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OPTIMIZATION

by

The Analytic Hierarchy Process

Thomas L. Saaty
The Wharton School
University of Pennsylvania

1. Introduction The purpose of optimization is to identify feasible alternatives which are preferred or indifferent to all other alternatives. In general, this part of applied mathematics is divided into (a) optimization subject to constraints and (b) optimization without constraints. In both cases when the function that meets the alternatives is known there exists a well developed theory to attack the problem. However, there is a large class of practical problems where a numerical function defining the preference relation among the alternatives is usually unknown. For example, at times military problems are not concerned with the maximization (or minimization) of a well defined and measurable quantity (such as dollar expenditure), but rather with the optimization of vague concepts such as military worth, loss of life, and the like. Mathematical techniques are useless in these areas unless we have a method to measure these concepts.

Even business corporations face such problems when they must decide among intangibles that do not have a measurable impact in terms of dollars. In complex situations like these the object is still to find a best possible alternative.

Classical optimization which has its origins in physics and engineering is concerned with the maximization or minimization of

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A-1

a function (or functional) subject to equality and/or inequality constraints. In programming theory the constraints also include the nonnegativity of the variables. A simple optimization problem as one encounters it in the calculus requires that an unconstrained function of one or several variables be maximized and if the function is differentiable, calculus methods may be applied. Well known other general formulations of optimization are: maximize or minimize $f(x_1, \ldots, x_n)$ subject to $g_i(x_1, \ldots, x_n) \leq 0$, $i=1,\ldots,m$. If all the g_i are identically zero we have an ordinary calculus problem (assuming differentiability). If the constraints are all equalities we have a lagrange multiplier problem. If $x_j \geq 0$, $j=1,\ldots,n$ we have a programming problem which is linear if f and all the g_i are linear. The optimum solution required may or may not be in integers.

The next class of problems is variational. A general formulation of an optimization problem in function space subject to constraints takes the form: maximize or minimize

$$\int f(t,x,y,y^{(1)},\ldots y^{(n)})\, dx,$$

(where $y^{(h)}$ $h=1,\ldots,n$ is the hth order derivative of y with respect to x) subject to differential, difference or more generally functional constraints of the form

$$g_i(t,x,y,y^{(1)},\ldots,y^{(n)}) \leq 0 \quad i=1,\ldots,m$$

or

$$\int k_i(t,x,y^{(1)},\ldots,y^{(n)})\, dx \leq 0$$

where the limits of the integrals may also be prescribed functions.

The problem may involve several parameters along with t, several independent variables along with x, and several functions along with y of these parameters and variables.

There are also recursive optimization techniques such as those exemplified in dynamic programming.

It is useful to consider how rare it is that an individual attempts to maximize or minimize anything. Early in life we are trained for balance, "the happy medium", to be satisfied, "you cannot take it with you". Some optimization experts have defined rationality in terms of "if offered more take more", although this seems to contradict our tendency to seek sufficiency. We intend to illustrate a more natural approach to behavioral optimization which is better attuned to the study of sufficiency.

The concept of sufficiency can be found not simply in some modern works in economics and operations research but also in ethics and philosophy. The problem is not that people object to the concept, rather, the study of this mode of behavior has been overpowered by traditional mathematical structures of optimization which have been carried over to the social and behavioral fields. Even risk avoidance is only minimized in theory. In practice people have acceptable levels of risk with which they deal.

→ The Analytic Hierarchy Process which we study here serves as a framework for people to structure their own problems and provide their own judgments based on knowledge, reason or feelings, to derive a set of priorities for activities to which they, for example, wish to allocate effort or resources. In this process transitivity of preference is studied through a new approach to consistency - which need not always strictly hold for the results to be acceptable. Also since hierarchic structures may not be complete, not all alternatives need to be directly comparable. [See Saaty, J. Math. Psych., 1977].

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The following is a brief outline of the steps followed by the Analytic Hierarchy Process(A

1. Define the problem and specify the solution desired.

2. Structure the hierarchy from the overall managerial purposes (the highest levels) through relevant intermediate levels to the level where control would alleviate -- or solve -- the problem.

3. Construct a pairwise comparison matrix of the relative contribution or impact of each element on each governing objective or criterion in the adjacent upper level. In such a matrix of the elements by the elements, the elements are compared in a pairwise manner with respect to a criterion in the next level. In comparing the i,j elements, people prefer to give a judgment which indicates the dominance as an integer. Thus, if the dominance does not occur in the i,j position while comparing the ith element with the jth element then it is given in the j,i position as a_{ji} and its reciprocal is automatically assigned to a_{ij}. (j,i,l)

4. Obtain all $\frac{n(n-1)}{2}$ judgments required to develop the set of matrices in 3 .

5. Having collected the pairwise comparison data and entered the reciprocals together with n unit entries down the main diagonal, the eigenvalue problem $Aw = \lambda_{max}w$ is solved and consistency is tested, using the departure of λ_{max} from n.

6. Steps 3,4 and 5 are done for all levels and clusters in the hierarchy.

7. Hierarchical composition is now used to weight the eigenvectors by the weights of the criteria and the sum is taken over all weighted eigenvector entries corresponding to those in the next lower level and so on, resulting in a composite priority vector for the lowest level of the hierarchy.

8. Consistency is then evaluated for the entire hierarchy by simply multiplying each consistency index by the priority of the corresponding criterion and adding overall such products. The result is divided by the same type of expression using the random consistency index corresponding to the dimensions of each matrix weighted by the priorities as before. The ratio should be about 10% or less for acceptable overall consistency. Otherwise, the quality of the judgmental data should be improved.

The following kinds of questions come to mind in using the AHP for optimization purposes.

- While in classical optimization one is restricted to measures of money, time, weight, temperature and a few other measurables, with the AHP one can use these measures along with newly derived measures of intangibles which the AHP generates.
- The structure of the problem is not set in advance, but is generated by those who experience the problem.
- No assumptions of orthogonality and independence need to be made as the variables may be interdependent and hence the use of a cartesian coordinate system may be inappropriate.
- One need not be restricted to the use of a euclidean metric.
- Different weights may be attached to different constraints or criteria through the use of the AHP.
- As in real life the objective function is so intermeshed with the constraints that optimization is regarded as a process of interaction between objectives and constraints.
- One can optimize any number of objectives, and not simply one.

On reflection it seems that the adoption of a model to represent a problem may be a matter of 1) Convention, through wide interpretation and usage thus in a sense giving the model precedence over the problem (conventions have a tendency to change with the times although it takes much longer in science than in the world of women's fashions); 2) Convenience, by having quick and technically adequate methods to get at an answer. Convenience may and often does skirt the problem; 3) Aesthetic, in that real world problems seem messier than some practitioners think is warranted and by making imaginative and convining assumptions we structure the problems as we "can", sometimes knowing that we are sacrificing relevance but always remembering that a thoughtful life is a compromise between logical clarity, simplicity, consistency, and tradition on the one hand and relevance to the reality we perceive on the other. But as we learn to deepen our logical perspective and its pragmatic reach by turning it into a dualistic interaction (adaptive process) the line of demarcation between these two fundamental areas of logical perception and reality loses its sharpness, as it might if we were to better understand and control the world in which we live.

2. Two Roles for the AHP in Optimization

There are two ways in which the AHP can be used in optimization. The first is to use it as a tool for measuring priorities of intangibles that should be considered in optimization such as relative preference among foods in diet problems to the contribution of an industry to social and environmental factors. The resulting priorities are then incorporated as coefficients of the objective function or constraints or the payoff matrix as the situation may be and the solution is carried out.

The other way in which the AHP can be used to "optimize" is to follow through the AHP process itself to get what is thought to be the correct or sufficient mix of factors in a problem. This process avoids imposing a structure - that of the model - on the problem and leaves the formulation and solution to those who have experience with the problem. We shall discuss both these types of approaches.

Optimization is a goal directed activity. Goals depend on the people involved. Thus in the final analysis optimization is an attempt to satisfy needs and desires. One wonders then why optimization is carried out in terms of measurements and structures more suitable to natural science. When we use cartesian geometry we assume that we are dealing with a set of independent variables represented in an orthogonal framework, and if some variables are more important than others we multiply them by appropriate constants or raise them to powers to reflect this fact. Even when we have a large number of variables we assume linearity and people have been known to deal with linear programming problems involving 5000 variables. Why? It is doubtful that these traditional methods give a good reflection of the real world.

The question of independence and dependence among the entities being treated becomes academic in the AHP if we can treat the criteria and objectives as separate mutually exclusive meaningful concepts. We must define them carefully to see what each one means. There is no harm in finding that there is overlap among them. We can essentially look at both their independent features and their overlapping characteristics.

3. Brief Descriptions of Past Applications of the AHP in Optimization

The measurement approach to optimization by the AHP has been used

1. In conjunction with linear programming and input-output constraints to study the rationing of energy to industries [see Saaty and Mariano].

2. To develop the benefits of transport projects in the form of priorities in the Sudan which together with their estimated costs provided benefit to cost ratios for allocating resources [see Saaty 1977 ... (Best Available) ...

determination of benefits in one hierarchy, costs in [illegible] and combining the results of the two for proportionate or 0 - 1 allocation.

3. To develop a general theory for multiple resource allocation of the knapsack type [forthcoming paper with J.P. Bennett] .

4. To develop a method to compute the payoff matrix of a nonzero sum game under conditions of incomplete information and its application to study the U.S.-OPEC energy options [forthcoming International Journal of Game Theory, 1979].

5. To construct input-output tables [Saaty and Vargas, Socio-Economic Planning Sciences Journal, will appear]. The same approach may be used to study flow of other than physical material among the activities [Vargas, Ph.D. dissertation, Wharton School, 1979] .

4. Textbook Type of Illustrations of the AHP

Below we illustrate how to use the AHP to deal with problems of optimization without constraints, a market basket example, optimization subject to constraints, cost-benefit resource allocation, and determining the payoff matrix in a two-person game.

a. Optimization Without Constraints

We take a simple example in order to determine the kind of steps to follow for more complicated problems.

A farmer wishes to sell his crop of 120 bushels of potatoes. If he sells it now he would get $1.00 per bushel. However, if he waits his crop will increase, due to growth, by 20 bushels per week, but the price would decrease by $.10 per bushel per week. When should he sell to maximize his profit?

To solve the problem we construct a hierarchy as in Figure 1. The overall purpose of the hierarchy is to maximize the farmer's profit. Profit is a function of production and the price per bushel that the farmer gets at a certain period of time. Ultimately, profit is a function of the number of weeks that the farmer waits to sell his crop. The different number of weeks represent his alternatives. The total harvesting period is assumed to last about five weeks. These alternatives comprise the bottom (fourth) level of the hierarchy.

The second and third levels are the production and the price per bushel, respectively. Since the price per bushel depends on the total amount produced, we consider each criterion at a different level of the hierarchy.

Production is considered in three different degrees: low, medium, and high. The price per bushel falls in three categories. It can be very small if the farmer waits too long and sells late, medium if he sells at an intermediate period and large if he sells early.

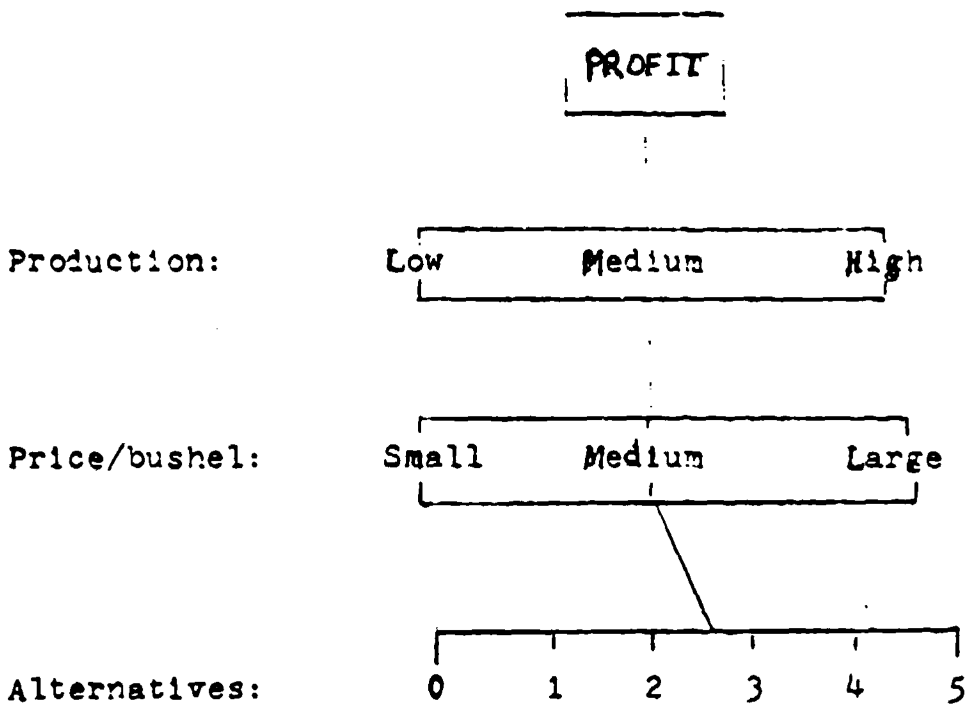

Figure 1.

Clearly, to obtain the maximum profit, crop size will have to be relatively large. However, eventually a very large harvest could be worthless. Therefore, a medium size crop would be desirable.

According to this observation, the pariwise comparisons among the different degrees of production can now be given by answering the following question: Comparing crop sizes, which one produces more profit and how strongly?

PROFIT	LOW	MEDIUM	HIGH	Weights
LOW	1	3	1/2	.3793
MEDIUM	1/3	1	3	.3313
HIGH	2	1/3	1	.2894

Let us assume that potato production is low. Given two category prices per bushel, which category is more likely to obtain?

LOW	S	M	L	Weights
S	1	1/3	1/5	.1095
M	3	1	1/2	.3090
L	5	2	1	.5816

MEDIUM	S	M	L	Weights
S	1	1/5	1/2	.1238
M	5	1	2	.5954
L	2	1/2	1	.2764

LARGE	S	M	L	Weights
S	1	4	5	.6908
M	1/4	1	1	.1603
L	1/5	1	1	.1488

To fill in the other two matrices for medium and large production similar questions are asked.

It remains to compare the number of weeks with respect to the price per bushel. The question asked here is: Given two weeks at which he can sell his product, if the price per bushel is small (medium or large) which week provides more profit?

SMALL	0	1	2	3	4	5	Wts.
0	1	1	1/2	1/3	1/4	1/5	.0640
1	1	1	1	1/2	1/3	1/4	.0850
2	2	1	1	1	1/2	1/3	.1201
3	3	2	1	1	1	1/2	.1730
4	4	3	2	1	1	1	.2430
5	5	4	3	2	1	1	.3156

MEDIUM	0	1	2	3	4	5	Wt
0	1	1/4	1/5	1/5	1/3	1/2	.05
1	4	1	1/3	1/3	1/2	1	.11
2	5	3	1	1	1	1	.23
3	5	3	1	1	1	1	.23
4	3	2	1	1	1	1	.19
5	2	1	1	1	1	1	.16

LARGE	0	1	2	3	4	5	Wts.
0	1	1	2	3	4	5	.3154
1	1	1	1	2	3	4	.2430
2	1/2	1	1	1	2	3	.1730
3	1/3	1/2	1	1	1	2	.1201
4	1/4	1/3	1/2	1	1	1	.0850
5	1/5	1/4	1/3	1/2	1	1	.0640

Alternatives (Weeks)	0	1	2	3	4	5
Composite Weights	.1488	.1535	.1790	.1752	.1752	.1726

The composite vector of weights multiplied by the row vector (0,1,2,3,4,5) yields the expected number of weeks that the farmer should wait until he sells the crop.

$$\text{ENW} = 0\text{x}.1488+1\text{x}.1535+2\text{x}.1790+3\text{x}.1790+4\text{x}.1752+5\text{x}.1726$$
$$= 2.6.$$

The calculus formulation of this problem is to maximize $(120 + 20x)(1-0.1x)$, whose solution is $x=2$ weeks.

b. Market Basket Example

Coreen L. Mett of Radford College, Virginia, has studied the problem of how to allocate food dollars according to preference and cost. Her hierarchy is as in Figure 2.

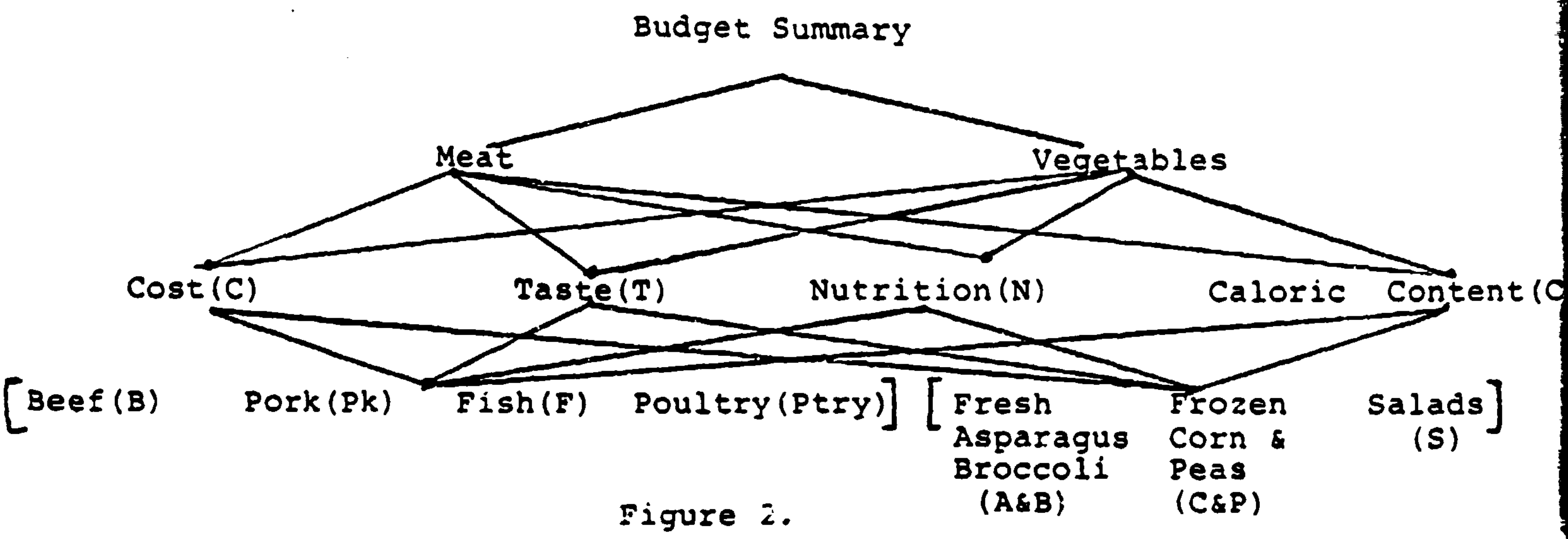

Figure 2.

Dr. Mett assigned the weights .4 to meat, and .6 to vegetables. Her matrices for the elements in the third level with respect to those in the second and those in the fourth with respect to those in the third follow. The meats and the vegetables listed in the fourth level are treated as two separate clusters with respect to each attribute in the third level.

MEAT	C	T	N	CC	Weights
C	1	1/3	1/3	1/3	.099
T	3	1	2	2	.413
N	3	1/2	1	1	.244
CC	3	1/2	1	1	.244

VEG	C	T	N	CC	Weights
C	1	1/5	1/3	1/3	.079
T	5	1	2	3	.477
N	3	1/2	1	2	.270
CC	3	1/3	1/2	1	.174

We now move to the fourth level pairwise comparisons with respect to third level attributes. Note that there are no comparisons necessary for cost as actual store prices are available.

COST

COST	Per Serving Store Price	Weights (Reciprocal of Cost, Normalized)
B	$1.049	.152
Pk	$ .55	.292
F	$ .995	.161
Ptry	$.404	.395

COST	Per Serving Store Price	Weights (a for meats)
A&B	$.99	.088
C&P	$.126	.694
S	$.402	.218

Note that for cost we take the reciprocal of the price to measure relative benefits as we do for the other attributes in the third level. In other words, cheaper is better.

TASTE

TASTE	B	Pk	F	Ptry	Weights
B	1	3	5	2	.466
Pk	1/3	1	3	1/3	.160
F	1/5	1/3	1	1/3	.08
Ptry	1/2	3	3	1	.294

TASTE	A&B	C&P	S	Weights
A&B	1	4	3	.623
C&P	1/4	1	1/2	.137
S	1/3	2	1	.24

NUTRITION

NUTRIT	B	Pk	F	Ptry	Weights
B	1	3	1/5	1/3	.125
Pk	1/3	1	1/7	1/4	.061
F	5	7	1	3	.563
Ptry	3	4	1/3	1	.251

NUTRIT	A&B	C&P	S	Weights
A&B	1	3	1/2	.334
C&P	1/3	1	1/3	.142
S	2	3	1	.525

CALORIC CONTENT

CC	B	Pk	F	Ptry	Weights
B	1	3	1/4	1/3	.141
Pk	1/3	1	1/5	1/4	.071
F	4	5	1	3	.52
Ptry	3	4	1/3	1	.268

CC	A&B	C&P	S	Weights
A&B	1	4	2	.557
C&P	1/4	1	1/3	.123
S	1/2	3	1	.32

On composing weights we find that each dollar spent should be divide as follows: A&B, 29.2¢; S, 19.7¢; F, 12.3¢; Ptry, 11.4¢; B, 10.8¢; C&P. 10.8¢; Pk. 5.1¢

C. Optimization Subject to Constraints

Consider an individual who has available three types of food: meat (beef), bread, and vegetables (broccoli). He must determine the optimal mixture of these foods to eat minimizing cost and satisfying minimum daily requirements for vitamins A, B_2 and the amount of calories needed.

Suppose that the cost of each food is as follows: meat, \$2.50/lb (\$0.0055/gm); bread, \$.55/lb (\$0.0012/gm); vegetable, \$.62/lb (\$0.0014/gm). The individual knows, approximately, the amounts of vitamin A, B_2, and calories per gram of each food. They are

Food	Vit.A (IU)	Vit. B_2(mg.)	Calories (kcal.)
Meat	.3527/gm	.0021/gm	2.86/gm
Bread	.0000	.0006	2.76
Veg.	25	.002	.25

The minimum daily requirements are: Vitamin A: 7,500 (I.U.), Vitamin B_2: 1.6338 mg. (This amount varies from individual to individual, and it is measured in mg/kgr. The minimum requirement for an individual who weighs 147 lbs is 1.6338 mg.); Calories: 2,050 kcal (again for the 147 lb individual).

We have the linear programming problem

$$\text{Minimize } z: (5.5\, x_1 + 1.2\, x_2 + 1.4\, x_3) \times 10^{-3} \text{ dollar}$$

Subject to:

$$\begin{aligned} .3527\, x_1 + + 25\, x_3 &\geq 7.500 \\ .0021\, x_1 + .0006\, x_2 + .002\, x_3 &\geq 1.6338 \\ 2.86\, x_1 + 2.76\, x_2 + .25\, x_3 &\geq 2.050 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Its solution is given by x_1(meat) = 0, x_2(bread) = 687.44 gms, x_3 (vegetables) = 610.67. And the cost per day of this diet would be z = \$1.67. Clearly, in this model the individual did not indicate his actual food preferences.

Now let us consider that the individual expresses his preferences, for which we construct the following hierarchy (Figure 3).

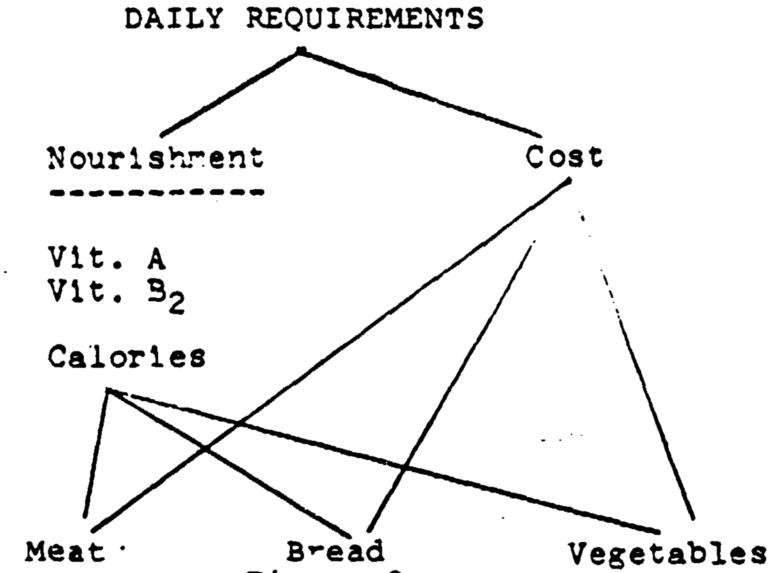

Figure 3

For a person with an average income, to satisfy his nourishment needs, is more important than the cost of the food, as long as cost is not prohibitive. Here we have for N, nourishment, and C, cost :

D.R.	N	C	Weights
N	1	3	0.75
C	1/3	1	0.25

He also assumes that vitamin A and B_2 are equally important and in turn more important than calories. Thus we assign the following weights to them:

NOURISHMENT	Vit A	Vit B_2	Cals	Weights	(x.75 for N)
Vit A	1	1	2	.4	(.3)
Vit B_2	1	1	2	.4	(.3)
Cals	1/2	1/2	1	.2	(.15)

The amount of Vitamin A, B_2 and calories per gram of meat bread and vegetables are obtained from the first table. They are

	Vitamin A	B_2	Kcal.
Meat	0.0139	0.4468	0.4872
Bread	0.0000	0.1277	0.4702
Vegetables	0.9861	0.4255	0.0426

According to cost we have

	Cost
M	0.6790
B	0.1481
V	0.1728

However, since we wish to minimize cost, we use the reciprocals of the amounts given above.

	Cost^{-1}
M	0.1051
B	0.4819
V	0.4130

The composite vector of priorities is

Foods	Meat	Bread	Vegetables
Priorities	.2376	.2293	.5331

The total amount of food that an individual must eat to minimize cost and satisfy the daily requirements obtained from the linear programming program is 1,298.11 gms or 2.86 lbs. If we distribute this total according to the above priorities the number of grams of each food eaten per day are

	Meat	Bread	Vegetables
Amount (gms)	308.43	297.66	692.02

Our concern is to ensure that the daily requirements are satisfied and determine the total cost. Multiplying the matrix of requirements by the solution obtained hierarchically we have

$$
\begin{array}{l}
 \\ \text{Vit A} \\ \text{Vit } B_2 \\ \text{Cals}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{Meat} & \text{Bread} & \text{Veg} \end{array} \\
\begin{pmatrix} 0.3527 & 0.0 & 25 \\ 0.0021 & .0006 & 0.002 \\ 2.86 & 2.76 & 0.25 \end{pmatrix}
\end{array}
\begin{array}{c} \\ \begin{pmatrix} 308.43 \\ 297.66 \\ 692.02 \end{pmatrix} \end{array}
\begin{array}{c} \\ = \end{array}
\begin{array}{c} \\ \begin{pmatrix} 17{,}409.28 \\ 2.2103 \\ 1{,}876.66 \end{pmatrix} \end{array}
\begin{array}{l} \text{Daily Requirements} \\ \geq 7{,}500 \text{ I.U.} \\ \geq 1.6338 \text{ mgs} \\ \leq 2{,}050 \text{ Kcal} \end{array}
$$

The total cost associated with this solution is \$3.0224. As a separate exercise we assumed that the nourishment factors are equally important. In that case the cost was \$2.92 and the consumption was 290.80 gms of meat, 419.30 gms of bread and 588.50 gms of begetables. The daily requirements were all satisfied as shown below.

	Contribution
Vit A	14,815.07
Vit B_2	2.0393
Kcal	2136.10

Let us now assume that cost is absolutely more important than nourishment. Thus

	Nourishment	Cost	Weights
N	1	1/9	.1
C	9	1	.9

and that Vit A, Vit B_2 and Kcal are equally important. We have

Foods:	Meat	Bread	Vegetable
Priorities:	.1210	.4607	.4184
gms of food eaten per day:	157.07	598.04	543.13

Contributions to daily requirements

Vit A	Vit B_2	Kcal
13,633.65	1.7749	2235.5931

The total cost is z = \$2.3419

Let us fix the total amount of food that can be eaten by the individual during a day, i.e.,

$$x_1 + x_2 + x_3 = 2 \text{ lbs}$$

The solution obtained by the AHP which, with the assumption made earlier that nourishment is more important than cost (a split of .75, .25) gives

Foods	Meat	Bread	Vegetables
Amount (gms)	203.57	293.28	411.69

Total cost z = \$2.05

providing the following amounts of vit A, B_2 and Kcal.

	Contribution	Daily Requirement
Vit A	10,364.05	7,5000
Vit B_2	1.4268	1.6338
Kcal	1494.59	2,050

The solution derived from the L.P. model, assuming that the right-hand side vector of coefficients is the vector of daily requirements obtained by the AHP, is

$$x_1\text{(meat)} = 0$$

$$x_2\text{(Bread)} = 566.33$$

$$x_3\text{(vegetable)} = 490.22$$

$$z\text{(cost)} = \$1.38$$

Let us assume that an individual expresses his food preferences in terms of the amounts of each he desires to eat per day, i.e.,

$$x_1\text{(meat)} \geq 140 \text{ gms (5 ozs)}$$

$$x_2\text{(bread)} \leq 56 \text{ gms (2 slices)}$$

$$x_3\text{(vegetable), no constraint}$$

then the solution of the first linear programming problem is

$$x_1\text{(meat)} = 637.30 \text{ gms}$$

$$x_2\text{(bread)} = 56 \text{ gms}$$

$$x_3\text{(vegetables)} = 291.01$$

$$z \text{ (cost)} = \$3.80$$

d. Benefits and Costs in Crossing a River

A governmental agency (such as the New York Port Authority) which has jurisdiction over the building of bridges, tunnels, etc. in a certain area must decide on whether to build or not to build a tunnel and/or a bridge across a river presently served by a privately owned ferry.

The factors which affect both the benefits and the costs of crossing a river are given in two hierarchies below (Figures 3, 4). These factors fall into three categories: economic, social, and environmental. The decision is made in terms of the ratios of benefits to costs.

Benefits

The economic factors affecting the choice consist of the benefit derived from the time saved in using a new bridge or tunnel rather than using the existing ferry. The increased traffic from outside of the area could bring in toll revenue which can add to the general income of the local government. The rise in commerce caused by this increased flow of traffic is seen as being beneficial to the community in general. Additionally, the traffic will aid the commerce nearby (such as gas stations, restaurants, etc). There is also economic benefit from the construction jobs generated. If they were the only ones to consider, most of these factors could be calculated quantitatively. The associated cost could also be computed quantitatively and a benefit/cost ratio could be used to make the decision. But we have to consider social and environmental factors which do not translate in any reasonable way to dollars.

The social benefits of the project are viewed to represent the benefits which the society as a whole will derive from the presence of a bridge or tunnel. They would provide greater safety and reliability than the ferry. They would also contribute to a greater number of trips across to visit relatives, friends, museums, etc. Finally they could generate community pride not present to the same degree in using the ferry.

Environmental factors are viewed in terms of their contribution to individual personal benefits. They differ from benefits to society, in that society often considers benefits to an abstract collection which does not represent the interest of any particular individual. The environmental factors of interest to an individual are the comfort of using the bridge, tunnel or ferry, the ease of accessibility of one over the others, and the aesthetics affecting the choice of alternative for crossing the river.

Costs

As with benefits, the costs of crossing a river also involve economic, social, and environmental factors. The three economic costs considered were the capital costs of the alternatives, the operating and maintenance costs associated with the three projects, and the economic consequence of not having a ferry boat business.

The social costs again represent costs to society. The degree to which lifestyles are disrupted using the alternatives to cross the river was thought to be important. The congestion of traffic differs between the various modes of crossings and is also deemed an important cost. The final social cost is the effect on society of the dislocation of people from their homes according to the alternate chosen.

environmental costs differ from environmental benefits in that they represent possible harm done to the ecosystem by the various alternates. The various ways of crossing the river add to the amount of auto emissions in the area. Additionally, pollution of the water and the general disruption of the ecology were thought to contribute to environmental costs.

Results

In the calculation of both benefits and costs, economic factors outweighed the other factors. The benefits derived from the commerce across the bridge, the added safety and reliability, and quick accessibility of crossing the river all received high priorities.

As for costs, the capital required, the dislocation of people from their homes and the amount of auto emissions all received high priorities.

The composite benefits and costs are as follows:

	Bridge	Tunnel	Ferry
Benefits (b_i)	.57	.36	.07
Costs (c_i)	.36	.58	.05

The criteria used in benefit/cost analysis are:

1) $b_i/c_i > 1$; That is, the benefits must exceed the costs.

2) $\max\limits_i b_i/c_i$; That is, choose the project with largest benefit to cost ratio.

For this example we have

Bridge	Tunnel	Ferry
$\frac{b_1}{c_1} = 1.58$	$\frac{b_2}{c_2} = .62$	$\frac{b_3}{c_3} = 1.28$

Both criteria favor the construction of a bridge across the river. Note that this has taken into consideration the capital requirements.

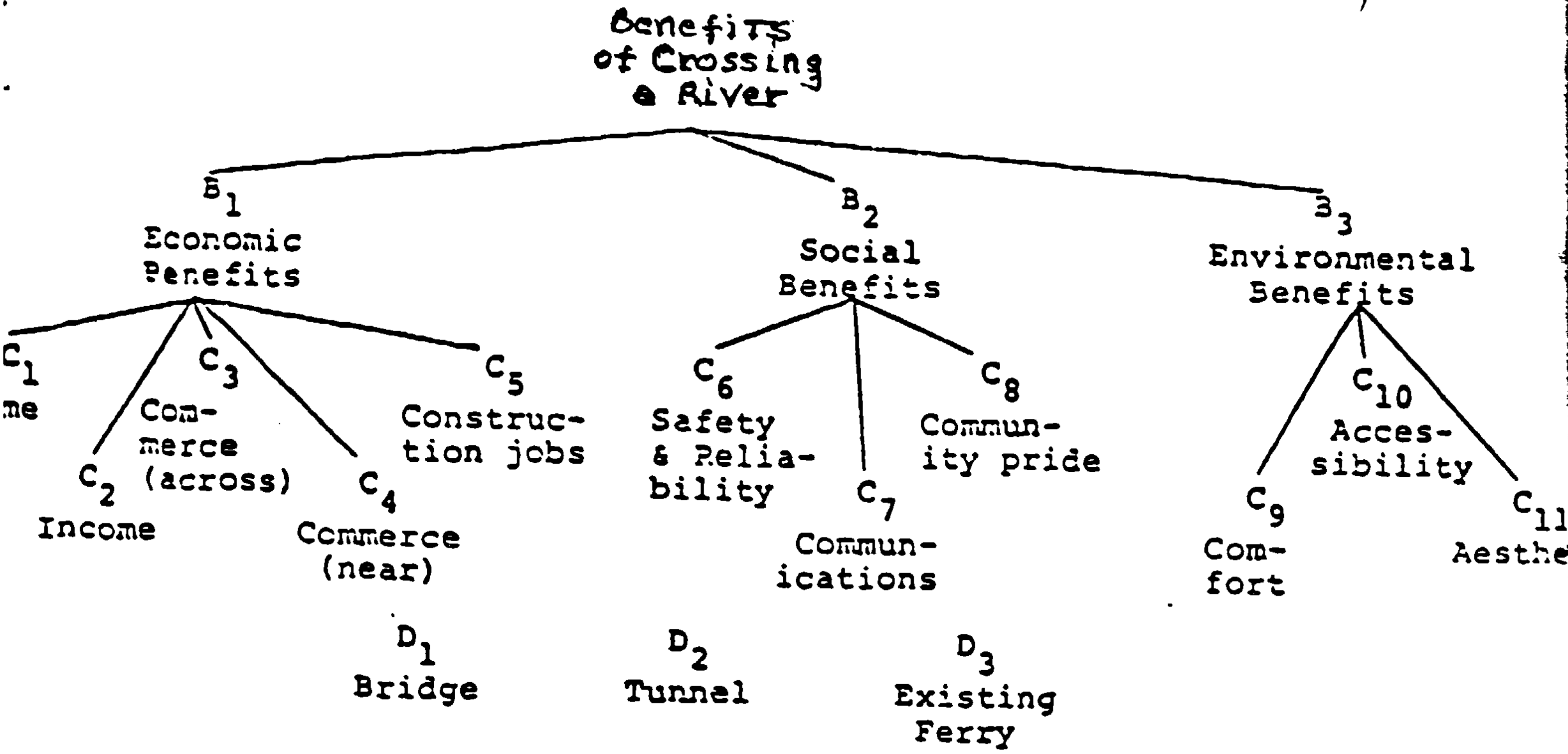

Figure 3

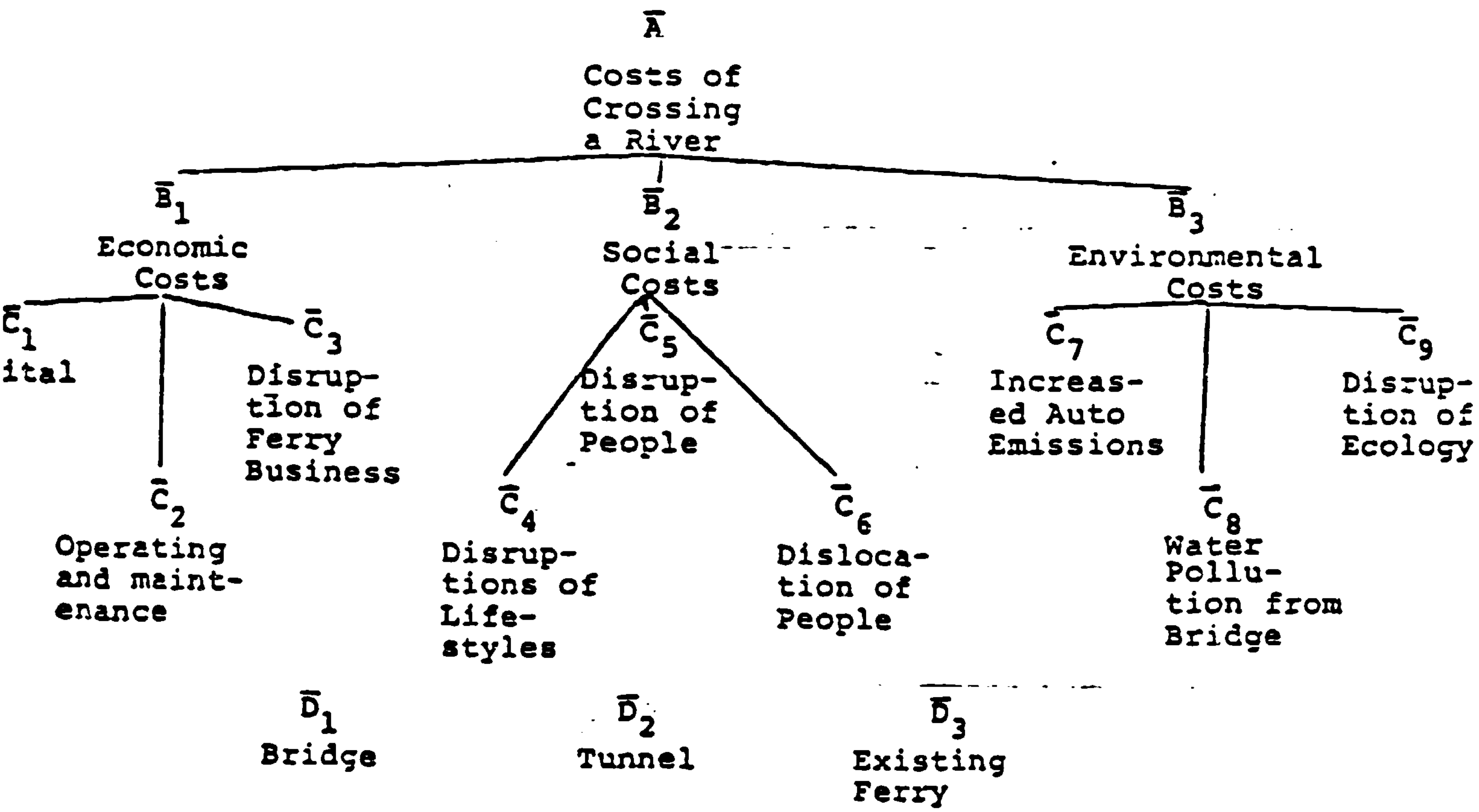

Figure 4

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
B_1	1	3	6	.67
	1/3	1	2	.22
B_3	1/6	1/2	1	.11

C.I.=0

B.	C.	C.	C_3	C_4	C_5	Eig
C_1	1	1/3	1/7	1/5	1/6	.04
C_2	3	1	1/4	1/2	1/2	.09
C_3	7	4	1	7	5	.54
C_4	5	2	1/7	1	1/5	.11
C_5	6	2	1/5	5	1	.23

C.I.=.14

B.	C.	C.	C.	Eig
C_6	1	6	9	.7[illegible]
C_7	1/6	1	4	.1[illegible]
C_8	1/9	1/4	1	.0[illegible]

C.I.=.05

B_2	C_9	C_{10}	C_{11}	Eig
C_9	1	1/4	6	.25
C_{10}	4	1	8	.69
C_{11}	1/6	1/8	1	.06

C.I.= .07

C_1	D_1	D_2	D_3	Eig
D_1	1	2	7	.58
D_2	1/2	1	6	.35
D_3	1/7	1/6	1	.07

C.I.= .02

C_2	D_1	D_2	D_3	Eig
D_1	1	1/2	8	.36
D_2	2	1	9	.59
D_3	1/8	1/9	1	.05

C.I.=.02

C_3	D_1	D_2	D_3
D_1	1	4	8
D_2	1/4	1	6
D_3	1/8	1/6	1

C.I. =

C_4	D_1	D_2	D_3	Eig
D_1	1	1	6	.46
D_2	1	1	6	.46
D_3	1/6	1/6	1	.08

C.I. = 0

C_5	D_1	D_2	D_3	Eig
D_1	1	1/4	9	.27
D_2	4	1	9	.67
D_3	1/9	1/9	1	.05

C.I. =.11

C_6	D_1	D_2	D_3	Eig
D_1	1	4	7	.68
D_2	1/4	1	6	.26
D_3	1/7	1/6	1	.06

C.I. =.09

C_7	D_1	D_2	D_3	E[illegible]
D_1	1	1	5	[illegible]
D_2	1	1	5	[illegible]
D_3	1/5	1/5	1	[illegible]

C.I. = 0

C_8	D_1	D_2	D_3	Eig
D_1	1	5	3	.64
D_2	1/5	1	1/3	.11
D_3	1/3	3	1	.26

C.I. = .02

C_9	D_1	D_2	D_3	Eig
D_1	1	5	8	.73
D_2	1/5	1	5	.21
D_3	1/8	1/5	1	.06

C.I. = .07

C_{10}	D_1	D_2	D_3	Eig
D_1	1	3	7	.64
D_2	1/3	1	6	.29
D_3	1/7	1/6	1	.07

C.I. = .05

C_{11}	D_1	D_2	D_3	E[illegible]
D_1	1	6	1/5	[illegible]
D_2	1/6	1	1/3	[illegible]
D_3	5	3	1	[illegible]

C.I. =[illegible]

Total Benefits Hierarchy C.R.H < .1 (good)

(Poor consistency does not affect final result becaus[e] of low priorit[y]

Best available Page

Best available Page

[illegible] Costs

$\bar{A}$	$\bar{B}_1$	$\bar{B}_2$	$\bar{B}_3$	Eig
$\bar{B}_1$	1	5	7	.74
$\bar{B}_2$	1/5	1	2	.17
$\bar{B}_3$	1/7	1/2	1	.09

C.I. = .01

$\bar{B}_1$	$\bar{C}_1$	$\bar{C}_2$	$\bar{C}_3$	Eig
$\bar{C}_1$	1	7	9	.77
$\bar{C}_2$	1/7	1	5	.17
$\bar{C}_3$	1/9	1/5	1	.06

C.I. = .1

$\bar{B}_2$	$\bar{C}_4$	$\bar{C}_5$	$\bar{C}_6$	Eig
$\bar{C}_4$	1	1/3	1/5	.11
$\bar{C}_5$	3	1	1/3	.26
$\bar{C}_6$	5	3	1	.64

C.I. =.02

$\bar{B}_3$	$\bar{C}_7$	$\bar{C}_8$	$\bar{C}_9$	Eig
$\bar{C}_7$	1	3	4	.62
$\bar{C}_8$	1/3	1	1/3	.13
$\bar{C}_9$	1/4	3	1	.25

C.I. =.11

$\bar{C}_1$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	E
$\bar{D}_1$	1	1/3	8	[illegible]
$\bar{D}_2$	3	1	9	[illegible]
$\bar{D}_3$	1/8	1/9	1	[illegible]

C.I. = [illegible]

$\bar{C}_2$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	Eig
$\bar{D}_1$	1	1/3	8	.30
$\bar{D}_2$	3	1	9	.65
$\bar{D}_3$	1/8	1/9	1	.05

C.I. = .05

$\bar{C}_3$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	Eig
$\bar{D}_1$	1	1	9	.47
$\bar{D}_2$	1	1	9	.47
$\bar{D}_3$	1/9	1/9	1	.05

C.I. = 0

$\bar{C}_4$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	Eig
$\bar{D}_1$	1	4	9	.69
$\bar{D}_2$	1/4	1	8	.26
$\bar{D}_3$	1/9	1/8	1	.05

C.I. =.09

$\bar{C}_5$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	E
$\bar{D}_1$	1	1	9	.4[illegible]
$\bar{D}_2$	1	1	9	.4[illegible]
$\bar{D}_3$	1/9	1/9	1	.0[illegible]

C.I. = 0

$\bar{C}_6$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	Eig
$\bar{D}_1$	1	1	9	.47
$\bar{D}_2$	1	1	9	.47
$\bar{D}_3$	1/9	1/9	1	.05

C.I. = 0

$\bar{C}_7$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	Eig
$\bar{D}_1$	1	3	8	.65
$\bar{D}_2$	1/3	1	6	.29
$\bar{D}_3$	1/8	1/6	1	.06

C.I. = .04

$\bar{C}_8$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	Eig
$\bar{D}_1$	1	3	7	.65
$\bar{D}_2$	1/3	1	5	.28
$\bar{D}_3$	1/7	1/5	1	.07

C.I.= .03

$\bar{C}_9$	$\bar{D}_1$	$\bar{D}_2$	$\bar{D}_3$	E
$\bar{D}_1$	1	1/6	7	[illegible]
$\bar{D}_2$	6	1	8	.7[illegible]
$\bar{D}_3$	1/7	1/8	1	.0[illegible]

C.I. = .1

Total Cost Hierarchy C. R. H. < .1 (good)

e. Determining the Payoff Matrix in a Two-person Game

The payoff matrix of a general two person game assumes that each player can provide a relatively good (rational) estimate of the payoffs he expects to receive from each of his strategies in countering each of the opponent's strategies. In some exceptional problems the rationale for assigning the payoffs is easier to provide, particularly if the utilities represent money. But what is needed in general is an analytical method for linking the payoffs directly to the strategies of both players and more specifically to the overall effectiveness of each strategy against all of the opponent's "active" strategies.

It is essential for our purpose that a player be able to assess in qualitative terms the relative dominance or effectiveness of each of his strategies when compared with all the others against each strategy of the opponent's. In principle if a player does not have this modicum of sophistication, he is obviously either not interested in winning or incapable (for the moment) of defining the object of his playing the game.

Now an intriguing problem in game theory is the assumption that it is possible to estimate payoffs for strategies in a game before the strategies of one player have been matched against those of the opponent in actual competition. Except for the simplest and most transparent situations it is impossible to spell out all the moves and tactics of a real-life strategy to really get a good idea of how well it would fare in competition. Some broad qualitites of a strategy may be known, but exact prescriptions of its effectiveness may encounter such unanticipated difficulties in practice that it may be difficult to get a "good" estimate of its worth when compared with other strategies. Thus a major problem is to obtain sufficiently reliable payoffs that can be used in the computation of optimal strategies.

We would like to elaborate on the idea of how to estimate payoffs and actually provide a method for estimating them. When a player in a game goes through the process of assessing the relative strengths of his strategies against those of the opponent, he must have some basic properties of strategies in mind which indicate merit in choosing one over another before the start of competition. But again we ask what can he be thinking of when he does this assessment since he has not played the game to find out how his strategies fare in the competition?

We believe that there are two steps which one is inclined to follow. First he evaluates his own strategies according to some intrinsic set of properties to assess their relative strengths. He would lso have to decide as to which of these properties is more important for winning. The process leads to a vector of the relative weights of his strategies with respect to all the properties. Having done this he may, if he has sufficient information, do the same for his opponent's strategies.

He next goes through a simulation exercise as to the effectiveness of his strategies when matched against each strategy of the opponent. Thus he obtains a vector of the relative strength of his strategies against each strategy of his opponent. The result is a matrix whose columns are these vectors of relative weights of the second step. He now weights each row of this matrix by the corresponding weight of the strategy from the first step to obtain his payoff matrix. It is a composition of the first step in which he has evaluated the constant value of his strategies and the second step in which he has analyzed the current (engagement) value of the strategies. Of course if he has no knowledge about the strategies of his opponent he would rely on the evidence of the first step alone. How does one carry out this numerical evaluation and does it make sense is the next task which we tackle. We also believe that the procedure can be generalized to n-person games.

Of course a player can estimate the payoffs to his opponent in light of whatever information he has about the opponent. He can then proceed to determine his optimum strategies without reference to what the opponent's assessment of the situation is, although he can also use such information in making his estimates.

When we deal with incomplete information, our behavior is largely based on constant values (which is the case for example when we assume constant prices in the economy); current values on the other hand, are conditional to the present (as are current prices in the economy). An example of a situation where one has to deal with both constant and current values, arises in the game of chess. In this game, a player faces a situation in which he can formulate the problem with certainty; but where he ignores the final result of a present action. Chess players always hope to move to a position which offers them "good" positions. After the opening game, the middle game in chess is determined by constant values to gain good position on the board (such as the center squares) and the final game by current values depending on the moves of the opponent.

- Summary of How to Compute Payoffs

1 - To compute the constant value

a) Construct a hierarchy of attributes of strategies.

b) Evaluate each strategy of a player with respect to these attributes to get the composite eigenvector of relative "constant" importance of strategies of a player. This corresponds to the a priori strengths of the strategies.

2 - To compute the current value

a) Match the strategies of a player in a pairwise comparison matrix according to their strength against each strategy of the opponent and get a matrix of eigenvectors.

Thus we consider a 2-person game in normal form. Denote the players by L and I, respectively and let $L_1, \ldots, L_n$ be the strategies of player L and $I_1, \ldots, I_m$ be those of player I. We then construct a matrix of pairwise comparisons of the strategies $L_1, \ldots, L_n$ according to their effectiveness against each of the strategies I_j, $j=1, \ldots, m$. A typical such matrix has the form:

$$\begin{array}{c|ccc} \multicolumn{4}{c}{I_j} \\ & L_1 & \cdots & L_n \\ \hline L_1 & & & \\ \cdot & & & \\ \cdot & & (a_{pq}) & \\ \cdot & & & \\ L_n & & & \end{array}$$

whose entries (a_{pq}) with $a_{pq} > 0$ indicate how much more effective strategy L_p is estimated to be over L_q against the opponent's strategy I_j. We also have $a_{pq} = 1/a_{pq}$, i.e. we use the reciprocal value in the transpose position - a reasonable criterion to improve the overall consistency. We may similarly construct matrices of pairwise comparisons for I_j, $j=1, \ldots, m$, for each L_i, $i=1, \ldots, n$.

Solve the principal eigenvalue problem for each of these matrices. The resulting (column) eigenvectors give the realtive effectiveness on a ratio scale from zero to one, of the strategies being compared with respect to each of the opponent's strategies. Arrange these eigenvectors as columns of a matrix.

b) Weight each row of this matrix by the constant value weight of the corresponding strategy. This gives the payoff matrix to that player. Do the same for the opponent's payoff matrix.

5. Conclusion

The use of the AHP in optimization is relatively new and requires both theoretical and practical investigations to deepen its contribution to problems in which there are several factors for which measures have not been developed. We feel that this paper attempts to give some understanding of the role that the AHP might play in optimization, particularly to areas where judgment and experience are important and must be included in the formulation and use of the model.

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