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# ADAPTIVE MULTIPLE-BAND CFAR DETECTION OF AN OPTICAL PATTERN WITH UNKNOWN SPECTRAL DISTRIBUTION

University of Southern California

Robert Gagliardi and Irving S. Reed

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19. ABSTRACT (Continue on reverse if necessary and identify by block number) A generalized constant-false-alarm-rate (CFAR) algorithm is developed for detecting the presence of an optical signal of non-zero intensity in J signal-plus-noise bands or channels. For many applications this new algorithm is more flexible and practical than previous ones. If J=1, the resulting test reduces to the standard normalized matched filter test for finding a signal in clutter of unknown and varying intensity. Both theoretical and computer simulation results show that the SNR improvement gain of this new algorithm using multiple band scenes over the single scene of maximum SNR is always greater than one and in some cases it can be substantial. The data base used to simulate this new adaptive CFAR test are actual LANDSAT image scenes. The present results for optical detection are extendable to radar target detection and to other related detection problems.					
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hypothesis test. The test in [5] for automatic synchronization was derived as a least mean square criterion.

In order to analyze the performance of this new CFAR test, the probability density function of the MLR test for both hypotheses is found exactly in Section III. These probability densities are used then to calculate the probability of a false alarm (PFA) and the probability of detection (PD) as a function of the detection threshold in a manner similar to that utilized in [1]. The PFA of the test is computed in a closed formula which is independent of the covariance matrix of the actual residual clutter noise encountered. The probability density obtained here for a real optical signal in residual clutter noise is similar to that found in [5] for complex communication channels. However, the method of derivation is different and can be extended to other more complex detection problems.

## II. Formulation Of The Problem

The present detection problem is formulated in a manner to that used in [1]. First let the column vector,  $\{ \underline{x}(n) = [ x_1(n), x_2(n), \dots, x_J(n) ]^T, \text{ for } n = 1, 2, \dots, N \}$  be the  $J$  correlated image scenes which contain an optical signal with known shape and unknown position. Let  $\underline{s} = [ s(1), s(2), \dots, s(N) ]^T$  be the signal pattern  $N$ - vector, and  $\underline{b} = [ b_1, b_2, \dots, b_J ]^T$  be a  $J$ -vector of signal intensities corresponding to the  $J$  scenes or channels, respectively. The two hypotheses which the adaptive detector must distinguish are given by

$$H_0 : \underline{x}(n) = \underline{x}^0(n) \tag{1}$$

$$H_1 : \underline{x}(n) = \underline{x}^0(n) + \underline{b}s(n)$$

for  $n = 1, 2, \dots, N$  where  $\underline{x}^0$  is the vector of residual clutter noise-only processes.

Thus, under hypothesis  $H_0$  defined in Eq. (1), the joint probability density function of the Gaussian clutter vector  $\underline{x}(n)$  is given by

$$p(\underline{x}(n) | H_0) = \frac{1}{(2\pi)^{(J+K)/2} |M|^{1/2}} e^{-\frac{1}{2} \underline{x}^T(n) M^{-1} \underline{x}(n)} \quad \text{for } n = 1, 2, \dots, N \quad (2)$$

where

$$M = \mathbf{E} [(\underline{x}(n) - \mathbf{E} \underline{x}(n))(\underline{x}(n) - \mathbf{E} \underline{x}(n))^T] \quad (3)$$

is the unknown covariance matrix of  $\underline{x}(n)$  and  $|M| \neq 0$  is its determinant.

It was demonstrated previously [3] that the subtraction of a space-varying local mean from the image can yield an approximate zero-mean, near-white, Gaussian process with a slowly space-varying covariance matrix. Thus for  $N$  sufficiently small the subimage size,  $N$ , can be chosen so that matrix  $M$  is approximately a constant. Experiments indicate that such residual clutter noise is also approximately independent and Gaussian from pixel to pixel [3]. Therefore, it is reasonable to assume that the residual clutter is independent from spatial sample to sample (see appendix A [3]).

Next define  $J \times N$  matrix of subimage data as follows:

$$X = [\underline{x}(1), \dots, \underline{x}(N)] \quad (4)$$

The  $j$ -th row of matrix  $X$  is  $\underline{x}_j = [x_j(1), x_j(2), \dots, x_j(N)]$  for  $j = 1, 2, \dots, J$ . These are the  $N$  observation samples of the  $j$ -th image scene or channel. All of these  $J$  scenes or channels may be obtained either from different frequency bands of the same image or from sequential observations of the same scene. Hence, for  $N \geq J$  the rows of matrix  $X$  are assumed to be linearly independent. Then because of the mutual independence of the components of  $X$  in Eq. (4), the joint probability density of  $X$  under  $H_0$  is given by

$$p_0(X) = p(\underline{x}(1), \underline{x}(2), \dots, \underline{x}(N) | H_0) = \prod_{n=1}^N p(\underline{x}(n) | H_0)$$

$$= \frac{1}{(2\pi)^{NJ/2} |M|^{N/2}} e^{-\frac{1}{2} \left( \sum_{n=1}^N \underline{x}^T(n) M^{-1} \underline{x}(n) \right)} \quad (5)$$

In terms of the trace function, which consists of the sum of the diagonal entries of a square matrix, the exponent in Eq. (5) can be re-expressed as

$$\sum_{n=1}^N \underline{x}^T(n) M^{-1} \underline{x}(n) = \text{Tr} \left( M^{-1} \sum_{n=1}^N \underline{x}(n) \underline{x}^T(n) \right) \quad (6)$$

Hence, in terms of (6),  $p_0(X)$  in (5) becomes

$$\begin{aligned} p_0(X) &= \frac{1}{(2\pi)^{NJ/2} |M|^{N/2}} e^{-\frac{1}{2} \text{Tr} \left( M^{-1} \sum_{n=1}^N \underline{x}(n) \underline{x}^T(n) \right)} \\ &= \frac{1}{(2\pi)^{NJ/2} |M|^{N/2}} e^{-\frac{N}{2} \text{Tr} \left( M^{-1} \hat{M} \right)} \end{aligned} \quad (7)$$

where

$$\hat{M} = \frac{1}{N} \sum_{n=1}^N \underline{x}(n) \underline{x}^T(n) = \frac{1}{N} X X^T \quad (8)$$

For the signal-plus-noise hypothesis  $H_1$ , described in Eq. (1), one computes the conditional mean value of  $\underline{x}(n)$ , given hypothesis  $H_1$ , as follows :

$$\begin{aligned} E [ \underline{x}(n) | H_1 ] &= E [ \underline{x}^0(n) + \underline{b} s(n) ] \\ &= \underline{b} s(n) \end{aligned} \quad (9)$$

for  $n = 1, 2, \dots, N$ . Hence, the joint probability density function of the residual signal-plus-Gaussian clutter noise vectors,  $\underline{x}(n)$  for  $n=1, 2, \dots, N$  under hypothesis  $H_1$ , is found similarly to be

$$\begin{aligned} p_1(X) &= p(\underline{x}(1), \underline{x}(2), \dots, \underline{x}(N) | H_1) \\ &= \prod_{n=1}^N p(\underline{x}(n) | H_1) = \frac{1}{(2\pi)^{NJ/2} |M|^{N/2}} e^{-\frac{N}{2} \text{Tr} \left( M^{-1} \hat{M}_b \right)} \end{aligned} \quad (10)$$

where

$$\underline{x}_b(n) = [\underline{x}(n) - \underline{b} s(n)] \quad \text{and} \quad (11-1)$$

$$\hat{M}_b = \frac{1}{N} \sum_{n=1}^N \underline{x}_b(n) \underline{x}_b^T(n) = \frac{1}{N} (X - \underline{b} \underline{s}^T)(X - \underline{b} \underline{s}^T)^T. \quad (11-2)$$

Note that  $\hat{M}$  and  $\hat{M}_b$  are nonsingular if  $N \geq J$ , ( e.g. see [5] ).

The generalized maximum likelihood ratio test (e.g. see [1]) is shown readily to be

$$\Lambda = \frac{|\hat{M}|^{N/2}}{\min_b |\hat{M}_b|^{N/2}} \begin{matrix} \geq k, \text{ then } H_1 \\ < k, \text{ then } H_0 \end{matrix} \quad (12)$$

Clearly, the test in Eq. (12) is equivalent to

$$l = \frac{|\hat{M}|}{\min_b |\hat{M}_b|} \begin{matrix} \geq c, \text{ then } H_1 \\ < c, \text{ then } H_0 \end{matrix} \quad (13)$$

where  $c = k^{2/N}$

$$|\hat{M}| = |XX^T| \quad (14-1)$$

and

$$|\hat{M}_b| = |(X - \underline{b} \underline{s}^T)(X - \underline{b} \underline{s}^T)^T|. \quad (14-2)$$

Next a substitution of Eqs. (14) into Eq. (13) yields

$$l = \frac{|XX^T|}{\min_b |(X - \underline{b} \underline{s}^T)(X - \underline{b} \underline{s}^T)^T|} \begin{matrix} \geq c, \text{ then } H_1 \\ < c, \text{ then } H_0 \end{matrix}$$

or

$$l = \frac{|XX^T|}{\min_b |F_b|} \begin{matrix} \geq c, \text{ then } H_1 \\ < c, \text{ then } H_0 \end{matrix} \quad (15)$$

where

$$F_{\underline{b}} \triangleq (X - \underline{b} \underline{S}^T)(X - \underline{b} \underline{S}^T)^T \quad (16)$$

To find  $\min_{\underline{b}} |F_{\underline{b}}|$  one can start by expanding  $F_{\underline{b}}$  as follows:

$$\begin{aligned} F_{\underline{b}} &= XX^T - \underline{b} \underline{S}^T X^T - X(\underline{b} \underline{S}^T)^T + \underline{b} \underline{S}^T (\underline{b} \underline{S}^T)^T \\ &= XX^T - \underline{b} \underline{S}^T X^T - X\underline{S} \underline{b}^T + \underline{b} \underline{b}^T (\underline{S}^T \underline{S}) \end{aligned} \quad (17)$$

Then by adding and subtracting the term  $\frac{(X\underline{S})(X\underline{S})^T}{(\underline{S}^T \underline{S})}$  simultaneously in Eq. (17), one can decompose the  $J \times J$  matrix  $F_{\underline{b}}$  into the sum of three matrices in such a manner that only one of them contains the unknown vector  $\underline{b}$ . This is accomplished by first expressing Eq. (17) as follows:

$$F_{\underline{b}} = (\underline{S}^T \underline{S}) \left[ \underline{b} \underline{b}^T - \frac{\underline{b} \underline{S}^T X^T}{\underline{S}^T \underline{S}} - \frac{X\underline{S} \underline{b}^T}{\underline{S}^T \underline{S}} + \frac{(X\underline{S})(X\underline{S})^T}{(\underline{S}^T \underline{S})^2} - \frac{(X\underline{S})(X\underline{S})^T}{(\underline{S}^T \underline{S})^2} + \frac{(XX^T)}{\underline{S}^T \underline{S}} \right] \quad (18)$$

Hence, by a "completion of squares" operation Eq. (18) becomes

$$F_{\underline{b}} = (\underline{S}^T \underline{S}) \left[ \left( \underline{b} - \frac{X\underline{S}}{\underline{S}^T \underline{S}} \right) \left( \underline{b} - \frac{X\underline{S}}{\underline{S}^T \underline{S}} \right)^T + \frac{(XX^T)}{\underline{S}^T \underline{S}} - \frac{(X\underline{S})(X\underline{S})^T}{(\underline{S}^T \underline{S})^2} \right] \quad (19)$$

It is proved in Theorem A. 2 of Appendix A that  $F_{\underline{b}}$  in Eq. (19) is positive definite matrix for any vector  $\underline{b}$ , including the special vector  $\underline{b} = \frac{X\underline{S}}{\underline{S}^T \underline{S}}$ . Hence by

setting  $\underline{b} = \frac{X\underline{S}}{\underline{S}^T \underline{S}}$  in Eq. (19), one obtains

$$B \triangleq F_{\underline{b} = \frac{X\underline{S}}{\underline{S}^T \underline{S}}} = \frac{XX^T}{\underline{S}^T \underline{S}} - \frac{(X\underline{S})(X\underline{S})^T}{(\underline{S}^T \underline{S})^2} \quad (20)$$

where  $B$  is positive definite. Thus  $B^{-1}$  and the square roots  $B^{\pm 1/2}$  exist. Note that  $B$  is by Eq. (20) independent of the magnitude of vector  $\underline{b}$ .

To calculate the determinant  $|F_{\underline{b}}|$  one can establish from Eqs.(19) and (20) the following relationship:

$$F_{\underline{b}} = (\underline{S}^T \underline{S}) B^{1/2} \left[ I_J + B^{-1/2} \left( \underline{b} - \frac{X\underline{S}}{\underline{S}^T \underline{S}} \right) \left( \underline{b} - \frac{X\underline{S}}{\underline{S}^T \underline{S}} \right)^T B^{-1/2} \right] B^{1/2} \quad (21)$$

Thus the the determinant of  $F_{\underline{b}}$  is

$$|F_{\underline{b}}| = (\underline{\mathcal{S}}^T \underline{\mathcal{S}})^J |B| \left| I_J + B^{-1/2} \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right) \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right)^T B^{-1/2} \right| \quad (22)$$

Hence finally, the denominator of Eq. (15) is given in a new form by

$$\min_{\underline{b}} |F_{\underline{b}}| = (\underline{\mathcal{S}}^T \underline{\mathcal{S}})^J |B| \min_{\underline{b}} \left| I_J + B^{-1/2} \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right) \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right)^T B^{-1/2} \right| \quad (23)$$

Since  $B^{-1/2} \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right)$  is a  $J \times 1$ -column vector, by a matrix identity proved in

Appendix of [3], one obtains

$$\left| I_J + B^{-1/2} \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right) \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right)^T B^{-1/2} \right| = 1 + \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right)^T B^{-1} \left( \underline{b} - \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \right) \quad (24)$$

where  $B^{-1}$  is positive definite. Therefore, the minimum in Eq. (23) is obtained when the second term in the right side of Eq. (24) vanishes. But this happens if and only if,

$$\underline{b} = \frac{X\underline{\mathcal{S}}}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} \quad (25)$$

Thus, a substitution of Eq. (24) into Eq. (23) yields finally,

$$\begin{aligned} \min_{\underline{b}} |F_{\underline{b}}| &= (\underline{\mathcal{S}}^T \underline{\mathcal{S}})^J |B| \\ &= (\underline{\mathcal{S}}^T \underline{\mathcal{S}})^J \left| \frac{XX^T}{\underline{\mathcal{S}}^T \underline{\mathcal{S}}} - \frac{(X\underline{\mathcal{S}})(X\underline{\mathcal{S}})^T}{(\underline{\mathcal{S}}^T \underline{\mathcal{S}})^2} \right| \\ &= \left| XX^T - \frac{(X\underline{\mathcal{S}})(X\underline{\mathcal{S}})^T}{(\underline{\mathcal{S}}^T \underline{\mathcal{S}})} \right| \quad (26) \end{aligned}$$

It is proved in Theorem A. 2 of Appendix A for  $\underline{b} = 0$  that the  $J \times J$  matrix  $XX^T$  is positive definite, so that the square roots  $(XX^T)^{1/2}$  exist. Hence, the denominator of Eq. (15) can be expressed further as

$$\min_{\underline{b}} |F_{\underline{b}}| = |XX^T| \left| I_J - \frac{(XX^T)^{-1/2} (X\underline{\mathcal{S}})(X\underline{\mathcal{S}})^T (XX^T)^{-1/2}}{(\underline{\mathcal{S}}^T \underline{\mathcal{S}})} \right| \quad (27)$$

Thus, since  $(XX^T)^{-1/2}(X\underline{\xi})$  is a  $J \times 1$  -column vector, Eq. (27) becomes

$$\min_b |F_b| = |XX^T| \left( 1 - \frac{(X\underline{\xi})^T (XX^T)^{-1} (X\underline{\xi})}{(\underline{\xi}^T \underline{\xi})} \right) \quad (28)$$

Finally, a substitution of Eq. (28) into Eq. (15) yields

$$l = \frac{1}{1 - \frac{(X\underline{\xi})^T (XX^T)^{-1} (X\underline{\xi})}{(\underline{\xi}^T \underline{\xi})}} \begin{array}{l} \geq r_0, \text{ then } H_1 \\ < r_0, \text{ then } H_0 \end{array} \quad (29)$$

as the likelihood ratio test function. Clearly by Eq. (15) the test in Eq. (29) is equivalent to

$$r = \frac{(X\underline{\xi})^T (XX^T)^{-1} (X\underline{\xi})}{(\underline{\xi}^T \underline{\xi})} \begin{array}{l} \geq r_0, \text{ then } H_1 \\ < r_0, \text{ then } H_0 \end{array} \quad (30)$$

where  $r$  is related to  $l$  by  $l = \frac{1}{1-r}$ , and  $r_0 = 1 - \frac{1}{c}$ . In the case  $J = 1$  the test  $r$  reduces to the normalized matched filter test,

$$r = \frac{(\underline{x}^T \underline{\xi})^2}{(\underline{x}^T \underline{x})(\underline{\xi}^T \underline{\xi})} \begin{array}{l} \geq r_0, \text{ then } H_1 \\ < r_0, \text{ then } H_0 \end{array} \quad (31)$$

a well-known CFAR test for one frequency band or channel.

### III. Detection and False Alarm Probabilities of Test

In order to find the probability density function of the test  $r$  in Eq. (30) on both hypotheses  $H_0$  and  $H_1$ , one partitions matrix  $X$  as follows:

$$X = [ \underline{x}(1) \mid \underline{x}(2) \mid \dots \mid \underline{x}(N) ] \quad (32)$$

where  $\underline{x}(n) \triangleq [x_1(n), x_2(n), \dots, x_J(n)]$  is the  $J \times 1$ -column vector as defined in Eq. (1).

By (1) and (3) one has

$$\text{Cov} [ \underline{x}(n) \mid H_i ] = M \quad \text{for } i = 0, 1.$$

Also

$$\mathbf{E} [ \underline{x}(n) | H_0 ] = \mathbf{E} [ \underline{x}^0(n) ] = 0 \quad \text{and}$$

$$\mathbf{E} [ \underline{x}(n) | H_1 ] = \mathbf{E} [ \underline{x}^0(n) + \underline{b}s(n) ] = \underline{b}s(n) ,$$

or in terms of  $X$  as defined in Eq. (32),

$$\mathbf{E} [ X | H_0 ] = 0 \quad \text{and} \quad \mathbf{E} [ X | H_1 ] = \underline{b} \underline{S}^T \quad (33)$$

Next perform a whitening procedure on  $\underline{x}(n)$  by defining

$$\underline{z}(n) = M^{-1/2} \underline{x}(n) , \text{ for } n = 1, 2, \dots, N , \quad (34-1)$$

i.e. let

$$Z = [ \underline{z}(1) | \underline{z}(2) | \dots | \underline{z}(N) ] = M^{-1/2} X \quad (34-2)$$

The whitening procedure in Eq. (34) and the assumption, that the residual clutter samples in the spatial coordinates are mutually independent, produces the result,

$$\text{Cov} [ z_i(m) z_j(n) ] = \delta(i-j, m-n) \quad (35)$$

for  $i, j = 1, 2, \dots, J$  and  $m, n = 1, 2, \dots, N$ . Here  $\delta(n, m)$  is the Kronecker delta function defined by

$$\delta(n, m) = \begin{cases} 1 & \text{if } n = 0 \text{ and } m = 0 \\ 0 & \text{otherwise ,} \end{cases} \quad (36)$$

and  $z_i(m)$  is the  $i$ -th element of vector  $\underline{z}(m)$ . Then by Eqs. (33) to (36),

$$\mathbf{E} [ Z | H_0 ] = 0 , \quad (37)$$

$$\mathbf{E} [ Z | H_1 ] = M^{-1/2} \underline{b} \underline{S}^T \quad \text{and} \quad (38)$$

$$\text{Cov} [ \underline{z}(n) | H_i ] = I_J \quad \text{for } i = 0, 1 . \quad (39)$$

Evidently by the transformation in the Eq. (34) the test function in Eq. (30) becomes in terms of  $Z$  in Eq. (34-2) the expression,

$$r = \frac{(\underline{Z}\underline{S})^T (\underline{Z}\underline{Z}^T)^{-1} (\underline{Z}\underline{S})}{(\underline{S}^T \underline{S})} \begin{cases} \geq r_0 , \text{ then } H_1 \\ < r_0 , \text{ then } H_0 \end{cases} \quad (40)$$

Now let

$$\underline{t} = \frac{\underline{S}}{(\underline{S}^T \underline{S})^{1/2}} \quad (41)$$

Then the test function in Eq. (40) becomes, using Eqs (40) and (41) ,

$$r = (\underline{Z} \underline{t})^T (\underline{Z} \underline{Z}^T)^{-1} (\underline{Z} \underline{t}) \quad (43)$$

Also by Eq. (41) the sum-of-squares norm of  $\underline{t}$  is given by  $\|\underline{t}\| = 1$  so that  $\underline{t}$  is a unit vector in the "direction" of vector  $\underline{S}$ .

Now consider the  $N \times N$  orthonormal matrix  $U_1$ , which carries out rotations in  $N$ -dimensional space, in such a manner that unit vector  $\underline{t}$  is transformed into the new unit vector,

$$\tilde{\underline{t}} = U_1 \underline{t} = [1, 0, \dots, 0]^T \quad (44)$$

Also, let

$$\underline{V} = \underline{Z} U_1^T = [\underline{v}(1), \underline{v}(2), \dots, \underline{v}(N)] \quad (45)$$

Then the test function  $r$  in Eq. (43) reduces to

$$r = \underline{v}(1)^T (\underline{V} \underline{V}^T)^{-1} \underline{v}(1) \quad (46)$$

The covariance matrix of  $\underline{v}(n)$ , for  $n = 1, 2, \dots, N$  is similar to that of  $\underline{z}(n)$ , the only change of the statistics of the  $\underline{v}(n)$  from that of the  $\underline{z}(n)$  is their mean values under hypothesis  $H_1$ . This mean is derived as follows:

$$\begin{aligned} \mathbf{E} [ \underline{V} | H_1 ] &= \mathbf{E} [ \underline{V} U_1^T | H_1 ] \\ &= M^{-1/2} \underline{b} \underline{t}^T U_1^T (\underline{S}^T \underline{S})^{1/2} \\ &= M^{-1/2} \underline{b} [1, 0, \dots, 0] (\underline{S}^T \underline{S})^{1/2} \\ &= [M^{-1/2} \underline{b} (\underline{S}^T \underline{S})^{1/2}, 0, \dots, 0] \end{aligned} \quad (47)$$

From Eqs. (44) and (47) a figure of merit or what might be termed, the generalized signal-to-noise ratio ( $GSNR$ ) of the test, is derived as follows:

$$\begin{aligned} (G\text{SNR}) &= \mathbf{E} [ \underline{v}^T(1) | H_1 ] \mathbf{E} [ \underline{v}(1) | H_1 ] \\ &= (\underline{b}^T M^{-1} \underline{b}) \| \underline{\Sigma} \|^2 \triangleq a \end{aligned} \quad (48)$$

Now consider a further simplification of the test function  $r$  in Eq. (46). First separate matrix  $V$  into two parts in such a manner that

$$VV^T = \underline{v}(1)\underline{v}^T(1) + \sum_{n=2}^N \underline{v}(n)\underline{v}^T(n) = \underline{v}(1)\underline{v}^T(1) + Q^T, \quad (49-1)$$

where

$$Q = \sum_{n=2}^N \underline{v}(n)\underline{v}^T(n) = [\underline{v}(2), \dots, \underline{v}(N)][\underline{v}(2), \dots, \underline{v}(N)]^T \quad (49-2)$$

is a non-singular  $J \times N$  matrix.

A well-known matrix inversion identity applied to Eq. (49-1) produces the result,

$$(VV^T)^{-1} = [\underline{v}(1)\underline{v}^T(1) + Q]^{-1} = [I - \frac{Q^{-1}\underline{v}(1)\underline{v}^T(1)}{1 + \underline{v}^T(1)Q^{-1}\underline{v}(1)}] Q^{-1}. \quad (50)$$

A substitution of Eq. (50) into the test function  $r$  in Eq. (46) yields the test function  $r$  as the new expression,

$$r = \frac{\underline{v}^T(1)Q^{-1}\underline{v}(1)}{1 + \underline{v}^T(1)Q^{-1}\underline{v}(1)} = \frac{r_1}{1+r_1}, \quad (51)$$

where

$$r_1 = \underline{v}^T(1)Q^{-1}\underline{v}(1). \quad (52)$$

It is desired now to find the probability density,  $f(r_1 | H_1)$ , of  $r_1$  in Eq. (52). First by Eq. (49-2), define  $D = [\underline{v}(2), \dots, \underline{v}(N)]$  and re-express Eq. (52) in the form,

$$r_1 = \|\underline{v}(1)\|^2 \left( \frac{\underline{v}^T(1)}{\|\underline{v}(1)\|} (DD^T)^{-1} \frac{\underline{v}(1)}{\|\underline{v}(1)\|} \right). \quad (54)$$

Then normalize the  $J$ -component vector  $\underline{v}(1)$  as follows :

$$\xi = \frac{\underline{v}(1)}{\|\underline{v}(1)\|} . \quad (55)$$

Hence by Eq. (55) one obtains  $r$  in Eq. (54) in the form,

$$r_1 = \|\underline{v}(1)\|^2 (\xi^T (DD^T)^{-1} \xi) = \|\underline{v}(1)\|^2 e , \quad (56-1)$$

where

$$e \triangleq (\xi^T (DD^T)^{-1} \xi) . \quad (56-2)$$

Now one can further process the term  $e$  in Eq. (56) by conditioning on the elements of  $\underline{v}(1)$  so that  $\xi$  can be treated as a normalized constant vector. Then since  $\xi$  has unity magnitude, there exists a  $J \times J$  orthonormal matrix  $U_2$  such that,

$$U_2 \xi = [1, 0, \dots, 0]^T . \quad (57)$$

Next apply this transformation to matrix  $D$ , defined before Eq. (54), by letting

$$H = U_2 D = U_2 [\underline{v}(2), \dots, \underline{v}(N)] . \quad (58)$$

Then the term  $e$  in Eq. (56) has the simple form,

$$e = \xi^T (DD^T)^{-1} \xi = [1, 0, \dots, 0] (HH^T)^{-1} [1, 0, \dots, 0]^T . \quad (59)$$

Clearly  $H$  in Eq. (58) has the exactly the same statistical properties as  $D$ , under the assumption that  $\underline{v}(1)$  is given.

Now partition  $H$  as follows:

$$H = \begin{bmatrix} \underline{h}_A^T \\ H_B \end{bmatrix} , \quad (60-1)$$

where  $\underline{h}_A$  is the  $N-1$ -column vector and  $H_B$  is the  $(J-1) \times (N-1)$  matrix. Then by Eq. (59),

$$(HH^T)^{-1} = \begin{bmatrix} \underline{h}_A^T \underline{h}_A & \underline{h}_A^T H_B^T \\ H_B \underline{h}_A & H_B H_B^T \end{bmatrix}^{-1} \triangleq \begin{bmatrix} R_{AA} & R_{AB} \\ R_{BA} & R_{BB} \end{bmatrix} . \quad (60-2)$$

According to the Frobenius relations, e.g. see [8] or [9], for a partitioned matrix,

$$R_{AA} = [h_A^T h_A - h_A^T H_B^T (H_B H_B^T)^{-1} H_B h_A]^{-1} = [h_A^T (I - H_B^T (H_B H_B^T)^{-1} H_B) h_A]^{-1} \\ = \frac{1}{h_A^T (I - H_B^T (H_B H_B^T)^{-1} H_B) h_A} = \frac{1}{h_A^T P_1 h_A} \quad (61)$$

A substitution of Eqs. (60) and (61) into Eq. (59) yields

$$e = \frac{1}{h_A^T P_1 h_A} \quad (62)$$

where  $P_1 \triangleq I_{N-1} - H_B^T (H_B H_B^T)^{-1} H_B$  is a projection operator such that  $P_1^2 = P_1$  and  $Tr(P_1) = N-J$ . In the same manner used for the projection matrix  $P$  in [1] it is not difficult to show that  $P_1$  has  $N-J$  unity eigenvalues and  $J-1$  zero eigenvalues. Thus  $P_1$  can be diagonalized to the form ,

$$U_3^T P_1 U_3 = \Lambda_1 = \begin{bmatrix} I_{N-J} & 0 \\ 0 & 0_{J-1} \end{bmatrix}, \quad (63)$$

where  $I_{N-J}$  is the  $(N-J) \times (N-J)$  identity matrix. By arguments similar to those used previously in [1], one finds also, under the assumption that  $\underline{v}(1)$  and  $P_1$  are given, that

$$h_A^T P_1 h_A = \underline{\eta}^T \underline{\eta} = \sum_{i=1}^{N-J} \eta_i^2, \quad (64-1)$$

where

$$\underline{\eta} \triangleq h_A^T U_3 \Lambda^{1/2} \quad (64-2)$$

is a  $1 \times (N-J)$ -column vector. The conditional joint probability density function of  $\underline{\eta}$  is subject to the normal density function,  $N(\underline{Q}, I_{N-J})$ , i.e.

$$p_{\underline{\eta}}(\eta_1, \dots, \eta_{(N-J)} | \underline{v}(1), P_1) = N(\underline{Q}, I_{N-J}). \quad (65)$$

Since  $p_{\underline{\eta}}(\eta_1, \dots, \eta_{(N-J)} | \underline{v}(1), P_1)$  in Eq. (65) does not depend on  $\underline{v}(1)$  and  $P_1$ , the vector  $\underline{v}(1)$  and matrix  $P_1$  must be statistically independent of vector  $\underline{\eta}$ . Also by

Eq. (62) one obtains the ratio  $r_1$  in the form,

$$r_1 = \|\underline{v}(1)\|^2 \cdot e = \frac{\underline{v}^T(1)\underline{v}(1)}{\underline{\eta}^T \underline{\eta}} = \frac{\sum_{j=1}^J v_j^2(1)}{\sum_{i=1}^{N-J} \eta_i^2}, \quad (66)$$

in term of the magnitudes of these two vectors only. Thus by the independence of vectors  $\underline{\eta}$  and  $\underline{v}(1)$  and a use of the Corollary 2 in [7, pp 52], one has the probability density function,

$$f(r_1 | H_1) = \frac{r_1^{\frac{J-2}{2}} e^{-\frac{a}{2}}}{B\left(\frac{N-J}{2}, \frac{1}{2}J\right) (1+r_1)^{\frac{N}{2}}} {}_1F_1\left(\frac{N}{2}; \frac{1}{2}J; \frac{ar_1}{2(1+r_1)}\right), \quad (67)$$

of  $r_1$  in Eq. (51) under hypothesis  $H_1$ .

Finally by using the relationship of  $r_1$  to  $r$  in Eq. (51) the probability density function of the test function  $r$  under hypothesis  $H_1$  is given by

$$f(r | H_1) = \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-J}{2}\right)\Gamma\left(\frac{J}{2}\right)} (1-r)^{\frac{N-J-2}{2}} r^{\frac{J-2}{2}} e^{-\frac{a}{2}} {}_1F_1\left(\frac{N}{2}; \frac{J}{2}; \frac{ar}{2}\right) \quad (68)$$

for  $0 \leq r \leq 1$  where  $a$  is the generalized SNR in Eq. (48) and  $F_1(a; b; x)$  is the confluent hypergeometric function. Clearly, if no signal is present, then  $a = 0$ . Thus Eq. (67) reduces in the  $H_0$  hypothesis to a Beta-function probability density of form,

$$f(r | H_0) = \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-J}{2}\right)\Gamma\left(\frac{J}{2}\right)} (1-r)^{\frac{N-J-2}{2}} r^{\frac{J-2}{2}} \quad \text{for } 0 \leq r \leq 1. \quad (69)$$

Finally in terms of the above probability density functions in Eqs. (69) and (68) the probability of a false alarm is found by

$$P_{FA} = \int_{r_0}^1 f(r | H_0) dr \quad (70)$$

and the probability of detection by

$$P_D = \int_{r_0}^{\infty} f(r | H_1) dr \quad , \quad (71)$$

#### IV. Performance Analysis

Performance curves of the probability of detection in Eq. (71) versus SNR for a given false alarm probability with respect to various values of  $N$  and  $J$  are computed in this section. First, in Fig. 1 the probability of detection is calculated as a function of the generalized signal-to-noise ratio (GSNR)  $a$  in Eq. (48) for several different values of parameter  $N$ . These curves demonstrate that, for a fixed GSNR,  $a$ , the CFAR detector has a higher detection probability if more samples are used.

In order to compare the detection performance improvement for different numbers of signal-plus-noise bands, the probability of detection in the single scene of maximum SNR is compared with the probability of detection in two correlated scenes, i.e. for  $J = 2$ . To accomplish this, the GSNR in  $J$  correlated scenes is related to the maximum SNR of the  $J$  scenes. Consider first the case of  $J = 2$ .

For  $J = 2$  let the maximum signal-to-noise ratio in the 2 correlated scenes be given by

$$a' = \frac{b_1^2 \|\underline{\Sigma}\|^2}{\sigma_1^2} \quad (72)$$

Then the GSNR in Eq. (48) for  $J = 2$  is

$$a = [b_1 \ b_2] M^{-1} [b_1 \ b_2]^T \|\underline{\Sigma}\|^2 \quad , \quad (73)$$

where

$$M = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \quad (74)$$

Hence,

$$M^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2} \begin{bmatrix} \sigma_2^2 & -\sigma_{12}^2 \\ -\sigma_{21} & \sigma_1^2 \end{bmatrix}, \quad (75)$$

and as a consequence one has

$$a = [b_1 \ b_2] M^{-1} [b_1 \ b_2]^T \|\underline{S}\|^2 = \frac{(b_1^2 \sigma_2^2 + b_2^2 \sigma_1^2 - 2b_1 b_2 \sigma_{12}) \|\underline{S}\|^2}{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2} \quad (76)$$

$$\begin{aligned} &= \frac{1}{1 - \frac{\sigma_{12}^2}{\sigma_1^2 \sigma_2^2}} \left( \frac{b_1^2 \|\underline{S}\|^2}{\sigma_1^2} + \frac{b_2^2 \|\underline{S}\|^2}{\sigma_2^2} - \frac{2b_1 b_2 \sigma_{12} \|\underline{S}\|^2}{\sigma_1^2 \sigma_2^2} \right) \\ &= \frac{a'}{1 - \frac{\sigma_{12}^2}{\sigma_1^2 \sigma_2^2}} \left( 1 + \frac{b_2^2 \sigma_1^2}{b_1^2 \sigma_2^2} - 2 \frac{b_2 \sigma_1}{b_1 \sigma_2} \right). \end{aligned} \quad (77)$$

In terms of the normalized correlation coefficient,

$$\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}, \quad (78)$$

and the ratio,

$$\lambda^2 \triangleq \frac{(SNR)_2}{(SNR)_1} = \frac{b_2^2 \|\underline{S}\|^2 / \sigma_2^2}{b_1^2 \|\underline{S}\|^2 / \sigma_1^2}, \quad (79)$$

of SNR in the second scene to SNR in the primary scene of maximum SNR, Eq. (77) can be re-expressed as follows:

$$a = G a' \quad (80-1)$$

where

$$G = \frac{(1 + \lambda^2 - 2\lambda\rho)}{1 - \rho^2} \quad (80-2)$$

is the gain in signal-to-noise ratio of a detector which uses  $J = 2$  scenes over a single detector which uses that scene with the maximum SNR. The following lemma is now

proved.

**Lemma 1:**

$$G = \frac{(1 + \lambda^2 - 2\lambda\rho)}{1 - \rho^2} \geq 1$$

for all  $\lambda$  and  $\rho$  with  $0 \leq \lambda, \rho \leq 1$  such that condition  $\lambda = 1$  and  $\rho = 1$  is not true.

*Proof:* Start with inequality  $(\lambda - \rho)^2 \geq 0$ , with equality if and only if  $\lambda = \rho$ . This implies  $\lambda^2 - 2\lambda\rho \geq -\rho^2$  which in term yields  $1 + \lambda^2 - 2\lambda\rho \geq 1 - \rho^2$ . Thus if  $\lambda \neq \rho$ , then by above inequality,  $G = \frac{(1 + \lambda^2 - 2\lambda\rho)}{1 - \rho^2} \geq 1$  and Lemma is true. However if  $\lambda = \rho$  and  $0 \leq \rho < 1$ ,  $G = \frac{(1 + \rho^2 - 2\rho^2)}{1 - \rho^2} = 1$  and again Lemma is true. Finally if  $\lambda = \rho$  and  $\rho = 1$ . Clearly for this final case  $G$  is indeterminate. Thus Lemma 1 is proved. The above Lemma is generalized to any positive integer  $J$  in Appendix B.

The above Lemma shows that the new CFAR detector algorithm of a target in two correlated scenes is always better than a CFAR detector in the scene with maximum SNR. Fig. 2 illustrates the probability of detection for a false alarm probability of  $P_{FA} = 10^{-5}$  as a function of  $a'$  for  $J = 1, 2$ ,  $N = 49$ ,  $\lambda = 0.2$  and  $\rho = 0.95$ , i.e. for  $G = 6.77$ . This shows that this detector using 2 scenes with  $(SNR)_2 = \frac{1}{5}(SNR)_1$  has an approximate 8.5 dB SNR improvement over a detector which uses the single scene with maximum SNR.

A comparison of the SNR in Eq. (80-1) is now made with Eq. (46) in [1] for  $K = 1$  the single noise-only reference scene case in [1]. By Eq. (46) in [1] the SNR for a scene with signal, using one reference without signal, is given by

$$a = (\sigma_y^2 / \sigma_{y|x}^2) a' = \frac{a'}{(1 - \rho^2)} \triangleq G a' \quad (81)$$

in terms of  $\rho$  and  $a'$ , the SNR in the scene with signal. By Eq. (80-1) the GSNR depends on  $\rho$ , the correlation coefficient, but also on  $\lambda$ , the ratio of SNR's in both

scenes. It is evident from Eqs. (80-1) and (81), that as long as the inequality  $\frac{\lambda}{2} < \rho < 1$  holds, the gain in SNR in Eq. (80-1) is always less than or equal to the gain in the SNR given in Eq. (81).

Using the correlation coefficient  $\rho = 0.81$ , a computational comparison is made in Fig. 3 with the results given in [1] for  $K = 1$ . These curves in Fig. 3 illustrate that for the same  $\rho$  the probability of detection curves for the new CFAR detector are limited to the probability of detection curves given in [1] for  $K = 1$ . The leftmost curve in Fig. 3 is as derived under the assumption that there is no signal in the reference scene.

Figs. 4(a) and (b) show typical  $32 \times 32$  subimages in two different optical bands ( the green and red bands ) of the San Diego area. A  $5 \times 5$  signal with the pattern, given in Fig. 5, is implanted in both of these green and red images with  $(SNR)_1 = 0.37$  and  $(SNR)_2 = 0.2 \times 0.37$ . A local mean is subtracted from both of these subimages. The resulting residual images are approximately zero-mean Gaussian processes. The CFAR test given in Eq. (30) is calculated for each pixel. The test statistic of this CFAR test is plotted pixel by pixel in Fig. 4(c). A target is detected with a threshold determined by a  $P_{FA} = 10^{-5}$ . The results of this test are illustrated in Fig. 4(d) which shows that the target was, in fact, detected.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. 5 Target signal template

In order to demonstrate the theoretical SNR improvement in Eq. (80-1), a computer simulation was performed to determine the required SNR in the primary scene of maximum SNR needed to detect a target in single primary scene and in  $J = 2$  scenes. A computer simulation was made similar to that developed in [1]. The results of this new CFAR test are shown in Table 1, where  $\lambda = 0.2$ . Using 5 different subimages the average improvement of a detector using  $J = 2$  scenes over the single primary scene is 6.21 dB.

## V. Conclusions:

Under the same assumptions for optical noise clutter used in [1], a generalized constant-false-alarm-rate (CFAR) algorithm is developed for detecting the presence of an optical signal of non-zero intensity in  $J$  signal-plus-noise bands or channels. For many applications this new algorithm is more flexible and practical than the one given in [1]. If  $J = 1$ , the resulting test reduces to the standard normalized matched filter test for finding a signal in clutter of unknown and varying intensity.

Both theoretical and computer simulation results show that the SNR improvement gain of this new algorithm using multiple band scenes over the single scene of maximum SNR is always greater than one and in some cases it can be substantial. A comparison of SNR gain between this new detection algorithm and the one given in [1] illustrates, that for the same correlation coefficient  $\rho$  of related scenes the probability of detection curves for the new CFAR detector are limited to the probability of detection curves given in [1].

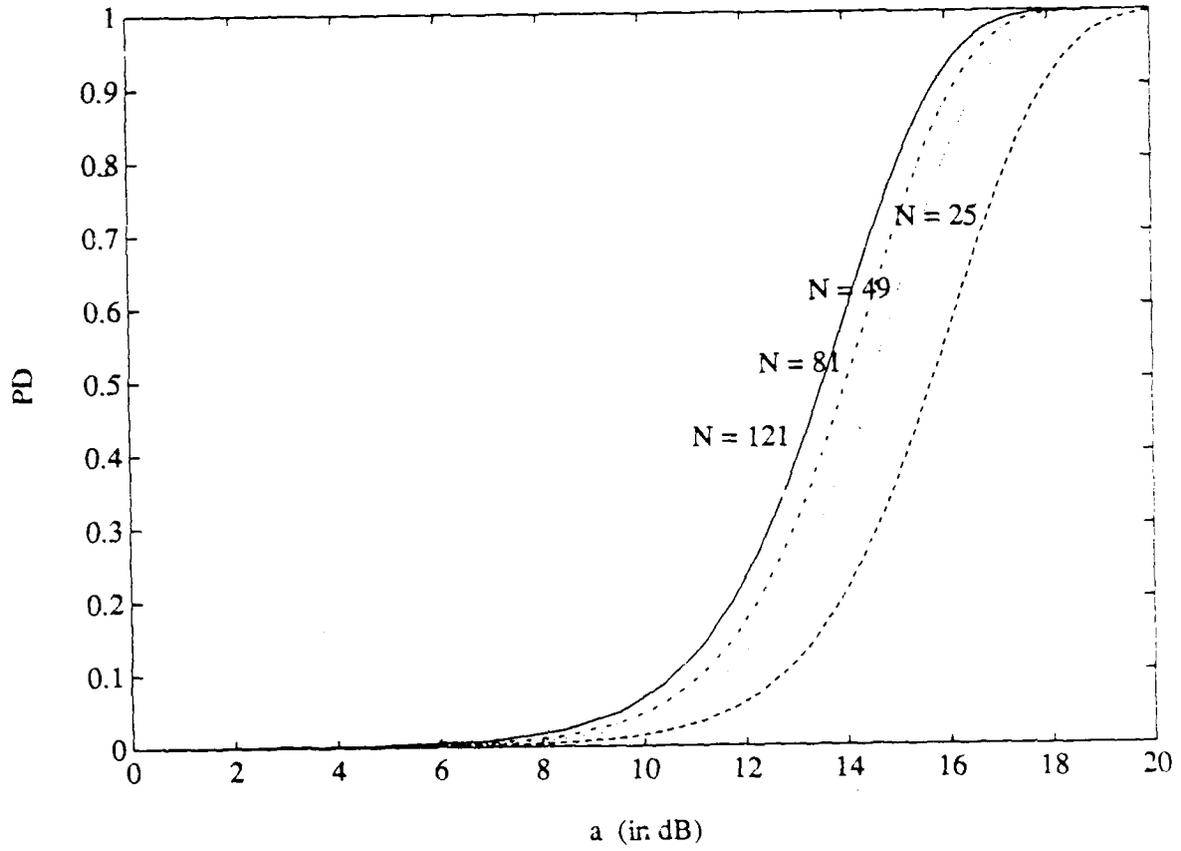


Fig. 1 Probability of detection versus GSNR.  $J=2$ ,  $p_{FA}=10^{-5}$

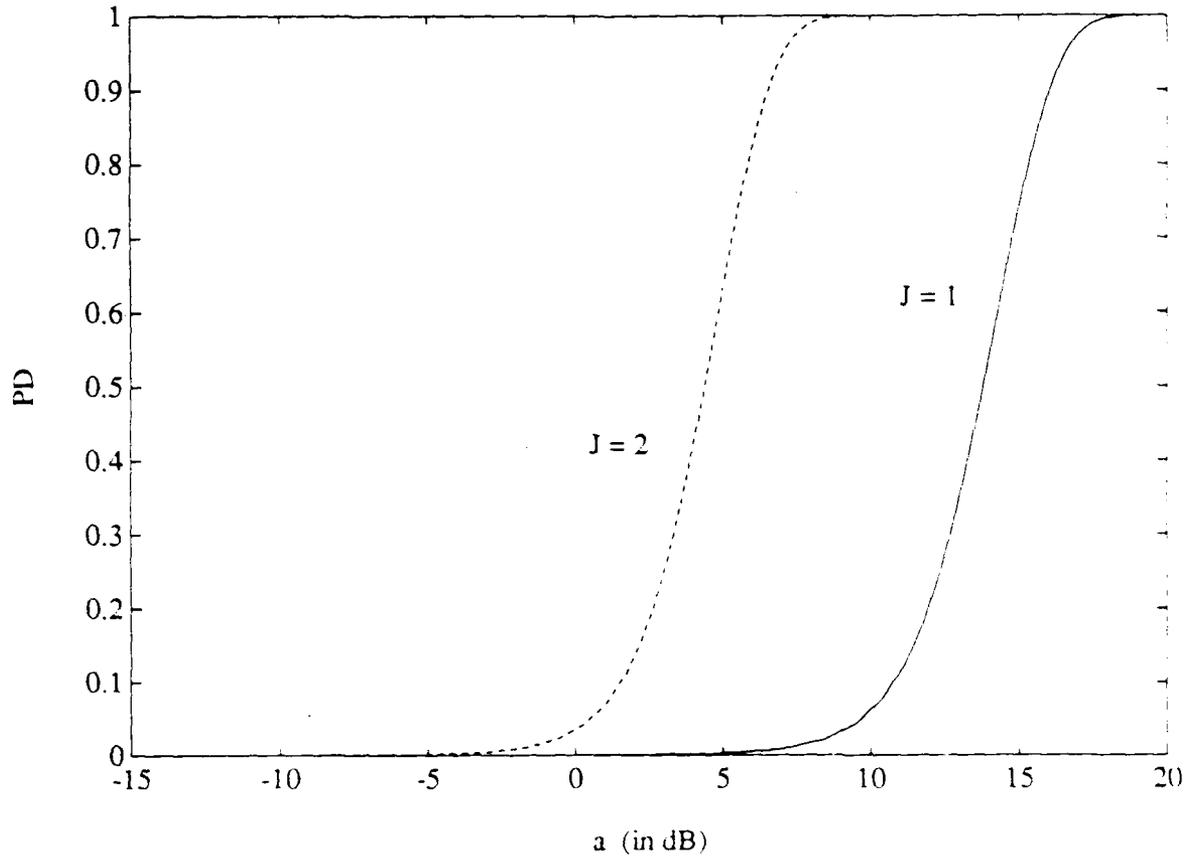


Fig. 2 Probability of detection versus GSNR.  $\rho=0.95$ ,  $p_{FA}=10^{-5}$

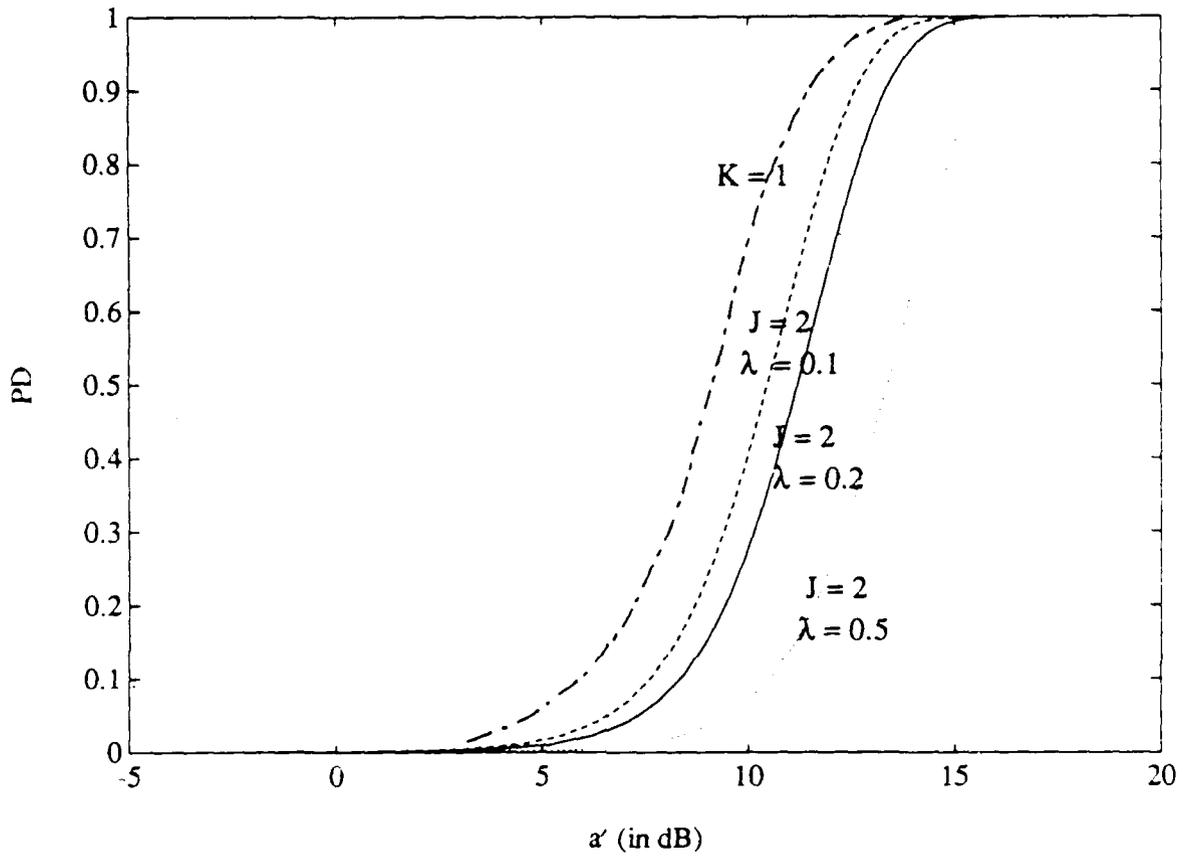


Fig. 3 Probability of detection versus SNR.  $\rho = 0.81$ ,  $P_{FA} = 10^{-5}$



Fig. (4a)

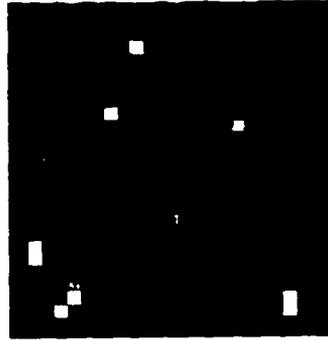


Fig. (4b)



Fig. (4c)

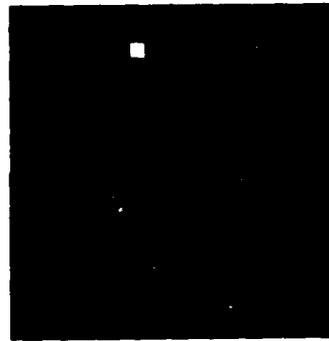


Fig. (4d)

**CFAR Algorithm Simulation Results for the Optical Color Image with  
N = 49 Samples for Covariance Estimation**

Subimage Location	J = 1 (dB)	J = 2 (dB)
( 100, 10 )	6.67	0.96
( 234, 30 )	9.34	2.04
( 50, 310 )	7.32	5.18
( 200, 200 )	2.30	-4.32
( 65, 1 )	3.80	-4.20
Average	5.88	-0.33
Improvement Factor	—	6.21

Table 1

## Appendix A

### Theorem A. 1:

$A$  is nonnegative definite if and only if there exist a matrix  $W$ , such that  $A = WW^T$ . For a proof, see [5, pp 257].

### Theorem A. 2:

The  $J \times J$  matrix  $F_{\underline{b}} \triangleq (X - \underline{b} \underline{S})(X - \underline{b} \underline{S})^T$  in Eq. (16) is positive definite for any arbitrary vector  $\underline{b}$ .

*Proof* : Since

$$\begin{aligned} F_{\underline{b}} &= (X - \underline{b} \underline{S}^T)(X - \underline{b} \underline{S}^T)^T \\ &= WW^T \end{aligned} \quad (\text{A-2})$$

where  $W = (X - \underline{b} \underline{S}^T)$ . By using Theorem A. 1 matrix  $F_{\underline{b}}$  is nonnegative definite, and all the eigenvalues of  $F_{\underline{b}}$  satisfy  $\lambda_j \geq 0$ , for  $j = 1, 2, \dots, J$ . But by previously showing that  $\hat{M}_{\underline{b}}$  were nonsingular,  $F_{\underline{b}}$  is also nonsingular by Eq. (14-2). This means that

$$\det(F_{\underline{b}}) = \lambda_1 \lambda_2 \dots \lambda_J \neq 0,$$

or  $\lambda_j > 0$  for  $j = 1, 2, \dots, J$ . Hence  $F_{\underline{b}}$  is a positive definite matrix. In particular if one chooses  $\underline{b} = \frac{X\underline{S}}{\underline{S}^T \underline{S}}$ , then by Eq. (26)

$$B \triangleq F_{\underline{b} = \frac{X\underline{S}}{\underline{S}^T \underline{S}}} = \frac{XX^T}{\underline{S}^T \underline{S}} - \frac{(X\underline{S})(X\underline{S})^T}{(\underline{S}^T \underline{S})^2} \quad (\text{A-3})$$

is a positive definite matrix.

## Appendix B

In this Appendix, Lemma 1 of Sec. IV is generalized to an arbitrary positive number  $J$ . By Eq. (48) the generalized signal-to-noise ratio (GSNR) for  $J$  bands is given by

$$a = [b_1 \ b_2 \ \dots \ b_J] M^{-1} [b_1 \ b_2 \ \dots \ b_J]^T \|\underline{\Omega}\|^2 \quad (\text{B.1})$$

where

$$M = E \left\{ \begin{bmatrix} x_1 - E x_1 \\ \underline{x} - E \underline{x} \end{bmatrix} \begin{bmatrix} x_1 - E x_1 \\ \underline{x} - E \underline{x} \end{bmatrix}^T \right\} = \begin{bmatrix} K_{x_1 x_1} & K_{x_1 \underline{x}} \\ K_{\underline{x} x_1} & K_{\underline{x} \underline{x}} \end{bmatrix} \quad (\text{B.2})$$

and

$$\underline{x} \triangleq [x_2 \ x_3 \ \dots \ x_J]^T.$$

In terms of the above notation the inverse of  $M$  in Eq. (B.2) can be put in the form,

$$M^{-1} = \begin{bmatrix} K_{x_1 x_1} & K_{x_1 \underline{x}} \\ K_{\underline{x} x_1} & K_{\underline{x} \underline{x}} \end{bmatrix}^{-1} \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (\text{B.3})$$

By the Frobenius relations in [8] or [9], one has the results

$$A = (K_{x_1 x_1} - K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1})^{-1} = \frac{1}{K_{x_1 x_1} - K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1}}, \quad (\text{B.4-1})$$

$$B = -(K_{x_1 x_1} - K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1})^{-1} K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} = \frac{-K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1}}{K_{x_1 x_1} - K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1}}, \quad (\text{B.4-2})$$

$$C = B^T = -(K_{x_1 x_1} - K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1})^{-1} K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1} = \frac{-K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1}}{K_{x_1 x_1} - K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1}}, \quad (\text{B.4-3})$$

$$D = K_{\underline{x} \underline{x}}^{-1} + \frac{K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1} K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1}}{K_{x_1 x_1} - K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1}}. \quad (\text{B.4-4})$$

for the submatrices defined in Eq. (B.3). Hence, one can rewrite Eq. (B.3) as

$$M^{-1} = \frac{1}{\Delta} \begin{bmatrix} 1 & -K_{\underline{x} \underline{x}}^{-1} K_{x_1 \underline{x}} \\ -K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1} & K_{\underline{x} \underline{x}}^{-1} \Delta + K_{\underline{x} \underline{x}}^{-1} K_{\underline{x} x_1} K_{x_1 \underline{x}} K_{\underline{x} \underline{x}}^{-1} \end{bmatrix}. \quad (\text{B.5})$$

where

$$\Delta \triangleq (K_{x_1 x_1} - K_{x_1 \bar{x}} K_{\bar{x} \bar{x}}^{-1} K_{\bar{x} x_1}). \quad (\text{B.6})$$

Note that  $\Delta > 0$  since  $M$  is the positive definite matrix ( see Sec. I Eq. (3). Next by a substitution of Eq. (B.5) into Eq. (B.1), one obtains

$$\begin{aligned} a &= [b_1 \ b_2 \ \dots \ b_J] M^{-1} [b_1 \ b_2 \ \dots \ b_J]^T \|\underline{\Omega}\|^2 \\ &= [b_1 \ \underline{b}^T] M^{-1} [b_1 \ \underline{b}^T]^T \|\underline{\Omega}\|^2 \\ &= \frac{1}{\Delta} [b_1^2 - b_1 \underline{b}^T K_{\bar{x} \bar{x}}^{-1} K_{\bar{x} x_1} - K_{x_1 \bar{x}} K_{\bar{x} \bar{x}}^{-1} \underline{b} b_1 + \underline{b}^T K_{\bar{x} \bar{x}}^{-1} \underline{b} \Delta + \underline{b}^T K_{\bar{x} \bar{x}}^{-1} K_{\bar{x} x_1} K_{x_1 \bar{x}} K_{\bar{x} \bar{x}}^{-1} \underline{b}] \|\underline{\Omega}\|^2, \end{aligned} \quad (\text{B.7})$$

where

$$\underline{b} \triangleq [b_2, b_3, \dots, b_J]^T.$$

In Eq. (B.7) define the general normalized correlate coefficient

$$\underline{\rho} \triangleq K_{\bar{x} \bar{x}}^{-1/2} K_{x_1 \bar{x}}^{-1/2} K_{\bar{x} x_1} \quad (\text{B.8})$$

and the ratio

$$\underline{\lambda}^T \underline{\lambda} \triangleq \frac{\underline{b}^T K_{\bar{x} \bar{x}}^{-1} \underline{b}}{K_{x_1 x_1}^{-1} b_1^2}. \quad (\text{B.9})$$

of the GSNR in the other scenes to SNR in the primary scene of maximum SNR.

Then Eq. (B.7) becomes

$$a = G a' \quad (\text{B.10})$$

where  $G$  is the generalized gain, given by

$$G = \frac{1 - 2\underline{\lambda}^T \underline{\rho} + \underline{\lambda}^T \underline{\lambda} (1 - \underline{\rho}^T \underline{\rho}) + (\underline{\lambda}^T \underline{\rho})^2}{1 - \underline{\rho}^T \underline{\rho}}. \quad (\text{B.11})$$

Note by Eqs. (B.6) and (B.8) and the comment following Eq. (B.6) that  $\underline{\rho}^T \underline{\rho} > 1$ .

**Lemma :** Let  $G$  be the generalized gain function defined in Eq. (B.10), then  $G \geq 1$ .

*proof :* Note first that

$$\begin{aligned} G \geq 1 \text{ iff } 1 - 2\lambda^T \rho + \lambda^T \lambda (1 - \rho^T \rho) + (\lambda^T \rho)^2 &\geq 1 - \rho^T \rho \\ \text{iff } \rho^T \rho - 2\lambda^T \rho + \lambda^T \lambda (1 - \rho^T \rho) + (\lambda^T \rho)^2 &\geq 0 \end{aligned} \quad (\text{B.12})$$

where " iff " denoted " if and only if ", the logical equivalence. Next let

$$\rho^T \rho = \alpha^2 \quad (\text{B.13-1})$$

$$\lambda^T \lambda = \beta^2 \quad (\text{B.13-2})$$

$$\lambda^T \rho = \alpha \beta \cos \gamma \quad (\text{B.13-3})$$

then using (B.13-1) to (B.13-3), the last statement in (B.12) becomes

$$G \geq 1 \text{ iff } \alpha^2 - 2\alpha\beta\cos\gamma + \beta^2(1 - \alpha^2) + \alpha^2\beta^2\cos^2\gamma \geq 0. \quad (\text{B.14})$$

There are two cases to consider.

case I : If  $\beta^2 \geq 1$ , then since  $1 - \rho^T \rho > 0$ , one has

$$\begin{aligned} \alpha^2 - 2\alpha\beta\cos\gamma + \beta^2(1 - \alpha^2) + \alpha^2\beta^2\cos^2\gamma &\geq \alpha^2 - 2\alpha\beta\cos\gamma + (1 - \alpha^2) + \alpha^2\beta^2\cos^2\gamma \\ &= 1 - 2\alpha\beta\cos\gamma + \alpha^2\beta^2\cos^2\gamma = (1 - \alpha\beta\cos\gamma)^2 \geq 0. \end{aligned} \quad (\text{B.15})$$

case II : If  $\beta^2 < 1$ , then

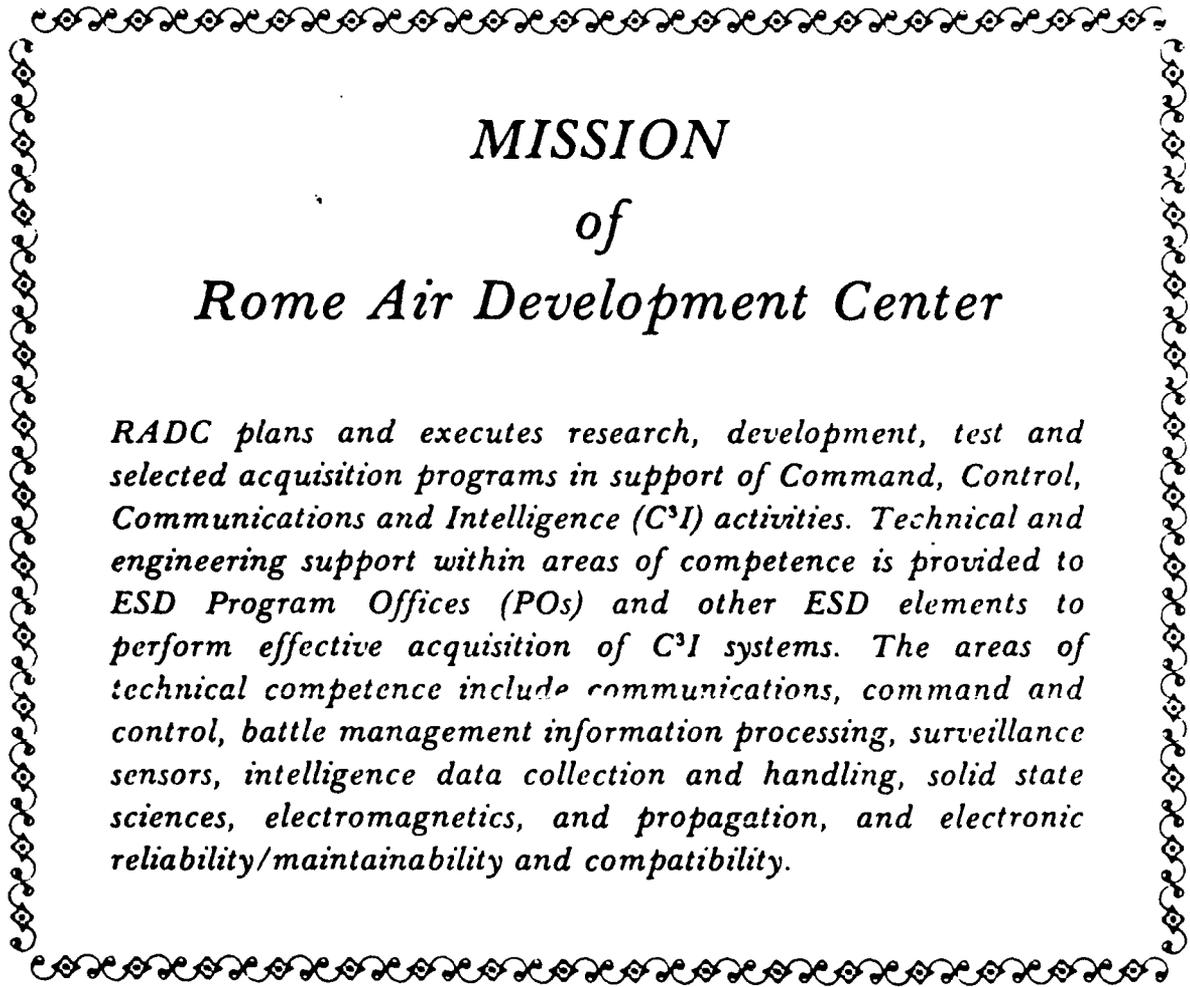
$$\begin{aligned} \alpha^2 - 2\alpha\beta\cos\gamma + \beta^2(1 - \alpha^2) + \alpha^2\beta^2\cos^2\gamma &= \alpha^2 - 2\alpha\beta\cos\gamma + \beta^2 - \alpha^2\beta^2(1 - \cos^2\gamma) \\ &\geq \alpha^2 - 2\alpha\beta\cos\gamma + \beta^2 - \alpha^2(1 - \cos^2\gamma) \\ &= (\alpha\cos\gamma - \beta)^2 \geq 0. \end{aligned} \quad (\text{B.16})$$

Thus  $G \geq 1$  and Lemma is proved.

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