n} = 0$  for  $i \notin I$ . Let  $\bar{x} = x_{v_n^0}, \bar{\lambda} = 2\lambda_{v_n^0}, \bar{y} = y_{v_n^0}$  and  $u = \bar{y} + \bar{\lambda}\bar{x}$ . Then clearly (4) holds. If  $x_i^* > 0$ , then  $\bar{x}_i > 0$ . So  $u_i = 0$ . Thus  $\langle x^*, u \rangle = 0$ . Finally, if  $\bar{x}_i > 0$ , then  $x_{v_n} > 0, \forall v_n \geq v_n^0$ . Then  $(y_{v_n} + 2\lambda_{v_n} x_{v_n})_i = 0, \forall v_n \geq v_n^0$ . By passing to a limit, we have  $v_i = 0$ . Consequently,  $\langle \bar{x}, v \rangle = 0$  and (5) holds.  $\square$

If  $F$  is positively homogeneous, then we have the following refined version of Theorem 4.2.

**Theorem 4.5.** *Let  $F : \mathbf{R}_+^n \rightarrow \mathbf{R}^n$  be an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in \mathbf{R}_+^n$ . Suppose  $F$  is positively homogeneous. If  $GCP(F, \mathbf{R}_+^n)$  has no solution, then there exist vectors  $x^*, \bar{x} \geq 0, y^* \in F(x^*), \bar{y} \in F(\bar{x})$  and scalars  $\bar{\lambda} > 0, \lambda^* \geq 0$ , such that*

$$v = y^* + \lambda^* x^* \geq 0, \langle x^*, v \rangle = 0, x^* \neq 0, \quad (9)$$

$$u = \bar{y} + \bar{\lambda}\bar{x} \geq 0, \langle \bar{x}, u \rangle = 0, \bar{x} \neq 0, \quad (10)$$

$$\langle x^*, u \rangle = \langle \bar{x}, v \rangle = 0. \quad \square \quad (11)$$

It is interesting to observe that if we consider  $D(t) = \{x \geq 0 : \langle x, d \rangle \leq t\}, \forall t \in \mathbf{R}_+$ , where  $d$  is a positive vector in  $\mathbf{R}^n$ , then we get the following version of Theorem 4.4.

**Theorem 4.6.** *Let  $F : \mathbf{R}_+^n \rightarrow \mathbf{R}^n$  an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in \mathbf{R}_+^n$ . Let  $z^* \in F(0)$ ,  $d$  be a positive vector in  $\mathbf{R}^n$  and suppose  $F$  is convex. If  $GCP(F, \mathbf{R}_+^n)$  has no solution, then there exist  $x^*, \bar{x} \geq 0$ ,  $y^* \in F(x^*), \bar{y} \in F(\bar{x})$ , scalars  $\bar{\lambda} > 0, \lambda^* \geq 0$ , such that*

$$v = y^* - z^* + \lambda^* d \geq 0, \langle x^*, v \rangle = 0, x^* \neq 0, \quad (12)$$

$$u = \bar{y} + \bar{\lambda} d \geq 0, \langle \bar{x}, u \rangle = 0, \bar{x} \neq 0, \quad (13)$$

$$\langle x^*, u \rangle = \langle \bar{x}, v \rangle = 0. \quad \square \quad (14)$$

We note that if  $F$  is an affine mapping, that is  $F(x) = Mx + q$ , where  $M$  is a  $n \times n$  matrix and  $q$  is a vector in  $\mathbf{R}^n$ , then Theorem 4.6 reduces to a main result due to Parida and Roy [11, Theorem 2].

Let  $K$  be a convex cone in  $\mathbf{R}^n$ . A point-to-set mapping  $F : K \rightarrow \mathbf{R}^n$  is said to be *homogeneous convex* on  $K$  if it is both convex and positively homogeneous on  $K$ , or equivalently if  $G(F) \cap (K \times \mathbf{R}^n)$  is a convex cone. we have the following existence result for homogeneous convex mappings.

**Theorem 4.7.** *Let  $F : \mathbf{R}_+^n \rightarrow \mathbf{R}^n$  an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in \mathbf{R}_+^n$ . Suppose  $F$  is homogeneous convex on  $\mathbf{R}_+^n$ . If  $\langle x, y \rangle$  is bounded below for all  $x \in \mathbf{R}_+^n$  and all  $y \in F(x)$ , then  $GCP(F, \mathbf{R}_+^n)$  has a solution.*

**Proof.** Suppose  $GCP(F, \mathbf{R}_+^n)$  has no solution. Then by Theorem 4.5, there exist  $x^*, \bar{x} \geq 0$ ,  $y^* \in F(x^*), \bar{y} \in F(\bar{x})$ , scalars  $\bar{\lambda} > 0, \lambda^* \geq 0$ , such that

$$v = y^* + \lambda^* x^* \geq 0, \langle x^*, v \rangle = 0, x^* \neq 0, \quad (15)$$

$$u = \bar{y} + \bar{\lambda} \bar{x} \geq 0, \langle u, \bar{x} \rangle = 0, \bar{x} \neq 0, \quad (16)$$

$$\langle x^*, u \rangle = \langle \bar{x}, v \rangle = 0. \quad (17)$$

So we have

$$\langle x^* + \lambda \bar{x}, v + \lambda u \rangle = 0, \forall \lambda > 0.$$

Thus for all  $\lambda > 0$ ,

$$\langle x^* + \lambda \bar{x}, y^* + \lambda \bar{y} \rangle / 4 + \langle x^* + \lambda \bar{x}, \lambda^* x^* + \lambda \bar{\lambda} \bar{x} \rangle / 4 = 0. \quad (18)$$

Since  $y^* \in F(x^*), \bar{y} \in F(\bar{x})$  and  $F$  is homogeneous convex,

$$(y^* + \lambda \bar{y}) / 2 \in F(x^*) / 2 + F(\lambda \bar{x}) / 2 \subset F((x^* + \lambda \bar{x}) / 2), \forall \lambda > 0.$$

Note that  $\bar{\lambda} > 0$  and  $\langle x, y \rangle$  is bounded below for all  $x \geq 0, y \in F(x)$ . Hence (18) is impossible to hold for all  $\lambda > 0$ . Therefore the result follows.  $\square$

**Remark.** The conclusion of Theorem 4.7 still holds if  $\mathbf{R}_+^n$  is replaced by any other orthant of  $\mathbf{R}^n$ .

We call  $F : K \rightarrow \mathbf{R}^n$  an *affine* point-to-set mapping on a convex set  $K \subset \mathbf{R}^n$  if  $F$  is both convex and concave on  $K$ , i.e. if, for all  $x, y \in K$  and  $0 \leq \lambda \leq 1$ ,

$$\lambda F(x) + (1 - \lambda)F(y) = F(\lambda x + (1 - \lambda)y).$$

$F$  is said to be *homogeneous affine* on a convex cone  $K$  if it is both affine and positively homogeneous on  $K$ . The following corollary is immediate.

**Corollary 4.8.** *Let  $F : \mathbf{R}_+^n \rightarrow \mathbf{R}^n$  an upper continuous point-to-set mapping such that  $F(x)$  is contractible and uniformly compact near  $x$  for all  $x \in \mathbf{R}_+^n$ . Suppose  $F$  is homogeneous affine on  $\mathbf{R}_+^n$ . If  $\langle x, y \rangle$  is bounded for all  $x \in \mathbf{R}_+^n$ ,  $y \in F(x)$ , then  $GCP(F, \mathbf{R}_+^n)$  is solvable.  $\square$*

We note that the above results may be obtained for more general complementarity problems, say, e.g.,  $GCP(F, \theta, K, C)$ .

## References

- [1] R. W. Cottle, "Nonlinear programs with positively bounded Jacobians," *SIAM Journal on Applied Mathematics* 14(1966), 147-158.
- [2] B. C. Eaves, "On the basic theorem of complementarity," *Mathematical Programming* 1(1971), 68-75.
- [3] S. C. Fang and E. L. Peterson, "Generalized variational inequalities," *Journal of Optimization Theory and Applications* 38(1982), 363-363.
- [4] P. Hartman and G. Stampacchia, "On some nonlinear elliptic differential functional equations," *Acta Mathematica* 115(1966), 153-188.
- [5] S. Karamardian, "Generalized complementarity problem," *Journal of Optimization Theory and Applications* 3(1971), 161-168.
- [6] M. Kojima, "A unification of the existence theorems of the nonlinear complementarity problem," *Mathematical Programming* 9(1975), 257-277.
- [7] A. Mas-Colell, "A note on theorem of F. Browder," *Mathematical Programming* 6(1974), 229-233.
- [8] O. H. Merrill, *Applications and Extensions of an Algorithm that Computes Fixed Points of Certain Upper Semicontinuous Point-to-Set Mappings*, University of Michigan, Ann Arbor, Ph.D. Thesis, 1972.
- [9] J. J. Moré, "The application of variational inequalities to complementarity problems and existence theorems," Technical Report 71-90, Department of Computer Sciences, Cornell University, Ithaca, New York, 1971.
- [10] J. S. Pang, *Least-Element Complementarity Theory*. Ph.D. dissertation, Department of Operations Research, Stanford University, Stanford, California, 1976.
- [11] J. Parida and K. L. Roy, "A note on the linear complementarity problem," *Opsearch* 18(1981), 229-234
- [12] J. Parida and A. Sen, "A variational-like inequality for multifunctions with applications," *Journal of Mathematical Analysis and Applications* 124(1987), 73-81.
- [13] R. T. Rockafellar, "Monotone processes of convex and concave type," *American Mathematical Society Memoir* 77 (1967), Providence, Rhode Island.
- [14] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
- [15] R. Saigal, "Extension of the generalized complementarity problem," *Mathematics of Operations Research* 1(1976), 260-266.
- [16] J. C. Yao, "Generalized quasi-variational inequalities and implicit complementarity problems," Technical Report SOL 89-15, System Optimization Laboratory, Department of Operations Research, Stanford University, Stanford, California, 1989.

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**SOL 89-16: A Basic Theorem of Complementarity for the Generalized Variational-like Inequality Problem, Jen-Chih Yao (November, 17 pp.).**

*established*

In this report, a basic theorem of complementarity is established for the generalized variational-like inequality problem introduced by Parida and Sen. Some existence results for both generalized variational inequality and complementarity problems are established by employing this basic theorem of complementarity. In particular, some sets of conditions that are normally satisfied by a nonsolvable generalized complementarity problem are investigated.