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DYNAMICS OF SPACE VEHICLES AND SPACE RESEARCH

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Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*<u>ye</u> initially, after vowels, and after ъ, ь; <u>e</u> elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

GRAPHICS DISCLAIMER

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DYNAMICS OF SPACE VEHICLES AND SPACE RESEARCH.

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Page 4.

Collection of articles is dedicated to memory of prominent Soviet scientist, Lenin prize winner, Dr. of physico-mathematical sciences G. S. Narimanov. Are included four thematic sections: space research, libration motion of space vehicles (KA) with liquid during small and sizable oscillations of its free surface, rotational motion of KA with the liquid. In the collection the contemporary analytical and numerical methods of the solution of the problems of the dynamics of the complex systems are reflected, including the methods of machine graphics, and also experimental.

For scientific workers, who carry out questions of rocket-space and aviation equipment, body being deformed, and also calculating methods of mathematical physics.

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GEORGIY STEPANOVICH NARIMANOV.

In service record of Georgiy Stepanovich is counted teaching in department of applied mechanics of Moscow University, where our joint operation occurred, together with many other forms of activity.

Georgiy Stepanovich in many respects contributed to development of research, useful for practice, and gained love and respect of both associates on department and graduate students and students. His personal scientific results on the theory of the joint oscillations of solid bodies and liquids filling them proved to be the valuable contribution to mechanics of the systems being deformed and rocket engineering.

I had the occasion repeatedly to encounter and frequently to work together with Georgiy Stepanovich in now already distant times of first steps of cosmonautics. Occurred the difficult days of failures. And then especially were manifested the personal delay of Georgiy Stepanovich, the skill to be dismantled/selected at the reasons, which determine the undesirable course of events, and to plan the ways of their elimination.

Georgiy Stepanovich Narimanov was keen expert of mechanics. Yes even not only mechanics, but also space technology as a whole. One cannot fail to say about his devotion to the great ideals of our society, understanding of human relations and values of culture, personal modesty, kindness, in a word - high culture.

I am grateful to fate for acquaintance with this remarkable person.

Academician A. Yu. Ishlinskiy.

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PREFACE.

Thematic collection "Dynamics of space vehicles and space research: it is dedicated to memory of well-known Soviet scientist, who made noticeable contribution to development of cosmonautics, Lenin prize winner, Doctor of Physics and Mathematics, professor Georgiy Stepanovich Narimanov (1922-1983).

Collection is opened by introductory article, which contains short survey/coverage of multifaceted activity of G. S. Narimanov, and consists of four parts, which correspond to main scientific directions in his creativity, which continue to develop his students and followers.

Each part, except the first, begins from determining its thematics basic work of G. S. Narimanov, published earlier, and includes articles of other authors, connected with this thematics. These works - the result of research of the Moscow, Kiev and Tomsk scientists, who rightfully can be related to the scientific school of G. S. Narimanov.

First part contains works of general character, which relate to space research with the help of space vehicles, and also to some mechanical and physical aspects of structure and evolution of solar system - one of subjects of this research.

Second part is dedicated to linear problems of dynamics of flight vehicles, which have sections, partially filled with incompressible fluid. In the specific frequency band the adequate model of these objects is solid, absolutely rigid body with the cavity, which contains liquid with the free surface. It is assumed that the field of the mass forces of the undisturbed motion can, in the first approximation, be considered potential (although unsteady), and liquid - ideal (its motion, as a result, irrotational). As the disturbed motion the low oscillations of body and liquid are examined (low deflections from the undisturbed state).

Specifically, in this setting this problem was in its time was formulated and solved by G. S. Narimanov.

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Articles, included in second part, contain series of original results, which expand limits of the applicability of initial mathematical model of G. S. Narimanov (in particular, along line of phenomenological account of eddies of low-viscosity liquid and elasticity of walls of tank and housing of space vehicle during longitudinal vibrations), and which also relate to methods of calculation of parameters of corresponding mathematical models. The results, obtained during the numerical application of these methods and during the experimental research are given.

Third part is dedicated to nonlinear problems of dynamics of the same objects; basic nonlinearity is caused by the fact that deflections of free surface of liquid from undisturbed state are "sizable". It is here represented as the new conclusion/output of the nonlinear equations of G. S. Narimanov from the variation principle, which makes it possible to construct the structural/design algorithm of the calculation of their coefficients for the cavities of the rotation of arbitrary configuration, and also the theoretical and experimental perturbation analyses of body and liquid.

Last, one fourth of collection is dedicated to rotational motions of axisymmetric flight vehicles with sections, which contain liquid. For the undisturbed motion is accepted the "rapid" quasi-stationary rotation of flight vehicle or solid body simulating it relative to longitudinal axis.

In G. S. Narimanov's article, which opens/discloses this part, concept of theory of long waves (fine/small liquid) is used and it is assumed that cavity is cylindrical. In the subsequent articles, together with this case, is examined opposite - cavity, wholly filled with liquid.

Are given some results of theoretical and experimental analyses of stability of rapidly revolving body with liquid filling, which can be considered as model of space vehicle, stabilized by rotation.

Thus, collection is essentially "problematically oriented" to those problems of space research and dynamics of space vehicles, into solution of which G. S. Narimanov introduced considerable personal creative contribution.

Editorial board and entire collective of writers hope that this collection will contribute to further development of actual directions, connected with scientific heritage of G. S. Narimanov, and to solution of new applied problems, which lie on forward edge of space technology.

Responsible editor is Hero of Socialist Labor, doctor of technical sciences, professor.

G. A. Tyulin.

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BRIEF OUTLINE OF SCIENTIFIC ACTIVITY OF G. S. NARIMANOV.

G. A. Tyulin.

Is given short survey/coverage of scientific activity of prominent Soviet scientific G. S. Narimanov in region of theory of flight and dynamics of carrier rockets (RN) and space vehicles (KA). Are in more detail examined the basic work of G. S. Narimanov in the dynamics of solid body, which contains the sections, partially filled with the ideal incompressible fluid, that is model of RN and KA with ZhRD [ЖРД - liquid propellant rocket engine].

Activity of Lenin prize winner, Doctor of Physics and Mathematics Georgiy Stepanovich Narimanov can be represented in the form of following directions: first, these are scientific research in region of mechanics of flight of carrier rockets (RN) and space vehicles (KA), dynamics of systems, which contain being deformed in process of motion elements (liquid, elastic), general problems of space research; in the second place, scientific organizational work in field of cosmonautics; in the third, pedagogical activity.

Speaking about first direction, should be stressed special role of development of adequate mathematical models of RN and KA, which consider mobility of liquid in tanks and elasticity of housing, in scientific research of G. S. Narimanov. Research of the dynamics of solid bodies with the liquid filling, simulating RN and KA with ZhRD in the powered flight, carried pioneer character and had high value in the solution of the practical problems of stability and automatic stabilization of RN and KA. Hundreds of works, dedicated to such research [1], are published during the recent three decades.

FOOTNOTE [1]. The approximate representation about this can be composed on the bibliography, given in work [9]. ENDFOOTNOTE.

Author did not place to himself to entirely compare results, obtained by G. S. Narimanov and other authors (many authors, especially foreign, worked in parallel and independently). Will here deal the discussion only with the work of G. S. Narimanov himself, moreover in essence about those, which carry priority character.

First research of G. S. Narimanov in field of dynamics of bodies with liquid filling was carried out in 1950-1951 and connected with solution of problems of developing of rocket-space technology. Reflecting about the unsuccessful attempts to agree on some special features of the dynamic behavior of objects, which were being observed during the flight tests, with the mathematical description of the disturbed motion of these objects as absolute solids of variable mass, Narimanov arrived at the conclusion that the reason for these failures lies in the inadequacy of the utilized for the analysis mathematical model of RN as solid body.

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Specifically, then he posed the problem about the review of the conceptual basis of research of rocket dynamics in the powered flights. In 1951 he successfully solved it, after proposing the first internally matched mathematical model of flight vehicles with ZhRD, in which was considered within the framework of the concept of flat/plane free surface the mobility of liquid ("floating mass-free rigid cover/cap").

This model made it possible not only to match results of mathematical simulation with flight test data, but also to obtain sufficiently good coincidence of a priori and a posteriori dynamic characteristics.

In 1951 G. S. Narimanov obtained more complete mathematical model of three-dimensional/space disturbance of motion of solid body with cavity, partially filled with ideal incompressible fluid, in potential variable field of mass forces with low deviations of all generalized coordinates and speeds from appropriate quiescent values under precise (linearized) boundary free-surface conditions of liquid.

In this work, published in 1956 [3], G. S. Narimanov, after introducing certain hypothetical solid body with liquid, formulated basic assumptions, which ensue from character of undisturbed and disturbed motions in powered flight and special features of layout and construction/design of stabilized flight vehicles with ZhRD, which remained virtually constant to this day. This determined the wide application of a mathematical model, proposed by G. S. Narimanov, during the solution of the applied problems of rocket-space technology.

Let us enumerate most important results, published by G. S. Narimanov in work [3] and having fundamental character: obtaining common mathematical model of three-dimensional/space disturbed motion of system body - liquid (linear approximation/approach) in the form of system of ordinary differential equations of infinitely high order; proof of existence and uniqueness of solution of this system of equations; proof of applicability of method of its reduction to system of finite order; obtaining by method of separation of variables of structural/design algorithm of calculation of hydrodynamic coefficients in the case of cylindrical cavities; demonstration of efficiency of application of methods of operational calculus for solving wide circle of problems of dynamics of bodies with liquid filling in the case of stationary field of mass forces of undisturbed motion; solution of two important model problems, which relate to plane-parallel motion of body with one (progressive/forward or rotary) degree of freedom.

Following cycle of work of G. S. Narimanov on this thematics was dedicated to problem of nonlinear vibrations of liquid in mobile cylindrical sections, i.e., to generalization of mathematical model [3] to case of "sizable" oscillations of free surface of liquid.

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For author it was possible to work out efficient method of synthesis of mathematical model of systems body - liquid on basis of reducing of nonlinear boundary-value problem to sequence of linear boundary-value problems. This idea was in detail developed in work [4], which pertains to the year 1957, in which were given the complete system of equations of the three-dimensional/space disturbed motion and the algorithm of the calculation of its coefficients, and also the numerical values of coefficients for the cavity in the form of straight/direct circular cylinder.

In the same year left work G. S. Narimanov [5], in which author, using equations, obtained in work [4], explained whole series of substantially nonlinear effects, which were being observed during flat/plane harmonic oscillations of body in vicinity of major resonance, which are accompanied by "sizable" oscillations of free surface of liquid: limitedness of amplitude in region of major resonance, bias/displacement of resonance frequency of fundamental unsymmetric harmonic to side of lower frequencies, i.e., "soft nonlinearity", asymmetry of protuberance and indentation in maximum wave in the presence of resonance (height of protuberance 1.5 and even 2 times more than depth), bias/displacement of nodal curve from center-line plane to side of protuberance. The same work gives the evaluations of the limits of the applicability of linear theory according to the frequency of the induced harmonic oscillations of body with the liquid.

In 1958 leaves first and only in its kind monograph of G. S. Narimanov [7], dedicated to the problems of the dynamics of bodies with the liquid filling (model of RN or KA with the rigid housing) and which includes also chapter about the equations of the flat/plane disturbed motion of the elastic crux of variable/alternating cross section with the free ends/leads (model of elastic housing of RN with the hardened liquid during the motion in one of the stabilization planes). In the book were included the basic results on the problem in question, obtained by that time by the author and which retain validity in their majority on the present time. Into it the expanded and substantially supplemented materials of works entered [3...5]. The main things from these additions are: the solution of boundary-value problems and the calculation of all hydrodynamic coefficients for the cylindrical cavities with the cross sections in the form of circle, rectangle, circular sector; the solution of the model problem about the plane-parallel motion of body with the liquid with two degrees of freedom in the presence of the servo force (model of thrust of ZhRD) - demonstration of emergence with some values of the parameters of the dynamic instability of system, which is impossible in the case of the hardened liquid; qualitative description, on the basis of nonlinear equations, the new experimentally observed effect - rotation of the free surface of liquid during resonance oscillations with the excess of a certain critical amplitude of oscillations.

In 1957 appeared besides [5] one additional original work of G. S. Narimanov [6], dedicated to class of problems of dynamics of solid bodies with cavities, which contain liquid, namely - to disturbed motion of rapidly revolving body with liquid.

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The author considered in this work the task about the symmetrical gyroscope with the partially filled with liquid cavity in the form of the circular cylinder, whose axis coincides with the axis of gyroscope. Using with the large art the hypothesis of the theory of long waves (i.e. "shallow water"), the author obtained the closed system of equations of the disturbed motion of gyroscope with the liquid, close in structure to the system of equations, given in work [3].

Infinite system of ordinary differential equations obtained in work [6] also allows/assumes application of method of reduction, which makes it extremely attractive for solving applied problems. In particular, the role of gyroscope with the liquid can play the stabilized by rotation space vehicle, furnished control motors, whose dynamics is completely described by the equations, given in work [6]. By this work G. S. Narimanov knew how to move the bridge between the tasks, connected with the "libration" motions of objects with the liquid aboard and the "rotational" motions.

Work of G. S. Narimanov in region of special sections of mechanics of systems (in connection with tasks of cosmonautics) being deformed caused to life vast cycle of research both general theoretical [2] and applied character.

As a result, by works of many authors, among whom there were numerous students and followers of G. S. Narimanov, was worked out completed theory, which made it possible not only to explain dynamic special features of behavior of objects with liquid filling, that are exhibited in flight, but also create reliable theoretical basis for design of stabilization systems of these objects, on the basis of assigned requirements for factors of stability of systems housing - liquid - automatic machine of stabilization.

Theory relating to "sizable" oscillations of liquid in tanks, proved to be especially fruitful during solution of problems of dynamics of KA in sections of motion with low thrust, stage separation, correction of orbit, landing on planets and series/row of other complex problems. The tasks, connected with the rotational motion of KA, gained special urgency in recent years in connection with the wide acceptance of the apparatuses, stabilized by rotation, with the periodic correction of rotational axis in the direction in the sun (for guaranteeing the normal mode of the work of solar batteries).

In 1977 new monograph [9], written by G. S. Narimanov together with L. V. Dokuchayev and I. A. Lukovskiy, left. In it the early work of Narimanov found further development, and the long-term investigations of all three authors, which were the continuation of these works, were also reflected.

Work [9] in detail examines not only axisymmetric cylindrical cavities, but also axisymmetric cavities with arbitrary piecewise-smooth contour of diametric cross section. Thus, it was for the first time taken into consideration in the nonlinear equations of the disturbed motion of system body - liquid "geometric nonlinearity".

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Is further presented variational method of solving the linear boundary-value problems, to sequence of which is reduced as in work [4], initial nonlinear boundary-value problem for the axisymmetric cavity of arbitrary configuration. Are given the results of solving the boundary-value problems, in particular the numerical values of coefficients for the cylindrical, conical, spherical, ellipsoidal, parabolic cavities of rotation.

In book are in detail examined problems of stability of steady states of "sizable" oscillations of free surface of liquid during induced harmonic progressive/forward and angular oscillations of body with cavity, partially filled with liquid. The stability regions and instability of plane and circular waves with respect to frequency and amplitude of forced oscillations of the free surface of liquid are for the first time strictly obtained, i.e., the program, planned in work [7], is completely realized.

Carried out comparison with similar results, obtained with the help of mechanical analog in the form of spherical pendulum, and is given evaluation of damping effect on amplitude-frequency system characteristics in region of major resonance.

Scientific interests of G. S. Narimanov contained many problems of space research and mechanics of space flight: external ballistics, including experimental (determination of trajectory of KA and law of its motion on basis of results of measurements, consumption/production of requirements for composition and accuracy of measurements); theory of automatic control of RN and KA; nonclassical problems of stability of motion (stability of essentially unsteady undisturbed motion, stability when external disturbances/perturbations of complex spectral composition, etc. are present).

G. S. Narimanov succeeded in, in particular, working out original method of integrating linear ordinary differential equations with variable coefficients, which was efficiently used for solving series of problems of dynamics of RN and KA.

Although many works of G. S. Narimanov, unfortunately, remained unpublished, certain representation about range of problems, which were many years in his field of view, give this fundamental labor, which left under its editorship (together with M. K. Tikhonravov) as [11].

In popular outline [8] G. S. Narimanov gives review of achievements and prospects for development of cosmonautics (1981).

On latter/last work of G. S. Narimanov, connected with space research with the help of automatically controlled KA, they give representation of work of collectives of authors with his participation [1, 12].

G. S. Narimanov paid considerable attention tc teaching activity, result of which was creation of scientific school, whose many representatives grew into serious independent scientists.

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The large order of Soviet engineers and scientists, who work in the area of rocket-space technology, learned on the works of G. S. Narimanov; they all also rightfully can be related to the representatives of this school. Some of them are the authors of the articles, included in present collection.

Speaking about special features of scientific style of G. S. Narimanov, it is necessary to stress clear understanding and broad coverage of problem, skill to reduce complex engineering problem to strictly formalized problem of mechanics, freshness of ideas and free possession of entire contemporary apparatus of analytical and numerical mathematical methods. Its works are characterized by the combination of the high strictness of the solution and presentation with the practical directivity and the applied character of research itself, in which is always outlined connection with the specific problems of the theory of flight and dynamics of RN and KA. This connection is reflected both in the initial axiomatics and in the form of the representation of final results, in necessary obtaining of reliable numerical evaluations and in the selection of sufficiently representative model problems.

One of parts, which characterize scientific honesty of G. S. Narimanov, was his constant tendency to subject obtained results to comprehensive checking both for internal coordination (conformity to laws of conservation, symmetry or antisymmetry of matrices, etc.) and external (coincidence with exact solutions when latter/last is known, conformity to data of experiments on physically similar models, and also to results of full-scale, including flight, tests). These features of his style of scientific work G. S. Narimanov knew how to inculcate both in his direct students and representatives of his scientific school.

In conclusion one should say that G. S. Narimanov indisputably inscribed by his research bright page in history of Soviet mechanics and, in particular, mechanics of space flight.

Works of G. S. Narimanov, placed in present collection, give sufficient representation about their author as about talented scientist, who placed bases of new important directions of applied research in cosmonautics and theories of flight RNIKA.

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SPACE RESEARCH.

G. S. NARIMANOV AND HIS CONTRIBUTION TO THE DEVELOPMENT OF COSMONAUTICS.

B. A. Pokrovskiy.

Brief biographical information about G. S. Narimanov is given. Are examined the shaping directions of activity of G. S. Narimanov in the field of cosmonautics: the solution of the series of the complex scientific-technical problems, connected with navigational-ballistic support and the flight control of space vehicles, the creation of the series of rocket-space systems and control-measuring complex; work in the boards for the launching of space vehicles, including according to the programs of international collaboration, and also the propaganda of achievements and prospects for cosmonautics.

In 1983 died prominent Soviet scientist - Lenin prize winner, Doctor of Physics and Mathematics, professor Georgiy Stepanovich Narimanov. His name, scientific and organizational activity is well known in the wide circles of Soviet and foreign scientists and specialists in the region of rocket-space technology and space research. He is awarded many government rewards.

G. S. Narimanov was born on 13 February, 1922, in Tbilisi city.

Soon family moved to Moscow. In 1939 Narimanov finished the Moscow secondary school No 110, successfully put entrance examinations and he was accepted to the physics department of the Moscow State University im. M. V. Lomonosov. But war broke studies in 1941; G. S. Narimanov refused to be evacuated with the university into the rear, he went by volunteer to the construction of defensive reinforcements first on the distant, and then on the near approaches to Moscow.

After crushing defeat of Fascist-German troops in the environs of Moscow its many defenders from number of student youth were directed to schools of higher education. Among them proved to be G. S. Narimanov. In 1948 he finished the military engineering air academy of the name of prof. N. Ye. Zhukovskiy and was directed toward the work to one of the scientific research institutes, created after war. This was the time of the rapid development of rocket engineering. Questions of the dynamics of the motion of carrier rockets and space vehicles, navigational trajectory consideration of their flight and experimental ballistics were investigated in one of the leading divisions of this institute, to which was directed young specialist G. S. Narimanov.

He successfully combined work in division with studies in mechanicomathematical department of Moscow State University, which finished "with difference" in 1950. The intimate knowledge of mathematics, mechanics and physics, natural abilities and diligence, skill freely to be oriented in complicated scientific and technical questions helped G. S. Narimanov rapidly to enter into the number of chief/leading colleagues of institute. In incomplete 27 years he was already the manager of laboratory.

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In 1952 he successfully defended the candidate dissertation, which was characterized by the depth of the conducted investigations and played large role in development of one of the new directions of applied mechanics. Already this work characterized G. S. Narimanov as the completely formed independent scientist and highly skilled specialist. In 1953 he was appointed as the leader of division, and then - the assistant of the director of institute. In these years, leading the number of comprehensive scientific research, Georgiy Stepanovich entirely considered the possibilities of calculating mathematics and electronic computing technology and much he did for the practical realization of these possibilities in the solution of the scientific and applied problems of rocket-space technology.

In period of direct preparation/training for launching of the first in the world Soviet artificial Earth satellites institute dealt with whole series of theoretical problems and practical problems in dynamics of flight of RN and KA and control of their flight. In the solution of many of them G. S. Narimanov participated. Together with the doctor of technical sciences P. Ye. El'yasberg he dealt with the development of the ballistic proof of the arrangement/position of measuring means in the territory of the Soviet Union, with determination and forecasting the parameters of orbits of space vehicles from the results of trajectory measurements; he participated in the creation of control-measuring complex (KIK) for control of their flight.

Points/items of tracking of set of measuring means, connection/communication and single time, that entered into KIK, were placed in territory of USSR in such a way as by its zones of radio visibility to maximum degree "to overlap" space, in which were planned flights of first artificial Earth satellites. One should stress that the arrangement/position of measuring points/items proved to be so deeply substantiated that they thus already almost three decades efficiently work in their initial locations.

Worked out under management and with direct participation of G. S. Narimanov programs and methods of navigational-ballistic support successfully were used long years during flight control of artificial Earth satellites, manned ships and automatic interplanetary space stations; they were constantly improved by his students and followers and successfully are used now in space research.

Sizable contribution to development of cosmonautics was introduced by G. S. Narimanov, also, after he occupied post of deputy chairman of one of branch scientific and technical councils, where he fruitfully worked in 1965-1971. Georgiy Stepanovich was closely related to the scientific research and experimental design works of a whole series of NII [НИИ - Scientific Research Institute] and KB in the field of the creation of new promising space and control-measuring systems. Putting into operation these systems made it possible to do a new considerable step on the path of further study and mastery/adoption of outer space.

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Should be especially noted the delivery/procurement to the earth of lunar soil during September 1970 by station "Luna-16", worked out under guidance of well-known Soviet designer of space automatic machines G. N. Babakin; many-month work on the surface of the Moon of the controlled from the Earth self-propelled apparatus - "Lunokhod-1"; prolonged test flight in the automatic and manned modes of the first in the history of cosmonautics permanent orbiting scientific space station "Salyut". Into the realization of all these programs G. S. Narimanov introduced a weighty contribution.

In period from 1971 through 1983 Soviet cosmonautics achieved new borders in further expansion and deepening of study of universe and mastery/adoption of near-earth space in peaceful purposes. The multifunctional space vehicles of scientific and applied designation/purpose were created; was created the series/row of the space systems, such as the state system of communications and television of the USSR, meteorological system; was realized assembly in orbits of two, and then of three manned and automatic apparatuses of large scientific research space complexes. The same time is characterized by further broadening and deepening of the international collaboration of the USSR with the fraternal socialist countries, and also with other states in study and mastery/adoption of space.

G. S. Narimanov's activity was multifaceted: guidance of scientific research work, participation in boards for launching of space vehicles, scientific-organizational work within the framework of programs of international collaboration, pedagogical activity and propaganda of achievements and prospects for Soviet cosmonautics. However, everything was subordinated to single target - further development of cosmonautics.

Being assistant of director of Institute of Space Research AS USSR for scientific work Georgiy Stepanovich headed series of major scientific research projects, many of which had not only important theoretical, but also high applied value for development of cosmonautics.

In the same years G. S. Narimanov directly participated in work of boards for launching of space vehicles of scientific designation/purpose (he was chairman of many boards). As is known, in preparation/training, launching, the flight control of space vehicles and in information processing, obtained with KA, participate the numerous collectives of scientists, testers and other specialists of spaceports, Mission Control Center, information-computing, control-measuring and search and rescue complexes, and also many scientific research and experimental-design organizations. The coordination of the work of all these organizations realizes boards for launching KA. To the success of the work of boards and, consequently, also very space experiments in the sizable degree contributed the organizational abilities of G. S. Narimanov, his intimate and comprehensive knowledge in many fields of cosmonautics and large experience of guidance of important collectives.

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Under guidance of boards with G. S. Narimanov's participation were realized successful launches of artificial Earth satellites of series "Kosmos", high-apogean - "Prognoz", automatic universal orbital stations, research rockets "Vertikal'" - in all more than sixty of these and other KA.

High scientific and applied value of space physics determines the important place, which it occupies in space research, and in particular, according to programs of international collaboration. Professor Narimanov took direct part in their development and realization. He also took active part in the work of council "Interkosmos" for the Academy of Sciences of the USSR from the first days of the creation of this international organization for study and mastery/adoption of outer space in the peaceful purposes.

In 1967-1980 Georgiy Stepanovich headed Soviet part of working group in space physics in this international organization and actively worked in boards for starting/launching of space vehicles within the framework of program of "Interkosmos".

By scientists and by specialists of socialist countries is carried out wide circle of research and experiments in interests of science and national economy, including in course of flights of representatives of nine fraternal countries with Soviet cosmonauts on orbital complex of "Salyut-6" - "Soyuz ", on board which was established/installed equipment, worked out by scientists of these states. G. S. Narimanov introduced large contribution to the organization of the first joint flights according to the program of "Interkosmos" of Soviet cosmonauts and cosmonauts from Czechoslovakia, GDR, Bulgaria, Hungary.

G. S. Narimanov returned numerous energy, experience and knowledge to noble/precious matter of international collaboration in study and use of outer space in peaceful purposes.

He participated in development and realization of series/row of international scientific programs. In 1966-1970 he headed the Soviet part of the Franco-Soviet joint working group on the space research. He participated in the work of boards for the launching of the series of Indian and French satellites by Soviet carrier rockets. He took part in the realization in 1975 of the joint flight of the Soviet "Soyuz" spacecraft and American "Apollo". In 1980 he worked in the research group of international astronautical federation (MAF) in

France on the problems of the use of geostationary orbits for the output on them of ISZ [ИСЗ - artificial earth satellite] of different designation/purpose. In the composition of Soviet delegations he participated in the XXIX and XXX congresses of MAF in 1978 in Yugoslavia and in 1979 - in the FRG.

Teaching activity occupied large place in G. S. Narimanov's life. The distinct knowledge of object/subject, depth, clarity and sequence of presentation and active creative contact with the audience - here are those characteristic features of G. S. Narimanov as teacher, which remember his former students and graduate students.

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It is difficult to overestimate the role, which played his lectures, written to them the articles and the books in training of the whole generation of research engineers, who work in the most varied areas of rocket-space technology.

Propaganda of achievements and prospects for development of cosmonautics is closely related to teaching. Besides scientific works personally G. S. Narimanov and in the co-authorship with other specialists wrote the series/row of the popular science books and brochures, addressed to the wide circle of readers. Among them let us note the small by the volume, but very informative brochure "Achievements and the prospects for cosmonautics", which in the beginning of the 80's was essential help to lecturers and to the propagandists of society "Knowledge" [3].

Many articles of G. S. Narimanov were published in newspapers and journals. The series of his original articles was published in Bulgaria, GDR, Poland, Czechoslovakia. For all his works are characteristic the wide spectrum of the included questions, the skillful introduction of reader to the circle of the most complex scientific-technical problems, figurative and intelligible language. These qualities differed his appearances by the radio and the television.

Much time Georgiy Stepanovich gave to scientific editing. Long years he was the member of the editorial board of the journal "Earth and the Universe" and the chairman of the section of cosmonautics of the Editorial and Publishing Council of publishing house "Mashinostroyeniye". The scientific editing of literature on cosmonautics was the object/subject of his special attention. In this work especially distinctly was exhibited the inherent in G. S. Narimanov intimate knowledge of the essence of the question both in the retrospective and in long-range plans, cautious and deferential attitude to the thoughts and the literary style of the author, that was being combined with the lack of compromise in fundamental scientific and technical questions. Let us refer only to three books, very dissimilar in the form and the content, in the work on which especially vividly were showed his best qualities as scientific editor. In the book "Pages of Soviet cosmonautics" [4], which rightfully can be named the peculiar pcpular history of cosmonautics, are lively figuratively reflected not only the years of the space age, but also its prehistory, or forseeable prospects for further development. The book "Reliability of complex systems" was original pioneer labor in this region [5]. It was the result of the many-year scale and in-depth research, carried out by its authors in different stages of design and manufacture of the units of new technology, its tests, and also storage and operation in different climatic zones of the country. And finally the third book - "From the spacecraft to the orbital stations" [6], for editing of which professor G. S. Narimanov was awarded the diploma of society "Knowledge" by the USSR for the sums of All-Union competition to the best products of popular science literature (1972).

Finishing this outline, one cannot fail to speak several words about Georgiy Stepanovich Narimanov as about man.

First of all should be stressed his formation and multiplicity of interests. He knew well classical and contemporary literature both Soviet, and foreign, he was interested in painting and theater, thinly he felt and loved classical music, he was interesting collocutor.

Georgiy Stepanovich's life far from always stored/added up easily and simply. There were in it the difficult periods, which he transferred with large courage. Georgiy Stepanovich was characterized by enormous composure and delay, he was very entire person and solidly adhered to the specific vital principles. All, who knew G. S. Narimanov, beginning from his close friends and ending with the unfamiliar people, always noted his benevolence, obligation, correctness to word, readiness to aid in the difficult situation, regardless of the fact, in which state there were his inherent matters.

In a word, Georgiy Stepanovich was man from capital letter, and he will forever remain similar in our memory.

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RESONANCE PHENOMENA IN ROTATIONS OF ARTIFICIAL AND NATURAL CELESTIAL BODIES.

V. V. Beletskiy.

Are examined resonance rotations of artificial and natural celestial bodies, in particular Moon, Mercury, Venus, taking into account gravitational, tidal and magnetic interactions.

Are discussed "generalized laws of Cassini" for resonance rotations and extremum properties of resonance motions.

Resonance rotations of Moon, Mercury, Venus and other natural and artificial celestial bodies are examined. Gravitational, magnetic and tidal interactions are taken into consideration. Are considered the "generalized laws of Cassini" for the resonance rotations and the extremum properties of resonance motions [1].

FOOTNOTE [1]. As the basis of this work was used the review of the author at the XVI international congress of theoretical and applied mechanics (Copenhagen, August of 1984), and also at the session of the national committee of the USSR on theoretical and applied mechanics (Moscow, on 16 March, 1984). ENDFOOTNOTE.

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Introduction. Let the motion being investigated contain a certain set of frequencies ω_i. Let us name motion resonance, if

$$\sum_i n_i \omega_i \approx 0, \qquad (1)$$

where n_i - the integers, small by hypothesis. Let $\kappa(t)$ - phase deflection of motion from the resonance, so that $\dot{\kappa}=0$ with precise satisfaction of condition (1). If motion $\kappa=\kappa_0$, $\dot{\kappa}=0$ is stable, then they indicate the presence of the phase stability of resonance motion.

Special and nonrandom role of resonance motions now, apparently, is universally recognized [1, 11, 13, 26, 34]. Such motions not only frequently are encountered in nature (Fig. 1a - orbital resonance in the celestial mechanics), but also by a special form are used in the technology, including - in the space. Examples of this use: the phenomenon of the self-synchronization of the rotors of different machines and technical devices/equipment [26] (Fig. 1b); the system of the passive stabilization of artificial celestial bodies [27] (Fig. 1c). Resonance can be controlled motions. In particular, the process of walking of man [16] is resonance (Fig. 1d).

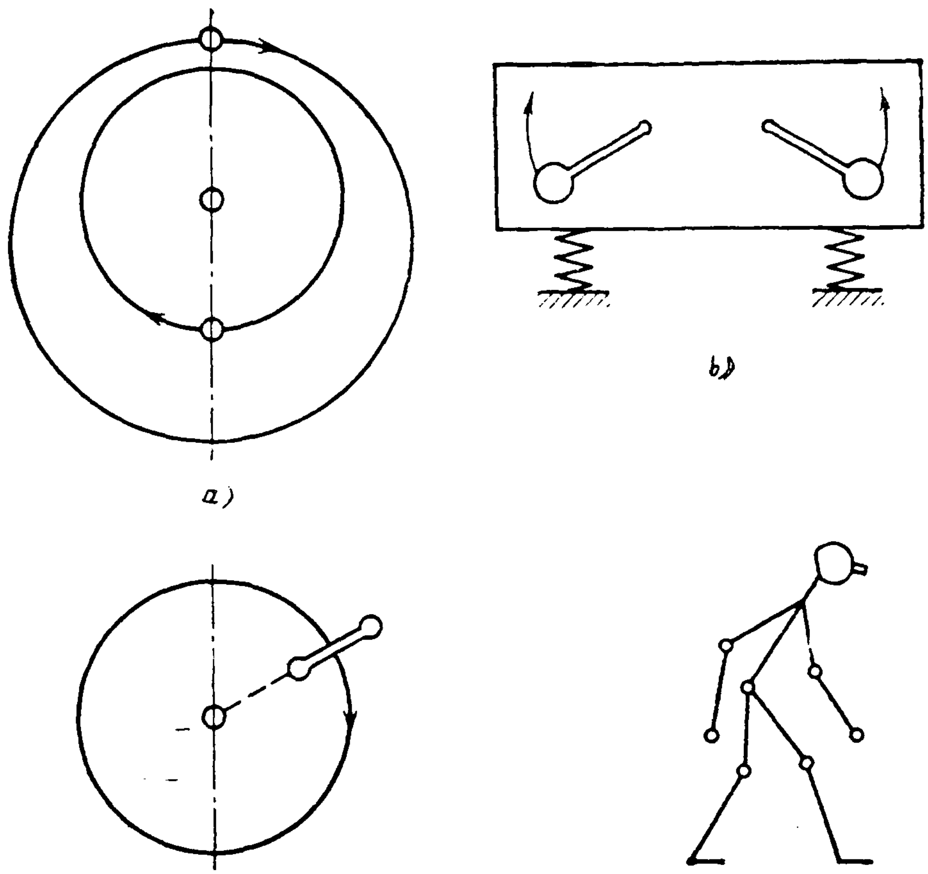

Fig. 1. Resonances in nature and technology: a) orbital resonance; b) self-synchronization of rotors; c) passive stabilization of artificial satellites; d) walking of man.

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There is even hypothesis (A. M. Molchanov, [47]) about complete resonance of solar system as corollary of its evolutionary maturity. Without considering this hypothesis, let us note the unconditional abundance of resonance motions in the solar system, including with the phase stability. The presence of phase stability testifies in favor of the fact that this resonance is actually caused by physical causes - "resonance interaction". On the other hand, the absence (or nonobservation) of phase stability completely does not testify about the chance of this resonance.

Confident examples of orbital resonances with phase stability are motions of following celestial bodies [1, 34]:

I. Triple resonance $\omega_I - 3\omega_{EИ} + 2\omega_G = 0$ between the frequencies of revolution of the satellites of Jupiter Io, Europa, Ganymede.

II. Resonances in the system of the satellites of Saturn:

1. $2\omega_D - \omega_E - \omega_{\pi E} = 0$ between the frequencies of revolution of Dione and Enceladus (into resonance relationship/ratio enter angular frequency $\omega_{\pi E}$ of pericenter of Enceladus);

2. $4\omega_T - 2\omega_M - \omega_{\Omega T} - \omega_{\Omega M} = 0$ between the frequencies of revolution of Tethys and Mimas (into resonance relationship/ratio enter angular frequencies $\omega_{\Omega T}$, $\omega_{\Omega M}$ of the units of the orbits of these satellites);

3. $4\omega_H - 3\omega_{TI} - \omega_{\pi H} = 0$ between the frequencies of revolution of Titan and Hyperion (into resonance relationship/ratio enters angular frequency $\omega_{\pi H}$ of the pericenter of the orbit of Hyperion).

III. Resonances of the asteroids of the group of Trojans at the points of the libration of system Sun-Jupiter (resonance 1:1).

IV. Resonances of the asteroids of the group Hilda with the average period of orbit $T_H = 2T_J/3$ (T_J - the period of the orbit of Jupiter).

V. The resonance of system Neptune-Pluto: the phase detuning

$$\varkappa = 3\lambda_P - 2\lambda_N - \delta_{\pi P} - 180°$$

oscillates within the close limits (although low $|\Delta\varkappa| \leqslant 76°$). Here λ_P, λ_N, $\delta_{\pi P}$ respectively the longitude/length of Pluto, Neptune and pericenter of the orbit of Pluto.

It is possible to name other examples. The system of the satellites of Uranus Miranda-Ariel-Umbriel is found in the resonance of the type Io-Europa-Ganymede, but with the slowly growing phase; famous resonance 5:2 between the periods of the orbit of Saturn and Jupiter, apparently, does not possess phase stability, although the effect of this resonance in orbits of planets is undoubted.

Combining different sets of celestial bodies, it is possible to fit several ten quasi-commensurabilities in orbital motions and even to show that probability of random appearance of these quasi-commensurabilities is much less than observed [34, 47]. But far from always after this stands clearly discovered physical interaction.

Especially many stable resonances are encountered in rotary planetary motions.

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In this case after all resonances of rotations the precise physical picture of interactions stands.

This survey is dedicated to dynamics of resonance rotations in connection with natural and artificial celestial bodies.

1. Resonances in rotary planetary motions. The necessary conditions of the realizability of resonance motion are:

a) the existence of conservative factor in the system, creating resonance "traps" - stability region in the near-resonance region:

b) the existence of the dispersive factor, which creates conditions for capturing the motion by resonance "trap".

In solar system gravitational interaction is basic conservative factor, basic dispersive factor - tidal braking.

In dynamics of artificial celestial bodies is observed great variety of conservative and dispersive factors.

Realizability of resonance motion depends on character of conservative and dispersive forces in many respects.

Research and construction of realizability conditions of resonance rotations in many respects determines contemporary theory of passive stabilization of artificial celestial bodies [9, 33, 35].

Let us name some types of resonances in rotations (Ω - angular velocity of axial rotation of body, ω - orbital angular velocity):

1. $\Omega-\omega\approx0$; resonance of type of Moon (1:1). The angular velocities of axial and rotations are equal to each other. This

motion is called still "relative equilibrium": body rests in the revolving orbital coordinate system and because of this always is converted by one side to the center of attraction as the Moon to the Earth. This type of resonance is the basis of the gravitational systems of passive stabilization.

2. $2\Omega-3\omega\approx0$; resonance of type of Mercury (3:2). Mercury makes exactly three revolutions around its axis exactly during two orbital periods.

3. $\Omega-\omega\approx0$; resonance of magnetic aging (2:1). The magnetized satellite, tracking the line of force of the magnetic field of the Earth, makes two revolutions in the rotary movement for one orbital period.

Phenomenal resonance in rotation of Venus will be considered later.

Resonance 1:1 has that unique special feature, which lies/rests, so to speak, "on surface" of theory of rotations. There is a corresponding explicit, final, particular solution of sufficiently, strictly assigned mission. This fact was known still to classics of celestial mechanics (Lagrange, Laplace), who studied the stability of this motion in lax linear setting [50].

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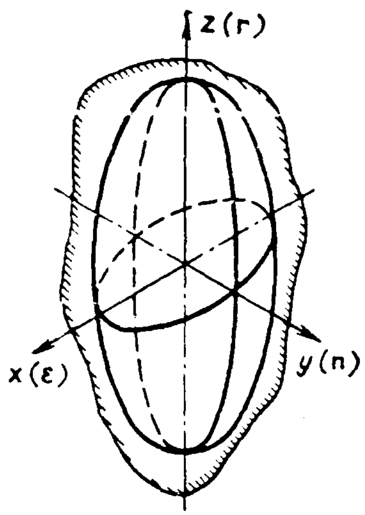
Strict, sufficiently general/common, nonlinear research of problem of relative equilibrium was carried out in connection with launches of artificial Earth satellites. The main results of this analysis can be formulated so [6, 9, 12]:

in free solid body in the noncentral Newtonian field of forces there is a motion with the circular orbit of the center of mass of body and the arrangement of main central inertia axes along the orbital (revolving) axes. For the stability of this motion (in a strict sense according to Lyapunov) it is sufficient so that the greatest axis of the ellipsoid of inertia of body would be directed along the radius-vector of orbit, and smallest - along the normal to the plane of orbit (Fig. 2).

Numerous artificial satellites with gravitational-gradient orientation system and many of natural satellites satisfy this criterion. Of 33 natural satellites of the solar system in 10 rotation is not resonance, in 10 more - what is unknown, but in remaining 13 - resonance (in resonance 1:1). These are Earth satellite (Moon); four satellites of Jupiter (Io, Europa, Ganymede, Callisto); five satellites of Saturn (Enceladus, Iapetus, Rhea, Tethys, Dione); the satellite of Neptune (Triton); both Martian satellites (Phoebus and Deimos).

2. On laws of Cassini motion of Moon. The situation of resonance 1:1, however, is not described by the completely presented above situation of "stable relative equilibrium".

Actually, disturbances/perturbations of other celestial bodies lead to more complicated dynamic effects. Thus, in the case of the Moon the "third body" - the Sun - introduces strong disturbances/perturbations into the orbit of the Moon.



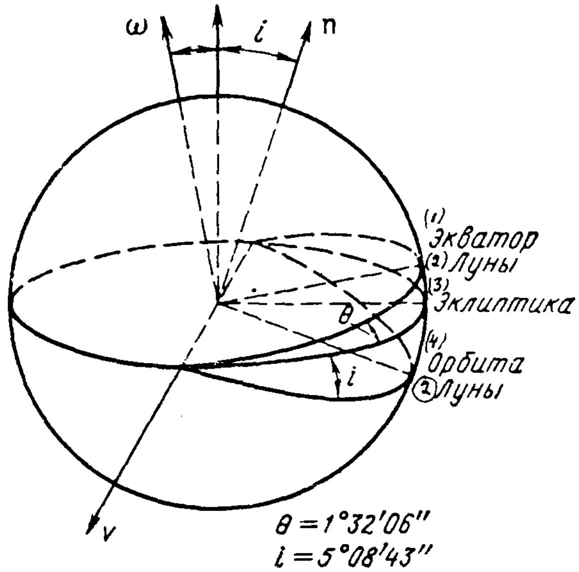


Fig. 2. Fig. 3.

Fig. 2. Arrangement of axes of ellipsoid of inertia in stable relative equilibrium (resonance 1:1).

Fig. 3. Diagram of laws of Cassini rotation of Moon.

Key: (1). Equator. (2). Moon. (3). Ecliptic. (4). Orbit.

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The plane of the orbit of the Moon precesses with period $T_{\delta} = 18{,}6$ years. This, in turn, are caused disturbances/perturbations in rotation of the Moon.

Real laws of rotation of Moon empirically established D. D. Cassini in 1693 in the following form.

1. Moon revolves evenly around axis, Moon remaining fixed in

body; period of rotation of Moon coincides with period of its orbit in orbit around Earth.

2. Equatorial plane of Moon retains constant inclination/slope toward ecliptic (equal to 1°32').

3. Ascending node of equator of Moon on ecliptic always coincides with descending unit of orbit of Moon on ecliptic.

Fig. 3 gives appropriate diagram. Since the orbit of the Moon is inclined to angle of $i \approx 5°9'$ toward the ecliptic, then from the second and third laws of Cassini it follows that the vector of the axial angular velocity of the Moon ω is normal to the nodal line and composes angle $\rho \approx 6°41'$ with normal n to the plane of lunar orbit.

Let us stress that nodal line (Fig. 3) - is mobile in space; it precesses with above-indicated period $T_{\delta} = 18{,}6$ years.

Laws of Cassini - empirical. They are not the final solution of precise equations of motion. The classical libration theory of Moon [50] and its subsequent development were based on the linearization of equations of motion about the close to the motion, described by the empirical laws of Cassini. Therefore almost 300 years stood a question about a stricter theoretical proof of Cassini's laws as the real laws of nature.

Solution of this question became possible only in our time in connection with general/common progress of theory and development of space research.

On rotation of Mercury. Schiaparelli in 1889 interpreted a series of his observations of Nercury in such form, that Mercury revolves around its axis with the period T=88 days, equal to orbital period. However, it was established by the radar methods (1965) that the period of the rotation of Mercury was close to 2/3 orbital periods (58, 65 days). It was discovered after this, that the optical observations of Schiaparelli allow/assume ambiguous interpretation, and period about 59 days is placed well in these observations. Resonance 3:2 in the rotation of mercury now is acknowledged by all.

Flat/plane rotation of celestial body, center of mass of which moves along elliptic orbit with eccentricity e, is described by equation [6]

$$(1+e\cos\nu)\frac{d^2\delta}{d\nu^2}-2e\sin\nu\frac{d\delta}{d\nu}+3\frac{A-C}{B}\sin\delta=4e\sin\nu;\ \delta=2\theta. \qquad (2)$$

Here ν - true anomaly; θ - angle between inertia axis of body and radius-vector of its orbit; A, B, C - main central moments of inertia of body.

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Even before discovery/opening of resonance 3:2 in rotation of Mercury in work [39] it was shown that equation [2] contains resonance solutions, including of type of Mercury (3:2). The zone of the phase stability of this resonance has sizes/dimensions of ~e. Since the orbit of Mercury has not low eccentricity ($e \approx 0{,}206$), then this raises the probability of the existence of resonance of the type 3:2.

Research of precise resonance (3:2) solutions of equation (2) is carried out in work [18]. In Fig. 4 in the plane of parameters $n^2 = 3(A-C)/B$, are shaded the stability regions of such solutions.

However, in actuality Mercury moves along disturbed, not inertial orbit; "flat/plane" model (2) - only step/stage for research of general case.

Thus, ripened need for creation of generalizing theory of resonance rotations of celestial bodies taking into account systematic disturbances/perturbations of their orbits.

Achievements of mathematics of XX century - theory of periodic solutions of Poincare, asymptotic methods of nonlinear vibrations - made it possible to construct this theory. To new development stage of the theory of rotary planetary motions the launching of artificial space vehicles and the demands of the practice of space flights were jerk/impulse.

4. Generalized laws of Cassini. The result of research was the theory of the so-called "generalized laws of Cassini", which describes

laws governing the resonance rotations of celestial bodies [10, 43]. The consecutive (sometimes parallel) stages of research, which led to setting of these laws, are contained in works [6], [5, 78, 30, 31, 36, 40, 45, 49]. The description of these works exists, for example, in works [9, 11, 12, 13].

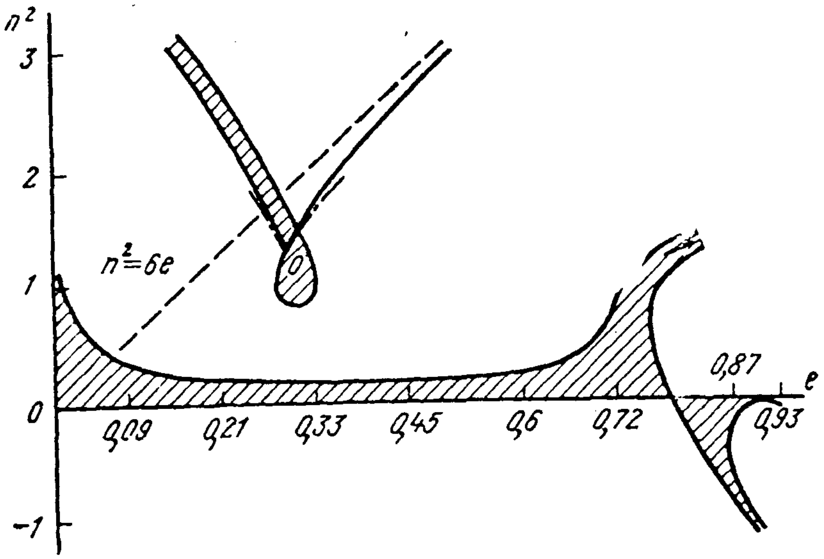

Fig. 4. Stability regions of resonance solutions (resonance 3:2).

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Cassini's laws are determined by superposition of stabilizing effect of gravitational field in resonance situation (see Section 1, 3) and laws of evolution of vector K of moment of momentum of rotation of body relative to processing orbit [5], [9].

Fig. 5a, b depicts trajectories of terminus of vector for two versions of simplified nonresonant situation [5, 7, 9]. Is considered the average/mean action of the moment of gravitational forces and the evolution of orbit. The construction of the theory of resonance motions requires the substantially more complicated equations, which do not possess the integral curves, shown in Fig. 5. Are retained (in the generalized form) only stationary points 1, 2, 3, 4 (Fig. 5), which, however, pass from two-dimensional phase space to six-dimensional.

Generalized laws of Cassini follow from stationary points of some nonlinear autonomous equations [10, 12, 43], obtained from general/common equations of motion by procedure of asymptotic methods for resonance situation.

Hamiltonian of these equations takes form

$$H=\frac{L^2}{2}\left[\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right)\sin^2\vartheta+\frac{\cos^2\vartheta}{C}\right]-(\Omega+k_\omega)L-[V]-$$
$$-\left[k_\Omega\left(\cos i\cos\rho+\sin i\sin\rho\sin\Sigma\right)\right]. \tag{3}$$

Here $[V]=f(\vartheta, \varphi, \varkappa, \rho, \Sigma)$ - specific in resonance situation force function of gravitational moments, which function on body; canonical variables are $\varkappa$, φ, Σ, and corresponding canonical pulses - $L, L\cos\vartheta, L\cos\rho$; $\varkappa$ - phase deflection from resonance rotation; φ, ϑ - Eulerian angles of spin and nutation (in system, connected with vector of moment of momentum); L - modulus/module of vector of moment of momentum; ρ and Σ - its angular coordinates relative to plane of orbit of body; Ω - resonance value of angular rate of rotation of body, k_ω, k_Ω - respectively speeds of processions of pericenter and unit of orbit of body; i - orbit inclination.

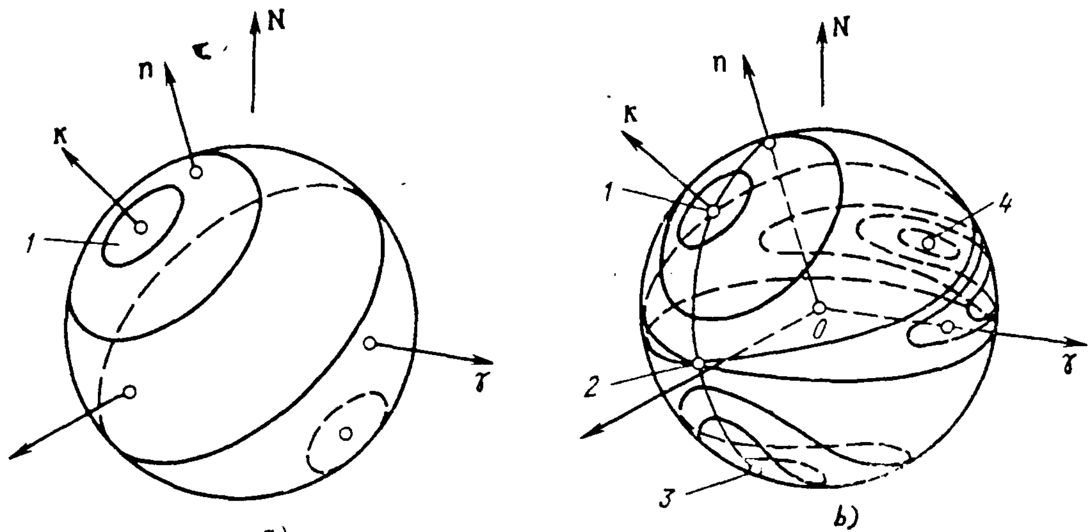

Fig. 5. Trajectories of terminus of vector of moment of momentum.

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Concrete/specific form [V] is given in works [10, 12, 43].

Extreme points of function H(L, ϑ, φ, κ, ρ, Σ) answer stationary points of equations of motion, which present generalized laws of Cassini.

These laws take following formal form.

1. Body revolves evenly around its principal central inertia axis with angular velocity

$$\Omega_0 = \Omega + k_\omega + k_\Omega \cos(i \pm \rho_0), \qquad (4)$$

close to one of resonance values: $\Omega=\omega$ (Moon), $\Omega=1.5\omega$ (Mercury) and so forth.

Here ω - orbital angular velocity.

2. Axis of angular rotation of body, normal to orbital plane and axis of precession of orbit lie/rest at one plane.

3. Axis of angular rotation of body and normal to orbital plane compose constant angle ρ_0, determined by equation

$$\cos\rho^* \mp \sin\rho^* \operatorname{ctg}\rho_0 + \rho\cos\rho_0 = 0, \qquad (5)$$

where parameters ρ^*, χ unambiguously are computed through orbital parameters, moments of inertia of solid body Ω_0 (4).

2 or 4 solutions have equation (5). Respectively when $\cos^{2/3}\rho^* + \sin^{2/3}\rho^* > \chi^{2/3}$ and $\cos^{2/3}\rho^* + \sin^{2/3}\rho^* < \chi^{2/3}$ (Fig. 6).

4. Phase coincidence of rotary and orbital motions with passage of pericenter: angle between radius-vector and nodal line coincides with angle between inertia axis of body and nodal line.

From second law it follows that axis of angular rotation of body precesses in space with the same period, as orbit.

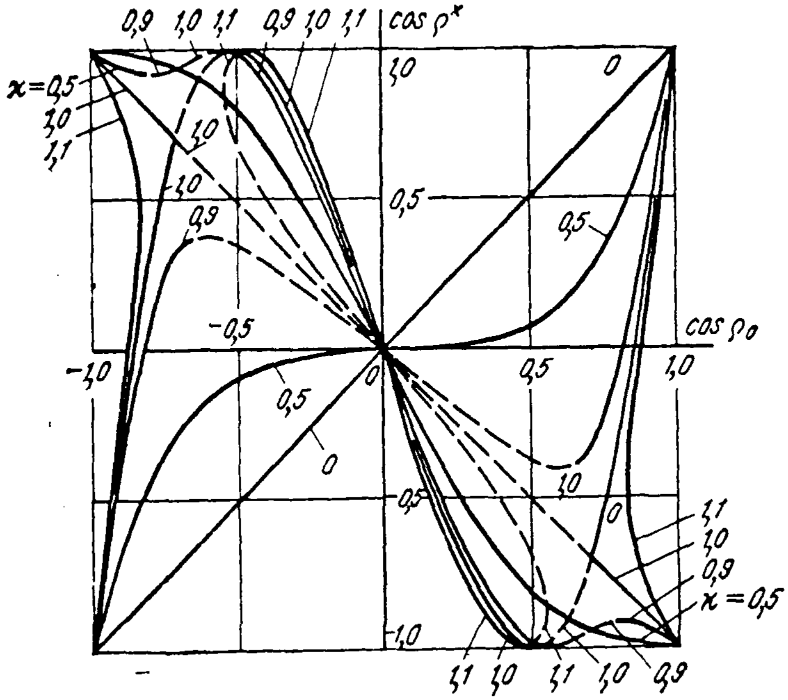

Fig. 6. Graphical solution of equations (5.3) (are indicated values of parameter $\varkappa=\chi/2$).

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Thus, the generalized laws of Cassini (the first and the second) include two resonances.

Basic from stability conditions of generalized laws of Cassini is described in Section 1 arrangement of inertia axes (case of Moon); in the case of Mercury for this arrangement of axes it has to be with passage of pericenter of orbit.

Described theory in work [30, 31] is supplemented by account of tidal effects and by picture of asymptotic stability of Cassini's laws.

It follows from characteristic equation of low oscillations about motions, described by generalized laws of Cassini [10, 12, 43], that these oscillations occur with frequencies of

$$\Omega_1 = \omega \frac{\sqrt{3}}{2}(1+\cos\rho_0)\sqrt{\frac{A-C}{A}};$$

$$\Omega_2 = \omega \frac{B-A}{8A}\sqrt{\frac{B-C}{A-B}(1+6\cos\rho_0-15\cos^2\rho_0)(5+6\cos\rho_0+21\cos^2\rho_0)};$$

$$\Omega_3 = \omega \frac{3(B-A)}{16A}\sqrt{-\varkappa\frac{\sin i}{\sin\rho_0}[\varkappa\cos(\rho_0-i)-2(4-3\delta)+2\delta\cos\rho_0};$$

$$\varkappa = \frac{16k_{\Omega}A}{3\omega(A-B)};\ \delta = \frac{A-C}{A-B};\ A>B>C,$$

which corresponds (in the case of Moon) to periods T_1=2.88 yr.; T_2= 75,20 yr.; T_3=24,68 yr. [3, 4]; periods T_1 and T_2 are observed in real motion of Moon.

Finally, it is shown [3] that "generalized laws of Cassini" are motions (in Poincare's sense) for precise periodic solutions of precise initial equations.

Thus, is closed question about conformity of observed motions to any exact solutions of precise equations of motion. Such - periodic - solutions can be constructed with Poincare's method, and the initial high-precision approximation/approach of these solutions is precisely the "generalized laws of Cassini".

5. Cosmogonic theory of Eneyev-Kozlov and evolution of rotations and inclinations of celestial bodies. One of the factors, which affect the evolution of rotary planetary motions, is tidal effects.

As is known [34], planets in their contemporary state are extremely little subjected to action of tidal evolution of rotations. The effect of tidal moments is more substantial for the earth-type planets and is negligible for giant planets. However, position was not always similar. It is possible to assume that at the specific stage of the cosmogonic process of the formation of planets from the proto-planet cloud the quantitative picture of tidal evolution impressively differed from contemporary.

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And actually, according to new cosmogonic theory of Eneyev-Kozlov [29, 42, 46] several giant globular clusters, which were moving along planetary orbits, were result of evolution of proto-planet cloud. Each accumulation - this is the future planet (it can be, with its satellites), which possesses the appropriate mass, but which differs from contemporary planet in terms of low density and enormous sizes/dimensions (order of the sizes/dimensions of Hill's sphere). Thus, the diameter of "Proto-Venus" was $\sim 0{,}6 \cdot 10^6$ km, and "Proto-Jupiter" $-0{,}5 \cdot 10^8$ km. The moment of tidal forces is proportional to the fifth degree ($-R^5$) of radius of planet [34], the moment of inertia is proportional to R^2, therefore, the speed of the evolution of rotation is proportional to R^3. This means that the proto-planets under the action of solar inflows evolved in their rotations many orders more rapidly than in the contemporary epoch ($10^5 ... 10^9$ times more rapid) and, therefore, the role of the tidal evolution of rotations of planets was very great.

This idea was expressed in 1977 by T. M. Eneyev, who based it, in particular, on results of work [12]. Analysis [14, 44] carried out subsequently showed that, apparently, basic part of the evolution occurred in the first $10^4 ... 10^6$ years after the formation of proto-planets - time very low on the cosmogonic scales.

Essential factor of evolution was also process of unavoidable compression of proto-planets from initial giant sizes/dimensions on contemporary final. The effect of compression in the absence of other effects leads to an increase in the angular velocity of planet during its constant/invariable inclination. The combination of the effect of compression with the tidal evolution can lead (and actually led) to the most diverse final results, observed in the contemporary epoch.

Referring after parts of analysis to works [11, 14, 44], let us describe its main results.

With contemporary meanings of tidal factor Q ($Q \simeq 10 ... 100$ for earth-type planets and $Q \simeq 10^5 ... 10^6$ for giant planets) characteristic time of evolution of rotation for proto-Earth $\tau \simeq 10^5$ years, for proto-Jupiter $\tau \simeq 10^7$ years; if in proto-planets value Q would be less at least by an order, then characteristic time of evolution would be reduced up to $10^4...10^5$ years.

Fig. 7 depicts picture of tidal evolution in plane of parameters ρ, Ω. Here ρ - angle between vectors of moment of momentum and normal to the orbital plane; Ω - the standardized value of this vector. To straight/direct rotation answers $\rho=0$, reverse/inverse $\rho=\pi$.

Most interesting effects of tidal evolution are such:

1. Tendency of all motions toward the straight/direct rotation, in particular, roll/revolution of initially reverse/inverse rotations into the straight lines. This process is passed in different ways under the different initial conditions.

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Extreme versions:

a) the strong evolution of inclination during the almost constant/invariable rotation (Uranus);

b) the evolution of rotation during the almost constant/invariable inclination (Venus).

2. Essential evolution of inclinations and in the case of initially straight/direct rotations: initially low inclinations can achieve high values, and then again decrease. For some planets (Earth, Mars) the process of a slow increase in the inclination

continues even now.

3. Possible decrease of angular velocity to value, smaller than orbital angular velocity (and even to values, close to zero, but also by subsequent restoration/reduction up to orbital).

4. Tendency of all motions toward maximum straight/direct rotation with zero inclination and with specific angular velocity, which depends on orbit eccentricity. In orbits, close to the circular, maximum angular velocity is close to the orbital (Moon, Martian satellites, series of the satellites of Saturn, Jupiter); with the orbit eccentricity e~0.2 maximum angular velocity is close to 3/2 orbital (Mercury).

In connection with these results T. M. Eneyev expressed following considerations [29]. The contemporary position of the axis of Uranus almost in the plane of its orbit can be explained by the fact that in the cosmogonic process of the formation of planets Proto-Uranus gained rapid reverse/inverse rotation around the axis, almost normal to the plane of orbit. Because of the powerful/thick tidal protuberances the axis of its rotation was strongly evolutionized (in accordance with that presented above) and "was inverted" to the orbital plane. Analysis [14, 44] confirmed this consideration.

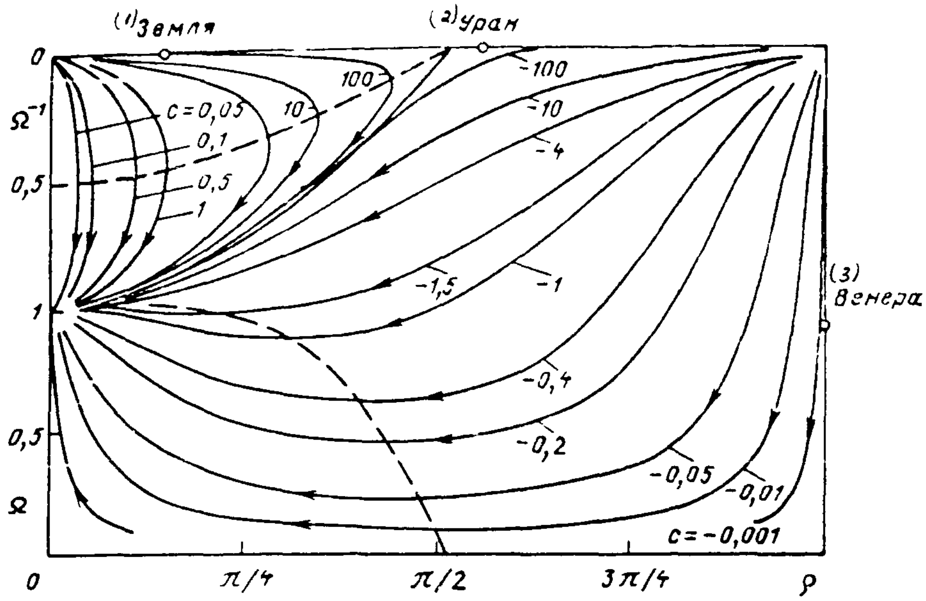

Fig. 7. Picture of tidal evolution in plane of parameters Ω. ρ (are indicated values of integration constant C).

Key: (1). Earth. (2). Uranus. (3). Venus.

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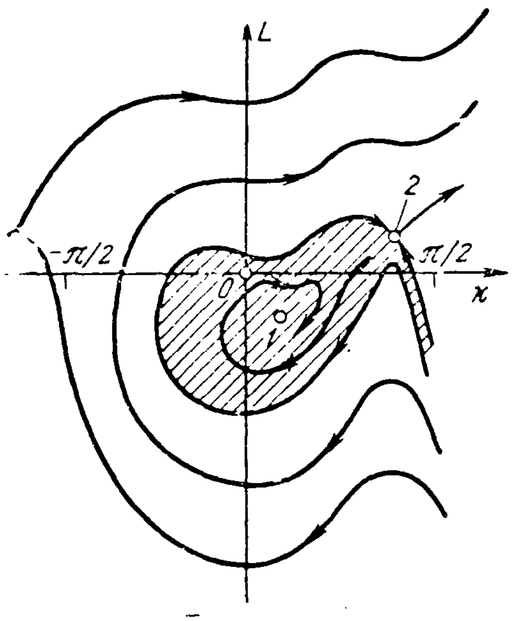
As far as Venus is concerned, its reverse/inverse rotation in contemporary epoch around axis, virtually normal to plane of orbit, can be explained by cosmogonic origin of reverse/inverse rotation and by fact that evolution of Venus occurs according to type, sharply different from evolution of Uranus. Venus within the time of evolution did not have time "to be inverted". The detailed theory of the rotation of Venus (taking into account the observed resonance effects) is described in [19, 23, 24, 25].

Establishment of possibility of reverse/inverse rotation of Venus and Uranus up to moment of formation of planets from proto-planet cloud is one of most important results of cosmogonic theory of Eneyev-Kozlov [29, 42, 46].

It follows also, perhaps, to note that in contemporary epoch Moon stronger (to two orders), than sun, affects tidal evolution of rotation of Earth. But the evolution of the ancient proto-earth under the action of tidal moment from the sun proceeded 10^3-10^4 times more rapidly taking into account this fact.

Process of compression of proto-planet, apparently, most strongly affected evolution of inclination of Jupiter: compensation for tidal decrease of angular velocity by its increase due to compression of proto-Jupiter led to freezing of evolution of inclination.

Should be noted also significant role in evolution of rotation of conservative gravitational moments: precisely they construct "traps", which ensure under the effect of tidal moment capture/grip of number of celestial bodies in resonance rotations (Moon, Martian satellites, some satellites of Saturn, Jupiter; Mercury; Venus). The diagram of this capture/grip in the resonance rotation with the phase stability is depicted in Fig. 8.



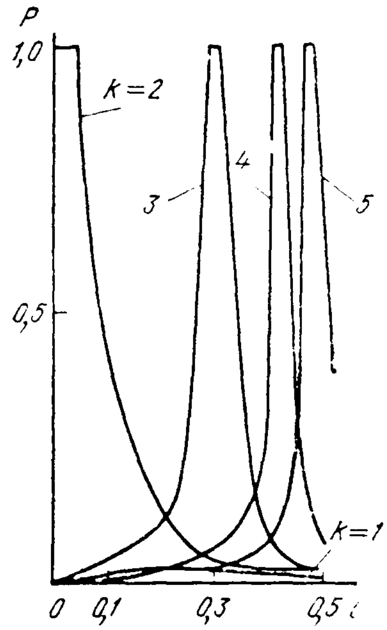


Fig. 8.

Fig. 9.

Fig. 8. Diagram of tidal capture/grip in stable resonance motion in phase space κ, L: 1, 2 - stationary points.

Fig. 9. Probability P of capture/grip into resonance

$\frac{k}{2}\omega$ depending on eccentricity e of orbit.

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Fig. 9 gives the dependences of the probability of capture/grip to the resonance

$\frac{k}{2}\omega$, on eccentricity e of orbit [38] [1].

FOOTNOTE [1]. Deep results according to the general theory of capture/grip in the resonance and the passages through the resonance are contained in dissertation of A. I. Neyshtedt "About some resonance problems in nonlinear systems", MGU, 1975. The presentation of some of them is in [2]. ENDFOOTNOTE.

Let us conduct basic sums. According to the cosmogonic theory of Eneyev-Kozlov [29, 42, 46] the proto-planets, which initially possessed very large sizes/dimensions, were the result of the evolution of proto-planet cloud In view of this large role played the tidal evolution of rotations of planets, which occurred several orders more rapidly than in the contemporary epoch. Determined - sometimes prevailing - role played the process of the compression of proto-planets to the contemporary sizes/dimensions. "Gravitational traps" contributed to "sticking" of some planets and satellites in the resonance rotations. The combination of these factors led to the contemporary diversity of inclinations and rotations of celestial bodies.

About rotations of Venus. In 1962 with the help of means of radar it was established that Venus has reverse/inverse rotation, and was determined period of rotation. According to contemporary data, this period $T_{\text{♀}}=243{,}0\pm0{,}03$ days, which is close to the resonance value of 243.16 days. Venus during period $\tau=583{,}92$ days of connections with the Earth will do exactly 5 revolutions around its axis relative to direction Sun-Venus and exactly 4 revolutions relative to direction Sun-Earth. In each connection (1 and 2) Venus is converted to the Earth by one and the same side (Fig. 10). The angular velocity Ω of the axial rotation of Venus is connected with the orbital angular velocities of Venus ω_V and Earth ω_E with the relationship/ratio

$$\Omega = 4\omega_V - 5\omega_E. \quad (6)$$

Calculation of this phenomenal resonance within the framework of flat/plane model of motion was carried out by Goldreyn and Pinl [34]. However, three-dimensional effects are most interesting in this problem.

In series of publications [19, 23, 24, 25] theory of three-dimensional/space resonance rotation of Venus was constructed. This theory considers gravitational interaction of the Sun and Venus, Earth and Venus, evolution of the orbit of Venus, tidal effects (under the effect of the Sun) in rotation of Venus.

Resonance zone in phase space is created on the average by gravitational field of Earth. Gravitational moment from the Sun on the average does not affect the creation of resonance zone.

Tidal instability of rotation and fact of reverse/inverse rotation of Venus are, at first glance, in contradiction. This gave rise to hypothesis about the existence (in the past) of the reverse/inverse satellite of Venus, which stabilized Venus in reverse/inverse rotation [37].

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Conclusion of new theory is more natural, that reverse/inverse rotation of Venus has cosmogonic origin [29, 42, 46].

Theory of rotation of Venus shows that in this case for time of tidal evolution (less than 10^{6} years) Venus did not have time "to be inverted". The process of "tilting/reversal" proceeds several orders slower than the process of capture/grip in the resonance rotation. Fig. 11 schematically depicts the process of the tidal evolution of Venus in the three-dimensional phase space of coordinates κ, L, ρ.

Steady resonance rotation of Venus is described by laws of type of generalized laws of Cassini. The Hamiltonian of problem takes form (3), only with considerably more complicated expression for [V].

Let us note, however, very low probability of capturing Venus into resonance in comparison with probability of capturing of Moon or Mercury into their resonances.

A. A. Khentov focused attention on following fact. Between the orbital frequencies of Venus ω_V, Earth ω_E, Jupiter ω_J there is a sufficiently accurately made resonance relationship/ratio $6\omega_V-10\omega_E+3\omega_J=0$. Therefore resonance frequency Ω_V of the axial rotation of Venus is subordinated not only to relationship/ratio (6), but also to relationship/ratio $2\Omega_V=2\omega_V-3\omega_J$. This fact again attests to the fact that the formally registered resonance relationships/ratios are poorly informative; such formulas are ambiguous. Single-valued selection can be done via the analysis of the mechanics of phenomenon. Is interaction sufficiently strong, in order to create resonance zone and to ensure the stability of resonance? Is sufficiently "wide" resonance zone and is sufficiently great the probability of capture/grip?

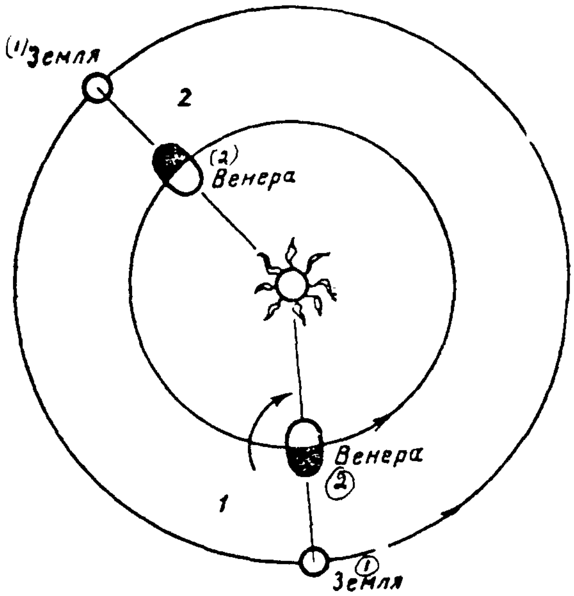

Fig. 10.

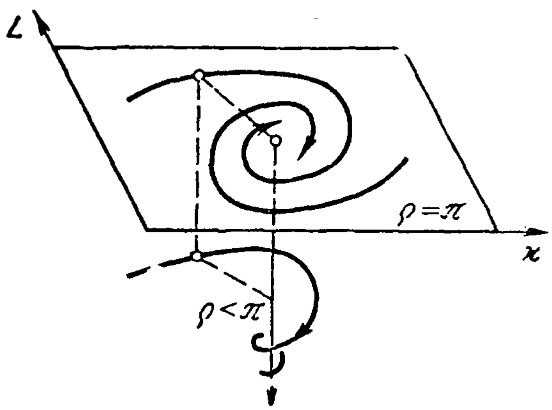

Fig. 11.

Fig. 10. Resonance rotation of Venus.

Key: (1). Earth. (2). Venus.

Fig. 11. Diagram of process of tidal evolution of Venus in three-dimensional phase space of coordinates κ, L, ρ.

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Only responses/answers to such (and similar) questions give information - is resonance physical or formal. In the specific case of Venus preference, apparently, must be given to interaction with the Earth, but not with Jupiter. This question, however, still is subject to research. If two (or several) resonances will prove to be uniformly strong during the testing for physicalness, a question about their superposition (interaction) difficult for the research arises.

7. Extremum properties of resonance motions. On set of possible motions resonance motions are special. It is possible to assume therefore that the characteristics of fields of force reach outer limits on the resonances.

There is series of theoretical and empirical principles, which confirm this thesis. In the development of ideas and methods of Poincare in book [26] the extreme principle established/installed in 1960 is presented and used for the specific problems. Thus one of its formulations.

Let us consider generating motion (in Poincare's sense: with zero value of low parameter), described by frequencies ω_s. And let us introduce number $\sigma_s = \mathrm{sign}(d\omega_s/dh_s)$. Here h_s - energy, which corresponds to frequency ω_s. If $\sigma_s = 1$, then it is said that object, is rigidly anisochronic, if $\sigma_s = -1$, then the object is softly anisochronic. Let us consider the value Λ average/mean during the ger rating motion is Lagrangian L. Let us assume that all objects of system possess one type anisochronism $(\sigma_s = \sigma)$. Then the functional

$$D = -\sigma\Lambda = \min, \qquad (7)$$

i.e. has a minimum in the stable resonance states of motion.

For orbital problems $\sigma<0$. It is possible under some conditions to disregard in Λ the terms, caused by kinetic energy, and then condition (7) is converted in $\langle V_0\rangle$-min. Here $\langle V_0\rangle$ - average/mean during the generating motions value of the force function of gravitational forces. This formulation in some sense is equivalent to the known heuristic principle of "smallest interaction" [48]. For the case, for example, of rotary planetary motions $\sigma>0$ and from (7) we will obtain opposite principle $\langle V_0\rangle$-max in the stable resonance modes.

Principle (7) is formulated for generating motions and makes it possible to determine values of phases of motions in stable operations.

In work [21] was proposed extreme principle in the following form:

$$\overline{V}(x_0, \dot{x}_0) = \lim_{t\to\infty} \frac{1}{t}\int_0^t V(x(x_0, \dot{x}_0, t), t)\,dt = \max \qquad (8)$$

during stable resonance motions. Here the discussion deals already with the average/mean value of force function V during the true (but not on "generating") motions. In (7) is laid not the ideology of the low parameter, but, rather, the ideology of the Lyapunov functions.

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Principle (7) makes it possible to find initial conditions x_0, $\dot{x}_0$, which correspond to stable resonance mode.

Numerical experiment on checking of this principle was conducted in essence with equation (2) (but not only with it). Fig. 12...17 gives some results. Instead of the average/mean force function $\bar{V}$ was computed the average/mean dimensionless potential energy $\bar{u}=-\kappa\bar{V}$, $k>0$. The minimums of function $\bar{u}$ correspond to the maximums of function $\bar{V}$.

Fig. 12 shows standard case of circular orbit ($e=0$, $n^2=0.1$), when equation (2) is converted into autonomous. Resonance 1:1 is answered by relative equilibrium in the orbital system, i.e., the solution $\delta=0$, equations (2). Analytically it is possible to show that this solution is stable, and functional (8) actually has a maximum during this solution (functional $\bar{u}$ has a minimum). Actually, it proves to be that

$$\bar{u}=1-\frac{E(k^2)}{K(k^2)}(k^2\leqslant 1);\ \bar{u}=k^2\left(1-\frac{E(1/k^2)}{K(1/k^2)}\right)(k^2\geqslant 1);$$

$$k^2=\theta_0'^2/n^2+\sin^2\theta_0. \tag{9}$$

Here K and E - complete elliptic integrals; θ_0 θ'_0 - initial data for equation (2).

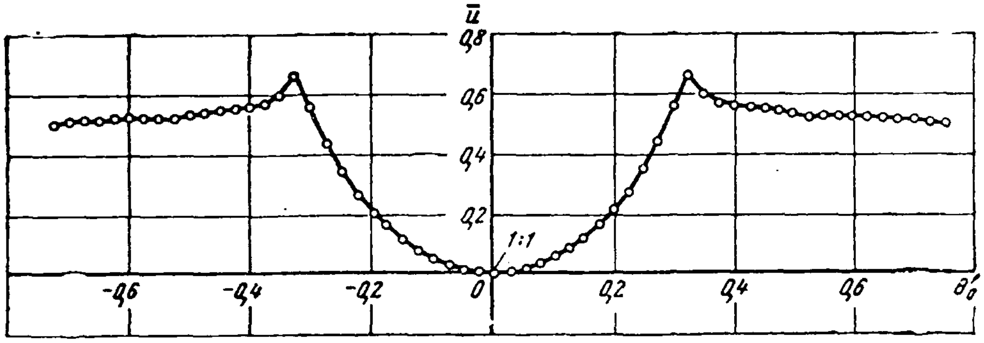

Fig. 12. Minimum of functional on resonance 1:1 (e=0, n^2=0.1).

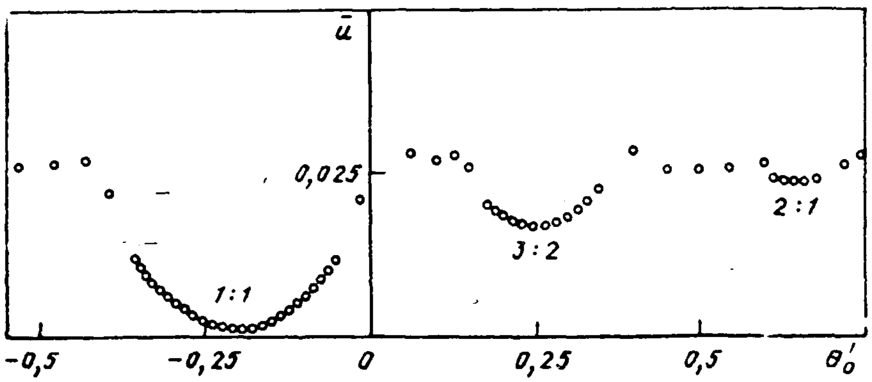

Fig. 13. Minimums of functional on resonances (e=0.1, n^2=0.1).

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Function $\bar{u}(k^2)$ with low k^2 behaves as $\bar{u} \sim k^2/2$, and when $k^2 \to \infty$ $\bar{u} \to 1/2$. These properties of function $\bar{u}(k^2)$ are visible in Fig. 12, which, however, is obtained by numerical method.

Fig. 13 similar pattern gives for e=0.1; n^2=0.1; are visible minimums for resonances 1:1; 3:2; 2:1. Fig. 14 shows the picture of the sharpened/turned representations during the orbital period on the phase plane $\theta'(\theta)$. This picture convinces us in the stability of the resonances, discovered in the previous figure.

In Fig. 15, 16 - similar pattern for other values of parameters $e=0.2$; $n^2=0.2$.

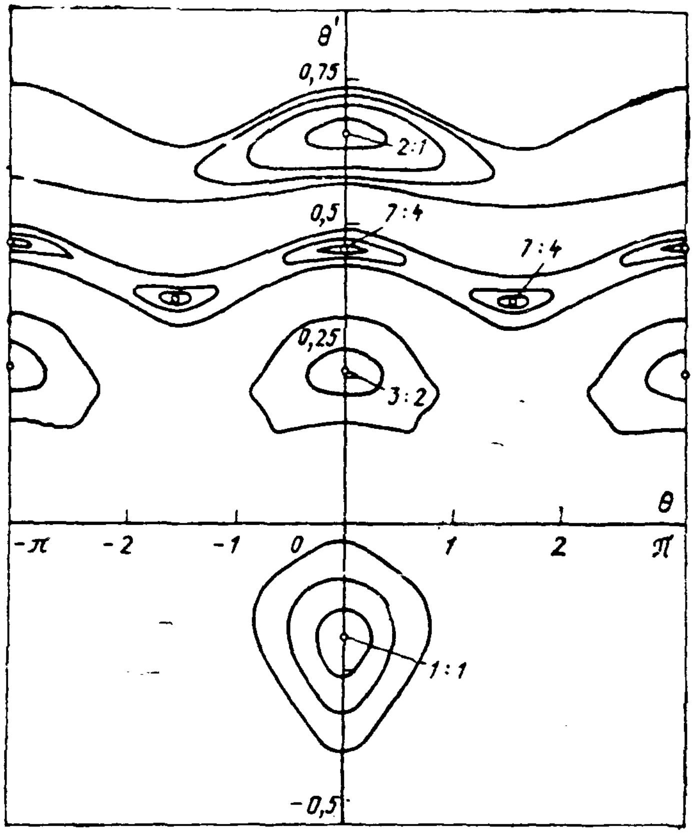

Fig. 14. Point representations (e=0.1, n^2=0.1).

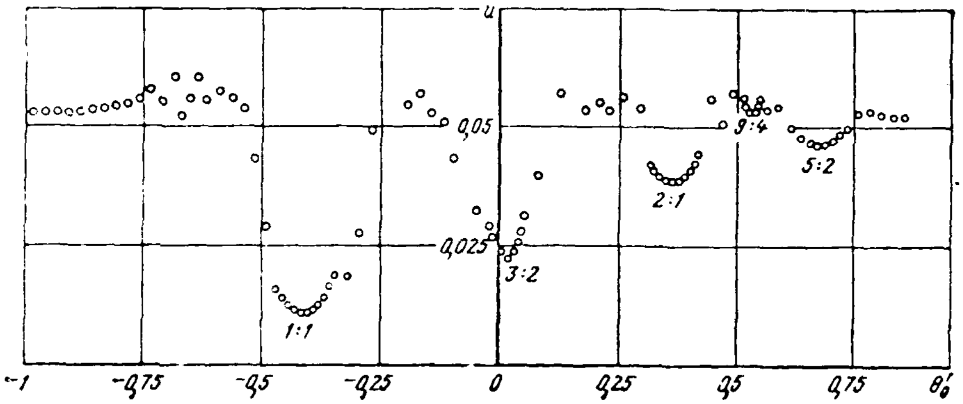

Fig. 15. Minimums of functional on resonances (e=0.2, n^2=0.2).

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Similar pattern occurs for problem about rotation of magnetized satellite in polar circular orbit [22]. This rotation is described by equation [20]:

$$(1+e\cos\nu)\theta''-2e\sin\nu\theta'+\frac{n^2}{2}\sin 2\theta-\frac{\alpha}{2}(3\cos(\theta-u)-$$
$$-\cos(\theta+u)]=2e\sin\nu.$$

Here ν - true anomaly (independent variable); $u=\nu+\omega$ - argument of the latitude, where $\omega=\text{const}$ - argument of the perigee of orbit; e - orbit eccentricity; $n^2=3(A-C)/B$ - parameter, which corresponds to the moment of gravitational forces (A, B, C - the main central moments of the inertia of satellite); $\alpha=I\mu_E/B\mu$ - parameter, which corresponds to the moment of magnetic forces; constant magnetic moment I is directed along the axis of satellite, which corresponds to the moment of inertia C; this axis forms angle θ with the current radius; the magnetic field of the Earth has magnetic moment μ_E; μ - gravitational constant. With $\alpha=0$ we obtain equation (2).

Let us consider circular orbit (e=0). In the case of the absence of the gravitational moment ($n^2=0$) are discovered (Fig. 17a) two stable resonance rotations: 2:1 - stabilization relative to the local line of force of magnetic field; 0:1 - stabilization relative to fixed direction. If in this problem is taken into account even gravitational moment ($n^2\neq 0$), then appears, as one would expect, one additional stable resonance 1:1 (Fig. 17b).

Resonances of type 0:1 serve as basis for construction of analytical theory of slow rotations of celestial bodies [15].

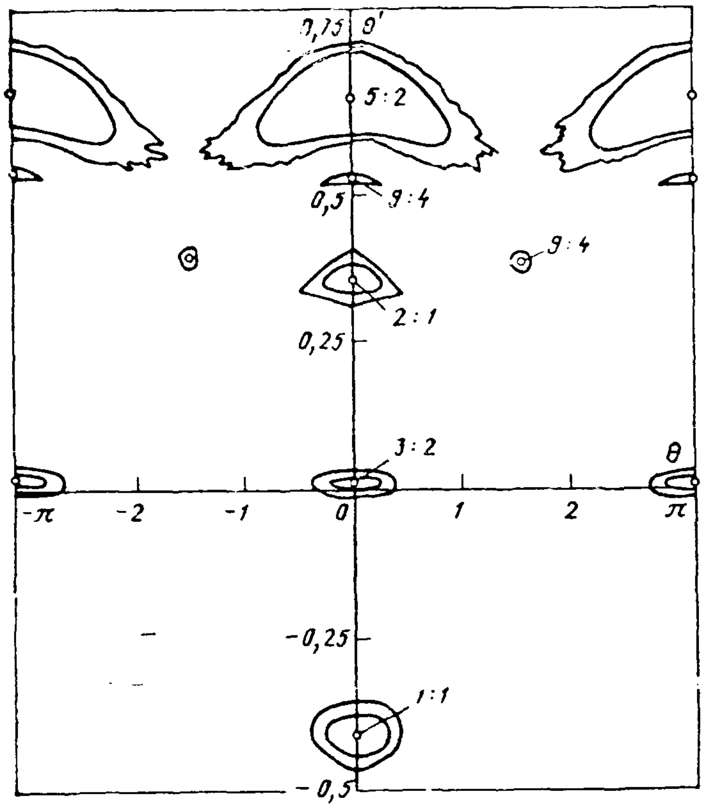

Fig. 16. Point representations (e=0.2, n²=0.2).

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Table 1 gives information about some extreme principles of resonance motions. In works [17, 28, 41], etc. the extremum properties of resonance motions were investigated at the level of precise theorems. Extreme principles can be formulated, also, for the motions, the more complicated, than resonance (principle of Percival for conditional-periodic motions [32]).

Final observations. Observed in nature resonances in the rotary planetary motions - low order (1:1; 3:2). In work [38] it is shown that this is deeply connected with the essence of the matter in the system of two gravitating bodies. The resonances of higher order can be caused by interaction more than of two bodies (Venus) or by presence of additional fields of force (magnetic resonance).

Nature arranged traps on paths of motion of celestial bodies. But trap - this yet not beast in the trap. In order to catch beast are necessary another patience and transportation.

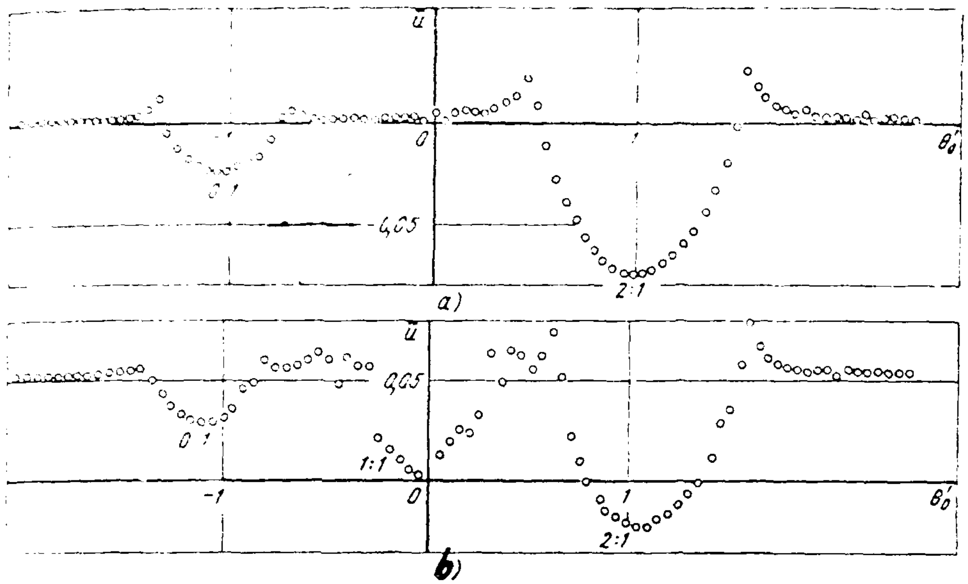

Fig. 17. Minimums of functional in problem of rotating magnetized satellite ($e=0$, $n^2=0,2$; $\alpha=0,05$; $\omega=-\pi/2$): a) functions only moment of magnetic forces; b) function moments of magnetic and gravitational forces.

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(1) Автор	(2) Экстремальная траектория	(3) Экстремальный принцип	(4) Устойчивость	(5) Доказательность
(6) Пуанкаре, 1892 г.	(7) Порождающая	$H = H_0(p) + \sum \mu^k H_k(p, q)$ $\frac{1}{\tau}\int_0^\tau H_1 dt = \text{extr}$	+	(8) Метод малого параметра
(9) Блехман, 1960 г.	–	$\Lambda = \frac{1}{\tau}\int_0^\tau (T+U)_{\mu=0}\, dt$ $D = -\sigma\Lambda, \ \sigma = \text{sign}\, d\omega/dh$ $\underline{D = \min}$ (10) Приближенно: (11) орбитальное движение $\sigma<0$ $\frac{1}{\tau}\int_0^\tau U_{\mu=0} dt = \min;$ (12) вращательное движение $\sigma>0$ $\frac{1}{\tau}\int_0^\tau U_{\mu=0} dt = \max$	+	(8) Метод малого параметра
(13) Овенден, 1973 г.	(14) Истинное движение	$H = T - U_н - U_{ij}$ $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t U_{ij} dt = \min$	+ $M = \text{fix}$	(15) Модельные расчеты
(16) Белецкий, 1975 г.	(14) Истинное движение	$\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t U dt = \max$	+	(17) Численный эксперимент

Key: (1). Author. (2). Extreme trajectory. (3). Extreme principle. (4). Stability. (5). Conclusiveness. (6). Poincare, 1892. (7). Generating. (8). Small parameter method. (9). Blekhman, 1960. (10). Approximately. (11). orbital motion $\sigma<0$. (12). rotation $\sigma>0$. (13). Ovenden, 1973. (14). Proper motion. (15). Model calculations. (16). Beletskiy, 1976. (17). Numerical experiment.

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INTEGRALS OF EQUATIONS OF MOTION IN A LIMITED CIRCULAR THREE-BODY PROBLEM.

I. M. Sidorov.

Analytical expressions for integral of conservative system with periodic excitation and analytical expressions for family of integrals of autonomous conservative system are constructed. Integrals are represented by the sum of the analytical expressions, arranged/located according to the degrees of the low parameter. A qualitative difference in the behavior of the solutions in the vicinity of equilibrium in the resonance and nonresonant cases follows from the analysis of integrals. Integrals for the system of equations, which corresponds to the disturbed motion near one of the points of libration for the limited circular three-body problem, are constructed. The obtained results will be coordinated with the known solutions of this problem.

Results given in work can be used both in celestial mechanics and dynamics of nonlinear multiple degree-of-freedom systems.

Further development and improvement of objects of rocket and space technology requires solution of whole series of problems of mechanics of space flight and dynamics of space vehicle taking into account such design features as mobility of liquid in fuel tanks and elastic vibrations of housing. G. S. Narimanov [3], [5] introduced the large contribution to the solution of these problems.

Forces, which function on KA, can be classed as follows. First, these are conservative forces with the symmetrical matrix of coefficients with the generalized coordinates, in the second place, circulation forces with the skew-symmetric matrix with the generalized coordinates, namely, the thrust of engines and controlling effects. Thirdly, dispersive and gyroscopic forces are considered.

Conservative forces are determining for describing structure of corresponding systems of equations. As far as circulation, dispersive and gyroscopic forces are concerned, for them is completely admissible assumption about the fact that the corresponding coefficients enter into equations with the low parameter.

In classical mechanics, including in celestial mechanics, are examined only conservative systems, but dispersive forces either in no way are considered or are considered as disappearing low. Hence it follows that the development of the methods of the analysis of conservative systems remains urgent problem, also, for the dynamics of KA.

Recently they will achieve considerable progress in development of calculating methods of analysis of conservative systems. Together with the development of calculating methods the methods of analysis, based on the research of general solutions, retain the specific value. For the conservative systems is known only one nontrivial integral - integral of kinetic energies. Such integrals in the general case were not obtained for the cooperative systems.

1. Integral for autonomous systems with periodic effect. Let us consider the system of the disturbed harmonic oscillators

$$\ddot{x}_k + \omega_k^2 x_k = \varepsilon v_k (x_1, \ldots, x_n, \ p(t)); \tag{1}$$

ϵ - low parameter; p(t) - periodic function with the period $2\pi/\omega_0$.

It is assumed that there is function V such, that

$$v_k (x_1, \ldots, x_n, \ p(t)) = \frac{\partial V}{\partial x_k} (x_1, \ldots, x_n, \ p(t)). \tag{2}$$

Let us consider first single degree-of-freedom system relative to variable x. Are chosen sufficiently small integers p_0, p_1 such, that $p_0 : p_1 \approx \omega : \omega_1$. Let us designate $\omega_0 = \omega/p_0$, $\omega_{10} = \omega_0 p_1$; $\Delta\omega = \omega_1 - \omega_{10}$. Value $\Delta\omega$ also is considered as the low parameter. Let us fulfill the replacement of variables according to asymptotic methods [1]:

$$x = a \cos (\omega_{10} t + \varphi); \ \dot{x} = -a\omega_{10} \sin (\omega_{10} t + \varphi). \tag{3}$$

On variables a, φ are placed conditions:

$$\dot{a} \cos (\omega_{10} t + \varphi) - \dot{\varphi} a \sin (\omega_{10} t + \varphi) = 0. \tag{4}$$

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System of equations is reduced to the form

$$\dot{a}=\Delta\omega(\omega_1+\omega_{10})\,a\sin(2\omega_{10}t+2\varphi)/2\omega_{10}-v_1(a\cos(\omega_{10}t+\varphi)\,p(t))\times \times\sin(\omega_{10}t+\varphi)\,\omega_{10}; \tag{5}$$

$$\dot{\varphi}=\Delta\omega(\omega_1+\omega_{10})\cos^2(\omega_{10}t+\varphi)/\omega_{10}-v_1(a\cos(\omega_{10}t+\varphi)\,p(t))\times \times\cos(\omega_{10}t+\varphi)/a\omega_{10}.$$

Equations (5) can be represented thus:

$$\dot{a}=\frac{\partial c_0}{a\partial\varphi}+\frac{\partial f}{a\partial\varphi};\ \dot{\varphi}=-\frac{\partial c_0}{a\partial a}-\frac{\partial f}{a\partial a}. \tag{6}$$

Function $c_0(a, \varphi)$ does not explicitly contain t, while function $f(a, \varphi, t)$ - periodic on variable t with period $2\pi/\omega_0$. The integral of system of equations (5) is obtained in the form of the sum of some analytical expressions, relative to functions c_0, f, arranged/located according to the degrees of the low parameter. Integral can be obtained with any of values ϵ; however, with the low ϵ it suffices to be bounded to first terms in the integral. Let us multiply first and second equations (5) respectively on

$\frac{\partial c_0}{\partial a}$, $\frac{\partial c_0}{\partial \varphi}$ and let us sum. Integrating both parts of the equality, we will obtain

$$c_0=\int\left(\frac{\partial c_0}{a\partial a}\frac{\partial f}{\partial \varphi}-\frac{\partial c_0}{a\partial \varphi}\frac{\partial f}{\partial a}\right)dt=\Delta_{11}+J_2;\ \Delta_{11}=\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial \varphi}-\frac{\partial c_0}{a\partial \varphi}\frac{\partial f^*}{\partial a};\qquad(7)$$

$$J_2=\int\left[\frac{\partial}{\partial a}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial \varphi}-\frac{\partial c_0}{a\partial \varphi}\frac{\partial f^*}{\partial a}\right)\dot a-\frac{\partial}{\partial \varphi}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial \varphi}-\frac{\partial c_0}{a\partial \varphi}\frac{\partial f^*}{\partial a}\right)\dot\varphi\right]dt.\qquad(8)$$

In expressions (7), (8) and in subsequent formulas sign * means that

$\frac{\partial f^*}{\partial \varphi}$ is integral of function

$\frac{\partial f}{\partial \varphi}$ on variable t with fixed values of a, φ. A similar procedure makes it possible to isolate from integral (7) fluctuating component of order ϵ^2. Remainder J_2 is integral of the sum of the products of three functions. In (8) instead of $\dot a$, $\dot\varphi$ we substitute their expressions according to (6)

$$J_2=\int\left[\frac{\partial}{\partial a}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial \varphi}-\frac{\partial c_0}{a\partial \varphi}\frac{\partial f^*}{\partial a}\right)\frac{\partial c_0}{a\partial \varphi}-\frac{\partial}{\partial \varphi}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial \varphi}-\frac{\partial c_0}{a\partial \varphi}\frac{\partial f^*}{\partial a}\right)\frac{\partial c_0}{a\partial a}\right]\times$$
$$\times dt+\int\left[\frac{\partial}{\partial a}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial \varphi}-\frac{\partial c_0}{a\partial \varphi}\frac{\partial f^*}{\partial a}\right)\frac{\partial f}{a\partial \varphi}-\frac{\partial}{\partial \varphi}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial \varphi}-\frac{\partial c_0}{a\partial \varphi}\frac{\partial f^*}{\partial a}\right)\frac{\partial f}{a\partial a}\right]dt.\qquad(9)$$

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First term in (9) is integral of periodic function on t, and for it it is possible to repeatedly use procedure of integration, after isolating periodic component of order ϵ^3

$$\Delta_{21} = -\frac{\partial}{\partial a}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^{**}}{\partial\varphi} - \frac{\partial c_0}{a\partial\varphi}\frac{\partial f^{**}}{\partial a}\right)\frac{\partial c_0}{a\partial\varphi} + \frac{\partial}{\partial\varphi}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^{**}}{\partial\varphi} - \frac{\partial c_0}{a\partial\varphi}\frac{\partial f^{**}}{\partial a}\right)\frac{\partial c_0}{a\partial a}. \qquad (10)$$

Integrand of second integral in (9) let us represent as in (6), in the form of two terms, after isolating components, not depending clearly on t

$$\frac{\partial f^*}{\partial\varphi}\frac{\partial f}{\partial a} = \left(\frac{\partial f^*}{\partial\varphi}\frac{\partial f}{\partial a}\right)_0 + \left(\frac{\partial f^*}{\partial\varphi}\frac{\partial f}{\partial a}\right)_1; \qquad (11)$$

$$\left(\frac{\partial f^*}{\partial\varphi}\frac{\partial f}{\partial a}\right)_0 = \frac{\omega_0}{2\pi}\int\limits_0^{2\pi/\omega_0}\frac{\partial f}{\partial\varphi}\frac{\partial f}{\partial a}\,dt. \qquad (12)$$

Second term in (11) - periodic function. Integral (12) is added to the fixed values of a, φ. Consequently, and from the second integral in (9) it is possible to isolate periodic component of order ϵ^3.

$$\Delta_{22} = -\left[\frac{\partial}{\partial a}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial\varphi} - \frac{\partial c_0}{a\partial\varphi}\frac{\partial f^*}{\partial a}\right)\frac{\partial f}{a\partial\varphi} - \frac{\partial}{\partial\varphi}\left(\frac{\partial c_0}{a\partial a}\frac{\partial f^*}{\partial\varphi} - \frac{\partial c_0}{a\partial\varphi}\frac{\partial f^*}{\partial a}\right)\frac{\partial f}{a\partial a}\right]_1^*. \qquad (13)$$

We convert noncyclic component in (9). In accordance with definition (11) we have

$$\left(\frac{\partial f^*}{\partial\varphi}\frac{\partial f}{\partial\varphi}\right)_0 = 0; \left(\frac{\partial f^*}{\partial\varphi}\frac{\partial f}{\partial a} + \frac{\partial f}{\partial\varphi}\frac{\partial f^*}{\partial a}\right)_0 = 0;$$

$$\left(\frac{\partial f^*}{\partial a}\frac{\partial f}{\partial a}\right)_0 = 0. \qquad (14)$$

Using (12), and also second equality (14) during other combinations of indices, we will obtain that under integral in (9) remains

$$dJ_2 = -\frac{\partial c_0}{a\partial a}\frac{\partial}{\partial\varphi}\left(\frac{\partial f^*}{\partial a}\frac{\partial f}{\partial\varphi}\right)_0 + \frac{\partial c_0}{a\partial\varphi}\left(\frac{\partial f^*}{\partial a}\frac{\partial f}{\partial\varphi}\right)_0. \qquad (15)$$

Further in view of equations (6) we have

$$dJ_2 = \frac{\partial}{\partial a}\left(\frac{\partial f^*}{a\partial a}\frac{\partial f}{\partial\varphi}\right)_0 \dot{a} + \frac{\partial}{\partial\varphi}\left(\frac{\partial f^*}{a\partial a}\frac{\partial f}{\partial\varphi}\right)_0 \dot{\varphi} - \frac{\partial}{\partial a}\left(\frac{\partial f^*}{a\partial a}\frac{\partial f}{\partial\varphi}\right)_0 \frac{\partial f^*}{a\partial\varphi} + $$
$$+ \frac{\partial}{\partial\varphi}\left(\frac{\partial f^*}{a\partial a}\frac{\partial f}{\partial\varphi}\right)_0 \frac{\partial f}{a\partial a}. \qquad (16)$$

Thus is selected noncyclic component Δ_{10} of order ϵ^2 and periodic component Δ_{23} of order ϵ^3

$$\Delta_{10} = \left(\frac{\partial f^*}{a\partial a}\frac{\partial f}{\partial\varphi}\right)_0; \ \Delta_{23} = -\frac{\partial}{\partial a}\left(\frac{\partial f^*}{a\partial a}\frac{\partial f}{\partial\varphi}\right)_0 \frac{\partial f^*}{a\partial\varphi} + \frac{\partial}{\partial\varphi}\left(\frac{\partial f^*}{a\partial a}\frac{\partial f}{\partial\varphi}\right)_0 \frac{\partial f^*}{a\partial a}. \qquad (17)$$

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Consequently, integral c_0 in (7) is represented as follows:

$$c_0 = \Delta_{10} + \Delta_{11} + \Delta_{21} + \Delta_{22} + \Delta_{23} + J_3. \qquad (18)$$

Here term c_0 is of the order ϵ; terms $\Delta_{10}, \Delta_{11} - \varepsilon^2$; $\Delta_{21}, \Delta_{22}, \Delta_{23} - \varepsilon^3$; J_3 - integral of sum of products of four factors. From integral J_3, using conversions, analogous to (9), (11), (16), it is possible to isolate noncyclic components Δ_{20} of order ϵ^3. During the following cycle of conversions from the remaining integral are selected the terms of order ϵ^4 and so forth. It is necessary to show for the unlimited continuation of the process of obtaining the integral that on each m cycle of conversions under the integral proves to be total differential from noncyclic component Δ_{m0} of order ε^{m+1}. The carried out analysis shows that this affirmation is made also for the component Δ_{30}, but general formula for Δ_{m0} thus far could not be obtained. Nevertheless the structure of the corresponding recursion formulas shows that, in all likelihood, the process of the isolations/liberations from the integral of components, the order of smallness of which grows by each cycle of conversions, can be continued unlimitedly.

Expression for integral is obtained analogously for system with n degrees of freedom. The replacement of variables in (1) is produced according to asymptotic method [4]. Small whole numbers are chosen so that $\omega:\omega_{10}:\ldots:\omega_{n0}=p_0:p_1:\ldots:p_n$. Let us designate $\omega_0=\omega/p_0$, $\omega_{k0}=p_k\omega_0$, $k=1,\ldots,n$. Values $\Delta\omega_k=\omega_k-\omega_{k0}$ are assumed to be small. The replacement of variables a_k, φ_k is determined by n relationships/ratios, analogous to (3), and on the variables is placed n conditions of the type (4). System of equations is reduced to the form

$$\dot{a}_k=\frac{1}{\omega_{k0}a_k}\left(\frac{\partial c_0}{\partial\varphi_k}+\frac{\partial f}{\partial\varphi_k}\right);\ \varphi_k=-\frac{1}{\omega_{k0}a_k}\left(\frac{\partial c_0}{\partial a_k}+\frac{\partial f}{\partial a_k}\right). \qquad (19)$$

Simplest analytical expression for integral is obtained, if we represent system (19) in the form

$$\begin{aligned}\dot{a}_k &= \frac{1}{\omega_{k0}a_k}\frac{\partial c_0}{\partial \varphi_k} + \frac{1}{\omega_{k0}a_k}\sum_{j=1}^{\infty}\left(\frac{\partial c_{j1}}{\partial \varphi_k}\cos j\omega_0 t + \frac{\partial c_{j2}}{\partial \varphi_k}\sin j\omega_0 t\right);\\ \dot{\varphi}_k &= -\frac{1}{\omega_{k0}a_k}\frac{\partial c_0}{\partial a_k} - \frac{1}{\omega_{k0}a_k}\left(\sum_{j=1}^{\infty}\frac{\partial c_{j1}}{\partial a_k}\cos j\omega_0 t + \frac{\partial c_{j2}}{\partial a_k}\sin j\omega_0 t\right).\end{aligned} \tag{20}$$

Integral of system (20) with an accuracy down to the terms of order ϵ^2 is following:

$$\begin{aligned}&-c_0(a_k, \varphi_k) + \sum_{k=1}^{n}\sum_{j=1}^{\infty}\frac{1}{2ja_k\omega_{k0}}\left(\frac{\partial c_{j1}}{\partial a_k}\frac{\partial c_{j2}}{\partial \varphi_k} - \frac{\partial c_{j2}}{\partial a_k}\frac{\partial c_{j1}}{\partial \varphi_k}\right) -\\ &-\sum_{k=1}^{n}\sum_{j=1}^{\infty}\left[\frac{\sin j\omega_0 t}{ja_k\omega_{k0}}\left(\frac{\partial c_0}{\partial \varphi_k}\frac{\partial c_{j1}}{\partial a_k} - \frac{\partial c_0}{\partial a_k}\frac{\partial c_{j1}}{\partial \varphi_k}\right) +\right.\\ &\left.+\frac{\cos j\omega_0 t}{ja_k\omega_{k0}}\left(\frac{\partial c_0}{\partial \varphi_k}\frac{\partial c_{j2}}{\partial a_k} - \frac{\partial c_0}{\partial a_k}\frac{\partial c_{j2}}{\partial \varphi_k}\right)\right].\end{aligned} \tag{21}$$

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Let us consider several examples of construction of integral for simple nonlinear systems with periodic excitation. It is illustrated based on these examples, that with different relationships/ratios of the frequency of excitation and frequency of system are possible various forms of the analytical expressions for the integral, which are characteristic for the more general systems, in particular, for the system of equations of the disturbed motion near one of the points of libration in the limited circular three-body problem.

Example 1.

$$\ddot{x}+(1+\beta)x=\alpha x^2\cos 2t. \tag{22}$$

After replacement of variables (3) system is reduced to form

$$\dot{a}=-\frac{\alpha a^2}{8}[\sin(t-\varphi)-\sin(t+3\varphi)-\sin(3t+\varphi)-\sin(5t+3\varphi)]+$$
$$+\frac{\beta a}{2}\sin(2t+2\varphi); \tag{23}$$
$$\dot{\varphi}=-\frac{\beta}{2}-\frac{\alpha a}{8}[3\cos(t-\varphi)+\cos(t+3\varphi)+3\cos(3t+\varphi)+$$
$$+\cos(5t+3\varphi)]+\frac{\beta}{2}\cos(2t+2\varphi).$$

Using representation of right sides of equations in the form (20), it is not difficult to write analytical expression for integral in the form (21) taking into account terms of order α^2

$$-\frac{\beta a^2}{4}+\frac{\alpha^2 a^4}{64}\left(\frac{2}{5}-\cos 4\varphi\right)-\frac{\beta^2 a^2}{16}+\frac{\beta a^2}{8}\Big[\beta\sin(2t+2\varphi)+\frac{\alpha a^2}{6}\times$$
$$\times\cos(3t+\varphi)+\frac{\alpha a}{2}\cos(t+3\varphi)+\frac{\alpha a^2}{16}\cos(5t+3\varphi)-\frac{\alpha a}{2}\cos(t-\varphi)\Big]=C. \tag{24}$$

If frequencies of system (22) relate as 1:2, then β=0 and in integral there remains only second term. With β=0 it is possible to write out following terms of order α^3 according to (18), since in this case is nonzero only expression for Δ_{23}. Integral will be following:

$$a^3\left(\frac{2}{5}-\cos 4\varphi\right)\left(a-4\frac{\partial f^*}{\partial\varphi}\right)-4a^3\alpha\sin 4\varphi\frac{\partial f^*}{\partial a}=C. \qquad (25)$$

For checking the numerical integration of system (23) was carried out with different values of parameter α. The results of integration show that obtained thus solution satisfies (25) with the accuracy α^2.

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When the relation of the frequencies of the system is not equal to 1:2, with the sufficiently low value of variable a the system retains stability, since in integral (24) determining is the first term, i.e., quadratic form. With $\beta\to 0$ the stability region is reduced, and if $\beta=0$, then the behavior of the solution is determined by the second term in (24). In this case is possible the unlimited increase of the variable a. Thus, the solutions of system in the presence of a precise resonance qualitatively differ from the nonresonant case. This occurs because the first term in integral (24) is negatively determined, and the second term can reverse the sign in vicinity a=0.

Example 2. System with the parametric resonance. In contrast to the integral of the system, examined in example 1, here the first term of the corresponding integral can change sign in vicinity of zero

$$\ddot{x}+(1+\beta)x=\alpha x\cos 2t. \qquad (26)$$

After replacement of variables (3) system is reduced to the form

$$\dot{a} = -\frac{\alpha a}{4}\sin 2\varphi - \frac{\alpha a}{4}\sin(4t + 2\varphi) + \frac{\beta a}{2}\sin(2t + 2\varphi);$$
$$\dot{\varphi} = \frac{\beta}{2} - \frac{\alpha}{4}\cos 2\varphi - \frac{\alpha}{4}\cos(4t + 2\varphi) - \frac{\beta}{2}\cos(2t + 2\varphi). \quad (27)$$

Integral of system (27), according to (21), takes following form:

$$-(\alpha a^2\cos 2\varphi - \beta a^2) + \frac{1}{4}\left[\left(\beta^2 a^2 + \frac{\alpha^2 a^2}{4} - \alpha\beta a^2\cos 2\varphi\right)\right] +$$
$$+\frac{\beta^2 a^2}{2}\cos(2t + 2\varphi) - \frac{\alpha^2 a^2}{2}\sin 2t\sin 2\varphi - \frac{\alpha\beta a^2}{2}\cos 2t +$$
$$+\frac{\alpha^2 a^2}{8}\cos 4t - \frac{a^2\alpha\beta}{8}\cos(4t + 2\varphi) = C. \quad (28)$$

If $\alpha \geq \beta$, then first term in integral (28) can change sign, i.e., is possible unlimited increase of variable a. If $\alpha < \beta$, then motion in vicinity a=0 is stable. The following terms in (28) do not vary qualitatively the behavior of the solution, but only make more precise stability limits.

Example 3. Let us consider the system, whose corresponding integral has the fixed-sign terms of the first and second order of the smallness

$$\ddot{x} + x = \alpha x^2 \cos 4t. \quad (29)$$

It suffices to consider system (29) for case of precise resonance, with β=0, since by analogy with example 1 with $\beta \neq 0$ first term in appropriate integral for system (29) is fixed-sign and is equal to $-\beta a^2/4$.

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In the variables a, φ system (29) is converted as follows:

$$\dot{a} = -\frac{\alpha a^2}{8}[\sin(5t+\varphi) - \sin(3t-\varphi) - \sin(t-3\varphi) + \sin(7t+3\varphi)];$$

$$\dot{\varphi} = -3\frac{\alpha a^2}{8}\left[\cos(5t+\varphi) + \cos(3t-\varphi) + \frac{1}{3}\cos(t-3\varphi) + \right.$$
$$\left. + \frac{1}{3}\cos(7t+3\varphi)\right]. \qquad (30)$$

Integral (30) taking into account terms of second order α^2

$$\frac{\alpha^2 a^4}{128}\left(1+1-\frac{3}{5}-\frac{1}{7}\right)=C,$$

i.e. retains constant sign in vicinity a=0.

Examination of examples 1, 2, 3 shows that with different relationships/ratios of frequency of system and frequency of excitation and depending on form of nonlinear function $v_1(x)$ substantially are changed analytical expressions of integral. For this very reason during the analysis of system by the methods of perturbation theory appear the difficulties, connected with the problem of small denominators.

Method of construction of integral further is applied to autonomous system.

2. Analytical expression of integral of conservative autonomous

system.

Nonlinear autonomous system

$$\ddot{x}_k + \omega_k^2 x_k = \varepsilon v_k (x_1, \ldots, x_n), \tag{31}$$

is examined, where ϵ - small parameter: function V satisfies conditions

$$\frac{\partial V}{\partial x_k}(x_1, \ldots, x_n) = v_k (x_1, \ldots, x_n). \tag{32}$$

Let us begin analysis from examination of simple nonlinear system

$$\ddot{x} + \omega^2 x = \alpha x^3. \tag{33}$$

With replacement of variables (3) under condition (4) we reduce equations to the form

$$\dot{a} = -\frac{\alpha a^3}{\omega} \cos^3(\omega t + \varphi) \sin(\omega t + \varphi); \; \dot{\varphi} = -\frac{\alpha a^2}{\omega} \cos^4(\omega t + \varphi). \tag{34}$$

With an accuracy to the terms of order α^2 the integral has the following expression:

$$C_1 = a^4 - \frac{\alpha a^6}{2\omega^2}\left[3 \cos(2\omega t + 2\varphi) + \frac{3}{4} \cos(4\omega t + 4\varphi) - \frac{17}{8}\right]. \tag{35}$$

Let us multiply first equation (34) by a and, after integrating it taking into account (4), we will obtain integral of kinetic energies

$$J_1 = a^2 - \frac{\alpha a^4}{2\omega} \cos^4(\omega t + \varphi). \tag{36}$$

Examination of (35), (36) shows that integrals C_1 and J_1 coincide with an accuracy to the terms of order α^2.

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Let us show that if we consider system (31) with two degrees of freedom, then the corresponding integral C_{21}, determined according to (21), will not be equivalent to the integral of kinetic energies J_2, i.e., this is new integral for the system. Further examination let us lead based on the sufficiently simple example of the system, which satisfies condition (32):

$$\ddot{x}_1+x_1=2\alpha x_1x_2;\ \ddot{x}_2+(2+\varepsilon)^2x_2=\alpha x_1^2. \tag{37}$$

Relation of frequencies in (37) is close to 1:2 (p_1=1; p_2=2). The replacement of the variables

$$x_1=a_{11}\cos(t+\varphi_{11});\ x_2=a_{12}\cos(2t+\varphi_{12}) \tag{38}$$

reduces system to following form ($\varepsilon_1=\varepsilon(4+\varepsilon)$):

$$\dot{a}_{11}=-\frac{\alpha a_{11}a_{12}}{2}[\sin(4t+2\varphi_{11}+\varphi_{12})+\sin(2\varphi_{11}-\varphi_{12})];$$

$$\dot{\varphi}_{11}=-\frac{\alpha a_{12}}{a}[\cos(4t+2\varphi_{11}+\varphi_{12})+\cos(2\varphi_{11}-\varphi_{12})+2\cos(2t+\varphi_{12})];$$

$$(39)$$

$$\dot{a}_{12}=-\frac{\alpha a_{11}^2}{8}[\sin(4t+2\varphi_{11}+\varphi_{12})-\sin(2\varphi_{11}-\varphi_{12})+2\sin(2t+\varphi_{12})]+$$
$$+\frac{\varepsilon_1 a_{12}}{4}\sin(4t+2\varphi_{12});$$

$$\dot{\varphi}_{12} = -\frac{\alpha a_{11}^2}{8a_{12}}\Big[\cos(4t+2\varphi_{11}+\varphi_{12})+\cos(2\varphi_{11}-\varphi_{12})+2\cos(2t+\varphi_{12}))+$$
$$+\frac{\varepsilon_1}{4}(1+\cos(4t+2\varphi_{12})\Big]$$

with satisfaction of the conditions

$$\dot{a}_{1k}\cos(p_k t+\varphi_{1k})-\dot{\varphi}_{1k}a_{1k}\sin(p_k t+\varphi_{1k})=0\quad (k=1,\ 2). \tag{40}$$

System of equations (39) satisfies conditions, necessary for construction of integral (21), which with an accuracy to α^2 takes following form:

$$-C_{21} = -\varepsilon_1 a_{12}^2 - \alpha a_{11}^2 a_{12}\cos(2\varphi_{11}-\varphi_{12})+\alpha^2 a_{11}^2 a_{12}^2\sin(2t+\varphi_{12})\times$$
$$\times\sin(2\varphi_{11}-\varphi_{12})+\frac{\alpha^2 a_{11}^4}{8}[\cos(2t+2\varphi_{11})+\cos(4t+4\varphi_{11})/4-9/8]+$$
$$+\frac{\alpha^2 a_{11}^2 a_{12}^2}{4}\left[\cos(4t+2\varphi_{12})-\frac{1}{2}\right]+\frac{\varepsilon_1^2 a_{12}}{8}\left[\cos(4t+2\varphi_{12})-\frac{1}{2}\right]+$$
$$+\frac{\alpha\varepsilon_1 a_{11}^2 a_{12}}{4}[\cos(2t+\varphi_{12})-\cos(2t+3\varphi_{12})/4+\cos(4t+2\varphi_{11}+\varphi_{12})/2]. \tag{41}$$

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Interval of kinetic energies in variables a_k, φ_k for system (37)

$$J_2 = -\left(\frac{a_{11}^2}{2}+2a_{12}^2\right)+\alpha a_{11}^2 a_{12}\cos^2(t+\varphi_{11})\cos(2t+\varphi_{12})+$$
$$+\frac{\varepsilon a_{12}^2}{2}\cos(2t+\varphi_{12}). \tag{42}$$

Integrals J_2, C_{21} are not equivalent. Actually, it follows from formula (42) for J_2 that under the low initial conditions the behavior of the solution is determined by the first term in (42), i.e., values a_{11}, a_{12} remain limited and from [41] it follows that is possible the unlimited increase of a_{11}. Integral C_{21} is constructed on the basis of the regulation frequency ω_1=1 of first equation (37). Analogously is constructed integral C_{22} on the basis of the frequency $\omega_2=2+\epsilon$ of the second equation. The replacement of the variables

$$x_1 = a_{21}\cos(p_1\omega_2 t/p_2 + \varphi_{21});\ x_2 = a_{22}\cos(\omega_2 t + \varphi_{22}) \tag{43}$$

reduces (37) to the system of equations relative to new variables a_{2k}, φ_{2k}. The obtained system is analogous to (39), but instead of two latter/last terms with the coefficient ϵ_1 in the equations for $\dot{a}_{12}$, $\dot{\varphi}_{12}$ in the new system appear corresponding terms in the equations for $\dot{a}_{21}$, $\dot{\varphi}_{21}$ with the coefficient

$\varepsilon_2 = -\varepsilon\left(1+\frac{\varepsilon}{4}\right)$. To the variables the conditions

$$\dot{a}_{2k}\cos(p_k\omega_2 t/p_2 + \varphi_{2k}) - \dot{\varphi}_{2k}a_{2k}\sin(p_k\omega_2 t/p_2 + \varphi_{2k}) = 0 \tag{44}$$

are superimposed.

Trigonometric relationships/ratios

$$\begin{gathered} a_{1k}\cos(p_k t + \varphi_{1k}) = a_{2k}\cos(p_k\omega_2 t/p_2 + \varphi_{2k}); \\ a_{1k}\sin(p_k t + \varphi_{2k}) = a_{2k}\omega_2/p_2\cos(p_k\omega_2 t/p_2 + \varphi_{2k}) \end{gathered} \tag{45}$$

follow from (38), (40), (44), (43).

With given values of parameters

$$a_{11}^2 = a_{21}^2 + \varepsilon\left(1+\frac{\varepsilon}{4}\right)\sin^2\left[\left(1+\frac{\varepsilon}{2}\right)t+\varphi_{21}\right];$$
$$a_{12}^2 = a_{22}^2 + \varepsilon(4+\varepsilon)\sin^2[(2+\varepsilon)t+\varphi_{22}]. \quad (46)$$

Like integral (41), integral C_{22} has two terms of order α, ϵ

$$C_{22} = -2\varepsilon a_{21}^2 - 2\alpha a_{21}^2 a_{22}\cos(2\varphi_{21}-\varphi_{22}) + 0(\alpha,\ \varepsilon). \quad (47)$$

Following terms in (47) of order α^2 can be obtained as in integral (41). Comparison of (41) and (47) shows that in the absence of a precise resonance $\epsilon \neq 0$, integrals C_{21} and C_{22} are not equivalent. However, if we join the integral of kinetic energies J_2, then only every two are independent of these three integrals.

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Thus, the reference system of equations is reduced to 6 analytical expressions (41), (47), (45) and to two first-order equations (40). Let us note that on the basis of integrals C_{21}, C_{22} by the appropriate selection of the coefficients of their linear combination it is possible to form the integral, whose dominant term is the positively determined quadratic form. With the resonance relationships/ratios of frequencies the behavior of the solution qualitatively is changed. If $\epsilon \neq 0$, then determining for the behavior of the solution near $a_{11}=a_{12}=0$ is the first term in C_{21}. All subsequent terms of order α^2 and above have higher degree relative to the variables a_{11}, a_{12} in comparison with the second term in C_{21}.

For system (1) with n degrees of freedom, if relation of frequencies of this system is not equal to relation of integers, can be constructed n integrals $C_{n,j}$ $(j=1, \ldots, n)$. The dominant term of each integral is the linear combination of n-1 terms $a^2{}_k$ $(k=1, \ldots, n;\ k \neq j)$.

3. Construction of integrals of equations of disturbed motion near one of points of libration for limited circular three-body problem. System of equations and corresponding designations are given in [2] and take the following form:

$$\ddot{\xi} - 2\dot{\eta} - \frac{3}{4}\eta - k\eta = \frac{\partial H}{\partial \xi};$$

$$\ddot{\eta} + 2\dot{\xi} - \frac{9}{4}\eta - k\xi = \frac{\partial H}{\partial \eta}; \qquad (48)$$

$$H = \frac{\sqrt{3}}{4}\left(-\frac{7}{9}k\xi^3 + \frac{3}{4}\xi^2\eta + \frac{11k}{3}\xi\eta^2 + \frac{3}{4}\eta^3\right);\ k = \frac{3\sqrt{3}}{4}(1-2\mu).$$

Structure of system of equations does not coincide with appropriate structure of system of equations (31), nevertheless, using more general approach to construction of asymptotic method [4], it is possible to reduce equations (348) to form required for of integral.

Let us assume ω_1, ω_2 - roots of characteristic equation of linear part of system

$$\begin{vmatrix} \omega^2 + \frac{3}{4} & k + 2i\omega \\ k - 2i\omega & \omega^2 + \frac{9}{4} \end{vmatrix} = 0; \qquad (49)$$

p_1, p_2 - small integers, selected so that $\omega_1 : \omega_2 \approx p_1 : p_2$.

Integrals of system (48) are constructed in the same way as this was shown based on example of system (31). Is chosen regulation frequency ω_1, the linear part of the system with the help of the identity transformation is changed so that the relation of changed frequencies $\omega_{10}:\omega_{20}$ would prove to be accurately equal to $p_1:p_2$; $\omega_{10}=p_1\omega_0$; $\omega_{20}=p_2\omega_0$.

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For this let us supplement into the second equation the term $-\epsilon_1\eta$, and let us replace function H by $H_1=H-\varepsilon_1\eta^2/2$. The parameter ϵ_1 is selected so that the characteristic equation

$$\begin{vmatrix} \omega^2+\frac{3}{4} & k+2i\omega \\ k-2i\omega & \omega^2+\frac{9}{4}+\varepsilon \end{vmatrix}=0 \qquad (50)$$

has roots ω_{10}, ω_{20}. It follows from (50) that

$$\left(\omega_{10}^2+\frac{3}{4}\right)\left(\omega_{20}^2+\frac{3}{4}\right)=3-k^2. \qquad (51)$$

Replacement of variables in converted thus system is following:

$$\xi=ka_1\cos(\omega_1 t+\varphi_1)+ka_2\cos(\omega_2 t+\varphi_2)-$$
$$-2a_1\omega_1\sin(\omega_1 t+\varphi_1)-2a_2\omega_2\sin(\omega_2 t+\varphi_2);$$
$$\eta=-\left(\omega_1^2+\frac{3}{4}\right)a_1\cos(\omega_1 t+\varphi_1)-\left(\omega_2^2+\frac{3}{4}\right)a_2\cos(\omega_2 t+\varphi_2). \qquad (52)$$

Coefficients in (52) correspond to coordinates of eigenvectors of linear part of system. To variables a_j, φ_j the conditions

$$\sum_{j=1}^{2}[\dot{a}_j(k\cos(\omega_j t+\varphi_j)-2\omega_j\sin(\omega_j t+\varphi_j))-$$
$$-\dot{\varphi}_j a_j(k\sin(\omega_j t+\varphi_j)+2\omega_j\cos(\omega_j t+\varphi_j))]=0.$$
$$\sum_{j=1}^{2}\left[\dot{a}_j\left(\omega_j^2+\frac{3}{4}\right)\cos(\omega_j t+\varphi_j)-a_j\dot{\varphi}_j\left(\omega_j^2+\frac{3}{4}\right)\sin(\omega_j t+\varphi_j)\right]=0. \quad (53)$$

are superimposed.

After substitution (52), (53) into (48) we obtain two additional equations

$$-\sum_{j=1}^{2}[\dot{a}_j\omega_j(k\sin(\omega_j t+\varphi_j)+2\omega_j\cos(\omega_j t+\varphi_j))+\dot{\varphi}_j a_j\omega_j(k\cos(\omega_j t+\varphi_j)-$$
$$-2\omega_j\sin(\omega_j t+\varphi_j))]=\frac{\partial H_1}{\partial\xi}; \quad (54)$$
$$\sum_{j=1}^{2}\left(\omega_j^2+\frac{3}{4}\right)[\dot{a}_j\omega_j\sin(\omega_j t+\varphi_j)+\dot{\varphi}_j\omega_j a_j\cos(\omega_j t+\varphi_j)]=\frac{\partial H}{\partial\eta}.$$

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From (53), (54) we determine equations for $\dot{a}_j, \dot{\varphi}_j$, which after series of conversions with use of (50) and (51) are reduced to following form:

$$\dot{a}_j=\frac{1}{\omega_j A_j a_j}\left(\frac{\partial H_1}{\partial\xi}\frac{\partial\xi}{\partial\varphi_j}+\frac{\partial H_1}{\partial\eta}\frac{\partial\eta}{\partial\varphi_j}\right)=\frac{1}{\omega_j A_j a_j}\frac{\partial H_1}{\partial\varphi_j}; \quad (55)$$
$$\dot{\varphi}_j=-\frac{1}{\omega_j A_j a_j}\left(\frac{\partial H_1}{\partial\xi}\frac{\partial\xi}{\partial a_j}+\frac{\partial H_1}{\partial\eta}\frac{\partial\eta}{\partial a_j}\right)=-\frac{1}{\omega_j A_j a_j}\frac{\partial H_1}{\partial a_j};$$
$$A_1=(\omega_2^2-\omega_1^2)\left(\omega_1^2+\frac{3}{4}\right); \quad A_2=(\omega_1^2-\omega_2^2)\left(\omega_2^2+\frac{3}{4}\right).$$

Consequently, system (48) is reduced to the form, for which is possible construction of integral. According to (20) let us represent right sides of (55) in the form

$$\frac{\partial H_1}{\partial \varphi_j}=\frac{\partial c_0}{\partial \varphi_j}+\sum_{l=1}^{\infty}\left(\frac{\partial c_{l1}}{\partial \varphi_j}\cos l\omega_0 t+\frac{\partial c_{l2}}{\partial \varphi_j}\sin l\omega_0 t\right);$$

$$\frac{\partial H_1}{\partial a_j}=\frac{\partial c_0}{\partial a_j}+\sum_{l=1}^{\infty}\left(\frac{\partial c_{l1}}{\partial a_j}\cos l\omega_0 t+\frac{\partial c_{l2}}{\partial a_j}\sin l\omega_0 t\right). \tag{56}$$

Integral C_{21} of system is determined by formulas (21). If the relationship/ratio of initial system $\omega_1:\omega_2$ is not equal to $p_1:P_2$, i.e., $\epsilon\neq 0$, then dominant term in the integral, which is determining for the behavior of the solution in the vicinity of the point of libration, is $\varepsilon_1 a_1^2/2$.

Further is constructed second integral for system (55) relative to regulation frequency ω_2. To the right and the left sides of the first equation of system is supplemented the term $-\epsilon_2\xi$, moreover ϵ_2 is selected so that the roots of the characteristic equation

$$\begin{vmatrix} \omega^2+\frac{3}{4}+\varepsilon_2 & k+2i\omega \\ k-2i\omega & \omega^2+\frac{9}{4} \end{vmatrix}=0 \tag{57}$$

have relatinship $\omega_{11}:\omega_{21}=p_1:p_2$.

Second integral C_{22}, which is defined just as C_{21}, has dominant term $\varepsilon_2 a_2^2/2$. From integrals C_{21}, C_{22} with the help of the linear combination it is possible to form the integral, whose dominant term is the positively determined quadratic form, and all the remaining terms have the higher order of smallness. Obtained thus integral proves stability of motion near the point of libration with the nonresonant relationships/ratios of frequencies.

When relationship/ratio of frequencies of reference system is accurately equal to resonance $\omega_1:\omega_2=p_1:p_2$, then $C_{21}=C_{22}=C$ and quadratic term is absent. In integral C determining is the following term:

$$\frac{1}{\omega_1 A_1}\left(\frac{\partial H}{a\partial a_1}\ \frac{\partial H}{\partial\varphi_1}\right)_0+\frac{1}{\omega_2 A_2}\left(\frac{\partial H}{a\partial a_2}\ \frac{\partial H}{\partial\varphi_2}\right)_0=C. \qquad (58)$$

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For determining integral according to formulas (21) should be represented right sides of equations in the form of sum of trigonometric functions, whose arguments are following: $\mp n_1(\omega_1 t+\varphi_1)\mp n_2(\omega_2 t+\varphi_2)$ $(n_1,\ n_2=0,\ 1,\ 2)$. In different relations of frequencies there can be three different types of the resonance relationships/ratios, which, in principle, correspond to examined above examples 1, 2, 3 for the single degree-of-freedom systems.

If $\omega_1:\omega_2=1:2$, then in right sides of equations (55) are contained terms, which explicitly do not depend on t. It is not difficult to show that when $\omega_2:\omega_2=1:2$ into integral enters as dominant term the component

$$C_0=a_1^2a_2\cos(2\varphi_1-\varphi_2).$$

This case corresponds to example 2. Consequently, C_0 can change sign, i.e., the analysis of integral C does not contradict known result of [2], then the motion near the point of libration is unstable when $\omega_1:\omega_2=1:2$. When $\omega_1:\omega_2=1:3$ into expressions for the right sides of the equations enter two sinusoids with arguments $\omega_1 t+\varphi_1$, $\omega_1 t-2\varphi_1+\varphi_2$. The instability is possible by analogy with example 1 in this case in the system. Case $\omega_1:\omega_2=1:1$ must be examined especially, since it is assumed in (52) that $\omega_1\neq\omega_2$. With all remaining integral relationships/ratios of frequencies $\omega_1:\omega_2$ into the right sides of the equations enter the harmonics, whose frequencies are not equal to zero, and these frequencies do not coincide with each other, which corresponds to example 3. In this case dominant term (58) of integral does not depend on φ_j and it is biquadratic form relative to variables a_j. It is necessary to show for the proof of stability of motion near the point of libration that this form is positively determined.

Summing up sums, let us formulate basic results and although let us briefly plan enumeration of unresolved problems. The procedure of obtaining analytical expression for the integral of conservative system with the periodic excitation and the family of the integrals of autonomous conservative systems is constructed. Integrals are the analytical expressions, arranged/located according to the degrees of small parameter. General formulas for the terms of this resolution are given. The examination of a whole series of examples shows that the basic special features of the behavior of the solution are determined by two first terms of expansion. Only at the worst, with some values of the parameters of system, the analysis also of the third term of expansion is necessary. However, for the affirmation about stability of motion it is necessary to show that the process of the formation of integral taking into account terms of ever higher order of smallness can be continued unlimitedly. On the basis of the obtained results this affirmation is valid, but for its complete proof it is necessary to obtain recursion formulas so that it would be possible to use the method of induction.

It is shown that with precise resonance relationships/ratios of frequencies character of solution qualitatively is changed, since in integrals dominant terms, which determine behavior of solution, vanish. In this case the behavior of the solution is determined by the terms of integral following in the order, whose analytical expression is substantially different. In contrast to the methods of perturbation theory, the method presented in the work does not encounter the difficulties, connected with the problem of low denominators.

Important problem is propagation of method of determining integral to more general conservative systems than examined in work canonical system of harmonic oscillators. In the specific case of the limited circular three-body problem the method of determining the integral presented succeeds in using despite the fact that the structure of matching systems is not canonical. On the basis of the integrals proposed is possible the creation of calculating algorithms not only for the conservative systems, but also for the systems of more general view. Very fact of the presence of the analytical expression of integrals offers new possibilities for the analysis of systems, but at the given moment the total number of questions, which appear in this case, substantially exceeds the number of responses/answers to them.

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WAVE STRUCTURE, QUANTIZATION AND MEGA-SPECTROSCOPY OF THE SOLAR SYSTEM.

A. M. Chechel'nitskiy.

Are presented short survey/coverage of some methods, ideas, results of mega-quantum wave astrodynamics and concept of wave universe, in particular, in application to observed dynamic structure of solar system.

It is shown that wide volume of new experimental and observational information about its structure can be correctly interpreted within the framework of representations of wave cosmogeonomy, quantization (in large) of solar system and wave resonance. Is discussed the possibility of the representation of the observed in nature wide spectrum of periodicities (in particular, orbital and rotary planetary motions) as the sets of the rhythms, whose genesis is connected with the wave structure of the solar system and the existence of mega-waves. It is shown that this spectrum belongs to the theoretically computed frequency spectrum of the solar system - its mega-spectroscopy.

In middle of 1980 publishing house "Mashinostroyeniye" let out to light/world work [18], which was opened by Georgiy Stepanovich Narimanov's preface. In this preface, in particular, the following speaks:

"The volume of experimental data rapidly grows. Already considerable scientific information is assembled. The study of this material will make it possible to answer many questions about the structure of the solar system and outer space surrounding us. The theoretical comprehension of many laws and phenomena in the solar system becomes ever more actual. In connection with the development/detection of some special features in the motion of planets and their satellites the necessity of theoretical examination and proof of some questions of astrodynamics appeared.

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The resonance properties of motion, the defined sequence of the values of planetary distances and other special features, observed in the structure of the solar system and in the motion of planets, cannot be considered as the random phenomena - they must find their explanation as the manifestation of a specific natural law. Therefore it is very tempting to investigate the possibility of the proof of some problems of astrodynamics, on the basis of other concepts, which use, for example, representation of wave dynamics".

Which is fate of ideas and propositions, about which is mentioned above?

In 1982 suddenly in foreign periodics arose unusual interest in theme, which usually did not stand in center of attention of science about planetary motion and structure of astronomical systems - to theme of quantization (in large) and wave structure of universe mega-systems. It is possible to state/establish this phenomenon as the virtually simultaneous entrance of two in principle similar works from different countries on one and the same theme and in one and the same journal "The Moon and the Planets" [44, 35]. The theme of these works can be described as the mega-quantum wave structure of astronomical systems. It is not difficult to note that it is new not only for the journal, but (judging by the previous publications) for authors themselves.

Then_appears whole series of works of Louise [36, 37] on considered theme, including in journal "Astrophysics and Space Science" [38]. One cannot fail to note the intensity of the flow of the publications, for which are characteristic generality from [18] approach to the problem, fundamental ideas, line of reasoning, proofs. Now and then are observed straight/direct coincidences with [18] (it suffices to compare, for example, Fig. 1 of work [44] with Fig. 17 of monograph [18]).

Oldershaw with latitude of scope/coverage of material inherent in it and with sincere gladness greets appearance of new evidence (new evidences) in favor of phenomenological unity of structure of physical systems of universe [40, 39]. But the well-known American scientist Greenberger in one of the most authoritative (judging to the composition of editorial board) physical journals "Foundation of

Physics" in his 49-page article with the enthusiasm calls astrophysicists to focus attention on the prospect of research of the fact that he calls "Quantization in the Large" (i.e. mega-quantum effects or quantization in the large) during the analysis of astronomical systems [33].

Let us let again word to G. S. Narimanov and will return tribute to his insight. In the preface to work [18] there are these words: "Without depending on the possible controversy of the series of the parts of the wave concept of the author one cannot fail to note attractiveness and the enormous heuristic charge, which these representations carry.

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But such concepts of mega-quantum wave astrodynamics as wave dynamic systems, astro-dynamic spectroscopy or new view on the wave genesis of the effects of resonance and commensurability in the solar system, have a right for the serious discussion.

Propositions of author can be considered as completely concrete/specific and structural/design attempt at resolution of those urgent problems, which stand before contemporary astrodynamics and cosmonautics. Should be greeted the same structural/design efforts and other authors, who defend others, including alternative, the point of view in the hope for the fact that the inflow of new ideas and fruitful discussions finally will lead to the resolution of the urgent fundamental problems".

G. S. Narimanov possessed ability to listen to and to understand, to look and to see. And, besides the fact that is not less important, he knew how to function. To the light memory of Georgiy Stepanovich Narimanov is devoted all that represented below, to what is judged to maintain/withstand the testing by time.

Wave universe and mega-quantum wave astrodynamics.

Within the framework of concept of wave universe [18] large astronomical systems can be considered as wave dynamic systems (WDS wave dynamic systems), that are in a sense analogs of system of atom [13]. From this point of view the solar system - is the wave dynamic system, components of which appear as celestial bodies themselves (Sun, planet, satellites, small bodies), and also its interplanetary, continuous filling - material medium (interplanetary plasma, electromagnetic fields, etc.), i.e., substance and field, described in the single dynamic context. The corresponding instrument, which gives the phenomenological and dynamic description of similar systems, is natural to call mega-quantum wave astrodynamics [18].

Fundamental wave equations and quantization of parameters of mega-objects. Central idea is here the fact that the mega-objects in question, being wave dynamic systems as everything WDS of the universe, are described by universal and single equations -

fundamental wave equations (latter in general form are given in work [18]). A special case of these equations is the wave equation

$$\nabla^2\psi + \frac{2}{\tilde{d}^2}[\mathscr{E} - U]\psi = 0, \qquad (1)$$

to which should be joined the boundary condition

$$\psi \neq \infty \overset{(1)}{\text{при}} \; r = \sqrt{x^2 + y^2 + z} \to \infty. \qquad (2)$$

Key: (1). with.

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Here ψ - certain function of space coordinates, limited at infinity; $\mathscr{E} = E^0/m$ - specific (standardized/normalized to mass m) energy E°; U=V°/m - specific (standardized/normalized to mass m) potential energy V°; ∇ - the operator of Hamilton;

$\tilde{d} = \frac{d}{2\pi}$ - fundamental constant of areal velocity, i.e., the constant of specific (standardized/normalized to mass m) action, or moment of momentum.

With particular value $\tilde{d} = d_e = \hbar/m_e = 1{,}16$ cm²/s and U=K/r, where $K = e^2/m_e$, m_e - mass of electron; h - Planck's constant, e - nuclear charge of hydrogen atom; r - radial coordinate of electron, equation (1) - not that different as Schroedinger equation for atom of hydrogen [13].

With other value $\tilde{d} \sim 10^{19}$ cm²/s and $U = K_\odot/r$, where $K = K_\odot$ - gravitational parameter of sun, equation (1) gives wave description of mega-systems of type of solar. From the fundamental wave equations (of type of that given above) follow the laws of the quantization of universe mega-systems, in particular: 1) the linear law of the quantization of areal velocity $L = L_{N=1} N$, where $L = v_a r$ - areal velocity, r - heliocentric radius, v_a - transversal speed; $L_{N=1}$ - level of quantization; $N = N(?)$ - quantum number of areal velocity (N=1, 2, ...); 2) the quadratic law of quantization (distances) of elites [18] - the physically chosen planetary (satellite) orbits - analog Bohr's law in microcosm [13].

If we postulate that solar system is wave dynamic system and, thus, is valid micromega-analogy (MM-analogy) [18], then from dynamic isomorphism of atomic structure and solar system it is possible to obtain series/row of corollaries, by using, in particular, diverse variants of quantization - according to Bohr, according to Sommerfield, according to de Broglie, Schroedinger, etc. Should be specially stressed the possibility of quantization in the mega-systems of nonoperation and moment of momentum (of type $K_m = m v_a r$), as in quantum mechanics of microcosm, and specific (standardized/normalized to the mass) action (moment of momentum), i.e., areal velocity [18]:

$$L = K_m/m = v_a r = [Ka(1-e^2)]^{1/2}; \quad (3)$$
$$L \approx (Ka)^{1/2} \overset{(1)}{\text{при}} \; e \approx 0,$$

Key: (1). with.

where a, e - semimajor axis and orbit eccentricity.

Specifically, this fundamental conclusion gives possibility to obtain picture of quantization in mega-world located in satisfactory agreement with observations, in particular, quantum numbers of elite orbits (semimajor axes of orbits of planets, satellites, asteroids and comets; eccentricities and inclinations), rings of Saturn, spin of celestial bodies. One of the corollaries is the genesis of the law of Titius-Bode and the possibility of the existence of intra-mercurial (problem of Vulcan) and transpluton planets.

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Mega-waves in solar system and trans-sphere. Important corollary (1), which carries fundamental character, is existence in any mega-system of the universe and, in particular, in the solar system, the mega-waves, which realize proximity effect on the scales, commensurate with the scales of system (MG-waves [18, 19, 20] - the analog of de Broglie's waves for the giant astronomical systems). The parameters of mega-waves are connected with following relationships/ratios [18, 19, 20], by the analogous relationships/ratios of Bohr and Planck-Einstein [13]:

$$v = \mathscr{A}\mathscr{K}; \quad \mathscr{E} = \mathscr{A}\Omega, \qquad (4)$$

where $\mathscr{K}$, Ω, $\mathscr{E}$ - wave number, angular frequency and specific (standardized/normalized to the mass) energy of MG-waves; v - speed of Keplerian motion; $\mathscr{A}$ - fundamental constant of the specific action (areal velocity) of the solar system. Specific energy $\mathscr{E}$ allows/assumes in the case in question representation $(K = K_{\odot})$

$$\mathscr{E} = \frac{E^0}{m} = \frac{1}{2} v^2 = \frac{K}{2a} \approx \frac{K}{2r}. \qquad (5)$$

Let us introduce value $C_{MG} = \text{const}$, which has dimensionality of speed

$$C_{MG} = v_* = K/L_*; \quad L_* = 2d, \qquad (6)$$

which we will identify with velocity of propagation of wave disturbances (mega-waves) [19, 20]. In view of (4)...(6) we will obtain

$$\Omega a = K/2d = G_{MG} = \text{const}; \quad \Omega a \approx \Omega r. \qquad (7)$$

Corresponding frequency θ, period $\mathscr{T}$, wavelength Λ and wave number $\mathscr{K}$ are connected with known relationships/ratios

$$\theta = \frac{\Omega}{2\pi}; \quad \mathscr{T} = \frac{1}{\theta}; \quad \Lambda = \frac{C}{\theta}; \quad \mathscr{K} = \frac{2\pi}{\Lambda}. \qquad (8)$$

In order to apply equation (1) to concrete/specific system, it is necessary to assign besides gravitational parameter $K = K_\odot$ one additional fundamental constant, for example d or C_{MG}. Latter/last value (velocity of propagation of mega-waves) is more preferable, since it allows/assumes more distinct physical interpretation.

Most natural is assumption that value C_{MG} lies/rests at area 155 km/s, which corresponds to velocity of propagation of solar wind and magnetosonic waves in interplanetary plasma [1, 5, 28, 30].

Possibility of agreement of calculated quantized values of parameters of solar system, obtained from equation (1), is criterion of correctness of this assumption with appropriate materials of observations. This question is examined below.

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In further analysis are used connected (self-consistent) numerical values of fundamental constants of quantization of solar system $æ$ and C_{MG}, which correspond to integer value $r_*/R_\odot = 8$, where r* - radius of trans-sphere (see below). In this case (see [19...22])

$$C_{MG} = \left(\frac{K_\odot}{r_*}\right)^{1/2} = \left(\frac{K_\odot}{8R_\odot}\right)^{1/2} = 154{,}386 \ \overset{(1)}{\text{км/с}}. \qquad (9)$$

Key: (1). km/s.

Phase speed of mega-waves is such. To it corresponds computed value of the fundamental constant of the action (areal velocity) of the solar system

$$æ = \frac{1}{2} L_* = K_\odot / 2C_{MG} = 0{,}42980 \cdot 10^9 \ \overset{(1)}{\text{км}^2/\text{с}}, \qquad (10)$$

Key: (1). km²/s.

where $K_\odot = 1{,}3272 \cdot 10^{11}$ km³/s² - gravitational parameter of the sun.

At characteristic speed $C = C_{MG}$, commensurate with speeds of rotation of celestial bodies in solar system, is completely probable presence in certain vicinity of sun with $r = r_* = a_*$ physically chosen surface, for which speed of Keplerian of orbital motion v=v* is equal to velocity of propagation of mega-waves C_{MG}:

$$C_{MG} = v_* = (K_\odot / r_*)^{1/2}. \qquad (11)$$

We will call this surface (certain analog of surface of transonic flow in aerohydromechanics), which divides regions $v > C_{MG}$ and $v < C_{MG}$ trans-sphere of solar system (assuming it spherical without depending on real form of this surface). Above value of the velocity of propagation of MG-waves accepted corresponds to the position of trans-sphere at the heliocentric distance of r=r*, accurately equal to 8 radii of the sun:

$$r_* = K_\odot / v_*^2 = K_\odot / C_{MG}^2 = 5567928 \text{ км} = 0{,}037219 \text{ a. e.} = 8 R_\odot. \qquad (12)$$

Here $r_* = a_*$ - radius of trans-sphere; $R_\odot = 696\,000$ км - radius of the sun.

Mega-spectroscopy of the solar system.

Astronomical system (in particular, solar system), considered as wave dynamic system, described by equation (1) with boundary condition (2), can be described by spectrum of natural frequencies. These frequencies can be calculated according to the eigenvalues of boundary-value problem (1), (2). On the other hand, to eigenvalues corresponds the set of stationary elite orbits - the analogs of the steady-state orbits of Bohr in atom [13].

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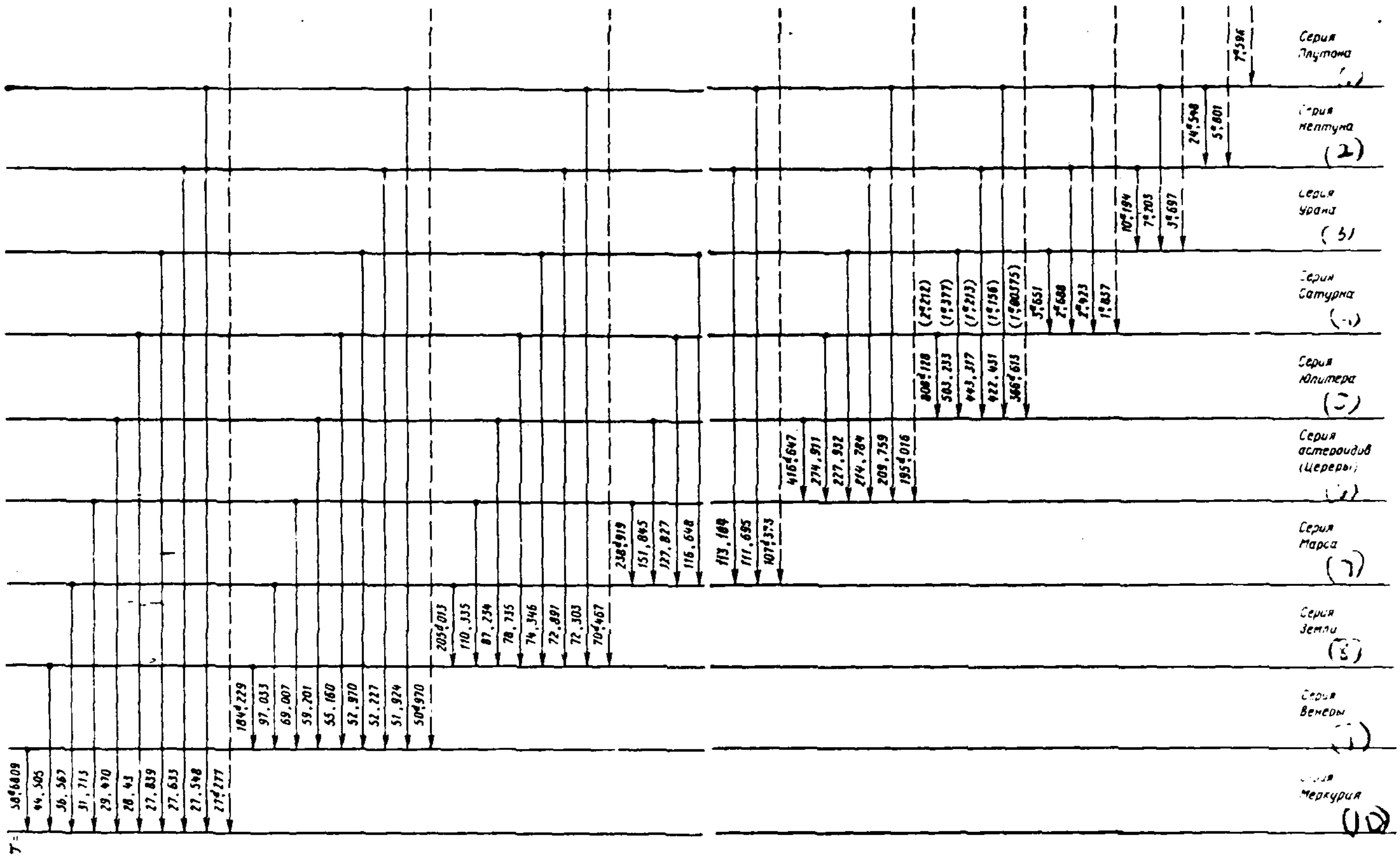
Table 1.



Key: (1). Series of Pluto. (2). Series of Neptune. (3). Series of Uranus. (4). Series of Saturn. (5). Series of Jupiter. (6). Series of asteroids (Ceres). (7). Series of Mars. (8). Series of Earth. (9). Series of Venus. (10). Series of Mercury.

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In view of MM-analogy should be together with other mega-quantum effects [18...25] expected the presence of the spectrum of characteristic for the solar system wave frequencies, including intercombinatory - analog of the frequency spectrum (spectroscopy) of atom [13]. This mega-spectroscopy of the solar system, in which wave periods are characterized already, it goes without saying, not by fractions of a second as in atom system, but by days and for years, it can be the fundamental wave spectrum of solar system [19, 20].

Wave frequency θ depending on linear orbit characteristics (semimajor axis a or its radius r=a in the case of main approximation/approach - circular orbit) can be by virtue of (5)...(7) represented form

$$\theta = \frac{D}{a} \approx \frac{D}{r}, \quad D = \frac{K_{\odot}}{4\pi d}. \qquad (13)$$

To eigenvalues of boundary-value problem (1), (2) correspond fundamental wave frequencies (therms of wave frequencies and some chosen (elite) planetary orbits with semimajor axes $a_i(=r_i)$ and frequencies $\theta_i = D/a_i (i=1, 2, \ldots)$. Together with the fundamental wave frequencies (therms) θ_i let us introduce into the examination different intercombinations θ_{ij} (beating) between them - intercombinatory wave frequencies, in the general case - difference and summation $\theta_{i,j}^{(\mp)} = \theta_i \mp \theta_j$.

Main intercombinations of wave frequencies we will call intercombinations between therms of adjacent elite orbits $\theta_{i,i+1}^{(\mp)} = \theta_i \mp \theta_{i+1}$. The set of the therms of wave frequencies and their intercombinations for the elite states (orbits) of the solar system we will call the fundamental spectrum of the wave frequencies of solar system [19, 20].

Table 1 depicts fundamental spectrum of wave periods $\mathcal{T}$ of solar system (more precise, its fragment), which corresponds to elite astro-dynamic levels (orbits of planets) of solar system.

Schematic of astro-dynamic levels is specially represented in the form, which reminds schematic of spectroscopic levels of hydrogen-like atom (diagram of Bohr-Grotrian) [13]. By analogy with known radiation series - therms and intercombinations in the atom - Lyman, Balmer, Paschen, etc., in the diagram are represented series of the wave periods of the solar system - therms and their intercombinations between the astro-dynamic elite levels, which correspond to the planets of the solar system; a series of asteroids (Ceres) is given conditionally (periods $\mathcal{T}$ are indicated in days (d) and years (a).

In contrast to wave period $\mathcal{T}$ (8) periods of orbital planetary motion T in two-body problem (Keplerian periods) are proportional to semimajor axes a of orbits to degree of 3/2:

$$T = \frac{2\pi}{K^{1/2}} a^{3/2}; \quad \nu = \frac{1}{T}. \tag{14}$$

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Here K as in (5), (7), gravitational parameter of central body, ν - frequency of orbital (Keplerian) motion, a - semimajor axis of planetary orbit [26, 2, 3].

Mega-spectroscopy, which corresponds to fundamental wave frequency spectrum of solar system, can serve as guiding filament for research of interaction of factors of endogenous (in particular, geophysical) and exogenous (cosmogenic, in particular, astro-dynamic) nature, offering possibility of more goal-directed research of boundary problems of astrodynamics, geophysics and physics of planets. The concept of wave resonance [18...20] is here the leading idea, it assumes tendency toward the commensurability of the wave periods, represented in the fundamental wave spectrum of the solar system, with the rhythms existing in it, in particular, with the observed Keplerian periods of orbital and rotary planetary motions, and also with the known rhythms of astrophysics. This concept is confirmed by many results of the direct observations, which are examined below.

Analysis of some observational data.

Localization of trans-sphere. The presence of special surface - special transition/transfer (jump of the physical parameters) in area $r = r_* = 8R_\odot$ (associated with the position of trans-sphere) directly is discovered by the characteristic fracture on the graph (Fig. 1) of experimental data (obtained with the use of KA "Helios-1, 2" and "Pioneer-6, 10, 11"), that describe the dependence of the parameters of interplanetary plasma on heliocentric distance [45].

Special surface, on which observed in space experiments speed u of solar wind [45] proves to be equal to Keplerian speed of orbital motion $u = (K_\odot / r)^{1/2}$, is placed also in area $r \approx 8R_\odot$ (Fig. 2), identified above with surface of trans-sphere. On this surface the velocity of the motion of the local disturbances/perturbations of interplanetary plasma (solar wind) gradually increasing in the corona of the sun crosses the critical value of v*, equal to C_{MG}.

Localization of velocity C_{MG}. Proton temperatures T_p of interplanetary plasma, according to the empirical dependence, discovered by Burlaga and Ogilvie [5, 28, 30] according to the results of experiments on the space vehicles (for example, "Explorer-34"), are proportional to the observed velocities u of the motion of interplanetary plasma (solar wind) (Fig. 3).

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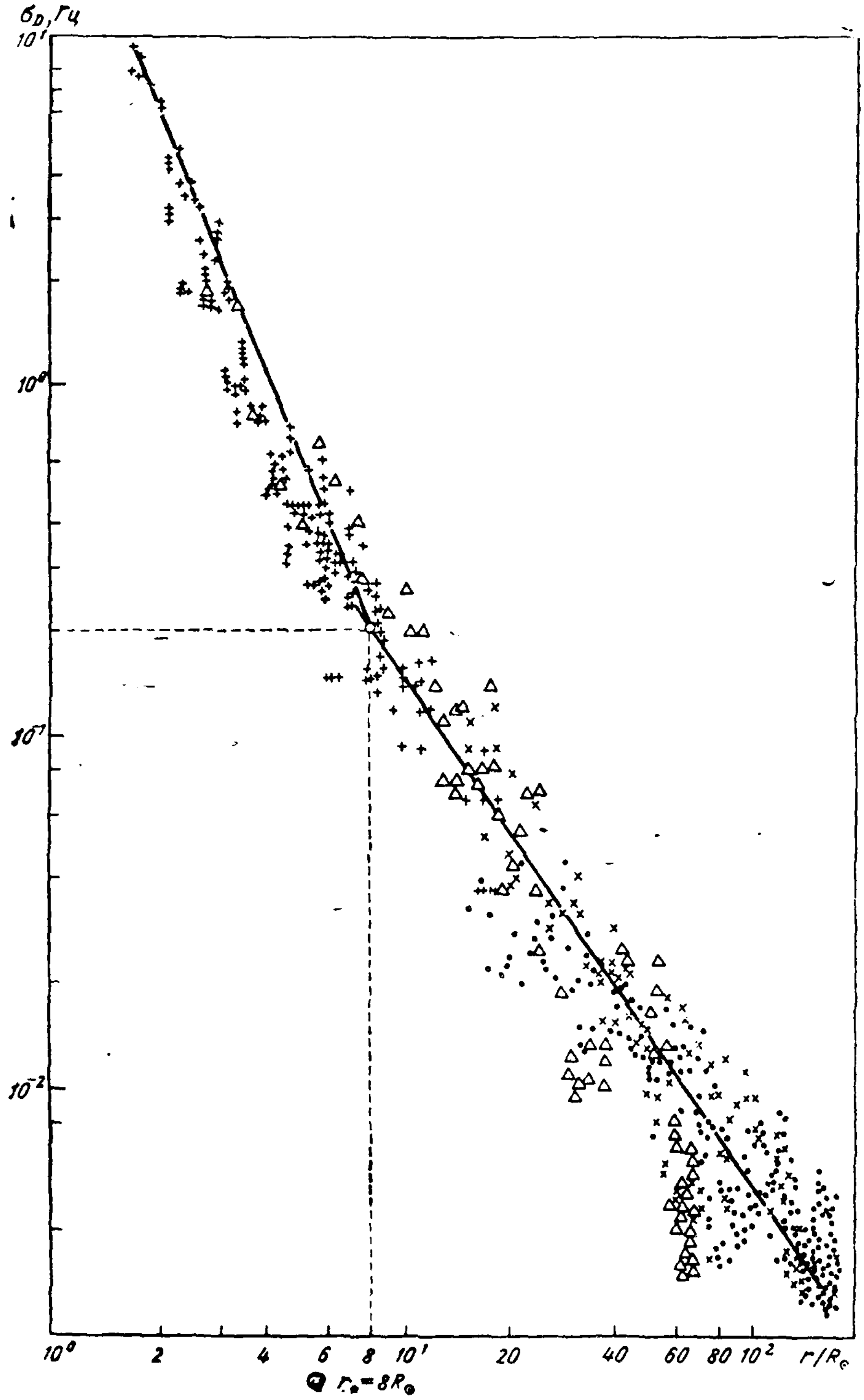
σ_D, Гц
10^1
10^0
10^{-1}
10^{-2}
10^0
2
4
6
8
10^1
20
40
60
80
10^2
$r/R_\odot$
$r_* = 8R_\odot$

Fig. 1.

Fig. 1. Dispersion σ_D (Doppler scintillations) of interplanetary plasma [45] and localization of trans-sphere in corona of sun. Evidence of experiment - characteristic fracture σ_D in the area of trans-sphere $r_* = 8 \cdot R_\odot$. The Doppler scintillations: ● - "Pioneer-11", × - "Pioneer-10"; △ - "Helios-1", + - "Helios-1, 2" and "Pioneer-6". Key: (1). Hz.

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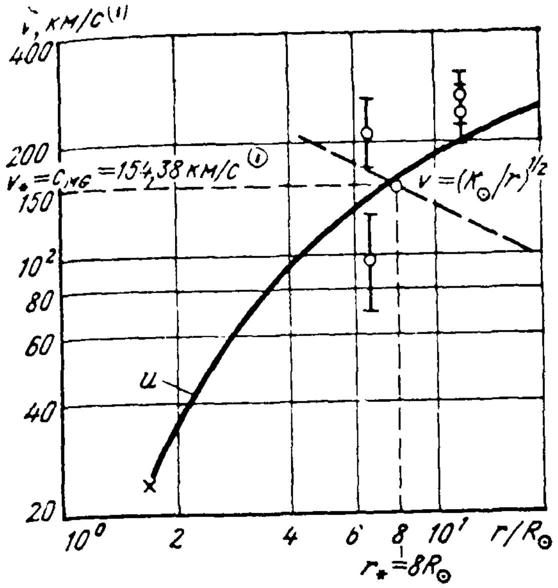
At smallest possible proton temperature $T_p \approx 0$ observed velocity of motions of interplanetary plasma is equal to u≈151 km/s, which follows also from propositions in [5, 28, 30] empirical dependence

$$(10^{-3} T_p)^{1/2} = 0{,}036\, u - 5{,}44. \quad (15)$$

Here T_p - temperature, K; u - velocity, km/s.

With $T_p \approx 0$ $u_0 = 151{,}11$ km/s. This experimental value is close to the velocity of propagation of mega-waves C_{MG} introduced above.

Mega-quantum effects. Let $\bar{a}_i = a_i/a_*$ - standardized/normalized to the value of radius of trans-sphere $a_* = r_* = 0{,}037219$ AU semimajor axes of planetary orbits (ephemeris DE19 JPL). Then standardized/normalized interplanetary distances $\overline{a_{i+1,i}} = \bar{a}_{i+1} - \bar{a}_i$ are equal for the earth-type planets to - 9.033, 7.433, 14.071; for the planets of Jupiter group - 116.069, 259.033, 292.999, 249.996, i.e., are (almost) integral, and interplanetary distance Venus-Earth - (almost) half-integral [21].



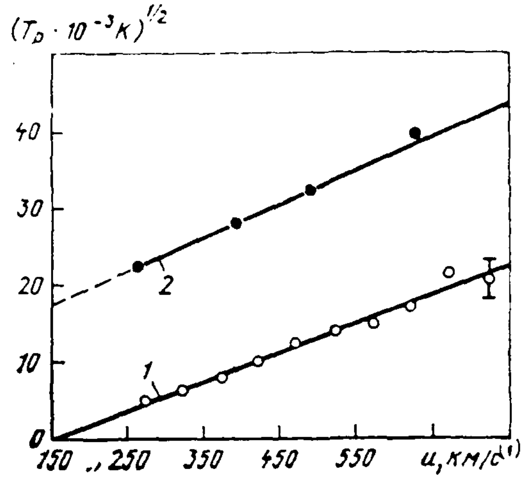


Fig. 2. Fig. 3.

Fig. 2. - Profile of solar wind (experimental data "Helios-1" [45] and localization of trans-sphere $(r_* = 8 \cdot R_\odot)$ in corona of sun. The singular solution $(v \approx C_{MG} = 154{,}38$ km/s, $r \approx r_* = 8 \cdot R_\odot)$, obtained from the experiment - point of intersection of observed speed u of solar wind [45] (solid line) and the Keplerian orbital speed

$$v_K = \left(\frac{K_\odot}{r}\right)^{1/2}$$ (dotted line).

Key: (1). km/s.

Fig. 3. Proton temperatures of interplanetary plasma on measurements on KA "Explorer-34" [5, 28]. Line 1, drawn through the experimental points, reflects the empirical dependence, proposed by Burlaga and Ogilvie [30]. For comparison the results of calculations according to the isothermal model of Parker (2) are shown. When $T_p \approx 0$ we have $u_0 \cong 151$ km/s, close to fundamental velocity $C_{MG} = 154{,}38$ km/s.

Key: (1). km/s.

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Let us consider now values of quantum numbers N_i, computed from formula [18, 22],

$$N_i=(2\pi)^{1/2}\frac{L_i}{L_*}=\left(2\pi\frac{a_i}{a_*}\right)^{1/2},\quad i=1,2,\ldots; \qquad (16)$$
$$L_i=L_{N=1}N_i;\quad L_{N=1}=L_*/(2\pi)^{1/2}$$

with substitution into it of experimental values of semimajor axes a_i of planetary orbits of terrestrial group. As a result we will obtain $N_i=7{,}9111;\ 11{,}050;\ 12{,}991;\ 15{,}969$ respectively for the orbits of Mercury, Venus, Earth, Mars. As is evident, they all are very close to integral eigenvalues of equation (1) with boundary condition (2) [22].

Existence of these mega-quantum effects is one of arguments in favor of wave structure of solar system and the fact that $r=r_*=a_*$ objectively plays role of certain of its significant dimension.

160^m oscillations of sun. Wave astrodynamics offers the possibility of interpretation of such phenomenon as 160^m oscillations of sun [42]. Let us consider wave interaction (beatings) of two physically chosen spherical surfaces: the surface of sun $r=R_\odot$ and trans-sphere $r=r_*=8R_\odot$. To them correspond the frequencies of the orbital Keplerian motions (d - days): $\nu_{R_\odot}=8.6275\ (d^{-1})$ and $\nu_*=0{,}38128\ (d^{-1})$. Hence we will obtain the following values of the intercombinations of frequencies $\nu^{(\pm)}=\nu_{R_\odot}\pm\nu_*$ and periods corresponding to them: summation $\nu^{(+)}=\nu_{R_\odot}+\nu_*=9{,}0088\ (d^{-1})$, period

$$T^{(+)}=\frac{1}{\nu^{(+)}}=0^d,11100=\underline{159,^m84\,(\approx 160^m)}$$

and difference $\nu^{(-)}=\nu_{R_\odot}-\nu_*=8{,}2462\ (d^{-1})$, period

$$T^{(-)}=\frac{1}{\nu^{(-)}}=0,^d12127=\underline{174,^m62\,(\approx 175^m).}$$

Thus, wave examination makes it possible to predict the existence of the conjugated/combined from $T=160^m$ characteristic period $T\simeq 175^m$. Oscillations with this period actually are observed [34]. Hence it follows that the zone of the corona of the sun, which stretches from its surface $r=R_\odot$ to trans-sphere $r_*=8R_\odot$, is if not generator, then, at least, efficient resonator for 160^m the oscillations of the sun - apparently, one of the prevailing modes of mega-waves.

Wave resonance. "I consider these fluctuations the most mysterious phenomenon, observed in the stellar motion. It is so difficult for the explanation by the action of any known reasons, which to us remains nothing, except the assumption that they are caused by action, until now, of unknown reasons.

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Variations in the change in the sea level, continental displacements/movements, thawing of ice in Arctic and other observed processes cannot be, in all likelihood, their reason", wrote Newcomb about the fixed/recorded by the astronomical methods (according to the observations of stars) fluctuations of the angular velocity of Earth [12]. In the spectrum of variations in the speed of rotation of the Earth are observed, in particular, the periods [29] $27^d,6$; $0^a,16 \ldots 0^a,17$ ($58^d,4 \ldots 62^d$); $0^a,20$ (73^d); $0^a,35$ (128^d); $0^a,5$ (183^d); 1^a; $1^a,8 \ldots 1^a,95$; $2^a,2$; $3^a,5$; 10^a.

Directing attention to Table 1, it is not difficult to discover that components of variations in speed of rotation of Earth in question belong to fundamental wave spectrum of solar system (in accordance with $\mathcal{T} = 27^d,548$; $58^d,680$ ($59^d,201$); $72^d,891$; $97^d,033$; $127^d,827$; $184^d,229$; 1^a; $1^a,837$; $2^a,212$; $3^a,651$; $10^a,194$.

Being in spectrum, represented in Table 1, period $\mathcal{T} = 205^d,013$ is observed in (transposed) near-daily nutation of pole of Earth and seismicity of Moon, periods $\mathcal{T} = 443^d,317$ and $\mathcal{T} = 24^a,548$ - in Chandler oscillations and nutation of pole of Earth [29, 7].

Resonances of observed oscillations with periods, which are elements of fundamental wave spectrum of solar system (see Table 1), it is natural to consider, as in the case 160^m of oscillations of sun, as the various concrete/specific forms of manifestation of some complicated dynamic processes, characteristic for solar system, connected with wave resonance.

Let us give still some examples from the same region. The wave periods of Mercury (see Tables 1) $\mathcal{T} = 27^d,277 \ldots 31^d,713$ are commensurate with the differential rotation of the sun; period $\mathcal{T} = 27^d,277$ - with the orbital motion of the Moon; $\mathcal{T} = 58^d,68$ - with the spin of Mercury; $\mathcal{T} = 366^d,613$ - with the orbital motion of the Earth, etc. The series of the periods, available in Table 1, for example monthly ($\mathcal{T} = 27^d,277 \ . \ . \ . \ 31^d,713$), semi-annual ($\mathcal{T} = 184^d,229$), yearly ($\mathcal{T} = 366^d,613$), many-year ($\mathcal{T} = 2^a,212;\ 5^a,801;\ 10^a,194;\ 24^a,548$), are characteristic for the observed in astrophysics rhythms - from the oscillations of the geomagnetic and meteorological parameters on the Earth [7, 16, 17] to variations in the interplanetary magnetic field (MMP) [6, 8, 11] and solar activity [6, 9, 10, 16]. This relates, for example, to the following observed variations of MMP [8, 6, 11, 16] $T = 87^d;\ 95^d;\ 127^d;\ 147^d;\ 180^d;\ 240^d;\ 360^d;\ 445^d;\ 510^d\ 2775^d$

and to the neutrino fluxes (tentatively $T = 25,5 \pm 1,5$ mo. ($2,125 \pm 0,125$ yr.), T=11 years, etc.).

Examples, given above, testify in favor of mega-quantum wave structure of solar system and presence of mechanism of synchronization of periodic processes of different physical nature with natural wave oscillations taking place in it, which correspond to its fundamental wave (discrete/digital) spectrum (i.e. in favor of wave resonance).

Development/detection of this mechanism is most important problem, which lies on joint of wave mechanics and astrophysics.

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However, central question thus far remains more detailed identification, phenomenological and dynamic description of quantum structure of solar system and systems with central body and satellites analogous to it.

Two systems of mega-waves in the solar system and the corresponding to them groups of planets and satellites.

Let us consider one additional approach to quantization, with which is considered existence of two (terrestrial and Jupiter) groups of planets; orbits are broken down into two groups and are examined respectively two systems of mega-waves [23...25]. The first is identified (in the sense of velocity of propagation) with the observed in the interplanetary plasma "rapid" magnetosonic waves (velocity $C^{[1]}_{M\sigma} = C_{MG}$, examined above), the second - with the "slow" magnetohydrodynamic waves (velocity $C^{[2]}_{M\sigma}$) [5, 30]. The numerical values of these velocities and their corresponding constants and radii of trans-spheres $r^{[s]}_{*} = a^{[s]}_{*}$ $(s=1,2)$ are such:

$$C^{[1]}_{M\sigma} = 154{,}39 \text{ км/с}^{(1)};\quad C^{[2]}_{M\sigma} = 42{,}105 \text{ км/с}^{(1)};\quad a^{[1]} = 0{,}4298 \cdot 10^{9} \text{ км}^2/\text{с}^{(2)};$$
$$a^{[2]} = 1{,}5759 \cdot 10^{9} \text{ км}^2/\text{с}^{(2)};\quad r^{[1]}_{*} = a^{[1]}_{*} = 8R_{\odot} = 0{,}037219 \text{ а. е.};\quad r^{[2]}_{*} = a^{[2]}_{*} =$$
$$= 107{,}56 R_{\odot} = 0{,}50039 \text{ а. е.}$$

Key: (1). km/s. (2). km²/s.

Until now the first of sets of parameters was used during calculations. Now let us act somewhat otherwise. Velocity $C^{[1]}_{M\sigma}$ and corresponding to it trans-sphere $a_*^{[1]}$ we will associate with the earth-type planets; velocity $C_{MO}^{[2]}$ and its trans-sphere $a_*^{[2]}$ - with the planets of the group of Jupiter.

Now quantum numbers of elite orbits can be calculated independently for each of groups according to formulas (in particular (16)) of Section 1-3 with use of corresponding constants of quantization.

Results of these calculations are represented in Table 2. As can be seen from this table, computed values N for the real orbits of each of the groups of planets are close to each other and to the whole or half integers. On the other hand, attention is drawn to the fact that, judging by the explosions in the series/row of integer values N, not all permissible elite orbits "are filled", i.e., there is a series of the orbits, in which the planets are absent. Here we, apparently, encounter new problem - stability problem of elite orbits, which allows/assumes special examination.

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Natural question arises, are some of parameters of quantization in question, namely velocities $C_{MO}^{[1]}$ and $C_{MO}^{[2]}$, fundamental constants for entire solar system as a whole, including satellite systems, or these constants carry local character (which, of course, would substantially decrease heuristic value of considered phenomenological and dynamic description of solar system as mega-wave system).

Response/answer to this question gives Table 2, in which are cited data also for satellite systems of planets (see quantum numbers $N^{[1]}$ and $N^{[2]}$).

One should stress that data according to quantum numbers $N^{[1]}$ and $N^{[2]}$ (see Table 2) for satellite systems are calculated with use of the same two constants $C_{MO}^{[1]}$ and $C_{MO}^{[2]}$, as during calculation of data for planetary system, moreover

$$a^{[s]} = K/2C_{MO}^{[s]}; \quad a_{*}^{[s]} = K/(C_{MO}^{[s]})^2 (s = 1,2); \qquad (18)$$

K - gravitational parameters of central bodies of satellite systems (planets).

As can be seen from table, occurs, apparently, nonrandom proximity of values N for real orbits of satellites to quantum numbers of planetary orbits.

This result, which has fundamental character, is important argument in favor of real existence of quantum orbits. On the other hand, the appearance in a number of cases together with wholes (half integral) and other numbers shows that, apparently, the true picture is more complicated than described by the analog of the equation of Schroedinger (1) with the simplest Hamiltonian, examined above.

In Table 2 is also series of whole (and half integral) values $N^{[1]}$, $N^{[2]}$, to which correspond "empty" elite orbits. Asserts itself the assumption that the "occupied" orbits are in a sense dominant, i.e., at least, more stable than "empty". As the indirect confirmation of latter/last assumption serves the presence in family of the rings of Saturn of slot near the internal edge of ring D^{in} and division of Cassini, close to the elite orbits with quantum numbers 15.5 or 16 and 21.5 respectively.

However, further analysis shows that not all components of satellite systems can be exhausting described within the framework only of two systems of quantum numbers, which correspond in planetary system to two groups of planets - terrestrial and Jupiter.

Shell structure of astronomical systems (WDS). Let us represent the basic constants of quantization in the form

$$C_{M\sigma}^{[s]} = \frac{C_{M\sigma}^{[1]}}{\varkappa^{s-1}}; \quad a^{[s]} = \varkappa^{s-1} a^{[1]}; \quad a_*^{[s]} = \varkappa^{2[s-1]} a_*^{[1]} \ (s = 1,2), \qquad (19)$$

where $\varkappa = C_{M\sigma}^{[1]}/C_{M\sigma}^{[2]} = 3{,}666$ - certain dimensionless parameter - fundamental constant of hierarchy [24].

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Table 2.

(N)	(1) Планетная (Солнечная) система				(2) Спутниковые системы планет																											
					Земля (E) (3)		Марс (Ma) (4)		Юпитер (J) (5)							Сатурн (S) (6)					Уран (U) (7)				Нептун (N) (8)						Плутон (P) (9)	
(10)	Планеты	$N^{(1)}$	Планеты	$N^{(2)}$	Спутники	$N^{(4)}$	Спутники	$N^{(3)}$	Спутники	$N^{(1)}$		$N^{(2)}$		$N^{(3)}$	Спутники, детали колец (12)	$N^{(1)}$		$N^{(2)}$		$N^{(3)}$	Спутники (11)	$N^{(2)}$		$N^{(3)}$	Спутники (11)	$N^{(2)}$		$N^{(3)}$		$N^{(4)}$	Спутники (11)	$N^{(5)}$
7																													N2	7,0368		
					E1	7,709					J2	7,680					S1	7,378					U3	7,894								
8	Me	8,083	J	8,072 ⇒																												
													J13	8,532			S2	8,358	S8	8,818												
													J6	8,656																		
													J10	8,762																		
9													J7	8,764							Кольца (13) u? α	8,976										
																					β	9,378	U4	9,129								
																	S3	9,300				9,506									P1	9,545
																					Кольца η	9,530										
10											J3	9,700									ε	9,894										
																	S4	10,526			U?	10,509										
11	V	11,050	S	10,919 ⇒																												
													J12	11,688																		
12									Кольцо	12,298			J11	12,144																		
									J14	12,318			J8	12,390			S5	12,440														
											J4	12,865	J9	12,518																		
13	E	12,992		⇒																												
							Ma1	13,413																								
14									J5	14,632																						
15			U	15,435 ⇒											(14) Внешний край облаков R^{out}	15,433					U5	15,793										
16	Mu	16,038		⇒					J15	16,214					(15) Внутренний край кольца D^{in}	16,263																
17															Внутренний край кольца C^{in}	16,997			S9	16,820												
18															(16) Французское деление S?	18,453																
19															C^{out}	18,849	S6	18,941														
			N	19,430 ⇒											B^{in}	19,076					U1	19,148										
20															Щель	19,829																
21																	S7	20,852														
	Ce (17)	21,614	P	21,535 ⇒			Ma 2	21,321							(18) Деление Кассини B^{out}	21,535																
22															A^{in} (19)	21,853																
									J1	22,325											U2	22,601										
23															(20) Люк Энке	22,956																
															(21) Деление Пионер A^{out}	23,186																
															S27	23,439																
24															F^{in}	23,360									N1	24,00						
															1980 S3	24,433																
25																																
26																											(N2)	(25,887)				
27															(S1)	(27,052)																

Key: (1). Planetary (Solar) system. (2). Satellite systems. (3). Earth. (4). Mars. (5). Jupiter. (6). Saturn. (7). Uranus. (8). Neptune. (9). Pluto. (10). Planets. (11). Satellites. (12). Satellites, part of rings. (13). Rings. (14). External edge of clouds. (15). Internal edge of ring. (16). French division. (17). Ceres. (18). Division. (19). Cassini. (20). Encke's hatch. (21). Pioneer.

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Asserts itself thought about possibility of existence of hierarchical series of velocities $C_{MG}^{[s]}$, constants of quantization of areal velocity $\text{đ}^{[s]}$, trans-spheres with radii of $r_*^{[s]}=a_*^{[s]}$ and other parameters from general/common constant hierarchy κ at s=2, 3, . . . (see in table 2 appropriate quantum numbers $N^{[3]}$, $N^{[4]}$, $N^{[5]}$). From a physical point of view this hypothesis assumes the presence of the sequence of the regions of interplanetary plasma, limited by shells $G^{[s]}$ ($G^{[1]}$and $G^{[2]}$ in the planetary system correspond to terrestrial and Jupiter planets), with some prevailing properties, which determine the observed dynamic structure of astronomical systems, in particular the solar system. Parameters $C_{MG}^{[1]}$ and $C_{MG}^{[2]}$ are from this point of view the elements of the hierarchy of the integral characteristics of the corresponding physical continua.

As can be seen from Table 2, set of quantum numbers $N^{[s]}$ $(s=1, 2, \ldots, 5)$ makes it possible to describe all observed satellite orbits similarly to how numbers $N^{[1]}$ and $N^{[2]}$ describe orbits of two groups of planets of solar system.

More detailed and more objective examination of hypothesis, formulated above, will be possible in obtaining of new experimental data about outer space, supplied/delivered, in the first place, with the help of artificial Earth satellites and planets, and also other space vehicles.

However, preliminary conclusion, which follows from examination of Table 2 (besides its obvious prognostic value), consists in the fact that $C_{MG}^{[1]}$ and $\varkappa = C_{MG}^{[1]}/C_{MG}^{[2]}$ can claim to role of fundamental constants for solar system as a whole, turning off/disconnecting satellite systems, which is important argument in favor of unity of its wave structure and presence of general/common for entire solar system physical factors, critical for this structure.

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SIMULATION OF THE DYNAMICS OF THE CARRIER ROCKETS OF SPACE VEHICLES.

M. M. Bordyukov.

Survey/coverage of methods and means of simulation, used during study of dynamic properties of contemporary carrier rockets of space vehicles, is given. The mathematical models of different dynamic ducts/contours and their couplings are represented in the form of overall structural diagram. A comparative analysis of means of computer technology, used during the simulation, is carried out. The special features of application for the dynamic investigations of physical models are shown.

Simulation both mathematical and physical is one of basic methods of studying dynamics of controlled flight of space carrier rockets. Mathematical simulation is realized with the help of different means of computer technology, which are divided into the analog and the digital. Physical models can be functional similar to real objects as, for example, the functioning mock-ups of units and aggregates/units, or the models, tested in the wind tunnels, or are the real assemblies and the instruments, tested under the conditions, close to the operational, for example according to the temperature, the g-forces, the vibrations, the character of external information, on the failures of constituent elements, etc. In the latter case modes of the work of instruments and external connections/communications are simulated. In this case often the instruments connect to the computer(s), which simulates the remaining part of the system, which makes it possible to create the conditions for the work of the equipment in the complex diagram being investigated and in accordance with the real program. Thus occurs the association/unification of physical and mathematical models in the single experiment.

In all cases simulation experiment is based on mathematical description of components/links of dynamic system and connections/communications between them. Each of the elements of the dynamic system of contemporary carrier rocket (RN) has sufficiently precise mathematical description, in other words, the mathematical model, which makes it possible to investigate it by means of simulation.

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Proof and analysis of mathematical models is the important independent section of dynamics of RN, to which is dedicated the vast literature [2, 10, etc.]. In this region a number of the basic research was carried out by G. S. Narimanov [4-8], moreover they were completed, as a rule, by mathematical simulation for purposes of the solution of urgent applied problems.

1. General/common structure of mathematical models of carrier rockets. In the review paper about the simulation there is no need

for giving the detailed description of one or the other dynamic components/links. Common appearance of RN and ZhRD as dynamic system can be represented by the block diagram (Fig. 1), which contains most typical elements and connections. Diagram characterizes also composition and interconnection of the corresponding research models. Of course they are not used everyone simultaneously. However, this general/common view can prove to be useful for the system analysis, the evaluation of separate particular models, determination of requirements for the means of simulation. The diagram is simplified for the clarity: it reflects motion only in the pitching plane. Therefore in it some cross couplings, inherent in spatial motion, are absent. At the same time the most typical components/links of different physical nature are shown in the diagram.

In spite of specific conformity between real units of rocket and blocks of diagram on it first of all is reflected not this structural/design analogy of model and unit, but analogy of interactions of dynamic elements.

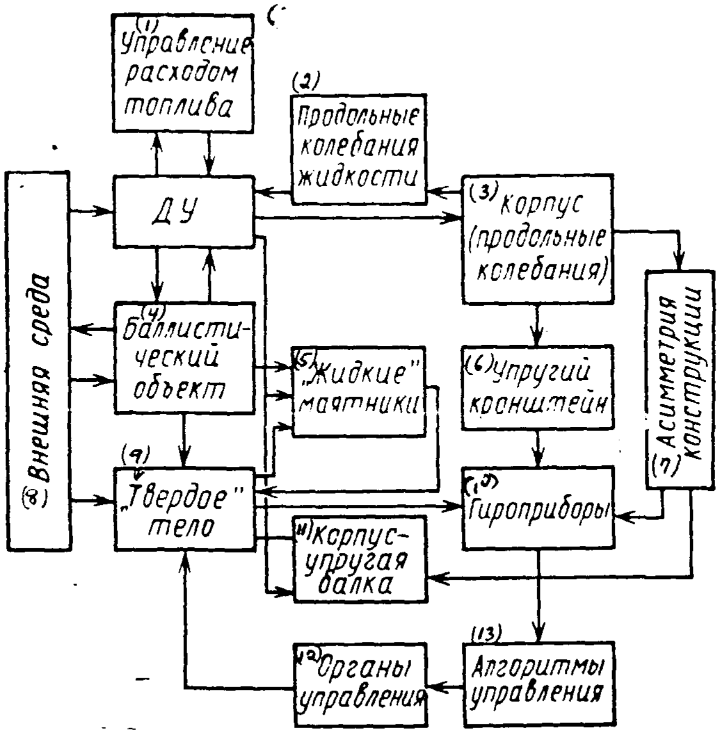

Fig. 1. Block diagram of carrier rocket as dynamic system.

Key: (1). Control of the fuel consumption. (2). Longitudinal vibrations of liquid. (3). Housing (longitudinal vibrations). (4). Ballistic object. (5). "Liquid" pendulums. (6). Elastic bracket. (7). Asymmetry of construction/design. (8). Environment. (9). "Solid" body. (10). Gyro instruments. (11). Housing-elastic beam. (12). Controls. (13). Algorithms of control.

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For example, basic structural element (housing) figures both in the block, which characterizes the angular motion of solid body and in the model of transverse, and also longitudinal vibrations. Engine installation enters into ballistic model and independent of this it is considered as the independent dynamic component/link, which participates both in the longitudinal and in transverse vibrations of RN. The liquid filler of tanks participates in the wave (transverse) and in the longitudinal vibrations.

Concrete/specific models for research of one or the other dynamic systems of RN are obtained by "cutting" of corresponding sections of overall diagram. The model of the disturbed motion around the center of mass, which describes also the deflections of the very center of mass from the calculated trajectory, relates to important for practice. It is at the same time one of complicated even larger in the dimensionality (number of degrees of freedom).

Another "running" model of dynamics is used for study of longitudinal vibrations of RN, in which participates housing, power-supply system and engine installation. In American scientific-technical literature it is accepted to call pogo which is explained by external similarity between the motion of housing of RN during these oscillations and by motion of jumping stilt (pogo stick) [12]. In contrast to the duct/contour of transverse vibrations, which contains control system and controlled by it, the duct/contour of longitudinal vibrations "is not guided". To ensure its stability with structural/design measures is very difficult, and often also it is impossible. The nonlinear effects of dissipation play the decisive role in the solution of this problem.

Practical requirements of design and final adjustment of RN lead also to other combinations of models, connected, for example, with control of thrust, and also describing dynamic processes in separate systems and aggregates/units: navigational instruments, combustion chambers, etc.

For research of dynamics of carrier rockets are used diverse means of simulations, whose characteristic is given in subsequent sections.

2. Digital computers as means of mathematical simulation. The mathematical models of dynamics of RN consist of differential equations and final dependences between the variables - algebraic and matrix. The high order of equations and the large number of nonlinearity make very labor-consuming the integration of such models by numerical methods.

Digital computers possess maximum universality with respect to utilized algorithms. With their aid a question about the mathematical simulation of dynamics of RN could be solved completely, if not stringent requirements on the speed. Let us consider both the requirements and real possibilities in this respect.

Let us designate duration of investigated phase of flight through τ, and duration of its reproduction on model - through $\mu\tau$.

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In the case of the combined models, when to the calculating means, which realize mathematical model, are connected some real instruments (drive of actuating elements, function generators, etc.), a question about the speed is solved unambiguously: the normal mode of work of the utilized onboard equipment can be provided only during the simulation at the rate of flight, i.e., scale of time μ must be equal to one.

If model is purely mathematical, then μ in principle is unconfined, although in this case time of simulation is desirable to make as far as possible less in interests of maximum "productivity" of research. Most sharply a question about an increase in the speed stands during the static research, which include the considerable number (N) of the realizations of the phase of flight being simulated. In this case appear serious difficulties in the guarantee of acceptable duration of entire experiment $T=\mu\tau N$.

Let us consider required high-speed operation of a digital computer for model of angular motion as one of large. Let the model describe flat/plane angular motion of RN as solid body, automatic machine of stabilization, oscillation of liquid propellant in the tanks and transverse elastic vibrations of construction/design. We will consider for certainty that RN has 12 tanks, and in each of them one tone of oscillations is considered, the model of elastic vibrations considers four tones, and the maximum frequency of processes in the system reaches 15 Hz. Under these conditions the degree of differential equation, which corresponds to the object of control, is equal to 36. Taking into account that the system has nonlinearity and must be locked by the algorithms of control, it is possible to establish that the simulation of angular motion by this RN at the rate of flight with the use for integrating Runge-Kutta method will require the application of TsVM with the speed about 10^7 operations per second. Application for this purpose of TsVM with the contemporary system of the automation of programming, which simplifies the development of the simulating program, but which simultaneously decreases the general speed, is possible only during the organization of the simultaneous work of several machines of class BESM-6 or YeS-1050. It is obvious that this version is virtually little real.

Example with very rigorous conditions examined shows limitedness of possibilities of TsVM; however, this does not exclude their application for targets in question. On the contrary, from the obtained quantitative estimation it is possible to draw useful practical conclusions.

First conclusion. Even for the fairly complicated model the application of contemporary TsVM can be appropriate. True, process it is necessary "to stretch" ($\mu=5\ldots10$), but this is acceptable with moderate N.

Second conclusion. Computer technology rapidly progresses, and it is possible to expect that TsVM with speed on the order of 10^7 operations per second is spread during the next decade. Further possibilities in this respect open/disclose multiprocessor computers.

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Thus, the application of digital means for the mathematical simulation of dynamic processes of RN has a good prospect already in the near future.

Third conclusion. Such bulky models, as the complete model of angular stabilization, far from always are investigated. Many of them much simpler can be reproduced on TsVM not only at the natural rate, but also $\mu<1$.

Is especially efficient the application of TsVM at early stage of research and design, when are chosen basic parameters and circuit solutions, model does not require elaboration, and although versions are diverse, each of them is examined/scanned "in general terms".

Application of TsVM for dynamic research can give further advantages during correct organization of programming. During calculations on the dynamics of RN are used the standard algorithms, which relate, for example, to the numerical integration, to transformations of coordinates, to the calculations of the eigenvalues of matrices and to other problems. This makes it possible to use the means of the automation of programming, in particular, the packages of applications programs (PPP), specially intended for the solution of the problems of dynamics. This simplifies simulation on TsVM. Furthermore, in PPP is widely represented the analytical apparatus of the stability theory, frequency response methods and other sections of the theory of automatic regulation, which makes it possible to carry out control tests or to duplicate/back up separate results, i.e., it is useful booster agent in the arsenal of simulation.

3. Simulations with the help of analog computers. Together with TsVM, and historically even earlier than them, for the simulation of dynamic objects were used and are used the now specialized calculating means, mainly, continuous (AVM) analog computers. The parameters of real physical processes are represented in AVM by the electrical values, which during the simulation are changed in the time analogously with their mechanical, thermal, and also electrical prototypes. Mathematically this means that the integrals of the differential equations of motion of the real object in question and equations, which describe the electrical decisive circuit, coincide with an accuracy to dimensional factor. This law is retained in the specific range of the parameters, in particular, the frequencies of the studied vibrations, which depends on frequency characteristics of AVM. The contemporary means of analog simulation reproduce well on real time (μ=1) processes with the frequencies to several ten hertz, which makes it possible to investigate the majority of the models of the dynamics of carrier rockets with the necessary accuracy.

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Analogy between object of research and AVM is exhibited also in their structural similarity, which, by the way, serves as one of factors of high speed operation of AVM, since its comprising calculating elements (integrators, adders, functional blocks) work, similarly to parts of real object, simultaneously, "in parallel". Respectively a quantity of equipment in this computing system directly from the volume of the solved problem: the number of variable magnitudes, the degree of differential equation, complexity of right sides.

AVM are spread not less than digital computers. In the recent three decades by our industry is released more than 40 types of each machines [1, 9]. AVM of different generations, utilized for the research of dynamics of RN, it is possible to describe by data of Table 1, from which it is evident that not all their characteristics progress to the identical degree. Progress concerns accuracy, element base, and also automation of programming. At the same time the composition of the decisive blocks varies little. This is explained by structural flexibility of AVM and, as a result, by possibility to comparatively simply raise its composition.

In spite of considerable universality AVM are inferior to digital computers in accomplishing of logical operations, flexibility of programming and accuracy. Latter/last characteristic is given in table; however, the errors given in it relate to one operational amplifier. Association/unification into the simulating system of many tens of amplifiers leads to the fact that its accuracy as a whole proves to be considerably lower.

Table 1.

(1) Тип АВМ	(2) «Электрон»	МН-17	МН-18	АВК-32	ЭМУ-200
(3) Год выпуска	1960	1967	1970	1979	1979
(4) Порядок дифференциальных уравнений	55	60	10	20	20
(5) Количество линейных операций	150	60	50	48	20
(6) Количество постоянных коэффициентов	975	160	120	160	170
(7) Количество нелинейных операций	165	78	23+50	38	83
(8) Количество логических операций	30	30	1	28	170
(9) Максимальное время интегрирования, с	1000	1000	1000	1000	—
(10) Погрешность интегрирования, % за 100 с	0,5	0,3	0,3	0,5	0,2
(11) Элементная база	(12) лампы электронные	(12) лампы электронные	(13) полупроводниковые элементы	(14) интегральные микросхемы	(14) интегральные микросхемы

Key: (1). Type of AVM. (2). "Elektron". (3). Year of issue. (4). Degree of differential equation. (5). Quantity of linear operations. (6). Quantity of constant coefficients. (7). Quantity of nonlinear operations. (8). Quantity of logical operations. (9). Maximum time of integration, s. (10). Error in integration, % for 100 s. (11). Element base. (12). tubes, electronic. (13). semiconductors. (14). integrated microcircuits.

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Furthermore, accuracy depends on the duration of the process of simulation, since in this case errors are accumulated. As a result errors in the reproduction of dynamic processes can reach several percentages.

4. Analog-digital complexes (ATsK). The greatest effect of the simulation of dynamic systems of RN is achieved by the application of computer complexes, which join in themselves analog and digital computers. Such complexes received wide acceptance in the last 20 years. Their advantages are revealed well during the concrete/specific analysis of the problems of dynamics of RN.

Superiority of AVM is most perceptible during simulation of systems, in which are inherent oscillations with frequencies of from ones hertz and above. Such systems include the duct/contour of the angular stabilization, the longitudinal vibrations and some others. At the same time for the simulation of ballistic motion or account of the effect of environment it is necessary to have TsVM, since here is required the "digital" accuracy, the memory, the calculation of nonlinear functions. When any models from these two groups must be examined together, unavoidably arises the question about the association/unification of the heterogeneous means of simulation.

As another example, which leads to the same conclusion, can serve problem of research of "analog" (with continuous processes) object of control (for example, in the same angular motion) with discrete/digital regulator, whose work it is natural to simulate with the help of digital means.

Digital computer in composition of simulating complex makes it possible to perform statistical processing of obtained dynamic parameters. ATsK substantially raise possibilities and productivity of simulation experiment because of the comprehensive automation of the process of experiments.

During creation of ATsK together with advantages appear additional difficulties. It is necessary to organize the exchange between the analog and digital parts, in which the information in principle is distinguished by the form. It is necessary to guarantee the high rate of exchange. Must be created single control of complex. For the solution of these problems is intended the specific type of equipment - converters analog-digital (ATsP) and digit - analog (TsAP). Those and others compose the basis of the third large part of ATsK - devices/equipment of conversion and connection/communication. Their characteristics for the serial Soviet samples are given in Table 2.

Table 2.

(1) Тип преобразователей		УП-6	УПС
(2) Количество каналов	ЦАП	24	32
	АЦП	23	72
(3) Время преобразования, мкс	ЦАП	6	5
	АЦП	15	20
(4) Разрядность		12	14
(5) Шкала напряжений, ±, В		50	10

Key: (1). Type of converters. (2). Quantity of channels of TsAP. (3). Conversion time, μs TsAP. (4). Precision. (5). Scale of voltages, ±, V.

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5. Role and place of physical models. Diversity and great possibilities of the computer(s) of different classes permit to investigate by the methods of mathematical simulation the very broad class of dynamic systems. However for the possibility of the very principle of mathematical simulation they are limited in view of the simplifications connected with it. When these simplifications concern separate units, the latter can be included in the stand of simulation directly. This is most frequently the actuating elements of control system, sensing elements, onboard computers. Utilized thus real equipment improves the adequacy of model and raises the accuracy of results. Creation for the real instruments of working conditions, close to the flight, is very efficient. Thus, it is possible to subject them to vibrations, to create varying loads on the operating

units, electromagnetic inductions in the electrical circuits, finally, to artificially create the malfunction of separate elements and units.

With entire temptation of enlistment to simulation of real instruments it is necessary to distinctly visualize limitedness and shadow sides of this path.

First of all it is necessary to keep in mind that some important factors of flight (for example, g-force) is difficult to reproduce physically on Earth, so that their study is possible only by means of good mathematical description. Since these factors are essential for the work of a number of devices, it is necessary to investigate also such instruments in the form of mathematical models. Even if there is a possibility in principle to create in the laboratory these or other physical conditions (vibration, temperature, vacuum), then technical difficulties connected with this are not always justified by the further information, which this physical model in comparison with the mathematical is capable of giving.

Another fact, inswept propagation of physical models, is their limitedness in variation of parameters. Mathematical model in this respect possesses indisputable advantages, since it makes it possible to optimize the circuit solutions and the parameters in the broad bands, to simulate all possible deflections and malfunctions.

And nevertheless physical simulation is often source of this information, which it is not possible to obtain by any other paths. As examples can serve the studies of the elastic vibrations of multiply connected constructions/designs and oscillations of fuel/propellant in the tanks of the complex form, whose target consists in obtaining of the substantiated mathematical description of the phenomena indicated. The simulation of this type exceeds the scope of this article. It is the independent theme, well reflected in the literature [3, 11].

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DISTURBED MOTION OF NONROTATING FLIGHT VEHICLE WITH LIQUID IN THE TANKS. THE SLIGHT DISTURBANCES OF FREE SURFACE.

On the motion of solid body, whose cavity is partially filled with liquid.

G. S. Narimanov.

Carried out in 1951 work of author, dedicated to compilation of equations of motion of solid body, which has cavities, partially filled with liquid, is briefly presented, is given proof of existence and uniqueness of solution of these equations and proof of application of method of reduction for obtaining solution.

Present article contains abbreviated/reduced presentation of carried out in 1951 work of author, dedicated to compilation of equations of motion of solid body, which has cavities, partially filled with liquid, and to analysis of solutions of these equations.

Analogous equations, derived somewhat by other means, were independently obtained later by N. N. Moiseyev [1, 2, 3].

In this article besides short conclusion/output of equations of problem proof of existence and uniqueness of solution of these equations is conducted, and proof of possibility of applying method of reduction for obtaining solution also is examined.

1. Coordinate system. Initial prerequisites/premises. We will assume that solid body possesses only the one cavity, filled with the liquid (generalization of results for the case of the larger number of cavities does not present any difficulties).

Motion of solid body and liquid we will examine relative to certain system of coordinates $O^*x_1^*x_2^*x_3^*$, which is not, generally speaking, inertial, but possessing the property, that field of inertial forces and gravitational forces has in it potential function.

Besides this coordinate system, let us introduce into examination another system $Ox_1x_2x_3$, rigidly connected with solid body (Fig. 1). The motion of the system of coordinates $Ox_1x_2x_3$, and therefore, solid body relative to the system of coordinates $O^*x_1^*x_2^*x_3^*$, we will determine with the help of velocity vector V_0 of point O and of angular velocity vector of rotation ω, passing through point O.

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For future reference let us introduce designations:

R, R* - radius-vectors relative to points O, O* respectively, R_0^* - radius-vector of point O relative to O*; n - unit vector of external normal to the surface of liquid; u_n - projection of relative speed of liquid on n; S - surface of liquid mass; ζ - hydrophilic surface of cavity; Σ_0 - undisturbed free surface of liquid, described by equation $x_3=C$; l - flat/plane closed curve of intersection of surfaces Σ_0 and ζ; Σ - disturbed free surface of liquid.

Let us represent equation of disturbed free surface of liquid in the following form:

$$x_3 - C = \sum_{i=1}^{\infty} a_i(t) f_i(x_1, x_2), \qquad (1)$$

where functions $f_i(x_1, x_2)$, with exception of constant value, which corresponds to $\lambda=0$, are normalized eigenfunctions of boundary-value problem:

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \lambda^2 f = 0; \ \frac{\partial f}{\partial n}\Big|_l = 0. \qquad (2)$$

Eigenfunctions of boundary-value problem (2) form complete and orthogonal set of functions on G region, for which should be taken part of plane $x_3=0$, limited by projection on it of curve l.

Let us give basic assumptions, which are accepted subsequently.

1. Liquid, which is located in cavity, is considered inviscid.

2. It is assumed that motion of liquid in system of coordinates $O^*x_1^*x_2^*x_3^*$ possesses velocity potential. Since in this coordinate system the field of mass forces has potential function, by virtue of the Lagrange theorem property of the potentiality of the motion of

liquid will be preserved in entire time of motion.

3. Only low motions of solid body and liquid are examined. Low is called such motion, in which values V_0, ω, R_0^*, angles between the similar/analogous axes of the systems of coordinates $Ox_1x_2x_3$ and $O^*x_1^*x_2^*x_3^*$, and also value $a_i, \dot{a}_i$ $(i=1, 2, \ldots)$ are so low that by products and their squares can be disregarded/neglected in comparison with the value of any of these values.

4. We will consider that potential function U of mass forces (inertia and gravity) in system of coordinates $O^*x_1^*x_2^*x_3^*$ can be represented as follows:

$$U=-\mathbf{j}\mathbf{R}^*, \qquad (3)$$

where j - total G-vector of field of mass forces.

5. Let us assume that vector j in all motions composes small angle with opposite direction of axis O^*x_3 and, that means value of projections j on axis $O^*x_1^*$ and $O^*x_2^*$ are low in sense of concept of smallness expressed above. Let us designate through Φ the velocity potential of liquid in the system of coordinates $O^*x_1^*x_2^*x_3^*$.

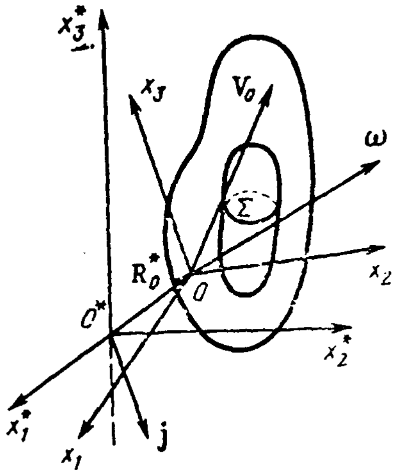

Fig. 1.

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Boundary conditions for function Φ take form:

$$\left[\frac{\partial\Phi}{\partial n}\right]_{S}=\mathbf{V}_0\mathbf{n}+[\boldsymbol{\omega}\times\mathbf{R}]\,\mathbf{n};\quad \left[\frac{\partial\Phi}{\partial n}\right]_{\Sigma}=\mathbf{V}_0\mathbf{n}+[\boldsymbol{\omega}\times\mathbf{R}]\,\mathbf{n}+u_n. \qquad (4)$$

In view of assumed about smallness of parameters of motion second boundary condition can be referred to plane x_3=C of undisturbed free surface of liquid.

Then we register conditions (4) in the form

$$\left[\frac{\partial\Phi}{\partial n}\right]_{S}=\mathbf{V}_0\mathbf{n}+\boldsymbol{\omega}[\mathbf{R}\times\mathbf{n}],\quad \left[\frac{\partial\Phi}{\partial x_3}\right]_{x_3=C}=\mathbf{V}_0\mathbf{n}+\boldsymbol{\omega}[\mathbf{R}\times\mathbf{n}]+$$
$$+\sum_{i=1}^{\infty}a_i f_i(x_1,\ x_2). \qquad (5)$$

It is obvious that function Φ, which satisfies conditions (5), can be represented as follows:

$$\Phi = V_0 R^* + \omega\Omega + \sum_{i=1}^{\infty} \dot{a}_i A_i + \text{const}, \tag{6}$$

where Ω - harmonic vector function $\Delta\Omega=0$, and A_i $(i=1, 2, \ldots)$ - harmonic functions, which satisfy with respect to conditions

$$\left[\frac{\partial\Omega}{\partial n}\right]_S = [\mathbf{R}\times\mathbf{n}]; \quad \left[\frac{\partial A_i}{\partial n}\right]_{\text{c}} = 0; \quad \left[\frac{\partial A_i}{\partial x_3}\right]_{x_3=C} = f_i(x_1, x_2). \tag{7}$$

2. Expression of momentum and angular momentum of liquid mass. Momentum vector of the liquid mass

$$\mathbf{K}_1 = \int_\tau \rho \operatorname{grad} \Phi d\tau, \tag{8}$$

where τ - volume, occupied by liquid; ρ - density of liquid ($\rho=\text{const}$).

Using expression (6), we will obtain

$$\mathbf{K}_1 = \rho \left\{\int_\tau \operatorname{grad}(V_0 R^*)\, d\tau + \int_\tau \operatorname{grad}(\omega\Omega)\, d\tau + \sum_{i=1}^{\infty} \dot{a}_i \int \operatorname{grad} A_i d\tau\right\}. \tag{9}$$

From physical considerations following equalities are obvious:

$$\int_\tau \operatorname{grad}(V_0 R^*)\, d\tau = V_a \tau; \int_\tau \operatorname{grad}(\omega\Omega)\, d\tau = \tau\, [\omega \times \mathbf{R}_{c1}], \tag{10}$$

where $\mathbf{R}_{c1}$ - radius-vector of center of inertia of liquid mass.

Let us further consider expressions, which stand under sign of sum in (9):

$$\int_\tau \operatorname{grad} A_i d\tau = \int_\tau (\operatorname{grad} A_i \nabla) R^* d\tau = \int_\tau (\nabla \operatorname{grad} A_i) R^* d\tau -$$

$$- \int_\tau R^* \Delta A_i d\tau = \oint_S R^* \frac{\partial A_i}{\partial n} dS.$$

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Identity is used for these conversions.

$$\mathbf{a} = (\mathbf{a}\nabla) \mathbf{R}^*, \left(\nabla = \sum_{k=1}^{3} e_k^* \frac{\partial}{\partial x_k^*} \right),$$

where $\mathbf{e}_k^*$ - unit vectors of the system of coordinates $O^*x_1^*x_2^*x_3^*$.

Using conditions (7), we will obtain

$$\oint_S R^* \frac{\partial A_i}{\partial n} dS = R_0^* \iint_G f_i(x_1, x_2)\, dG +$$

$$+ \iint_G R f_i(x_1, x_2)\, dG = \iint_G R_G f_i(x_1, x_2)\, dG, \qquad (11)$$

where $\mathbf{R}_G$ indicates radius-vector of point of G region. Let us designate through

$$\mathbf{L}^{(i)} = \iint_G \mathbf{L} f_i dG \qquad (12)$$

the Fourier coefficient of the vector function L. Then, using (11), it is possible to register

$$\int_{\tau} \operatorname{grad} A_i d\tau = \mathbf{R}_G^{(i)}. \qquad (13)$$

Using equalities (10) and (13), let us represent momentum vector of liquid in the form

$$\mathbf{K}_1 = m_1 \mathbf{V}_0 + m_1 [\omega \times \mathbf{R}_{c1}] + \rho \sum_{i=1}^{\infty} \dot{a}_i \mathbf{R}_G^{(i)}, \qquad (14)$$

where $m_1 = \rho\tau$ - mass of liquid.

Let us designate moment of momentum of liquid relative to point O_1 (O_1 - point in system of coordinates $O^*x_1^*x_2^*x_3^*$, with which at the given instant coincides point O) through G_1.

Analogous with vector K_1 vector G_1 can be represented in the form

$$\mathbf{G}_1 = \sum_{k=1}^{\infty} \mathbf{G}_{1k}, \qquad (15)$$

$$\mathbf{G}_{11} = \int_{\tau} \rho [\mathbf{R} \times \mathbf{V}_0] d\tau = m_1 [\mathbf{R}_{c1} \times \mathbf{V}_0]; \quad \mathbf{G}_{12} = \int_{\tau} \rho [\mathbf{R} \times \operatorname{grad}(\omega \mathbf{\Omega})] d\tau = (J_1, \omega),$$

where J_1, as follow from work of N. Ye. Zhukovskiy [4], symmetrical affine orthogonal tensor of 2nd rank, analogous to tensor, inertias of solid body, with components

$$J_{1kj} = \rho \int_{\tau} \operatorname{grad} \Omega_k \operatorname{grad} \Omega_j d\tau. \qquad (16)$$

Expression for G_{13} can be represented in the form

$$G_{13}=\rho\sum_{i=1}^{\infty}\dot{a}_i\int_{\tau}[\mathbf{R}\times\operatorname{grad}A_i]\,d\tau=\rho\sum_{i=1}^{\infty}\dot{a}_i\boldsymbol{\Omega}_G^{(i)};$$

$$\boldsymbol{\Omega}_C=\boldsymbol{\Omega}(x_1,\ x_2,\ x_3=C),$$

since

$$\int_{\tau}[\mathbf{R}\times\operatorname{grad}A_i]\,d\tau=\iint_S[\mathbf{R}\times\mathbf{n}]\,A_i dS=\iint_S\frac{\partial\Omega}{\partial n}A_i dS=$$

$$=\iint_S\frac{\partial A_i}{\partial n}\Omega\,dS=\iint_G\boldsymbol{\Omega}_C f_i\,dG=\boldsymbol{\Omega}_C^{(i)}.$$

Finally moment of momentum of liquid relative to point O_1 will be registered as follows:

$$G_1=m_1[\mathbf{R}_{c1}\times\mathbf{V}_0]+(J_1,\ \omega)+\rho\sum_{i=1}^{\infty}\dot{a}_i\boldsymbol{\Omega}_C^{(i)}. \qquad (17)$$

3. Equations of motion. Let us register the expressions of the momentum of solid body K_0 and its moment of momentum G_0 relative to point O_1:

$$\mathbf{K}_0=m_0\mathbf{V}_0+m_0[\omega\times\mathbf{R}_{c0}];\quad G_0=m_0[\mathbf{R}_{c0}\times\mathbf{V}_0]+(J_0,\omega), \qquad (18)$$

where m_0 - mass of solid body; J_0 - tensor of its inertia relative to point O, and $\mathbf{R}_{c0}$ - radius-vector of the center of inertia of solid body relative to point O. Using expressions (14), (17) and (18), let us compile the equations of the momentum and moment of momentum, which with an accuracy to the small first-order quantity will take the following form:

$$m\dot{\mathbf{V}}_0+[\dot{\omega}\times\mathbf{L}]+\rho\sum_{i=1}^{\infty}\ddot{a}_i\mathbf{R}_G^{(i)}=\sum_j\mathbf{P}_j; \quad (19)$$

$$[\mathbf{L}\times\dot{\mathbf{V}}_0]+(J,\ \dot{\omega})+\rho\sum_{i=1}^{\infty}\ddot{a}_i\mathbf{\Omega}_C^{(i)}=\sum_j\mathbf{M}_j, \quad (20)$$

where $m=m_0+m_1$ - general/common mass of system;

$J=J_0+J_1$ — tensor of the inertia of system relative to point O, and $\mathbf{L}=m_0\mathbf{R}_{c0}+m\mathbf{R}_{c1}$ - static torque of system relative to the same point; $\mathbf{P}_j$ - external forces and the inertial forces, which function on the system; $\mathbf{M}_j$ - moment of external and inertial forces relative to point O. We convert the right side of the second equation.

It is obvious that of all external moments only moment from mass forces $\mathbf{M}_g=[\mathbf{L}\times\mathbf{j}]$ depends on form of free surface of liquid, L - static moment with respect to point O of system solid body + liquid.

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Let us designate the difference between L and vector L_0, which is static moment in the case, when the equation of free surface takes form x_3=C, through ΔL_0:

$$\mathbf{L}=\mathbf{L}_0+\Delta\mathbf{L}_0, \quad (21)$$

It is obvious that it is possible to represent with an accuracy to small first-order quantity ΔL_0, substituting the value of x_3 on (1), in the form:

$$\Delta \mathbf{L}=\rho \iint_{G} \mathbf{R}\left(x_{3}-C\right) d G=\rho \sum_{i=1}^{\infty} a_{i} \mathbf{R}_{G}^{(i)}. \tag{22}$$

Moment from mass forces $\mathbf{M}_g$ can be represented in this form:

$$\mathbf{M}_{g}=\mathbf{M}_{g0}+\rho \sum_{i=1}^{\infty} a_{i}\left[\mathbf{R}_{G}^{(i)} \times \mathbf{j}\right], \tag{23}$$

where $\mathbf{M}_{g0}=[\mathbf{L}_0\times\mathbf{j}]$ - moment, which functions on system in the case, when free surface is plane, whose equation x_3=C.

After preserving from value $\mathbf{M}_g$ under sign of sum, which stands in right side of equation (20), only $\mathbf{M}_{g0}$, let us isolate that part of moment of mass forces, which depends on parameters a_i, i.e., on form of free surface. In this case equations (19) and (20) will take the form

$$m \mathbf{V}_{0}+[\dot{\boldsymbol{\omega}} \times \mathbf{L}_{0}]=\sum_{j} \mathbf{P}_{j}-\rho \sum_{i=1}^{\infty} \ddot{a}_{i} \mathbf{R}_{G}^{(i)}; \tag{24}$$

$$[\mathbf{L}_{0} \times \dot{\mathbf{V}}_{0}]+(J, \dot{\omega})=\sum_{j} \mathbf{M}_{j}-\rho \sum_{i=1}^{\infty}\left(\ddot{a}_{i} \boldsymbol{\Omega}_{C}^{(i)}-a_{i}\left[\mathbf{R}_{G}^{(i)} \times \mathbf{j}\right]\right). \tag{25}$$

Let us further compile equations, which determine change in parameters a_i. For this purpose we use conditions of pressure constancy on the free surface of liquid, which leads to the following equalities:

$$\iint_{G} p_{\Sigma} f_{i}\, d G=0 \quad (i=1,2, \ldots), \tag{26}$$

where p_Σ - pressure at the appropriate points of surface Σ.

Using Cauchy integral of Euler equations, is expressed p_Σ with an accuracy down to the terms of first-order of smallness as follows:

$$p_\Sigma = -\rho\left(\frac{\partial\Phi}{\partial t} + U\right)_\Sigma .$$

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After substituting this expression into (26), after excluding U and Φ with the help of (3) and (6) with an accuracy down to the terms of first-order of smallness, conditions (26) into this case are converted thus:

$$\sum_{k=1}^{\infty} \ddot{a}_k A_{kc}^{(i)} + a_i j = (\mathbf{j} - \dot{\mathbf{V}}_0)\mathbf{R}_G^i - \dot{\omega}\mathbf{\Omega}_C^{(i)}$$

$$(i = 1,2,\ldots), \qquad (27)$$

where $$A_{kc}^{(i)} = \iint_G A_{kc} f_i dG \quad j = |\mathbf{j}| .$$

Equations (27) together with (24) and (25) ose infinite system of equations, which describe solution of represented problem in the case of arbitrary form of cavity. Equations (27) substantially are simplified for the cavities, which have cylindrical form.

It will place system of coordinates $Ox_1x_2x_3$ so that its origin, point O, would be located in plane of symmetry of cylindrical column of liquid (Fig. 2), and axis Ox_3 would be parallel to generatrix. Let us consider for this case the boundary-value problem, which determines functions A_k:

$$\frac{\partial^2 A_k}{\partial x_1^2}+\frac{\partial^2 A_k}{\partial x_2^2}+\frac{\partial^2 A_k}{\partial x_3^2}=0; \quad \left[\frac{\partial A_k}{\partial n}\right]_\sigma=0;$$

$$\frac{\partial A_k}{\partial x_3}=\begin{cases} 0 \text{ при } x_3=-\frac{1}{2}h; \\ f_k(x_1, x_2) \text{ при } x_3=\frac{1}{2}h, \end{cases} \tag{28}$$

Key: (1). with.

where σ - lateral surface of cavity.

In this case functions A_k can be represented in the form

$$A_k=\frac{\operatorname{ch}\lambda_k\left(x_3+\frac{1}{2}h\right)}{\lambda_k \operatorname{sh}\lambda_k h} f_k(x_1, x_2), \tag{29}$$

where λ_k - eigenvalues of boundary-value problem (2). Then

$$A_{kc}=C_k f_k(x_1, x_2); \quad C_k=\frac{1}{\lambda_k}\operatorname{cth}\lambda_k h; \quad A_{kc}^{(i)}=\begin{cases} 0 \text{ при } i\neq k; \\ C_k \text{ при } i=k. \end{cases}$$

Key: (1). with.

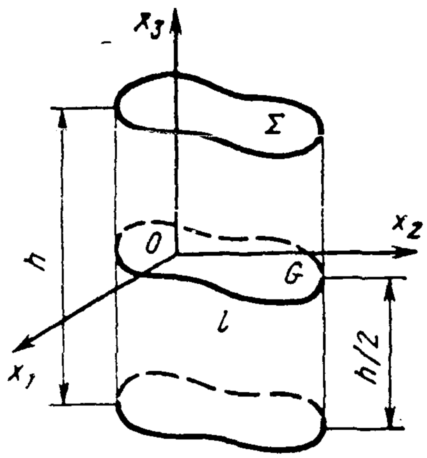

Fig. 2.

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Equations of motion of body, which has cylindrical cavity, partially filled with liquid, take form

$$m\dot{\mathbf{V}}_0+[\dot{\omega}\times\mathbf{L}_0]=\sum_j\mathbf{P}_j-\rho\sum_{i=1}^{\infty}\ddot{a}_i\mathbf{R}_G^{(i)}\quad(i=1,2,\ldots);$$

$$[\mathbf{L}_0\times\dot{\mathbf{V}}_0]+(J,\ \dot{\omega})=\sum_j\mathbf{M}_j-\rho\sum_{i=1}^{\infty}(\ddot{a}_i\mathbf{\Omega}_C^{(i)}-a_i\,[\mathbf{R}_G^{(i)}\times\mathbf{j}]);\qquad(30)$$

$$\ddot{a}_iC_i+a_ij=(\mathbf{j}-\dot{\mathbf{V}}_0)\mathbf{R}_G^{(i)}-\dot{\omega}\mathbf{\Omega}^{(i)}\quad(i=1,2,\ldots).$$

Initial values $a_i(t_0)$ and $\dot{a}_i(t_0)$ are determined from relationships/ratios

$$a_i(t_0)=\iint_G\xi f_i dG=\xi^{(i)},\quad \dot{a}_i(t_0)=\iint_G\chi f_i dG=\chi^{(i)},$$

$$x_3(x_1,\ x_2,\ t_0)-C=\xi(x_1,\ x_2);\quad u_n(x_1,\ x_2,\ t_0)=\chi(x_1,\ x_2).$$

It is possible to conclude from form of equation (30) that if Fourier coefficients from R_G^l and Ω_C^l, that correspond to any function f(l), are simultaneously equal to zero, then level, which determines change in parameter a_l, does not depend on other equations, i.e., change in this parameter does not depend on motion of vessel and, on the contrary, motion of vessel does not depend on change in this parameter.

4. Existence and uniqueness of solution of equations (30). Possibility of application for solving the method of reduction. We will examine only the plane motion of vessel, which possesses the arbitrary cavity, partially filled with liquid. The examination of plane motion simplifies computations, without reducing generality.

Let us assume that motion of vessel occurs in plane $O^*x_2^*x_3^*$ and that axis Ox_3 passes through center of gravity of system in undisturbed position of free surface, and vector j is parallel to axis $O^*x_3^*$. Let us designate through ϵ the angle between axes $O^*x_3^*$ and Ox_3.

Then system (30) can be represented in the form

$$m\ddot{x} - l\ddot{\varepsilon} = \sum_j P_j - \rho \sum_{i=1}^{\infty} \ddot{a}_i x_G^{(i)};$$

$$J\ddot{\varepsilon} - l\ddot{x} = \sum_j M_j - \rho \sum_{i=1}^{\infty} (\ddot{a}_i \Omega_C^{(i)} + a_i j x_G^{(i)}); \qquad (31)$$

$$\ddot{a}_i + \omega_i^2 a_i = -\frac{1}{C_i} (x_C^{(i)} \ddot{x} + \Omega_C^{(i)} \ddot{\varepsilon} + j x_G^{(i)} \varepsilon) \quad (i = 1, 2, \ldots).$$

Here $x^{*}_{02}=x;\ P_{j2}=P_j;\ x^{(i)}_{G_1}=x^{(i)}_G;$

$$\frac{1}{C_i}=\omega_i^2;\ L_3=l;\ J_{12}=J;\ M_{j1}=M_j;\ \Omega^{(i)}_{C_1}=\Omega^{(i)}_C.$$

Initial conditions we have in the form

$$\varepsilon=\varepsilon_0;\ \dot{\varepsilon}=\dot{\varepsilon}_0;\ x=x_0;\ a_i=\xi^{(i)};\ \dot{a}_i=\chi^{(i)}\ \overset{(1)}{\text{при}}\ t=t_0.$$

Key: (1). with.

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Let us consider in system (31) equations, which contain finite number of unknowns. Let us compile the homogeneous equations corresponding to them:

$$\ddot{a}_i+\omega_i^2 a_i=0\quad(i=1,2,\ldots).\tag{32}$$

Let us designate through y_i and z_i linearly independent solutions of equations (32), which satisfy initial conditions

$$y_i=1;\ \dot{y}_i=0;\ z_i=0;\ \dot{z}_i=1\ \overset{(1)}{\text{при}}\ t=t_0.$$

Key: (1). with.

Then, obviously, it is possible to register following expressions:

$$a_i(t)=\xi^{(i)}y_i+\chi^{(i)}z_i+\int_{t_0}^{t}K_i(t\tau)\frac{1}{c_i}\left(x^{(i)}_G\dot{x}+\Omega^{(i)}_G\ddot{\varepsilon}+jx^{(i)}_G\varepsilon\right)d\tau,\tag{33}$$

where $K_i(t\tau)=z_i(t)y_i(\tau)-z_i(\tau)y_i(t)$.

First of all we should be convinced of the fact that $|y_i(t)|$, $|z_i(t)|$, and therefore, and $|K_i(t\tau)|$ in any limited segment $t(t_0\leqslant t\leqslant t_1)$ cannot exceed certain positive number M, which can be assigned independent of index i.

Let us rewrite equations (32) in this form:

$$\ddot{a}_i+j(t)\beta_i^2 a_i=0\ \left(\beta_i^2=\frac{1}{c_1}=\lambda_1\operatorname{th}\lambda_i h\right)\quad (i=1,2,\ldots)\qquad (34)$$

and we will assume that function j(t), which characterizes change on time of fièld of mass forces, has continuous second derivative. Furthermore, let us assume that j(t)≥g, where g - certain positive number. After replacing in the equations the variables

$$\zeta=\int_{t_0}^{t}\sqrt{j}\,dt;\quad a_i^*(\zeta)=\sqrt[4]{j}\,a_i(t),\qquad (35)$$

we convert them to the following form:

$$\frac{d^2a_i^*}{d\zeta^2}+\left[\beta_i^2-q(\zeta)\right]a_i^*=0;\quad q(\zeta)=\left(\frac{1}{j}\right)^{1/4}\frac{d^2}{d\zeta^2}(j)^{1/4}.\qquad (36)$$

Let us consider solutions of equations (36) with fixed values of initial conditions

$$a_i^*=m;\quad \dot{a}_i^*=n\ \ \overset{(1)}{\text{при}}\ \zeta=0.$$

Key: (1). with.

Let us designate function $a_i^*(\zeta)$, which satisfies these initial conditions, through $a_i(\zeta)$. Following theorem [5], which let us give without the proof, occurs.

In any limited segment of change in variable $\zeta(0\leqslant\zeta\leqslant\zeta_1)$ solutions of equations (36) $a_i(\zeta)$ satisfy inequality $|a_i(\zeta)|<N$, where N - certain positive constant number (not depending on index i, but therefore, on parameter β_i).

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Let us designate

$$y_i^*(\zeta)=\sqrt[4]{jy_i(t)};\quad z_i^*(\zeta)=\sqrt[4]{jz_i(t)};$$

$$y_i^*=\sqrt[4]{j(t_0)};\quad \dot{y}_i^*=\left(\frac{1}{j}\right)^{1/4}\frac{d}{dt}(\sqrt[4]{j})_{t=t_0};\quad z_i^*=0,\quad \dot{z}_i^*=\left(\frac{1}{j}\right)^{1/4}$$
$$\overset{(1)}{\text{при}}\ \zeta=0\,(t=t_0),$$

Key: (1). with.

and since $j(t)\geqslant g>0$, then all these initial conditions are limited.

It follows from theorem given above that solutions of equations (36) $y_i^*(\zeta)$ and $z_i^*(\zeta)$ are limited in absolute value by certain number N:

$$|y_i^*|<N;\quad |z_i^*|<N\quad (0<\zeta<\zeta_1).$$

But then from formula (35) we obtain for $t_0\leqslant t\leqslant t_1$

$$\max|y_i(t)| < M; \ \max|z_i(t)| < M; \ \max|K_i(t\tau)| < 2M^2 \ (M = N/g^{1/4}), \quad (37)$$

where M - number, which does not depend on value of index i, which can take any values from 1 to ∞.

Further, using (33), let us find

$$\ddot{a}_i = -\frac{1}{C_i}\left(x_G^{(i)}\ddot{x} + \Omega_C^{(i)}\ddot{\varepsilon} + j x_G^{(i)}\varepsilon + j\xi^{(i)}y_i + j\chi^{(i)}z_i\right) - $$

$$-\frac{1}{C_i^2}\int_{t_0}^{t} K_i(t\tau)\left(x_G^{(i)}\ddot{x} + \Omega_C^{(i)}\ddot{\varepsilon} + j x_G^{(i)}\varepsilon\right) j\,d\tau. \quad (38)$$

Let us substitute expressions $a_i(t)$ and $\ddot{a}_i(t)$ from (33) and (38) into first two equations of system (31)

$$m\ddot{x} - l\ddot{\varepsilon} = \sum_j P_j + \ddot{x}\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i} + \ddot{\varepsilon}\rho\sum_{i=1}^{\infty}\frac{\Omega_C^{(i)}x_G^{(i)}}{C_i} + \varepsilon j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i} +$$

$$+ j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}}{C_i}\left(\xi^{(i)}y_i + \chi^{(i)}z_i\right) + j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}}{C_i^2}\int_{t_0}^{t} K_i(t\tau)\left(x_G^{(i)}\ddot{x} + \right.$$

$$\left. + \Omega_C^{(i)}\ddot{\varepsilon} + j x_G^{(i)}\varepsilon\right) d\tau; \quad (39)$$

$$J\ddot{\varepsilon} - l\ddot{x} = \sum_j M_j + \ddot{x}\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}\Omega_C^{(i)}}{C_i} + \ddot{\varepsilon}\rho\sum_{i=1}^{\infty}\frac{\Omega_C^{(i)2}}{C_i} +$$

$$+ \varepsilon j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}\Omega_C^{(i)}}{C_i} + j\rho\sum_{i=1}^{\infty}\left(\frac{\Omega_C^{(i)}}{C_i} - x_G^{(i)}\right)\left(\xi^{(i)}y_i + \chi^{(i)}z_i\right) +$$

$$+ j\rho\sum_{i=1}^{\infty}\frac{\left(\Omega_C^{(i)} - x_G^{(i)}C_i\right)}{C_i^2}\int_{t_0}^{t} K_i(t\tau)\left(x_G^{(i)}\ddot{x} + \Omega_C^{(i)}\ddot{\varepsilon} + j x_G^{(i)}\varepsilon\right) d\tau.$$

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We will thus far assume that series/rows under integral signs in equations (39) evenly descend (this will be proved subsequently). Then in equations (39) it is possible to take out integrals as the signs of sum; we will obtain

$$a_{12}\ddot{x}+b_{12}\ddot{\varepsilon}+b_{10}\varepsilon=\sum_j P_j+f_1(t)+\int_{t_0}^{t} A_{12}(t\tau)\ddot{x}d\tau+$$

$$+\int_{t_0}^{t} B_{12}(t\tau)\ddot{\varepsilon}d\tau+\int_{t_0}^{t} B_{10}(t\tau)\varepsilon d\tau; \quad (40)$$

$$a_{22}\ddot{x}+b_{22}\ddot{\varepsilon}+b_{20}\varepsilon=\sum_j M_j+f_2(t)+\int_{t_0}^{t} A_{22}(t\tau)\ddot{x}d\tau+$$

$$+\int_{t_0}^{t} B_{22}(t\tau)\ddot{\varepsilon}d\tau+\int_{t_0}^{t} B_{20}(t\tau)\varepsilon d\tau. \quad (41)$$

Here

$$a_{12}=m-\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i}; \quad a_{22}=-\left(l+\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}\Omega_C^{(i)}}{C_i}\right)=b_{12};$$

$$b_{22}=J-\rho\sum_{i=1}^{\infty}\frac{\Omega_C^{(i)2}}{C_i}; \quad A_{12}=j\rho\sum_{i=1}^{\infty}K_i(t\tau)\frac{x_G^{(i)2}}{C_i^2}; \quad B_{10}=jA_{12};$$

$$B_{12}=j\rho\sum_{i=1}^{\infty}K_i(t\tau)\frac{x_G^{(i)}\Omega_C^{(i)}}{C_i^2}; \quad b_{10}=-j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i};$$

$$b_{20}=-j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}\Omega_C}{C_i}; \quad A_{22}=j\rho\sum_{i=1}^{\infty}K_i(t\tau)\frac{x_G^{i}}{C_i}\left(\frac{\Omega_C^{(i)}}{C_i}-x_G^{(i)}\right);$$

$$B_{22}=j\rho\sum_{i=1}^{\infty}K_i(t\tau)\frac{\Omega_C^{(i)}}{C_i}\left(\frac{\Omega_C^{(i)}}{C_i}-x_G^{(i)}\right); \quad f_1(t)=j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}}{C_i}(\xi^{(i)}y_i+\chi^{(i)}z_i);$$

$$B_{20}=jA_{22}; \quad f_2(t)=j\rho\sum_{i=1}^{\infty}\left(\frac{\Omega_C^{(i)}}{C_i}-x_G^{(i)}\right)(\xi^{(i)}y_i+\chi^{(i)}z_i).$$

It is obvious that the system of integral-differential equations (40, 41) can be converted upon consideration of the initial conditions of moving the vessel (x_0, $\dot{x}_0$, ϵ_0, $\dot{\epsilon}_0$) to a certain final system of integral equations of Vol'erra's type, which, in turn, can be in general form represented as follows:

$$y_k=\varphi_k(t)+\sum_{s=1}^{m}\int_{t_0}^{t}K_s(t\tau)\,y_s d\tau \quad (k=1,\ldots,m). \qquad (42)$$

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Thus, it is shown that solution of infinite system of differential equations (31) is reduced to solution of homogeneous linear equations of second order (32) with subsequent solution of system of integral equations (42). In this case we assumed the convergence of numerical series/rows and the uniform convergence of function series in expressions (41). Thus, if this assumption proves to be justified, then in view of the existence theorem and uniqueness of the solution of equations (42) (with the limitedness of values $|\varphi_k(t)|$ and $|K_s(t\tau)|$) we are convinced of existence and uniqueness of the solution of the infinite system of differential equations (31). Let us demonstrate the uniform convergence of series/rows in expressions (41) in any limited interval of time t.

Hence will follow both limitedness $|\varphi_k(t)||K_s(t\tau)|$ in (42) and possibility of introduction of sign of summation under integral signs in equations (39). For this purpose let us formulate the following theorem.

Theorem. Vector functions $\operatorname{grad} f_i(x_1, x_2)$, where $f_i(x_1, x_2)$ are the eigenfunctions of boundary-value problem (2), are mutually orthogonal on G region.

Proof. Using a formula of Green and a boundary condition of boundary-value problem (2), we will obtain

$$\iint_G \operatorname{grad} f_k \operatorname{grad} f_m dG = -\iint_G f_k \Delta f_m dG.$$

But from (2) we have $\Delta f_m = -\lambda_m{}^2 f_m$. Hence, after taking into account the orthogonality of eigenfunctions, we obtain the proof necessary to us:

$$\iint_G \operatorname{grad} f_k \operatorname{grad} f_m dG = \lambda_m^2 \iint_G f_k f_m dG \quad (k \neq m). \qquad (43)$$

Now let us begin proof of uniform convergence in any limited interval of time t of function series, which stand in expressions (31). Let us consider the series/row, which stands in expression A...

$$\sum_{i=1}^{\infty} K_i(t\tau) \frac{x_G^{(i)\prime}}{C_i} . \qquad (44)$$

It is obvious from equality (37) that in any limited interval of time for this series/row it is possible to construct majorant series/row:

$$2M^2=\sum_{i=1}^{\infty}\frac{x_G^{(i)}}{C_i^2}.$$

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Let us consider convergence of series

$$\sum_{i=1}^{\infty}\frac{x_G^{(i)}}{C_i^2}=\sum_{i=1}^{\infty}\frac{\operatorname{th}^2\lambda_i h}{\lambda_i^2}\left(\iint_G x\lambda_i^2 f_i\,dG\right)^2.$$

It is obvious that all terms of series/row are positive. Let us note also that $\operatorname{th}\lambda_i h<1$. Using a formula of Green and a boundary condition for function f_i, we convert integral, which is located in the brackets:

$$\iint_G x\lambda_i^2 f_i\,dG=-\iint_G x\Delta f_i\,dG=\iint_G \operatorname{grad}x\operatorname{grad}f_i\,dG.$$

Using this conversion, it is possible to compose inequality

$$\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i^2}\leqslant\sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}\left(\iint_G \operatorname{grad}x\operatorname{grad}f_i\,dG\right)^2.$$

Above is shown orthogonality of functions $\operatorname{grad}f_i$ on G; this makes it possible to compose inequality of Bessel

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \left(\iint_G \operatorname{grad} x \operatorname{grad} f_i dG \right)^2 \leqslant \iint_G (\operatorname{grad} x)^2 dG = G.$$

Latter proves limitedness of sum of majorant series/row and, therefore, uniform convergence of series/row (44).

Let us consider series/row, which stands in expression B_{12}:

$$\sum_{i=1}^{\infty} K_i(t\tau) \frac{x_G^{(i)} \Omega_C^{(i)}}{C_i^2}. \qquad (45)$$

Proof of uniform convergence of this series/row analogous with that carried out for series/row (44) is reduced to proof of convergence of series

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \left(\iint \operatorname{grad} x \operatorname{grad} f_i dG \right) \left(\iint_G \operatorname{grad} \Omega_C \operatorname{grad} f_i dG \right). \qquad (46)$$

Let us register inequality of Bessel

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \left[\iint_G \left(\operatorname{grad} x + \operatorname{grad} \Omega_c \right) \operatorname{grad} f_i dG \right]^2 \leqslant \iint_G (\operatorname{grad} x + \operatorname{grad} \Omega_c)^2 dG.$$

Left side of this inequality can be represented in the form

$$\sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}\left(\iint_G \operatorname{grad} x \operatorname{grad} f_i dG\right)^2+\sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}\left(\iint_G \operatorname{grad}\Omega_c \operatorname{grad} f_i dG\right)^2+$$

$$+2\sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}\left(\iint_G \operatorname{grad} x \operatorname{grad} f_i dG\right)\left(\iint_G \operatorname{grad}\Omega_c \operatorname{grad} f_i dG\right).$$

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Hence we obtain, that summation of series (46) is less

$$\frac{1}{2}\iint_G (\operatorname{grad} x+\operatorname{grad}\Omega_c)^2\, dG,$$

that also proves uniform convergence of series/row (45).

Uniform convergence of all remaining series/rows, entering expressions (41), analogously is proven.

Let us pass to proof of possibility of applying method of reduction for solving infinite system of differential equations (31).

Instead of infinite system of equations (31) we will examine certain finite system of equations with n unknown parameters $u_i(t)$, that is obtained from (31), if we in it assume $a_i(t)=0$ for i<n.

Let us register this system of equations

$$m\ddot{x} - l\ddot{\varepsilon} = \sum_j P_j - \rho \sum_{i=1}^{n} \ddot{a}_i x_0^{(i)};$$

$$J\ddot{\varepsilon} - l\ddot{x} = \sum_j M_j - \rho \sum_{i=1}^{n} (\ddot{a}_i \Omega_C^{(i)} + a_i j x_0^{(i)});$$

$$\ddot{a}_i + \omega_i^2 a_i = -\frac{1}{C_i} (x_0^{(i)} \ddot{x} + \Omega_C^{(i)} \ddot{\varepsilon} + j x_0^{(i)} \varepsilon) \quad (i = 1, \ldots, n). \tag{47}$$

As initial conditions for unknown functions of system (47) let us take appropriate initial conditions of unknowns of system of equations (31), considering initial conditions for parameters a_i at i<n equal to zero.

Essence of method of reduction consists in finding of solution of infinite system of equations (31) with the help of process of successive approximations to solution of system (31) by solutions of finite systems of equations of type (47) with entire increasing number n of parameters of free surface $a_i(t)$.

Let us demonstrate that in any limited time interval solutions of systems of equations (47) with $n \to \infty$ evenly converge to solution of infinite system of equations (31).

It is completely obvious that with each given in advance number n solution of system of equations (47) can be reduced to integration of n linear homogeneous second order equations of type (32) with subsequent solution of system of integral equations of form

$$y_n^{[n]} = \varphi_k^{[n]}(t) + \sum_{s=1}^{m} \int_{t_0}^{t} K_s^{[n]}(t\tau)\, y_s^{[n]} d\tau \quad (k = 1, \ldots, m). \tag{48}$$

Equations (48) are similar to equations (42).

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If we demonstrate that the solutions of system (48) with $n \to \infty$ evenly converge to the solution of system (42), then hence there will be analogous affirmation relative to the solutions of systems (47) and (31).

For this purpose let us consider integral equations

$$\Delta y_k^{[n]} = \Delta\psi_k^{[n]}(t) + \sum_{s=1}^{m} \int_{t_0}^{t} K_s^{[n]}(t\tau)\,\Delta y_s^{[n]} d\tau \quad (k = 1, \ldots, m). \tag{49}$$

Theorem. If in any limited segment they are limited by the variable t of the series/row of equations (49) in the absolute value, then the solutions of equations in this interval will be as low as desired in the absolute value, as soon as absolute values $\Delta\psi_k^{[n]}(t)$ are sufficiently low.

Proof of this theorem we lower; it can be easily carried out on basis of consecutive iteration of solution and compilation of corresponding evaluations. Further let us designate

$$\Delta y_k^{[n]} = y_k - y_k^{[n]};\quad \Delta K_s^{[n]}(t\tau) = K_s(t\tau) - K_s^{[n]}(t\tau);$$

$$\Delta\psi_k^{[n]}(t) = \varphi_k(t) - \varphi_k^{[n]}(t) + \sum_{s=1}^{m} \int_{t_0}^{t} \Delta K_s^{[n]}(t\tau)\, y_s d\tau.$$

In view of proved above uniform convergence of series/rows in expressions (41) it is possible to claim following.

1. In any limited interval of time t

$$\lim |K_s^{[n]}(t\tau)| = 0; \quad \lim |\Delta\psi_k^{[n]}(t)| = 0 \text{ при}^{(1)} \ n \to \infty.$$

Key: (1). with.

2. Independent of value of n there is in limited interval of time t upper bound for $|K_s^{[n]}(t\tau)|$, i.e.

$$|K_s^{[n]}(t\tau)| < M.$$

Hence it follows that, since with sufficiently high value of number of value of all $|\Delta\psi_k^{[n]}|$ can be limited by as conveniently small number with limitedness of values $|K_s^{[n]}(t\tau)|$ independent of number n, in view of expressed above theorem $y_k = \lim y_k^{[n]}$ with $n \to \infty$ in limited interval of time t, i.e., solutions of equations (49) evenly converge to solutions of equations (42) which proves possibility of applying method of reduction for solving equations (30).

5. Case $j = \text{const}$. Some examples. When $j = \text{const}$ we have

$$y_i = \cos\omega_i (t - t_0); \quad z_i = \frac{1}{\omega_i} \sin\omega_i (t - t_0).$$

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Integral-differential equations (40) have kernels, which depend only on difference in arguments, since in this case

$$K_i(t\tau) = -\frac{1}{\omega_i}\sin\omega_i(t-\tau),$$

and therefore for their solution successfully can be used transform of Laplace or Carson with subsequent operation of convolution of representing functions. Let us consider some examples.

A. L. N. Sretnskiy's problem. Let us assume that the body with the cavity of cylindrical form can complete only forward motions along axis $O^*x_2^*$, being located under the action of elastic forces of some springs, working according to Hooke's law and arranged/located in the direction of axis $O^*x_2^*$. Analogous problem was solved by L. N. Sretenskiy for the case of cavity in the form of rectangular prism [6].

Here we will examine this problem for general case of cylindrical form of cavity, using diagram of solution of problem presented above.

Equations (42) in this case will take the form

$$m\ddot{x} + cx = -\rho\sum_{i=1}^{\infty}\ddot{a}_i x_O^{(i)};$$

$$\ddot{a}_i + \omega_i^2 a_i = -\frac{1}{C_i}x_O^{(i)}\ddot{x} \quad (i = 1, 2, \ldots).$$

Using initial conditions of problem, let us lead its solution to solution of integrodifferential equations of type (40). In this case we will obtain only the one equation of the following form:

$$\left(m-\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)*}}{C_i}\right)\ddot{x}+cx=j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}}{C_i}\left(\xi^{(i)}\cos\omega_i t+\chi^{(i)}\frac{1}{\omega_i}\sin\omega_i t\right)-$$

$$-\rho\int_0^t\sum_{i=1}^{\infty}\frac{x_G^{(i)}\omega_i}{C_i}\sin\omega_i(t-\tau)\ddot{x}d\tau. \qquad (50)$$

Let us compile representing equation for equation (50). In this case we will designate

$$F(p)\doteqdot x(t),\ \overset{(1)}{\text{если}}\ F(p)=p\int_0^{\infty}e^{-pt}x(t)\,dt.$$

Key: (1). if.

Using known relationships/ratios of operational calculus [7], and also using theorem of convolution, representing equation can be represented in the following form:

$$p^2\left(m-\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)*}}{C_i}\right)\left[F(p)-x_0-\frac{1}{p}\dot{x}_0\right]+cF(p)=$$

$$=j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}}{C_i}(\xi^{(i)}p+\chi^{(i)})\frac{p}{p^2+\omega_i^2}-\rho\sum_{i=1}^{\infty}\frac{\omega_i^2 x_G^{(i)*}}{C_i}\times$$

$$\times\left[F(p)-x_0-\frac{1}{p}\dot{x}_0\right]\frac{p}{p^2+\omega_i^2}. \qquad (51)$$

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From equation (51) we obtain following expression for image of unknown function:

$$F(p)=\frac{F_1(p)}{F_2(p)},$$

where
$$F_1(p)=\left(mp^2-\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i}\frac{p^4}{p^2+\omega_i^2}\right)\left(x_0+\frac{1}{p}\dot{x}_0\right)+$$
$$+j\rho\sum_{i=1}^{\infty}\frac{x_G^{(i)}}{C_i}\left(\xi^{(i)}p+\chi^{(i)}\right)\frac{p}{p^2+\omega_i^2};$$
$$F_2(p)=mp^2+C-\rho\frac{x_G^{(i)2}}{C_i}\frac{p^4}{p^2+\omega_i^2}.$$

Function F(p), obviously, is meromorphic. Let us register its expansion into series in terms of partial fractions:

$$F(p)=\sum_{k=1}^{\infty}\frac{F_1(p_k)}{p_kF_2'(p_k)}\frac{p}{p-p_k},\qquad (52)$$

where p_k $(k=1, 2, \ldots)$ - roots of level $F_2(p)=0$. We further find the initial function x(t):

$$x(t)=\sum_{k=1}^{\infty}\frac{F_1(p_k)}{p_kF_2'(p_k)}e^{p_k t}.\qquad (53)$$

Formula (53) completely solves assigned problem, as soon as will be determined roots of equation $F_2(p)=0$.

Let us consider this equation, after representing it in the form

$$\frac{mp^2+C}{p^4}=\frac{\rho}{j}\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{1+(C_i/j)\,p^2} \quad \text{(1)или} \quad \frac{C-m\lambda}{\lambda^2}=\frac{\rho}{j}\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{1-\lambda/\omega_i^2}$$

$$(\lambda=-p^2). \qquad (54)$$

Key: (1). or.

Equation (54), being analogous to equation, obtained by L. N. Sretenskiy for case of cavity in the form of rectangular prism, gives solution of problem for cavities of arbitrary cylindrical form.

We investigate roots of equation (54). For this let us consider two lines in plane $y\lambda$ determined by the equations

$$y_1=\frac{C-m\lambda}{\lambda^2}; \quad y_2=\frac{\rho}{j}\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{1-\lambda/\omega_i^2}. \qquad (55)$$

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Form of these two curves is depicted in Fig. 3. The abscissas of the points of intersection of the curves $y_1(\lambda)$ and $y_2(\lambda)$, obviously, will be the roots of equation (54).

Let us show that there exists not one negative root of equation (54). This means that numbers p_k in expression (53) still are pure imaginary. Let us consider the left side of the equation. With $\lambda \leq 0$, since $C>0$, we have

$$y_1=\frac{C-m\lambda}{\lambda^2}>-\frac{m}{\lambda}. \qquad (56)$$

It is analogous for right side of equation (54) with λ≤0

$$y_2 = \frac{\rho}{j}\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{1-(C_i/j)\lambda} < -\frac{\rho}{\lambda}\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i}.$$

Let us consider series/row $\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i}$.

Substituting values $x_G^{(i)}$, C_i, entering terms of this series/row, by appropriate expressions and using inequality $\operatorname{th}\lambda_i h < \lambda_i h$, we will obtain

$$\sum_{i=1}^{\infty}\frac{x_G^{(i)2}}{C_i} = \sum_{i=1}^{\infty}\left(\iint_G x f_i\, dG\right)^2 \lambda_i \operatorname{th}\lambda_i h < h\sum_{i=1}^{\infty}\lambda_i^2\left(\iint_G x f_i\, dG\right)^2.$$

After producing conversions, analogous to those, which were conducted in p. 4, and using Bessel's inequality for orthogonal on region G of vector functions $\operatorname{grad} f_i$, we will obtain

$$\sum_{i=1}^{\infty}\lambda_i^2\left(\iint_G x f_i\, dG\right)^2 = \sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}\left(\iint_G \operatorname{grad} x \operatorname{grad} f_i\, dG\right)^2 \leqslant$$
$$\leqslant \iint_G (\operatorname{grad} x)^2\, dG = G.$$

Hence we have

$$y_2 < -\frac{\rho h G}{\lambda} = -\frac{m_1}{\lambda} < -\frac{m}{\lambda}, \qquad (57)$$

where $m_1 = \rho h G$ - mass of liquid.

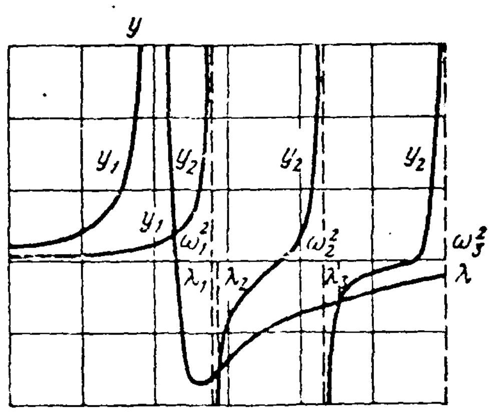

Fig. 3.

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Comparing inequalities (56) and (57), we will obtain $y_1>y_2$ with $\lambda\leq 0$, i.e., equation (54) cannot be satisfied with any negative value λ.

B. Problem about pendulum with liquid filling. Let us consider the plane motion of body with the cylindrical cavity, filled with liquid, about the fixed point. Let us take fixed point for point O of the origin of body coordinate system. Then the first equation of system (31) is excluded from the examination.

Let us assume that center of mass of system in undisturbed state of free surface of liquid is located below point of suspension O at a distance l_C.

Then, if no other forces, except gravity, act on system, we obtain

$$\sum_j M_j = -jml_C\varepsilon = -jl\varepsilon \quad (l = ml_C).$$

Equations (31) will take the form:

$$J\ddot{\varepsilon} + jl\varepsilon = -\rho \sum_{i=1}^{\infty} (\ddot{a}_i \Omega_c^{(i)} + a_{ij} x_G^{(i)});$$

$$\ddot{a}_i + \omega_i^2 a_i = -\frac{1}{C_i} (\Omega_C^{(i)} \ddot{\varepsilon} + jx_G^{(i)} \varepsilon) \quad (i = 1, 2, \ldots).$$

Lowering computations, analogous to carried out above, we obtain

$$\varepsilon(t) = \sum_{k=1}^{\infty} \frac{F_1(p_k)}{p_k F_2'(p_k)} e^{p_k t}, \tag{58}$$

where

$$F_1(p) = \left[J - \rho \sum_{i=1}^{\infty} \frac{\Omega^{(i)2} p^2}{C_i (p^2 + \omega_i^2)} - \rho_l \sum_{i=1}^{\infty} \frac{x_G^{(i)} \Omega_c^{(i)} \omega_i^2}{p^2 + \omega_i^2} \right] (\varepsilon_0 p^2 + \dot{\varepsilon}_0 p) +$$

$$+ j\rho \sum_{i=1}^{\infty} \left(\frac{\Omega_c^{(i)}}{C_i} + x_G^{(i)} \right) (\xi^{(i)} p + \chi^{(i)}) \frac{p}{p^2 + \omega_i^2};$$

$$F_2(p) = Jp^2 + jl + \frac{\rho}{j} \sum_{i=1}^{\infty} \frac{(\Omega_c^{(i)} p^2 + jx_G^{(i)2} \omega_i^2)}{p^2 + \omega_i^2}.$$

In this case p_k $(k = 1, 2, \ldots)$ - the roots of equation $F_2(p) = 0$.

Let us consider character of roots of equation $F_2(p) = 0$. For this purpose, after assuming $p^2 = -\lambda$, let us register this equation in the following form:

$$\frac{jl - J\lambda}{\lambda^2} = \frac{\rho}{\lambda} \sum_{i=1}^{\infty} \frac{\left(\Omega_c^{(i)} - \lambda^{-1} j x_G^{(i)}\right)^2}{1 - (C_i/j)\lambda}. \tag{59}$$

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Let us consider two curved lines in plane y, λ, determined by equations

$$y_1 = \frac{jl - J\lambda}{\lambda^2}; \quad y_2 = \frac{\rho}{j} \sum_{i=1}^{\infty} \frac{\left(\Omega_c^{(i)} - \lambda^{-1} j x_G^{(i)}\right)^2}{1 - (C_i/j)\lambda}. \tag{60}$$

Form of these curves is depicted in Fig. 4 and 5.

Let us explain conditions, with which system in question possesses instability.

Let us show that with sufficiently greater in absolute value negative λ value y_2 is less than value y_1. Actually, with the sufficiently high negative values λ expressions (60) can be replaced by the following approximation formulas:

$$y_1 = -\frac{J}{\lambda}; \quad y_2 = -\frac{\rho}{\lambda} \sum_{i=1}^{\infty} \frac{\Omega_c^{(i)2}}{C_i}.$$

Let us consider expressions, which stand in numerator and denominator of terms of series/row with y^2.

Using harmonic functions A_i, determined by conditions (28), and also using Green's formula, we convert these expressions as follows.

$$\Omega_c^{(i)} = \iint\limits_G \Omega_c f_i dG = \oint\limits_S \Omega \frac{\partial A_i}{\partial n} dS = \int\limits_\tau \operatorname{grad} \Omega \operatorname{grad} A_i d\tau;$$

$$C_i = \iint\limits_G A_i f_i dG = \oint\limits_S A_i \frac{\partial A_i}{\partial n} dS = \int\limits_\tau (\operatorname{grad} A_i)^2 d\tau.$$

Following equalities prove, that vector functions $\operatorname{grad} A_i$ $(i=1, 2, \ldots)$ are mutually orthogonal in region τ:

$$\int\limits_\tau \operatorname{grad} A_i \operatorname{grad} A_k d\tau = \oint\limits_S A_i \frac{\partial A_k}{\partial n} dS = \iint\limits_G A_i f_k dG = C_i \iint\limits_G f_i f_k dG.$$

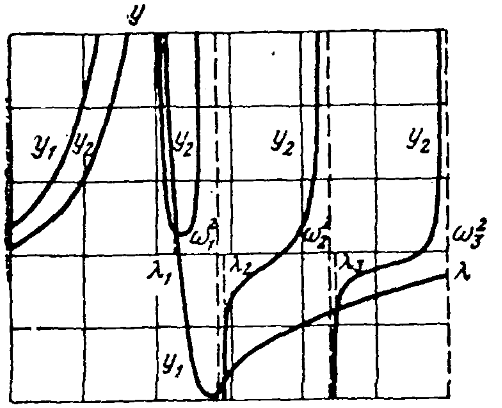

Fig. 4.

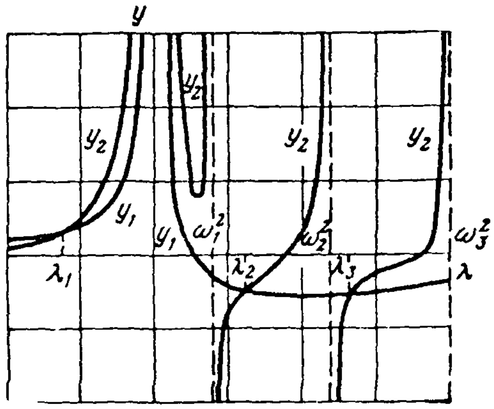

Fig. 5.

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Hence, using Bessel's inequality, we obtain following estimate of the magnitude y_2:

$$y_2=$$

$$=-\frac{\rho}{\lambda}\sum_{i=1}\frac{\left(\int_\tau \operatorname{grad}\Omega \operatorname{grad} A_i d\tau\right)^2}{\int_\tau (\operatorname{grad} A_i)^2 d\tau}\leqslant -\frac{1}{\lambda}\int_\tau \rho(\operatorname{grad}\Omega)^2 d\tau = -\frac{J_1}{\lambda} < -\frac{J}{\lambda}.$$

This inequality proves our affirmation, that $y_2<y_1$ with sufficiently large negative λ.

Let us find conditions, with which in certain sufficient low vicinity $\lambda=0$ with $\lambda<0$ will be fulfilled reverse/inverse inequality: $y_2>y_1$.

In sufficiently low vicinity of point $\lambda=0$ expressions (60) for y_1

and y_2 can be represented by following approximation formulas:

$$y_1 = \frac{jl}{\lambda^2}; \quad y_2 = \frac{j}{\lambda^2}\rho \sum_{i=1}^{\infty} x_G^{(i)2} = \frac{j}{\lambda^2}\rho \iint_G x^2 dG = \frac{j}{\lambda^2}\rho I \quad \left(I = \iint_G x^2 d\,G\right)$$

where I - moment of inertia of plane figure G region relative to axis, which is projection of rotational axis of solid body to G region.

From this expression for y_2 we can draw conclusion that $y_2 > y_1$ with negative λ in vicinity $\lambda=0$, if $\rho I > l$.

In this case pendulum with liquid filling will be deliberately unstable, although $l>0$, i.e., although center of mass of system in undisturbed state of free surface will lie/rest lower than point of suspension.

Thus, if for stability of solid pendulum it is necessary to satisfy only condition $l>0$, then for stability of position of equilibrium of pendulum with liquid filling, which has free surface, must be carried out more rigorous condition $l>I\rho>0$, which is in this case necessary stability condition. Fig. 4 illustrates curves $y_1(\lambda)$, $y_2(\lambda)$ and arrangement of the roots of equation in the case of the stable position of pendulum. Fig. 5 presents the case of the unstable position of pendulum.

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EQUATIONS OF THE DISTURBED MOTION OF SOLID BODY WITH CAVITIES, WHICH CONTAIN LOW-VISCOSITY SWIRLED LIQUID.

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General equations of disturbed motion of stabilized object with cavities, partially filled with liquid, are derived taking into account eddy motion of latter. It is assumed that Reynolds number is considerably more than one, but Strouhal number does not exceed 10.

In description of motion of object in stabilization planes is used model of solid absolutely rigid body with cavities, partially filled with liquid, which have internal narrow circular or radial edges/fins. The obtained equations are the generalization of G. S. Narimanov's equations for the line of the account of the eddy of liquid and pass into the latter with its irrotational motion.

During motion of object in direction of longitudinal axis as model axisymmetric elastic body with cavity, formed by thin-walled shell with internal rigid circular edges/fins, is examined.

In work [1], which belongs to G. S. Narimanov's pen, are for the first time published obtained by him general equations of dynamics of solid body, which has cavity of arbitrary configuration, partially filled with ideal fluid.

This work stimulated appearance of vast cycle of research in dynamics of objects with liquid filling. The literature, dedicated to these questions, which appeared in the years, which elapsed from the moment of publication [1], is virtually boundless and count many hundred designations.

Equations, proposed by G. S. Narimanov [1], were used as basis of mathematical models of disturbed motion of flight vehicles with sections, which contain components of liquid propellant, and played large role in solution of series of problems of dynamics of these objects. One of the aspects of this problem is examined in the present work, which the author would wish to express his respect for the memory of G. S. Narimanov and admiration by his basic research in the dynamics of bodies with the liquid.

Work, which is further continuation of [5] and [6], is dedicated to phenomenological description of eddying motion of low-viscosity liquid in cavities of solid body, stabilized in space, having damping devices in the form of radial or circular edges/fins, during its oscillations in stabilization planes and in direction of longitudinal axis (in the latter case are introduced into examination elastic thin-walled shells, which form walls of cavity).

On basis of proposed phenomenological mathematical model of eddy of liquid, which is further development of models, proposed in works

[5, 4, 6], are com iled equations of disturbed motion of system body - liquid, being generalization of equations of work [1] to new class of motions of liquid.

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1. Equations of disturbed motion of solid body with cavity, partially filled with liquid. Account of the eddy motion of the latter. Let us consider in the same setting, as in work [1], the problem about the disturbed motion of body with the cavity, furnished with dampers in the form of the radial or circular edges/fins, whose number let us designate K. We will consider liquid low-viscosity. Let us introduce as the characteristic dimensionless parameters Strouhal-and Reynolds numbers and the maximum relative width of edges/fins [4]:

$$\mathrm{Sh}=\frac{2\pi v^{\circ}}{\omega^{\circ} l}=\frac{2\pi u^{\circ}}{l}; \ \mathrm{Re}=\frac{\omega^{\circ} l^{2}}{\nu}; \ \bar{b}=\frac{b_{\max}}{l}, \qquad (1)$$

where v°, u° - characteristic speed and amplitude of the oscillations of edge/fin relative to liquid in the direction, perpendicular to its plane, during the disturbed motion with a characteristic frequency of ω°; l - significant dimension of cavity (for example, in the case of axisymmetric cavity mean radius r_0); ν - kinematic viscosity coefficient of liquid; $b_{\max}$ - maximum width of edge/fin. We will examine the motion of liquid with the high values of Reynolds number, the "intermediate" values of Strouhal number and the low relative width of the edges/fins:

$$\mathrm{Re} \gg 1; \ \mathrm{Sh} \leqslant 1 \ldots 10; \ \bar{b} \ll 1. \qquad (2)$$

Let us introduce "absolute" system of coordinates $O^*x^*y^*z^*$, whose axis O^*x^* is antiparallel to gradient j of field of mass forces of undisturbed motion, and pole O^* is connected with arbitrary point of body in its undisturbed motion, and "connected" system of coordinates Oxyz, rigidly fastened with body. The disturbed motion of body - this is the motion of the system of coordinates Oxyz relative to $O^*x^*y^*z^*$. The speed of pole O and the angular velocity of body in this motion let us designate v_0 and ω respectively.

With respect to character of undisturbed and disturbed motions let us take hypotheses, traditional for problems of dynamics of stabilized LA [4], we will in particular assume small values $|v_0|$ and $|\omega|$.

Volume, occupied with liquid, let us designate Q, moistened surface of cavity S. The "waveless" free surface of liquid Σ, following [1], let us relate to the body coordinate system (concept of "rigid cover/cap" Σ, coinciding with "waveless" surface of liquid).

Let us disregard/neglect, taking into account large Reynolds numbers, eddy of liquid in wall boundary layer; however, let us take into consideration eddying of entire mass of liquid, caused by powerful vortex-forming effect of edges/fins with sharp edges.

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We will use phenomenological description of eddy of liquid in cavity [4], after isolating eddying, averaged by volume Q:

$$\Gamma(t)=\frac{1}{2Q}\int_{Q}\operatorname{rot}\mathbf{v}^{*}(\mathbf{R},\ t)\,dQ=\omega+\boldsymbol{\Omega}, \tag{3}$$

where v* - absolute velocity of liquid taking into account eddy; Γ and Ω - average/mean angular rates of rotation of liquid, absolute and relative respectively; R - radius-vector of point, which belongs to region Q, which originates in O.

Let us represent field of velocities v*(R, t) in the form

$$\mathbf{v}^{*}(\mathbf{R},\ t)=\mathbf{v}_0+\operatorname{grad}(\omega,\ \Psi)+\sum_{n=1}^{\infty}\dot{s}_n\operatorname{grad}\varphi_n+\Gamma\times\mathbf{R}-\operatorname{grad}(\Gamma,\ \Psi);$$

$$\operatorname{div}\mathbf{v}^{*}=0;\ \operatorname{rot}\mathbf{v}^{*}=2\Gamma, \tag{4}$$

where Ψ and φ_n - are displacement potentials of particles of liquid taking into account irrotational and nonseparated flow of edges/fins, which are solutions of following boundary-value problems:

$$\Delta\Psi=0;\ \left.\frac{\partial\Psi}{\partial\nu}\right|_{S+\Sigma}=(\mathbf{R}\times\boldsymbol{\nu})\mid_{S+\Sigma}; \tag{5}$$

$$\Delta\varphi=0;\ \left.\frac{\partial\varphi}{\partial\nu}\right|_{S}=0;\ \left.\frac{\partial\varphi}{\partial\nu}\right|_{\Sigma}=\varkappa\varphi\mid_{\Sigma}; \tag{6}$$

ν - unit vector of external normal to the surface $S+\Sigma$; $\varkappa=\varkappa_n$ $(n=1,2,\ldots)$ - eigenvalues of boundary-value problem (6), connected with natural vibration frequencies of liquid in fixed cavity ω_n by relationship/ratio $\omega_n{}^2=j\varkappa_n$; $\varphi=\varphi_n$ $(n=1,\ 2,\ \ldots)$ - eigenfunctions of boundary-value problem (6) (form of natural oscillations of liquid).

We will use following expression [7] for flow forces $\mathbf{f}_\nu$, functioning on edge/fin perpendicularly to its surface, connected with vortex formation on edges of edge/fin, which is well coordinated with experimental data [2, 3] in range of Strouhal numbers in question:

$$\mathbf{f}_\nu = \rho k_s^* b^* \varphi(x') \sqrt{\frac{|\mathbf{v}_\nu|}{\pi}} \mathbf{V}_\nu;$$
$$k_s^* = 1{,}1 k_s^\circ; \; b^* = b^{3/2}; \; \mathbf{v}_\nu = (\mathbf{v}, \nu)\nu, \qquad (7)$$

where ν - unit vector of external normal to plane of edge/fin, forming sharp angle with v; ρ - mass density of liquid; b - width of edge/fin; k°_s - empirical constant, determined in the case of axisymmetric cavities with the help of Table 1 [3, 4].

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Further $\varphi(x') = \alpha e^{-\beta x'}$ - empirical function, introduced into [3], α=0.4; β=0.131 for axisymmetric cavities (x' - coordinate, calculated off free surface in depth of liquid, in reference to width of edge/fin); $\mathbf{v}_\nu$ - relative velocity of liquid at points of center line of edge/fin L, calculated in the absence of edges/fins (potentials Ψ° and φ°_n), and $\mathbf{V}_\nu$ - its form, determined by formulas

$$\mathbf{v}_\nu(\mathbf{R}, t) = \left[\Omega, \left(\mathbf{R}, \nu - \frac{\partial \Psi^\circ}{\partial t}\right) + \sum_{n=1}^{\infty} \dot{s}_n \frac{\partial \varphi_n^\circ}{\partial \nu}\right]\nu;$$
$$\mathbf{V}_\nu(\mathbf{R}, t) = \int_{-\infty}^{t} \frac{d\mathbf{v}_\nu}{d\tau} \frac{d\tau}{\sqrt{t-\tau}}. \qquad (8)$$

It is possible to show (see [4]) that sum of two last terms in (4) and expressions (8) do not depend on selection of pole O.

We pass to compilation of equations of disturbed motion of system body - liquid. We will use the variation principle, analogous to the principle of Hamilton-Ostrogradskiy

$$\int_{t_1}^{t_2} \delta \dot{W} dt = 0, \qquad (9)$$

in which the role of "action" plays "power" $\dot{W}$; t_1 and t_2 - arbitrary moments of time. Without breaking generality, it is possible to assume $t_1 = -\infty$. If we as $\dot{W}^{(p)}$ take change for 1 from the total energy of the system

$\frac{d\dot{W}}{dt}$ under the assumption of the ideality of liquid (irrotational motion), then the equations of work [1] are obtained from (9). In the case in question it is possible therefore to be bounded only to the calculation of generalized forces $\mathbf{M}^{(r)}$ and $\mathbf{P}_n^{(r)}$, which correspond to generalized velocities Ω and $\dot{S}_n$, connected with the eddy of liquid, and the compilation of further equation for Ω.

Table 1.

s	(1) Тип ребер	(2) Характер колебаний жидкости	k_s°
0	(3) Кольцевые	(4) Осесимметричные	4,10
1	(5) Радиальные	(6) Антисемметричные	4,57
3	Кольцевые(3)	Антисемметричные (6)	2,71

Key: (1). Type of edges/fins. (2). Character of oscillations of liquid. (3). Circular. (4). Axisymmetric. (5). Radial. (6). Antisymmetric.

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After designating counterpart of $\dot{W}$ through $\dot{W}^{(r)}$, let us write out two equivalent expressions:

$$\dot{W}^{(r)}=(\mathbf{M}^{(r)},\ \Omega)+\sum_{n=1}^{\infty} P_n^{(r)}\dot{s}_n=-(\mathbf{B},\ \Theta)\,\Omega+$$

$$+\sum_{n=1}^{\infty}\left[(\beta_{on},\Theta)\dot{s}_n+(\beta_{on},\Omega)S_n-\sum_{m=1}^{\infty}\beta_{nm}\dot{s}_nS_m\right]; \tag{10a}$$

$$\dot{W}^{(r)}=-\frac{\rho k_s^* b^*}{\sqrt{\pi}}\sum_{k=1}^{k}\int_{-\infty}^{t}\dot{\nu}\sqrt{|\mathbf{v}_\nu|}\,(\mathbf{V}_\nu,\mathbf{v}_\nu)\,\varphi(x')\,ds, \tag{10b}$$

where

$$\Theta(t)=\int_{-\infty}^{t}\frac{\dot{\Omega}(\tau)\,d\tau}{\sqrt{t-\tau}};\ S_n(t)=\int_{-\infty}^{t}\frac{\ddot{s}_n(\tau)\,d\tau}{\sqrt{t-\tau}}; \tag{11}$$

B - symmetrical tensor of second order; β_{on} - vectors; μ_n and β_{nm} - scalars (n, m=1, 2, ...):

$$\mathbf{B}=\{\beta^{\circ}_{ij}\};\ \beta_{on}=\sum_{j=1}^{3}\mathbf{i}_j\beta_{onj};$$

$$\mu_n=\rho\int_{\Sigma}\varphi_n\frac{\partial\varphi_n}{\partial\nu}\,dS. \tag{12}$$

Formulas for coefficients β°_{ij}, β_{onj}, β_{nm} are obtained after substitution of expressions v_ν and V_ν (8) into right side of (10b) and comparison of coefficients when Ω and $\dot{s}_n$ in (10a) and (10b):

$$\beta^{\circ}_{ji}=\beta^{\circ}_{ij}=\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{k=1}^{K}\int_{L_k}\sqrt{|\mathbf{v}_\nu|}\left[-(\mathbf{R}\times\nu)_i+\frac{\partial\Psi^{\circ}_i}{\partial\nu}\right]\left[-(\mathbf{R}\times\nu)_j+\frac{\partial\Psi^{\circ}_j}{\partial\nu}\right]b^*\varphi(x')\,ds; \tag{13}$$

$$\beta_{onj}=\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{k=1}^{K}\int_{L_k}\sqrt{|\mathbf{v}_\nu|}\left[-(\mathbf{R}\times\nu)_j+\frac{\partial\Psi^{\circ}_j}{\partial\nu}\right]\frac{\partial\varphi^{\circ}_n}{\partial\nu}\,b^*\varphi(x')\,ds;$$

$$\beta_{nm}=\beta_{mn}=\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{n=1}^{K}\int_{L_k}\sqrt{|\mathbf{v}_\nu|}\,\frac{\partial\varphi^{\circ}_n}{\partial\nu}\,\frac{\partial\varphi^{\circ}_m}{\partial\nu}\,b^*\varphi(x')\,ds,$$

where L_k - duct/contour, formed by center line of k edge/fin.

Potentials φ°_n in (13), in contrast to (12) as Ψ°_j, correspond to irrotational motion of liquid in cavity in the absence of edges/fins.

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As a result of substitution into (9) expressions of "potential" part $\dot{W}^{(p)}$ of power $\dot{W}$, determined by field of velocities (4), and "vortex/eddy" $\Psi^{(r)}$ from (10a), and also expression of moment of hydrostatic forces from [1], we will obtain following complete system of equations of disturbed motion of system in question:

$$(m^\circ + m)\dot{\mathbf{v}}_O + \dot{\boldsymbol{\omega}} \times (\mathbf{L}^\circ + \mathbf{L}) + \sum_{n=1}^{\infty} \lambda_n \ddot{s}_n = \mathbf{P};$$

$$(\mathbf{J}^\circ + \mathbf{J}^{(0)}, \dot{\boldsymbol{\omega}}) + (\mathbf{L}^\circ + \mathbf{L}) \times (\dot{\mathbf{v}}_O - \mathbf{j}) + (\mathbf{J}^*, \dot{\boldsymbol{\Omega}}) + \sum_{n=1}^{\infty} [\lambda_{on} \ddot{s}_n - (\lambda_n \times \mathbf{j}) s_n] = \mathbf{M}_O;$$

$$(\mathbf{J}^*, \dot{\boldsymbol{\Omega}}) + (\mathbf{J}^*, \dot{\boldsymbol{\omega}}) + \left(\mathbf{B}, \int_{-\infty}^{t} \frac{\dot{\boldsymbol{\Omega}}(\tau)\, d\tau}{\sqrt{t-\tau}}\right) - \sum_{n=1}^{\infty} \beta_{on} \int_{-\infty}^{t} \frac{\ddot{s}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0; \qquad (14)$$

$$\mu_n(\ddot{s}_n + \omega_n^2 s_n) + (\lambda_n, \dot{\mathbf{v}}_O) + (\lambda_{on}, \dot{\boldsymbol{\omega}}) - (\lambda_n, \mathbf{j}) +$$

$$+ \mu_n \sum_{m=1}^{\infty} \beta_{nm} \int_{-\infty}^{t} \frac{\ddot{s}_m(\tau)\, d\tau}{\sqrt{t-\tau}} - \left(\beta_{on}, \int_{-\infty}^{t} \frac{\dot{\boldsymbol{\Omega}}(\tau)\, d\tau}{\sqrt{t-\tau}}\right) = 0$$

$$(n = 1, 2, \ldots)$$

Here P and $\mathbf{M}_O$ - main vector and the main moment (relative to pole O) of the system of external forces; L° and L - vectors of static moments (relative to pole O) of solid body and hardened in the presence of "rigid cover/cap" liquid; $\mathbf{J}^\circ = \mathbf{J}^{(0)}$ - tensors of the inertia of body and hardened liquid; $\mathbf{J}^* = \mathbf{J}^{(0)} - \mathbf{J}$, where J - tensor of N. Ye. Zhukovskiy of the connected moments of the inertia of liquid in the cavity with the "rigid cover/cap":

$$\mathbf{J}^* = \{J^*_{ij}\}; \; \mathbf{J}^{(0)} = \{J^{(0)}_{ij}\}; \; \mathbf{J} = \{J_{ij}\}. \qquad (15)$$

Elements of tensors (15) and vectors λ_n and λ_{on} are determined by following formulas:

$$J_{ij}^{(0)}=J_{ji}^{(0)}=\rho\int_Q(\mathbf{R}\times\mathbf{i}_i)(\mathbf{R}\times\mathbf{i}_j)\,dQ;$$

$$J_{ij}=J_{ji}=\rho\int_Q\nabla\Psi_i\nabla\Psi_j dQ=\rho\oint_{S+\Sigma}\Psi_j\frac{\partial\Psi_i}{\partial\nu}dS;$$

$$J_{ij}^{*}=J_{ji}^{*}=J_{ij}^{(0)}-J_{ij};$$

$$\lambda_n=\rho\int_\Sigma\mathbf{R}\frac{\partial\varphi_n}{\partial\nu}dS;$$

$$\lambda_{On}=\rho\int_\Sigma\Psi\frac{\partial\varphi_n}{\partial\nu}dS. \qquad (16)$$

∇ - here and throughout - the operator of Hamilton.

Let us consider case of irrotational motion of liquid in absolute system of coordinates $O^*x^*y^*z^*$, to which corresponds $\Gamma\equiv0$; $\Omega=-\omega$ in (14) and $k_s^*=0$ in (13). Equations (14) take the form

$$(m^\circ+m)\dot{\mathbf{v}}_O+\dot{\boldsymbol{\omega}}\times(\mathbf{L}^\circ+\mathbf{L})+\sum_{n=1}^{\infty}\lambda_n\ddot{s}_n=\mathbf{P};$$

$$(\mathbf{J}^\circ+\mathbf{J}^{(0)},\ \dot{\boldsymbol{\omega}})+(\mathbf{L}^\circ+\mathbf{L})\times(\dot{\mathbf{v}}_O-\mathbf{j})+\sum_{n=1}^{\infty}[\lambda_{on}\ddot{s}_n-(\lambda_n\times\mathbf{j})s_n]=\mathbf{M}_O; \qquad (17)$$

$$\mu_n(\ddot{s}_n+\omega_n^2 s_n)+(\lambda_n\dot{\mathbf{v}}_O)+(\lambda_{on},\ \dot{\boldsymbol{\omega}})-(\lambda_n,\ \mathbf{j})=0 \quad (n=1,\ 2,\ldots).$$

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System of equations (17) coincides with accuracy to designations with total system of equations of disturbed motion of solid body with cavity, partially filled with ideal fluid, obtained for the first time by G. S. Narimanov [1].

Let us consider now opposite case, when relative angular velocity of liquid is identically equal to zero, to which corresponds motion of entire liquid as solid body. Equations (14) must in this case pass into the usual equations of the dynamics of solid body. Actually, after assuming in (14) $\Omega\equiv 0,\ s_n\equiv 0\ (n=1, 2, \ldots)$, we will obtain

$$\begin{aligned}&(m^\circ+m)\dot{\mathbf{v}}+\dot{\boldsymbol{\omega}}\times(\mathbf{L}^\circ+\mathbf{L})=\mathbf{P};\\&(\mathbf{J}^\circ+\mathbf{J}^{(0)},\ \dot{\boldsymbol{\omega}})+(\mathbf{L}^\circ+\mathbf{L})\times(\dot{\mathbf{v}}_O-\mathbf{j})=\mathbf{M}_O.\end{aligned}\tag{18}$$

As is known, these equations describe motion of system body - hardened liquid.

One should stress that if we do not consider eddy of entire mass of liquid in absolute coordinate system, then passage to the limit from appropriate equations to equations of dynamics of solid body, demonstrated above, is impossible.

Equations (14) correspond to arbitrarily selected pole O. If we select as the pole, as is done in [1], the center of mass G_0 of system body - hardened (in the presence of "rigid cover/cap") liquid, then total static moment $L^\circ+L$ will become zero, and equations (14) will pass into the following:

$$(m^\circ+m)\dot{\mathbf{v}}+\sum_{n=1}^{\infty}\lambda_n\ddot{s}_n=\mathbf{P};$$

$$(\mathbf{J}^\circ+\mathbf{J}^{(0)},\ \dot{\boldsymbol{\omega}})+(\mathbf{J}^*,\ \dot{\Omega})+\sum_{n=1}^{\infty}[\lambda_{on}\ddot{s}_n-(\lambda_n\times\mathbf{j})\,s_n]=\mathbf{M}_{G_0};$$

$$(\mathbf{J}^*, \dot{\Omega}) + (\mathbf{J}^*, \dot{\omega}) + \left(\beta, \int_{-\infty}^{t} \frac{\Omega(\tau)\, d\tau}{\sqrt{t-\tau}}\right) - \sum_{n=1}^{\infty} \beta_{on} \int_{-\infty}^{t} \frac{\ddot{s}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0; \qquad (19)$$

$$\mu_n(\ddot{s}_n + \omega_n^2 s_n) + (\lambda_n, \dot{\mathbf{v}}) + (\lambda_{on}, \dot{\omega}) - (\lambda_n, \mathbf{j}) +$$

$$+ \mu_n \sum_{m=1}^{\infty} \int_{-\infty}^{t} \frac{\ddot{s}_m(\tau)\, d\tau}{\sqrt{t-\tau}} - \left(\beta_{on}, \int_{-\infty}^{t} \frac{\dot{\Omega}(\tau)\, d\tau}{\sqrt{t-\tau}}\right) = 0$$

$(n = 1, 2, \ldots)$,

where v - velocity of point G_0 in the disturbed motion, and all coefficients of equations (19), which depend on the position of pole O, are given to the new pole G_0 as $\mathbf{M}_{G_0}$.

Let us consider axisymmetric object with axisymmetric section, which contains liquid, that moves in plane O*x*z*.

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Equations (19) acquire in this case the following form (if we drop/omit some unessential now indices):

$$(m^\circ + m)\dot{v} + \sum_{n=1}^{\infty} \lambda_n \ddot{s}_n = P_z;$$

$$(J^\circ + J^{(0)})\dot{\omega} + J^*\dot{\Omega} + \sum_{n=1}^{\infty} (\lambda_{on}\ddot{s}_n + j\lambda_n s_n) = M_{G_0 y};$$

$$J^*\dot{\Omega} + J^*\dot{\omega} + \beta \int_{-\infty}^{t} \frac{\dot{\Omega}(\tau)\, d\tau}{\sqrt{t-\tau}} - \sum_{n=1}^{\infty} \beta_{on} \int_{-\infty}^{t} \frac{\ddot{s}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0; \qquad (20)$$

$$\mu_n(\ddot{s}_n + \omega_n^2 s_n) + \lambda_n \dot{v} + \lambda_{on}\dot{\omega} + \lambda_n j \psi +$$

$$+ \mu_n \sum_{m=1}^{\infty} \beta_{nm} \int_{-\infty}^{t} \frac{\ddot{s}_m(\tau)\, d\tau}{\sqrt{t-\tau}} - \beta_{on} \int_{-\infty}^{t} \frac{\dot{\Omega}(\tau)\, d\tau}{\sqrt{t-\tau}} = 0$$

$(n = 1, 2, \ldots)$,

where all generalized velocities v, ω, Ω - now scalar quantities as the coefficients of equations (20), moreover $\omega=\dot{\psi}$; $J^*=J^{(0)}-J$ (J - here and throughout no longer tensor, but scalar - connected moment of the inertia of liquid, calculated according to N. Ye. Zhukovskiy for the cavity with the "rigid cover/cap"). If we assume in equations (20) $\dot{\Omega}=-\dot{\omega}$; $v=\dot{\zeta}$; $\beta=0$; $\beta_{on}\equiv 0$; $\beta_{nm}\equiv 0$, then equations (20) pass with accuracy to designations into the appropriate equations of work [1]:

$$(m^\circ+m)\ddot{\zeta}+\sum_{n=1}^{\infty}\lambda_n\ddot{s}_n=P_z;$$

$$(J^\circ+J)\ddot{\psi}+\sum_{n=1}^{\infty}(\lambda_{on}\ddot{s}_n+j\lambda_n s_n)=M_{G_0 y}; \quad (21)$$

$$\mu_n(\ddot{s}_n+\omega_n^2 s_n)+\lambda_n\ddot{\zeta}+\lambda_{on}\ddot{\psi}+\lambda_n j\psi=0$$

$(n=1, 2 \ldots,)$.

2. Elastic shell in the form of body of revolution. Axisymmetric motions of liquid. Let us consider the axisymmetric elastic thin-walled shell, which has within K rigid circular damping edges/fins, partially filled with low-viscosity liquid. Let us assume that the shell completes low oscillations in the direction of longitudinal axis, which is parallel to the field gradient of the mass forces of the undisturbed motion.

Let us preserve in force all basic hypotheses, which concern dimensionless parameters of problems (1), introduced above, i.e., we will assume that these parameters satisfy conditions (2).

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Let us connect with cavity with undeformed walls cylindrical coordinate system $Oxr\theta$ with pole at arbitrary point O, which lies on longitudinal axis of shell Ox.

Vector of displacements of points of median surface S° during its axisymmetric oscillations let us designate u (u, w), where u - tangential, and w - normal (on external normal to S°) components, which lie at radial plane Oxr.

We will use traditional hypotheses [3] about character of motion of shell and liquid. The eddy of liquid in field Q, generated by circular edges/fins, we will as in Section 1, describe in the integral sense with the help of averaged throughout entire liquid volume eddying.

In this case all vortices/eddies should be considered circular with centers on axis Ox and averaging carried out in radial plane Oxr through G region, which is cross section Q with half-plane Oxr. As a result we will obtain the axisymmetric eddy of liquid with the angular velocity, identical at all points of G region with any value of vectorial angle θ, directed along unit vector i_{θ} of tangent toward circle/circumference $r=\text{const}$

$$\Gamma=\frac{1}{2G}\int_{G}(\mathrm{rot}\,\mathbf{v}^{*},\ \mathbf{i}_{\theta})\,dS, \qquad (22)$$

where v* - absolute velocity of liquid.

Field of velocities v*(R, t) can be represented in the form, analogous to (4):

$$\mathbf{v}^{*}(\mathbf{R},\ t)=v_{O}\,\mathbf{i}_{x}+\sum_{j=1}^{\infty}\dot{q}_{j}\,\mathrm{grad}\,\Psi_{j}+\sum_{n=1}^{\infty}\dot{s}_{n}\,\mathrm{grad}\,\varphi_{n}+\Gamma\,(\mathbf{i}_{\theta}\times\mathbf{R}-\mathrm{grad}\,\Psi^{*});$$

$$\mathrm{div}\,\mathbf{v}^{*}=0;\ \mathrm{rot}\,\mathbf{v}^{*}=2\,\mathbf{i}_{\theta}\,\Gamma. \qquad (23)$$

Here R - radius-vector with beginning O at arbitrary point on the longitudinal axis of cavity; v_O - velocity of polar wandering O in the direction of axis Ox; Ψ_j, q_j - displacement potentials and the generalized coordinates, which correspond to the natural oscillations of system elastic shell - liquid with the flat/plane free surface: φ_n, s_n - displacement potentials and the generalized coordinates, which correspond to wave motions on the free surface during the oscillations of liquid in the cavity with the rigid walls; Ψ* - displacement potential, which ensures equality to zero of normal component of velocity on surface of S+Σ with the eddies of liquid.

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Potentials Ψ^{*}, Ψ_j, φ_n are solutions of following boundary-value problems:

$$\Delta\Psi^{*}=0;\ \left.\frac{\partial\Psi^{*}}{\partial\nu}\right|_{S+}=(\mathbf{R}\times\nu)\,\mathbf{i}_{\theta};\ \left.\frac{\partial\Psi^{*}}{\partial\nu}\right|_{r=0}=0; \qquad (24)$$

$$\Delta\Psi=0;\ \left.\frac{\partial\Psi}{\partial\nu}\right|_{S}=w(\mathbf{R});\ \left.\frac{\partial\Psi}{\partial\nu}\right|_{\Sigma}=-\frac{1}{\Sigma}\int_{S}w\,dS;\ \left.\frac{\partial\Psi}{\partial\nu}\right|_{r=0}=0; \qquad (25)$$

$$\Delta\varphi=0;\ \left.\frac{\partial\varphi}{\partial\nu}\right|_{S}=0;\ \left.\left(\frac{\partial\varphi}{\partial\nu}-\varkappa\varphi\right)\right|_{\Sigma}=0;\ \left.\frac{\partial\varphi}{\partial\nu}\right|_{r=0}=0, \qquad (26)$$

where Δ - three-dimensional operator of Laplace; $\kappa=\kappa_1, \kappa_2, \ldots$ - eigenvalues. Boundary-value problem (24) is closed by the equations, which describe the elastic deformations of shell by corresponding boundary conditions [3].

Vast literature is dedicated to solution of boundary-value problems of type (24)...(26), and we on them do not stop.

We will use again expression (7) for flow forces, which functions on edge/fin along the normal to its surface, caused by eddy of liquid. If we disregard/neglect the tangential displacements of elastic shell low in comparison with the displacements of liquid, then functions $v_\nu(\mathbf{R}, t)$ and $V_\nu(\mathbf{R}, t)$, which are scalar analogs (8), are expressed by the formulas

$$v_\nu(\mathbf{R},\ t)=\Gamma\left[(\mathbf{R}\times\mathbf{\nu})\,\mathbf{i}_\theta-\frac{\partial\Psi^*}{\partial\nu}\right]+\sum_{j=1}^{\infty}\dot{q}_j\,\frac{\partial\mathring{\Psi}_j}{\partial\nu}+\sum_{n=1}^{\infty}\dot{s}_n\,\frac{\partial\mathring{\varphi}_n}{\partial\nu}\ ; \qquad (27)$$

$$V_\nu(\mathbf{R},\ t)=\int_{-\infty}^{t}\frac{dv_\nu}{d\tau}\,\frac{d\tau}{\sqrt{t-\tau}}\,.$$

Right sides of expressions (23) and (27) do not depend on selection of pole O, since they have the same structure, as (4) and (8).

If we now use variation principle (9), then on basis of expressions (7), (23) and (27) it is possible to obtain equations of disturbed motion in direction of longitudinal axis of elastic axisymmetric shell with liquid taking into account eddies of latter (also axisymmetric).

Let us register, as above, two equivalent expressions for that part of power $\dot{W}^{(r)}$, which corresponds to eddies of liquid:

$$\dot{W}^{(r)} = -\left(\beta^{*}\theta_0 + \sum_{j=1}^{\infty} \gamma_{j0} Q_j + \sum_{n=1}^{\infty} {}_{n} S_{n}\right)\Gamma - \sum_{j=1}^{\infty}\left(\gamma_{jo}\theta_0 + \sum_{i=1}^{\infty} \gamma_{ji} Q_i + \right.$$

$$\left. + \sum_{n=1}^{\infty} \delta_{jn} S_n\right)\dot{q}_j - \sum_{n=1}^{\infty}\left(\beta_{no}\theta_0 + \sum_{j=1}^{\infty}\delta_{jn} Q_j + \mu_n \sum_{m=1}^{\infty} \beta_{nm} S_m\right)\dot{s}_n; \qquad (28a)$$

$$\dot{W}^{(r)} = -\frac{\rho k_s^{*}}{\sqrt{\pi}} \sum_{k=1}^{K} \int_{L_k} \sqrt{|v_\nu|}\, V_\nu v_\nu b^{*} \varphi(x')\, ds, \qquad (28б)$$

where $\theta_0 = \int\limits_{-\infty}^{t} \frac{\dot{\Gamma}(\tau)\, d\tau}{\sqrt{t-\tau}}$; $Q_j = \int\limits_{-\infty}^{t} \frac{\ddot{q}_j(\tau)\, d\tau}{\sqrt{t-\tau}}$; $S_n = \int\limits_{-\infty}^{t} \frac{\ddot{s}_n(\tau)\, d\tau}{\sqrt{t-\tau}}$; (29)

L_k - center line of k edge/fin.

Comparison of expressions (28a) and (28b) after substitution in (28b) v_ν and V_ν from (27) makes it possible to obtain formulas for coefficients $\gamma_{ji}, \beta_{nm}, \delta_{jn}$ etc., which will be given below.

Let us write out now some auxiliary relationships/ratios, which prove orthogonality of functions $\nabla\varphi_n$ and $\nabla\Psi_j$ to function

$\mathbf{i}_\theta \times \mathbf{R} - \nabla\Psi^*$ in region Q, which are corollary (24)...(26):

$$\int_Q [\mathbf{R} \times (\mathbf{i}_\theta \times \mathbf{R}) - (\mathbf{R} \times \nabla)\Psi^*]\, \mathbf{i}_\theta\, dQ = \int_Q [(\mathbf{R}\times\mathbf{i}_\theta)^2 - (\nabla\Psi^*)^2]\, dQ =$$

$$= \int_Q (\mathbf{R}\times\mathbf{i}_\theta)^2 dQ - \oint_{S+\Sigma} \Psi^* \frac{\partial\Psi^*}{\partial\nu} dS;$$

$$\int_Q (\mathbf{i}_\theta \times \mathbf{R} - \nabla\Psi^*)\, \nabla\varphi_n dQ = \oint_{S+\Sigma} (\mathbf{i}_\theta \times \mathbf{R})\, \boldsymbol{\nu}\, \varphi_n dS - \oint_{S+\Sigma} \Psi^* \frac{\partial\varphi_n}{\partial\nu} dS =$$

$$= \oint_{S+\Sigma} \frac{\partial\Psi^*}{\partial\nu} \varphi_n dS - \oint_{S+\Sigma} \Psi^* \frac{\partial\varphi_n}{\partial\nu} dS \equiv 0; \qquad (30)$$

$$\int_Q (\mathbf{i}_\theta \times \mathbf{R} - \nabla\Psi^*)\, \nabla\Psi_j dQ = \oint_{S+\Sigma} (\mathbf{i}_\theta \times \mathbf{R})\, \Psi_j dS - \oint_{S+\Sigma} \Psi^* \frac{\partial\Psi_j}{\partial\nu} dS =$$

$$= \oint_{S+\Sigma} \frac{\partial\Psi^*}{\partial\nu} \Psi_j dS - \oint_{S+\Sigma} \Psi^* \frac{\partial\Psi_j}{\partial\nu} dS \equiv 0.$$

Is consistent pole O with center of mass G_0 of system shell - liquid in undisturbed state (statically deformed shell, hardened liquid).

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Using (23) and (27)...(29), we will obtain with the help of (9) following mathematical model, which describes the eddies of liquid in Q region and wave motions on its surface together with the elastic vibrations of shell during its axisymmetric strains:

$$(m^\circ + m)\dot{v} = P_x;$$

$$a^*\dot{\Gamma} + \beta^* \int_{-\infty}^{t} \frac{\dot{\Gamma}(\tau)\, d\tau}{\sqrt{t-\tau}} + \sum_{j=1}^{\infty} \gamma_{j0} \int_{-\infty}^{t} \frac{\ddot{q}_j(\tau)\, d\tau}{\sqrt{t-\tau}} + \sum_{n=1}^{\infty} \rho \int_{-\infty}^{t} \frac{\ddot{s}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0; \qquad (31)$$

$$a_j(\ddot{q}_j+\sigma_j^2 q_j)+\sum_{n=1}^{\infty}\lambda_{jn}\ddot{s}_n+\gamma_{jo}\int_{-\infty}^{t}\frac{\dot{\Gamma}(\tau)\,d\tau}{\sqrt{t-\tau}}+\sum_{i=1}^{\infty}\gamma_{ji}\int_{-\infty}^{t}\frac{\ddot{q}_i(\tau)\,d\tau}{\sqrt{t-\tau}}+$$

$$+\sum_{n=1}^{\infty}\delta_{jn}\int_{-\infty}^{t}\frac{\ddot{s}_j(\tau)\,d\tau}{\sqrt{t-\tau}}=P_j;$$

$$\mu_n(\ddot{s}_n+\omega_n^2 s_n)+\sum_{j=1}^{\infty}\lambda_{jn}\ddot{q}_j+\beta_{no}\int_{-\infty}^{t}\frac{\dot{\Gamma}(\tau)\,d\tau}{\sqrt{t-\tau}}+\sum_{i=1}^{\infty}\delta_{jn}\int_{-\infty}^{t}\frac{\ddot{q}_j(\tau)\,d\tau}{\sqrt{t-\tau}}+$$

$$+\mu_n\sum_{m=1}^{\infty}\beta_{nm}\int_{-\infty}^{t}\frac{\ddot{s}_m(\tau)\,d\tau}{\sqrt{t-\tau}}=0$$

$$(j,\ n=1,\ 2,\ldots),$$

where P_x - projection on axis O_x of the main vector of the system of external forces; P_j - the generalized force, which corresponds to generalized coordinate q_j; m° - the mass of shell; m - mass of liquid; σ_j and ω_n $(j=1,\ 2,\ \ldots;\ n=1,\ 2,\ \ldots)$ - partial natural vibration frequencies of shell with the liquid when $s_n=0$ and wave motions of liquid when $q_j=0$; v - the velocity in the direction of axis Ox of point G_0.

Coefficients of equations (30), connected with eddies of liquid, are determined by formulas:

$$a^*=\rho\int_Q(\mathbf{R}\times\mathbf{i}_\theta)^2\,dQ-\rho\oint_{S'+\Sigma}\Psi^*\frac{\partial\Psi^*}{\partial\nu}\,dS;$$

$$\beta^*=\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{k=1}^{K}\int_{L_k}\sqrt{|v_\nu|}\left[(\mathbf{R}\times\boldsymbol{\nu})\,\mathbf{i}_\theta-\frac{\partial\Psi^{\circ *}}{\partial\nu}\right]^2 b^*\varphi(x')\,ds;$$

$$\beta_{n0}=\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{k=1}^{K}\int_{L_k}\sqrt{|v_\nu|}\left[(\mathbf{R}\times\boldsymbol{\nu})\,\mathbf{i}_6-\frac{\partial\Psi^{\circ *}}{\partial\nu}\right]\frac{\partial\mathring{\varphi}_n}{\partial\nu}\,b^*\varphi(x')\,ds; \qquad (32)$$

$$\beta_{nm}=\beta_{mn}=\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{k=1}^{K}\int_{L_k}\sqrt{|v_\nu|}\,\frac{\partial\mathring{\varphi}_n}{\partial\nu}\,\frac{\partial\mathring{\varphi}_m}{\partial\nu}\,b^*\varphi(x')\,ds;$$

$$\gamma_{j0}=\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{k=1}^{K}\int_{L_k}\sqrt{|v_\nu|}\left[(\mathbf{R}\times\boldsymbol{\nu})\,\mathbf{i}_6-\frac{\partial\Psi^{\circ *}}{\partial\nu}\right]\frac{\partial\mathring{\Psi}_j}{\partial\nu}\,b^*\varphi(x')\,ds;$$

$$\gamma_{ji}=\gamma_{ij}+\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{k=1}^{K}\int_{L_k}\sqrt{|v_\nu|}\,\frac{\partial\mathring{\Psi}_i}{\partial\nu}\,\frac{\partial\mathring{\Psi}_j}{\partial\nu}\,b^*\varphi(x')\,ds;$$

$$\delta_{jn}=\frac{\rho k_s^*}{\sqrt{\pi}}\sum_{k=1}^{K}\int_{L_k}\sqrt{|v_\nu|}\,\frac{\partial\mathring{\Psi}_j}{\partial\nu}\,\frac{\partial\mathring{\varphi}_n}{\partial\nu}\,b^*\varphi(x')\,ds.$$

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For the remaining coefficients we will obtain following expressions [3]:

$$a_j=\rho\int_Q(\nabla\Psi_j)^2\,dQ+\rho^\circ\delta^\circ\oint_{S^\circ}|\mathbf{u}_j|^2\,dS=\rho\oint_{S+\Sigma}\Psi_j\,\frac{\partial\Psi_j}{\partial\nu}\,dS+$$
$$+\rho^\circ\delta^\circ\oint_{S}|\mathbf{u}_j|^2\,dS;$$

$$\lambda_{jn}=\rho\int_Q(\nabla\Psi_j,\ \nabla\varphi_n)\,dQ=\rho\int_\Sigma\Psi_j\,\frac{\partial\varphi_n}{\partial\nu}\,dS; \qquad (33)$$

$$\mu_n=\rho\int_Q(\nabla\varphi_n)^2\,dQ=\rho\int_\Sigma\varphi_n\,\frac{\partial\varphi_n}{\partial\nu}\,dS.$$

Here S° - middle surface of shell; ρ° - mass density of material of walls; δ° - their average/mean thickness; vector u_j corresponds to j form of natural oscillations of shell when $s_n \equiv 0$.

When $\gamma_{j0} \equiv 0$, $\gamma_{ji} \equiv 0$; $\beta_{n0} \equiv 0$; $\beta_{nm} \equiv 0$; $\delta_{jn} \equiv 0$; $\Gamma \equiv 0$, $v = \xi$ (31) passes into usual system of equations of axisymmetric oscillations of shell with liquid with irrotational motion of latter [3]:

$$(m^{\circ} + m)\ddot{\xi} = P_x;$$
$$a_j \ddot{q}_j + \sum_{n=1}^{\infty} \lambda_{jn} \ddot{s}_n = P_j; \qquad (34)$$
$$\mu_n (\ddot{s}_n + \omega_n^2 s_n) + \sum_{j=1}^{\infty} \lambda_{jn} \ddot{q}_j = 0 \quad (n,\ j = 1,\ 2, \ldots).$$

3. Equations of disturbed motion of stabilized axially symmetrical body in stabilization planes and in direction of longitudinal axis. Let us consider the stabilized in the space object with the N cavities, which possesses the mass and geometric axial symmetry (cavities have shape of bodies of revolution with the common longitudinal axis, which coincides with the longitudinal axis of object). We will be bounded, as is done in the majority of applied research, to the account only of the one form of wave motions of liquid in each of the cavities. We will schematize the considered/examined object during the motion in the stabilization planes by solid, absolutely rigid body with liquid filling, and during the disturbed motion in the direction of longitudinal axis - by body, which includes N elastic thin-walled shells. In this case let us take into consideration M first forms of the longitudinal vibrations of

body, which are accompanied by the axisymmetric deformations of shells, which form the walls of cavities.

Thus, index n will now correspond to number of cavity, calculated off tail section and head, and j - to number of form of natural elastic axisymmetric oscillations of entire body with liquid (G_0 - center of mass of entire system in its undisturbed motion with hardened liquid).

All the remaining hypotheses, formulated above, remain valid. After taking for the basis mathematical models (19), (31), obtained with these hypotheses, it is possible to obtain the equations of the disturbed motion of system in the stabilization planes (from (20) - in the yaw plane) and in the direction of longitudinal axis. In the latter case it makes sense to examine two different models: a) not considering the elastic deformations of body (in the range of the "low-frequency" oscillations, connected with wave motions on the free surface of liquid); b) not considering wave motions of liquid (in the range of the "high-frequency" oscillations, connected with the longitudinal elastic vibrations of system housing - liquid).

(1) *Движение в плоскости рыскания $O^*x^*z^*$*

$$(m^\circ + m)\ddot{\zeta} + \sum_{n=1}^{N} \lambda_n \ddot{s}_n = P_z;$$

$$(J^\circ + J^{(0)})\ddot{\psi} + \sum_{n=1}^{N} (J_n^* \ddot{\vartheta}_n + \lambda_{0n}\ddot{s}_n + j\lambda_n s_n) = M_{G_0 y};$$

$$J_n^* \ddot{\vartheta}_n + J_n^* \ddot{\psi} + \beta_n^* \int_{-\infty}^{t} \frac{\ddot{\vartheta}_n(\tau)\, d\tau}{\sqrt{t-\tau}} - \beta_{on} \int_{-\infty}^{t} \frac{\ddot{s}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0;$$

$$\mu_n \left(\ddot{s}_n + \beta_n \int_{-\infty}^{t} \frac{\ddot{s}_n(\tau)\, d\tau}{\sqrt{t-\tau}} + \omega_n^2 s_n \right) + \lambda_n \ddot{\zeta} + \lambda_{on} \ddot{\psi} + j\lambda_n \psi - \beta_{on} \int_{-\infty}^{t} \frac{\ddot{\vartheta}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0 \tag{35}$$

$$(n = 1,\ 2, \ldots, N).$$

(2) *Вращение вокруг продольной оси Ox (крен)*

$$(I^\circ + I^{(0)}) \ddot{\varphi} + \sum_{n=1}^{N} I_n^* \ddot{\chi}_n = M_{G_0 x}; \tag{36}$$

$$I_n^* \ddot{\chi}_n + I_n^* \ddot{\varphi} + \beta_n^0 \int_{-\infty}^{t} \frac{\ddot{\chi}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0 \quad (n = 1,\ 2, \ldots, N).$$

Key: (1). Motion in the yaw plane O*x*z*. (2). Rotation around longitudinal axis Ox (bank).

In equations (35), (36) are introduced not requiring commentaries simplified designations for elements of tensors B, J°, $J^{(0)}$, J*, or for generalized velocities: $\dot{\zeta}, \dot{\psi}, \dot{\vartheta}_n$ (rudder channel) and $\dot{\varphi}, \dot{\chi}_n$ (channel of bank).

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(1) *Движение в направлении продольной оси Ox*

(1a) абсолютно жесткое тело ($q_j \equiv 0$)

$$(m^\circ + m) \ddot{\xi} = P_x;$$

$$a_n^* \ddot{\gamma}_n + \beta_n^* \int_{-\infty}^{t} \frac{\ddot{\gamma}_n(\tau)\, d\tau}{\sqrt{t-\tau}} + \beta_{no} \int_{-\infty}^{t} \frac{\ddot{s}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0; \tag{37}$$

$$\mu_n(\ddot{s}_n+\omega_n^2 s_n)+\beta_{no}\int_{-\infty}^{t}\frac{\ddot{\gamma}_n(\tau)\,d\tau}{\sqrt{t-\tau}}=0$$

$$(n=1,\ 2,\ldots,N);$$

(1b) отсутствие волновых движений $(s_n=0)$

$$(m^\circ+m)\ddot{\xi}=P_x;$$

$$a_n^*\ddot{\gamma}_n+\beta_n^*\int_{-\infty}^{t}\frac{\ddot{\gamma}_n(\tau)\,d\tau}{\sqrt{t-\tau}}+\sum_{j=1}^{M}\gamma_{jo}^{(n)}\int_{-\infty}^{t}\frac{\ddot{q}_j(\tau)\,d\tau}{\sqrt{t-\tau}}=0;$$

$$a_j(\ddot{q}_j+\sigma_j^2 q_j)+\gamma_{jo}^{(n)}\int_{-\infty}^{t}\frac{\ddot{\gamma}_n(\tau)\,d\tau}{\sqrt{t-\tau}}+\sum_{i=1}^{M}\gamma_{ji}\int_{-\infty}^{t}\frac{\ddot{q}_i(\tau)\,d\tau}{\sqrt{t-\tau}}=0$$

$$(n=1,\ 2,\ldots,\ N;\ j=1,\ 2,\ldots,M).$$

Key: (1). Motion in the direction of longitudinal axis Ox. (1a). absolutely rigid body .... (1b). the absence of wave motions ....

In comparison with (30) is here introduced new designation $\dot{\gamma}_n=\Gamma_n$; remaining designations do not require commentaries.

Mathematical models (35)...(38) can be assumed as basis of analysis of disturbed motion of axisymmetric objects with cavities, which contain liquid, taking into account eddies of latter.

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RESEARCH OF THE DYNAMICS OF SOLID BODY WITH CAVITIES, WHICH CONTAIN LIQUID, TAKING INTO ACCOUNT ITS EDDYING.

V. G. Lebedev, A. I. Mytarev.

Is examined mathematical model of solid body with cavities, which contain liquid taking into account its potential and eddy, analogous to mathematical model of unsteady skin effect in magnetic circuit of controlled electromagnet. This model takes the form of the system of nonlinear integrodifferential equations with the singular kernels. The numerical and analytical algorithms of the study of dynamics and stability of the objects, described by such mathematical models, are developed. The efficiency of the proposed algorithms is shown based on the example of the analysis of the disturbed motion of solid body (rotation around the longitudinal axis) and are revealed some dynamic effects, which are absent in the models, which do not consider kinetic energy of the eddies of liquid.

In work [4] it is shown that such different, at first glance, phenomena, as eddy currents in magnetic circuit of controlled electromagnet and eddies of liquid in cavities of mobile solid body, which have internal edges/fins, can be described with large Reynolds numbers and small Strouhal numbers within the framework of one and the same mathematical model. Some aspects of the use of this model for the perturbation analysis of objects with the liquid will be examined below.

Mathematical model, which considers irrotational motion of liquid, partially filling cavities of mobile solid body, were obtained by G. S. Narimanov [2]. Mathematical model [3...5], which is the system of nonlinear integrodifferential equations, is its further development in the part of the account of the effect of the eddy of liquid in the presence in cavities of internal edges/fins. In the work of G. S. Narimanov [2] it was shown that the system of the differential equations of infinite dimensionality, which describe the disturbed motion of system body - liquid, can be reduced to finite system of integrodifferential equations. These equations were used by him for the proof of existence and uniqueness of solution and proof of the possibility of the reduction of the reference system of differential equation to the system of final order. The same equations can be used directly for the construction of different calculating algorithms. In this sense the results, obtained below, possess the specific succession with [2].

1. Equations of disturbed motion of solid body with cavities, partially filled with liquid, upon consideration of its eddy. Let us give the equations of the disturbed motion of stabilized solid body in the stabilization planes upon consideration of the potential and eddy of liquid, which are borrowed from the article of this collection [5].

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(1)Движение в плоскости рыскания

$$(m^{\circ}+m)\ddot{\zeta}+\sum_{n=1}^{N}\lambda_n\ddot{s}_n=P_z;$$

$$(J^{\circ}+J^{(o)})\ddot{\psi}+\sum_{n=1}^{N}(J^{*}\ddot{\vartheta}_n+\lambda_{on}\ddot{s}_n+j\lambda_n s_n)=M_{G_0 y};$$

$$J_n^{*}\ddot{\vartheta}_n+J_n^{*}\ddot{\psi}+\beta_n^{*}\int_{-\infty}^{t}\frac{\ddot{\vartheta}_n(\tau)\,d\tau}{\sqrt{t-\tau}}-\beta_{on}\int_{-\infty}^{t}\frac{\ddot{s}_n(\tau)\,d\tau}{\sqrt{t-\tau}}=0; \qquad (1)$$

$$\mu_n\left(\ddot{s}_n+\omega_n^2 s_n+\beta_n\int_{-\infty}^{t}\frac{\ddot{s}_n(\tau)\,d\tau}{\sqrt{t-\tau}}\right)+\lambda_n\ddot{\zeta}+\lambda_{on}\ddot{\psi}+j\lambda_n\psi-\beta_{on}\int_{-\infty}^{t}\frac{\ddot{\vartheta}_n(\tau)\,d\tau}{\sqrt{t-\tau}}=0$$

$$(n=1,\ 2,\dots,N),$$

Key: (1). Motion in the yaw plane.

where ζ - coordinate of the lateral displacement of solid body; ψ - yaw angle; s_n - the generalized coordinate of the fundamental tone of the skew-symmetric oscillations of liquid in the n section; N - number of sections; ϑ_n - further coordinate, which characterizes the dynamics of liquid in its eddy, which is absent from the traditional equations upon consideration only of irrotational motion; m° - mass of body without the liquid; m - mass of the hardened liquid; J° - moment of the inertia of body without the liquid; $J^{(o)}$ - moment of the inertia of the hardened liquid; J_n^{*} - connected moment of the inertia of liquid, which corresponds to its eddy, μ_n - the apparent additional mass of liquid in the n section with its irrotational motion; P_z and $M_{G_0 y}$ - respectively external force and moment, applied to the body, including controlling and perturbing components; j - modulus/module of the field gradient of mass forces; ω_n - partial frequency of the wave vibrations of liquid in the n section; λ_n, λ_{on} - coefficients of inertial couplings; β°_n, β_{on}, β_n - coefficients, which characterize the effect of edges/fins, and which are the nonlinear functions of the relative velocity of the motion of liquid [3]:

$$\beta^{\circ}_n \approx \tilde{\beta}^{\circ}_n \sqrt{|\dot{\vartheta}_n|};\ \beta_{on} \approx \tilde{\beta}_{on} \sqrt{|\dot{s}_n|};\ \beta_n \approx \tilde{\beta}_n \sqrt{|\dot{s}_n|}. \qquad (2)$$

In pitching plane equations are written/recorded analogously.

(1) Вращение вокруг продольной оси (крен)

$$(I^{\circ} + I^{(0)})\ddot{\varphi} + \sum_{n=1}^{N} I^{*}_n \ddot{\chi}_n = M_{G_0 x};$$

$$I^{*}_n \ddot{\chi}_n + I^{*}_n \ddot{\varphi} + \beta^0_n \int_{-\infty}^{t} \frac{\ddot{\chi}_n(\tau)\, d\tau}{\sqrt{t-\tau}} = 0 \qquad (3)$$

$$(n = 1, 2, \ldots, N).$$

Key: (1). Rotation around the longitudinal axis (bank).

Here φ - roll attitude; χ_n - generalized coordinate, which corresponds to eddies of liquid. The coefficient, which characterizes the effect of edges/fins, is expressed as follows:

$$\beta^0_n = \tilde{\beta}^0_n \sqrt{|\dot{\chi}_n|}. \qquad (4)$$

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Remaining designations are analogous in sense to designation, accepted in system of equations (1). In contrast to the yaw planes and pitch in the rolling plane wave motions of liquid are absent.

Upon simplified consideration of eddy of liquid (with neglect of appropriate kinetic energy), as this was done usually [3, 6], equations (1) and (3) pass into following:

$$(m^{\circ}+m)\ddot{\zeta}+\sum_{n=1}^{N}\lambda_n\ddot{s}_n=P_z;$$

$$(J^{\circ}+J)\ddot{\psi}+\sum_{n=1}^{N}\left(\lambda_{on}\ddot{s}_n+\beta_{on}\int_{-\infty}^{t}\frac{\ddot{s}_n(\tau)\,d\tau}{\sqrt{t-\tau}}+j\lambda_n s_n\right)+\beta\int_{-\infty}^{t}\frac{\ddot{\psi}(\tau)\,d\tau}{\sqrt{t-\tau}}=M_{O_0 y};$$

$$\mu_n\left(\ddot{s}_n+\omega_n^2 s_n+\beta_n\int_{-\infty}^{t}\frac{\ddot{s}_n(\tau)\,d\tau}{\sqrt{t-\tau}}\right)+\lambda_n\ddot{\zeta}+\lambda_{on}\ddot{\psi}+j\lambda_n\psi+$$

$$+\beta_{on}\int_{-\infty}^{t}\frac{\ddot{\psi}(\tau)\,d\tau}{\sqrt{t-\tau}}=0\quad(n=1,\ 2,\dots,N).\tag{5}$$

$$(I^{\circ}+I)\ddot{\varphi}+\beta^{0}\int_{-\infty}^{t}\frac{\ddot{\varphi}(\tau)\,d\tau}{\sqrt{t-\tau}}=M_{O_0 x}.\tag{6}$$

In these equations J, I - connected moments of inertia of liquid, β, β° - coefficients, which characterize effect of edges/fins. They are expressed as follows:

$$J=J^{(0)}-\sum_{n=1}^{N}J_n^{*};\ I=I^{(0)}-\sum_{n=1}^{N}I_n^{*};\ \beta=\sum_{n=1}^{N}\beta_n^{0};\ \beta^{0}=\sum_{n=1}^{N}\beta_n^{0}.\tag{7}$$

Thus, disturbed motion of solid body with cavities, partially filled with liquid, upon consideration of its eddy is described by system of nonlinear integrodifferential equations. By analogous equations is described, in particular, the dynamics of the controlled electromagnet upon consideration of unsteady skin effect in the conducting material of magnetic circuit [4].

At present are absent detailed methods of study of dynamics and stability of systems of similar class, studies of objects analogous to methods, described by ordinary differential equations. In connection with this equations (5) and (6) are substituted by ordinary differential equations. This replacement is conducted as follows.

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Let us register Laplace's image under the zero initial conditions for one of the integral terms [1], by assuming/setting temporarily β=const:

$$\beta \int_{-\infty}^{t} \frac{\ddot{\psi}(\tau)\, d\tau}{\sqrt{t-\tau}} \doteq \beta \sqrt{\pi p}\, \dot{\psi}(p), \qquad (8)$$

where p - Laplace's variable.

Let us assume that process being investigated is close to single-frequency with frequency of ω_0. As ω_0 during the research of wave motions of liquid it is possible to take partial frequency ω_n, and during the research of dynamics at the characteristic frequencies of solid body the frequency, formed/shaped with the automatic machine of stabilization.

Assuming/setting in (8) $p=i\omega_0$, we will obtain

$$\beta\int_{-\infty}^{t}\frac{\ddot{\psi}(\tau)\,d\tau}{\sqrt{t-\tau}}=J'\ddot{\psi}+\beta'\dot{\psi}, \tag{9}$$

where $$J'=\beta\sqrt{\frac{\pi}{2\omega_0}};\quad \beta'=\beta\sqrt{\frac{\pi\omega_0}{2}}. \tag{10}$$

Usually in (9) they retain only dissipative term $\beta'\dot{\psi}$, disregarding inertia $J'\ddot{\psi}$. However, with the strongly developed internal edges/fins the further moment of inertia J' becomes commensurate $(J^\circ+J)$, and in this case arises the question about the account of the eddies of liquid within the framework of mathematical models (1) and (3) during the study of dynamics and stability of the objects of the class in question, to which is dedicated this work.

Integrodifferential equations (5) and (6) are frequently substituted for perturbation analysis, close in each of stabilization planes to single-frequency process, on basis of conversions (8), (9), by systems of ordinary differential equations, which we will subsequently call traditional:

$$(m^\circ+m)\ddot{\zeta}+\sum_{n=1}^{N}\lambda_n\ddot{s}_n=P_z;$$

$$(J^\circ+J)\ddot{\psi}+\sum_{n=1}^{N}(\lambda_{on}\ddot{s}_n+\beta'_{on}\dot{s}_n+j\lambda_n s_n)+\beta'\dot{\psi}=M_{O_0y}; \tag{11}$$

$$\mu_n(\ddot{s}_n+\beta'_n\dot{s}_n+\omega_n^2 s_n)+\lambda_n\ddot{\zeta}+\lambda_{on}\ddot{\psi}+j\lambda_n\psi+\beta_{on}\dot{\psi}=0\quad(n=1,2,\ldots,N);$$

$$(I^\circ+I)\ddot{\varphi}+\beta^{0\prime}\dot{\varphi}=M_{O_0x}, \tag{12}$$

where $$\beta'=\beta\sqrt{\frac{\pi\omega_0}{2}};\quad \beta'_{on}=\beta_{on}\sqrt{\frac{\pi\omega_0}{2}};\quad \beta'_n=\beta_n\sqrt{\frac{\pi\omega_0}{2}};$$
$$\beta^{0\prime}=\beta^0\sqrt{\frac{\pi\omega_0}{2}}. \qquad (13)$$

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Frequencies ω_0 and ω° in formulas (13) are chosen depending on form of motion being investigated, and coefficients themselves, which characterize effect of edges/fins, are nonlinear functions of corresponding generalized velocities:

$$\beta'\approx\tilde{\beta}\sqrt{|\dot{\psi}|}\sqrt{\frac{\pi\omega_0}{2}};\quad \beta'_{on}\approx\tilde{\beta}_{on}\sqrt{|\dot{s}_n|}\sqrt{\frac{\pi\omega_0}{2}};$$
$$\beta'_n\approx\tilde{\beta}_n\sqrt{|\dot{s}_n|}\sqrt{\frac{\pi\omega_0}{2}};\quad \beta^{0\prime}\approx\tilde{\beta}^0\sqrt{|\dot{\varphi}|}\sqrt{\frac{\pi\omega^\circ}{2}}. \qquad (14)$$

2. Methods of study of dynamics and stability of objects, described by nonlinear integrodifferential equations with kernel of form $(t-\tau)^{-1/2}$. Let us consider the object, described by integrodifferential equations with the kernel of form $(t-\tau)^{-1/2}$, which corresponds either to analysis of stability "in small" of the system of electromagnetic levitation upon consideration of skin effect or to determination of the parameters of the limiting cycles, caused by the nonlinearity of oscillation damping of liquid in the sections of solid

body [4].

Let us introduce variable of form $q=\sqrt{p}$. Relative to this variable, as can be seen from (8), the characteristic equation of closed system the object of control (for example, equation (1) or (3) when $\beta=\mathrm{const}$) + the automatic machine of stabilization in the description of the latter by ordinary differential equations contains only whole degrees. The solution in plane q can be registered as expansion into partial fractions of form $q/(q-q_s)$, where q_s - root of characteristic equation.

For transition/transfer from image to original we will use following inversion formula [1]:

$$\frac{q}{q-q_s} \doteq e^{q_s^2 t}\left(1-\frac{2}{\sqrt{\pi}}\int_0^{-q_s\sqrt{t}} e^{-u^2}du\right)=e^{q_s^2 t}\left[1-\Phi_s(\eta)\right];\ \eta=q_s\sqrt{t}. \tag{15}$$

Integral $\Phi_s(\eta)$ is not expressed as elementary functions. However, for the conclusion/output of stability condition it is possible to use its asymptotic representation with $t\to\infty$ [1]:

$$\Phi_s(-q_s\sqrt{t})=\begin{cases}\Phi_s(\eta)\sim 1+\dfrac{e^{-\eta^2}}{\eta\sqrt{\pi}}\left[1-\dfrac{1}{2\eta^2}+\dfrac{1\cdot 3}{(2\eta^2)^2}-\dots\right] \\ \qquad {}^{(1)}\text{при } \operatorname{Re} q_s>0,\ \eta=q_s\sqrt{t}; \\ \Phi_s(\eta)\sim 1-\dfrac{e^{-\eta^2}}{\eta\sqrt{\pi}}\left[1-\dfrac{1}{2\eta^2}+\dfrac{1\cdot 3}{(2\eta^2)^2}-\dots\right] \\ \qquad {}^{(2)}\text{при } \operatorname{Re} q_s>0,\ \eta=-q_s\sqrt{t}.\end{cases} \tag{16}$$

Key: (1). with.

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Combined analysis of expressions (15) and (16) makes it possible to obtain stability condition in plane q, which is written/recorded as follows:

$$\frac{\pi}{4} < \operatorname{Arg}(q_s) < \frac{7\pi}{4}. \qquad (17)$$

Obtained criterion (17) makes it possible to establish fact of stability or instability of system and it can be used for construction of stability regions in planes of parameters of controlled system and regulator, and also for determination of limiting cycles. However, the knowledge of a continuous change in the generalized coordinates in time is required for the complete analysis of the dynamic properties of the controlled system. This leads to the need for the direct numerical solution of the system of nonlinear integrodifferential equations (1) or (3), which is the only adequate method of the analysis of dynamics and stability of system when the expressed effect of nonlinear factors is present, and under the influence of the disturbances/perturbations of arbitrary composition.

Let us consider, without breaking generality, algorithm of solution of nonlinear integrodifferential equations with kernel of form $(t-\tau)^{-1/2}$ based on example of channel of bank for object, which has one cut off, partially filled with liquid, and developed intra-tank devices/equipment, when automatic machine of angular stabilization is present. The equations of closed system object - regulator take the form:

$$
\begin{aligned}
&(I^{\circ}+I^{(0)}_{\text{ж}})\ddot{\varphi}+I^{*}\ddot{\chi}=a_{\varphi\delta}\delta;\\
&I^{*}\ddot{\chi}+I^{*}\ddot{\varphi}+\beta^{0}\int_{-\infty}^{t}\frac{\ddot{\chi}(\tau)\,d\tau}{\sqrt{t-\tau}}=0;\\
&T\dot{\delta}+\delta=a_{\varphi}\varphi+a_{\dot{\varphi}}\dot{\varphi};\\
&\beta^{0}=\tilde{\beta}^{0}\sqrt{|\dot{\gamma}|}.
\end{aligned} \tag{18}
$$

Here δ - angle of deflection of control devices; $a_{\varphi\delta}$ - gradient of controlling moment; T - time constant of drive, which determines its inertness; a_{φ}, $a_{\dot{\varphi}}$ - coefficients of algorithm of stabilization.

After introducing designation $\chi(\tau)=\sigma(\tau)$ and considering β° and φ temporarily known functions of time, let us represent second equation of system (18) in the form of nonhomogeneous integral equation of Volterra with singular kernel:

$$I^{*}\sigma+\beta^{0}\int_{-\infty}^{t}\frac{\sigma(\tau)\,d\tau}{\sqrt{t-\tau}}=-I^{*}\ddot{\varphi}. \tag{19}$$

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Let us consider one of possible algorithms of solution of integral equation (19), based on application of quadrature formulas of calculation of definite integral and determination of functions β° and $\ddot{\varphi}$ in resolving system of equations (18) together with (19). Since the mathematical model in question cannot describe the initial process of forming the vortices/eddies, we will assume that in the system to moment of time t=0 there is no prehistory, i.e., any eddies of liquid, which makes it possible to take integral in (19) from 0 to t.

Let us compute values of integrand $\sigma(\tau)$ at particular moments of time $0, h, 2h, \ldots, ih, \ldots, nh$, where h coincides with step/pitch of integration of system (18), and nh=t. Within intervals $0\ldots h,\ h\ldots 2h,\ \ldots,\ (i-1)h\ldots ih,\ \ldots,\ (n-1)h\ldots nh$ we will consider function $\sigma(\tau)$ linear. The representation of integrand in the form of the product of piecewise-linear function to kernel $(t-\tau)^{-1/2}$ permits, first of all, to reduce the singularity of kernel, and in the second place, to do a method of calculating the integral for more precise than, let us say, the method of trapezoids, since to the method of trapezoids corresponds piecewise-linear interpolation of function $\sigma(\tau)(t-\tau)^{-1/2}$, which does not remove the singularity of kernel.

Taking into account that stated above and after making simple, but cumbersome calculations, we will obtain

$$\int_0^t \frac{\sigma(\tau)\,d\tau}{\sqrt{t-\tau}} = \int_0^t \frac{\sigma(\tau)\,d\tau}{\sqrt{t-\tau}} \simeq \sqrt{h}\left\{G_n + \frac{4}{3}\sigma(nh)\right\}, \tag{20}$$

where $G_0 = G_1 = 0;\ \ G_2 = \frac{4}{3}\sigma(h)\left[2\left(\sqrt{2}-1\right)\right];$

$$G_n=\frac{4}{3}\sigma(h)\left[n\sqrt{n}-\sqrt{n-1}\left(n+\frac{1}{2}\right)+\frac{4}{3}\sum_{i=2}^{n-1}\left\{\sigma((i-1)h)\times\right.\right.$$
$$\times\left[\sqrt{n-i}(n-i)-\sqrt{n-(i-1)}\left(n-\frac{2i+1}{2}\right)\right]+$$
$$\left.\left.+\sigma(ih)\left[\sqrt{n-(i-1)}(n-(i-1))-\sqrt{n-i}\left(n-\frac{2i-3}{2}\right)\right]\right\}+$$
$$+\frac{2}{3}\sigma((n-1)h)\quad(n>2). \tag{21}$$

After designating $G_n=G(t)$, and $\sigma(nh)=\sigma(t)$, which is correct at particular moments of time, multiple to step of integration, we will obtain, after substituting expression (20) into (18), system of equations, which can be integrated by one of standard numerical methods (Runge-Kutta, Adams-Stoermer, etc.):

$$\begin{aligned}
&(I^\circ+I^{(0)})\ddot{\varphi}(t)+I^*\sigma(t)=a_{\varphi\delta}\delta(t);\\
&\left[I^*+\frac{4}{3}\beta^0\sqrt{h}\right]\sigma(t)=-[I^*\ddot{\varphi}(t)+\beta^0\sqrt{h}G(t)];\\
&\ddot{\chi}(t)=\sigma(t);\\
&T\dot{\delta}(t)+\delta(t)=a_\varphi\varphi(t)+a_{\dot{\varphi}}\dot{\varphi}(t);\\
&\beta^0=\bar{\beta}^0\sqrt{|\dot{\chi}|}.
\end{aligned} \tag{22}$$

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In equations (22) at each step/pitch of integration G(t) is computed from formulas (21).

Analogous computing circuit can be written for integrating equations of disturbed motion in yaw planes and pitch, and also for any integrodifferential equations with kernel of form $(t-\tau)^{-1/2}$.

3. Systematic example. Let us consider as a systematic example the dynamics of axisymmetric solid body, stabilized relative to longitudinal axis. Based on this example it is possible to demonstrate efficiency of the proposed algorithms and to reveal some new qualitative special features, which are not exhibited within the framework of traditional mathematical models (11) and (12).

We will investigate and compare two versions of mathematical description of object of control: 1) traditional model (12) with factors of nonlinear damping, computed from formula (14); 2) mathematical model, which considers kinetic energy of eddy of liquid (18).

Let us pose problem of determining parameters of possible limiting cycles, caused by nonlinearity of oscillation damping of liquid. Let in the system with some relationships/ratios of the parameters be established/installed the stable auto-oscillations with a frequency of ω° and amplitudes of φ_0, $\dot{x}_0$, δ_0 (in model (12) parameter $\dot{x}_0$ - is absent). Let us lead on the basis of the method of harmonic balance the linearization of the nonlinear damping factor. Following [6], the damping factor is expressed as follows:

For model (12)

$$\beta^{0\prime}(\dot{\varphi}_0) = \varkappa \tilde{\beta}^0 \sqrt{\dot{\varphi}_0} \sqrt{\frac{\pi\omega^\circ}{2}}. \tag{23}$$

For model (18)

$$\beta^0(\dot{\chi}_0) = \varkappa \tilde{\beta}^0 \sqrt{\dot{\chi}_0}, \tag{24}$$

where $\varkappa = \frac{8 \cdot 08}{5\sqrt{\pi}} \approx 0{,}91.$

On boundary of oscillatory stability characteristic equation of closed system has two complex conjugate roots

$$q_{1,2} = \alpha(1 \pm i) \quad (\alpha > 0), \tag{25}$$

and remaining roots lie/rest within stability region, determined by criterion (17).

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To roots (25), as shows analysis of expressions (15) and (16), correspond in steady-state mode sustained oscillations with frequency of

$$\omega^\circ = q_{1,2}^2 = 2\alpha^2. \tag{26}$$

For mathematical models (12) and (18) were constructed stability regions in plane of time constant of drive T and coefficient β, by which for model (12) is understood coefficient $\beta^{\circ\prime}$ 23), while for model (18) coefficient β°. The construction of stability regions for model (18) was carried out by calculating the roots of the characteristic equation of closed system and determination of boundary value of the parameters in accordance with criterion (17) and for the comparison by direct numerical integration. These regions are presented in Fig. 1. Double shading is converted inside the stability region. The numbering of curves corresponds to the frequency of limiting cycle, obtained on the basis of the calculation of the roots of characteristic equation for formula (26). In the sections, where different curves virtually coincide (with an accuracy to three significant places), general/common numbering is given.

Analysis of represented stability regions makes it possible to make following conclusions.

1. Account of further degree of freedom, which corresponds to eddies of liquid, within the framework of most complete mathematical model (18) substantially narrows stability regions of closed system in comparison with regions, which correspond to traditional mathematical model (12).

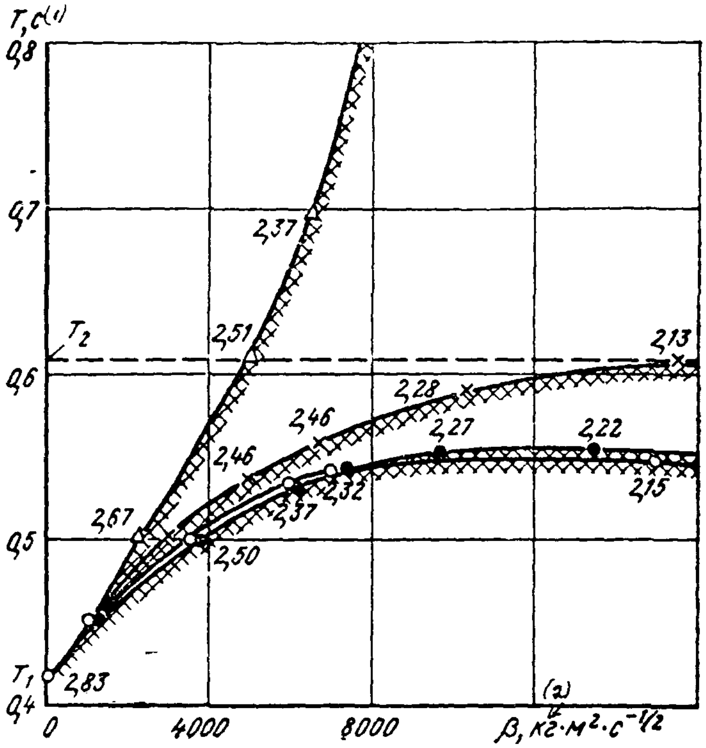

Fig. 1. Stability limits: △ - calculation according to traditional model; ○ - calculation on basis of characteristic equation of linear equivalent of system (18) (β=const for each point); ● - numerical solution of linear equivalent of system (18): * - numerical integration of system (18) with $\beta^0=\tilde{\beta}^0\sqrt{|x|}$

Key: (1). s. (2). $kg \cdot m^2 \cdot s^{-1/2}$.

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2. Good coincidence of stability regions, calculated on roots of characteristic equation with application of criterion (17) and method

of direct numerical integration of linear equivalent of system (18) testifies about high accuracy of developed method of numerical solution of integrodifferential equations with singular kernel of form $(t-\tau)^{-1/2}$ and at the same time about possibility of successful use of comparatively simple criterion (17).

3. Stability regions, obtained by integration of nonlinear system of integrodifferential equations (complete model (18)) virtually coincide with regions, calculated on roots of characteristic equation of equivalent linear system, to values $\beta^0 \approx 6000 \ldots 7000$. In this range β° with a sufficient degree of accuracy it is possible to calculate the parameters of limiting cycles, without resorting to the direct solution of the system of nonlinear integrodifferential equations.

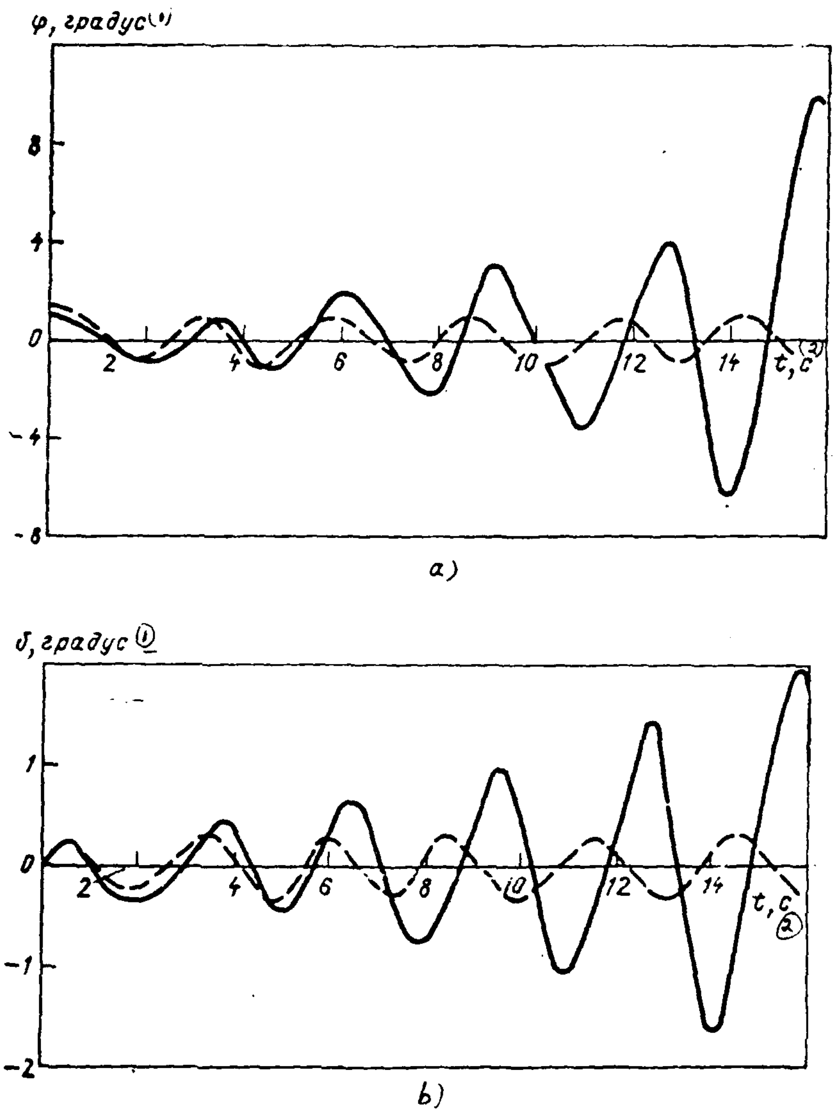

Fig. 2. Laws of motion of body around longitudinal axis: φ - angle of rotation of body; δ - angle of deflection of control device; - - - - - - model (12) (stable auto-oscillations); —— - model (18) (instability "in large").

Key: (1). degree. (2). s.

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With the more expressed nonlinear effects (larger values $\beta°$) straight/direct numerical integration is, evidently, the only adequate instrument of the analysis of the stability of the systems of the class in question.

4. There are two critical values of time constant of drive: $T_1 \simeq 0.415$ and $T_2 \simeq 0.6$ (see Fig. 1). With $T<T_1$ the system is stable "in the small" and "in the large". When $T_1<T<T_2$ the system is unstable "in the small", but it leaves to the stable limiting cycle, i.e., it is stable "in the large". With $T>T_2$ the system is unstable both "in the small", and "in the large".

It should be noted that within the framework of traditional mathematical model (12) cannot be revealed very dangerous from technical point of view mode of instability of system "in large" - system with any values $\beta°$ leaves to stable limiting cycle. Thus, a stricter mathematical description makes it possible to determine not only quantitative, but also very essential new qualitative effects.

Fig. 2 gives results of numerical integration with T=0.8 c>T_2 of equations (18) and (12), from which it is evident that depending on utilized mathematical model either appears stable limiting cycle with completely acceptable parameters (model (12)), or are developed dangerous sustained oscillations (model (18)).

In conclusion let us consider results of mathematical simulation of turn of considered/examined solid body around longitudinal axis for preset angle for the same two versions of mathematical model. The time constant of drive was selected as being equal to 0.2 s, which is less than T_1, i.e., was investigated the dynamics of the system, stable "in the small". The corresponding transient processes are represented in Fig. 3. As can be seen from figure, the transient processes, obtained for two versions of the mathematical model of the system in question, substantially are distinguished both by the value of overregulation and by the duration; model (18) corresponds in this case to heavier situation than (12), i.e., use of (12) does not give calculation "in the reserve".

Example examined sufficiently convincingly proves need for account of kinetic energy of eddy of liquid during analysis of dynamics of systems of class in question, which is reached, in particular, as a result of using refined mathematical model [3-5].

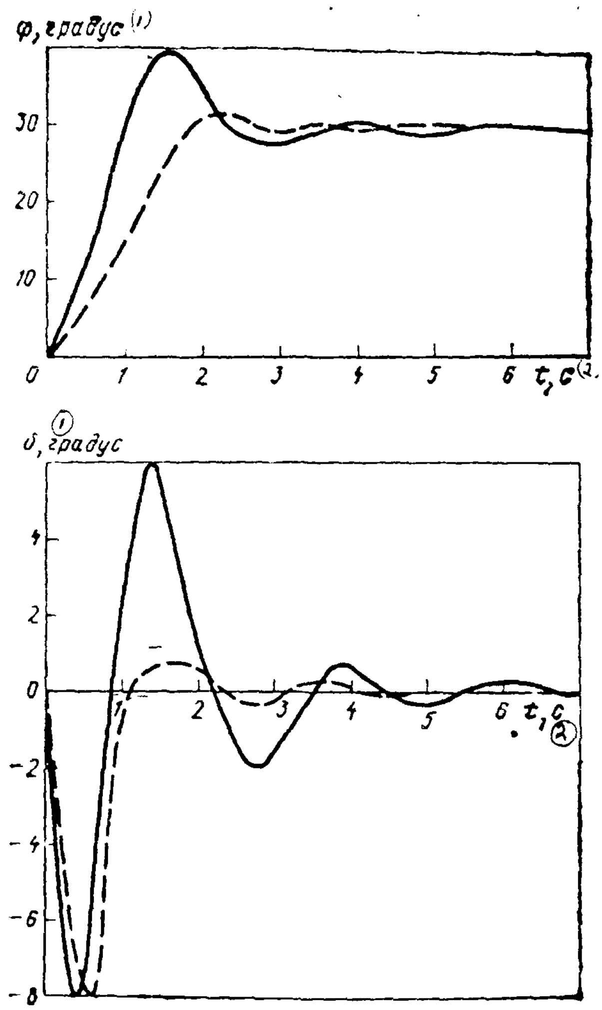

Fig. 3. Turn of body around longitudinal axis to preset angle: φ - angle of rotation of body; δ - angle of deflection of control device; - - - - - - - model (12); ———— - model (18).

Key: (1). degree. (2). s.

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SOLUTION OF THE PROBLEM ABOUT THE NATURAL OSCILLATIONS OF LIQUID IN A RIGID AXISYMMETRIC TANK BY ANALYTICAL CONTINUATION.

M. S. Galkin, N. I. Rudkovskiy.

Examples of solution of problem about natural oscillations of liquid in rigid axisymmetric tanks are examined by method of analytical continuation. The obtained results and their comparison with the appropriate solutions, obtained by other methods, confirm an accuracy of method sufficient for practical purposes.

Solution of problem about natural oscillations of liquid in rigid tanks is important intermediate stage in solution of problem about motion of body, whose cavity is partially filled with liquid [1].

For solution of problem about natural oscillations of liquid in rigid axisymmetric tanks method of natural forms (method, which uses natural modes of vibration in close problem) is used, variation and other methods. The method of natural forms makes it possible to solve problem for the very low and very large depths. With variational method it is necessary for each level of filling to solve problem at eigenvalues.

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In work is used method of analytical continuation for solution of problem about natural oscillations in entire region, when solution for part of region is known. Taking as the initial the solution, for example, for the spherical tank on the low levels of filling, it is possible to construct the solution for the arbitrary axisymmetric region, which has the spherical bottom. As the coordinate functions it is possible to take polynomials, and the generatrix of region to assign in the form of piecewise-smooth function.

1. Solution of problem by analytical continuation. In work [2] are obtained the following differential equations for frequencies and forms of the natural oscillations of liquid in the arbitrary region for parameter h (Fig. 1):

$$\frac{d}{dh}\left(\frac{\varkappa_k}{a}\right)=G_{kk}-\left(\frac{\varkappa_k}{a}\right)^2; \qquad (1)$$

$$a\,\frac{df_k}{dh}=\sum_{l=0}^{\infty} b_{kl}f_l \quad (k=1,2....),$$

where $b_{kl}=\begin{cases}\dfrac{a^2G_{kl}}{\varkappa_k-\varkappa_l} & k\neq l;\\[2mm] -\dfrac{a}{2}\Gamma_{kk} & k=l;\end{cases}$

$$\Gamma_{kl}=\int_{S_0} f_kf_l\,\mathrm{tg}\,\alpha\,ds_0+\int_{S_1} f_kf_l\,\mathrm{tg}\,\alpha_1\,ds_1; \qquad (2)$$

$$G_{kl}=\int_F \nabla f_k\nabla f_l\,dF-\frac{\varkappa_k}{a}\Gamma_{kl}, \qquad (3)$$

where $\varkappa_k=\omega_k^2a/g$ - dimen onless frequency of the k tone (subsequently simply natural frequency); f_k - displacement potential of the k tone.

Equations (1) have simple treatment. Let us assume that for the assigned region is known the solution of the corresponding boundary-value problem with some values of the parameters of region h, a, a_1. Then for the region, which corresponds to the level of the filling with liquid h+Δh, the value $\varkappa_h(h+\Delta h), f_h(h+\Delta h)$ it is possible to obtain on the basis of equations (1), without solving boundary-value problem.

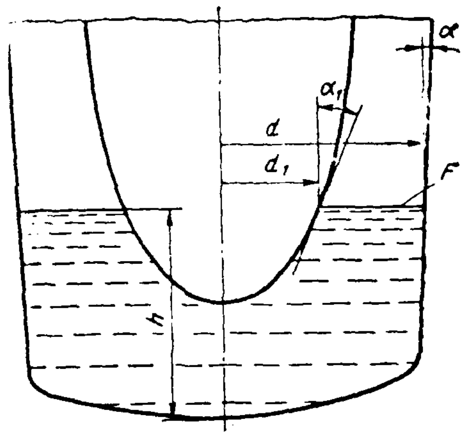

Fig. 1. Geometric characteristics of tank.

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Using the dimensionless parameter $\bar{h}=h/a$, equations (1) can be represented in the following form:

$$\frac{d}{d\bar{h}}\varkappa_k = a^2 G_{kk} - \varkappa_k^2; \quad (4)$$

$$\frac{d}{d\bar{h}} f_k = \sum_{l=0}^{\infty} b_{kl} f_l.$$

2. Numerical realization of algorithm.

a) axisymmetric oscillations of liquid in axisymmetric vessel. Using cylindrical coordinates, solution system on the free surface can be presented in the following form (see Fig. 1):

$$f_k = \frac{1}{\sqrt{\pi a^2}} \sum_{n=1}^{\infty} C_{kn} \left(\frac{r}{a}\right)^{2n-2} \quad (k=2,3,\ldots);$$

$$f_1 = \frac{1}{\sqrt{\pi a^2 (1-b^2)}}, \quad \text{(1) где } b = \frac{a_1}{a}; \qquad (5)$$

$$\int_F f_k f_l dF = 0 \text{ (2) при } k \neq l;$$

$$\int_F f_k f_k dF = 1 \text{ (2) при } k = l.$$

Key: (1). where. (2). with.

After substituting (5) into (4), we obtain differential equations for coefficients C_{kl}, and expressions (2), (3) will take the form

$$\Gamma_{kl} = \frac{2}{a} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{kn} C_{lm} (\operatorname{tg}\alpha + \operatorname{tg}\alpha_1 b^{2m+2n-3}); \qquad (6)$$

$$G_{kl} = \frac{4}{a^2} \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} C_{kn} C_{lm} \frac{(m-1)(n-1)}{m+n-2} (1 - b^{2n+2m-4}).$$

For integrating system let us use Euler's method. The finite number of coordinate functions and the finite series for f_k in (5) is taken during the numerical realization of algorithm. As corollary, during the calculations is broken orthonormalization of functions (5). Therefore at each step of integration is done orthonormalization of functions (5).

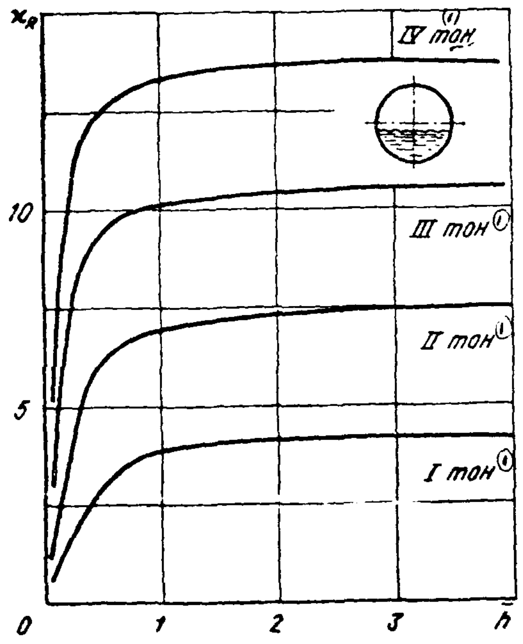

Fig. 2. Dependence of dimensionless natural frequencies of axisymmetric oscillations of liquid in spherical tank on relative level of filling.

Key: (1). ... tone.

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Spherical tank. The initial solution was undertaken from work [3] on the level of filling $\bar{h}$=0.1. The dependence of dimensionless natural frequencies κ for the first four tones on the level of filling is shown in Fig. 2. Table 1 gives the comparison of the obtained results with the results of calculations employing other procedures for the hemisphere. During the integration of equations (4) for obtaining the solution with $\bar{h}$=1 (hemisphere) the step of integration was equal to 0.001. Calculation on computer(s) BESM-6 of entire sphere occupied 60 min.

Cylinder with spherical bottom. Results for this region make it possible to consider the convergence of algorithm, since with $\bar{h} \geq 1$ the solution asymptotically approaches the solution for the cylinder. The dependence of the frequencies of the first four tones on the level of filling is given in Fig. 3. Table 2 gives the values of the frequencies of the first four tones with $\bar{h}=1$.

Table 1.

(1) Номер тона	(2) Метод конечного элемента	(3) Результат ONERA	(4) Эксперимент	(5) Метод аналитического продолжения
1	3,960	3,771	3,679	3,779
2	8,567	7,193	6,964	6,988
3	15,20	10,86	10,28	10,15

Key: (1). Number of tone. (2). Method of finite element. (3). Result ONERA. (4). Experiment. (5). Method of analytical continuation.

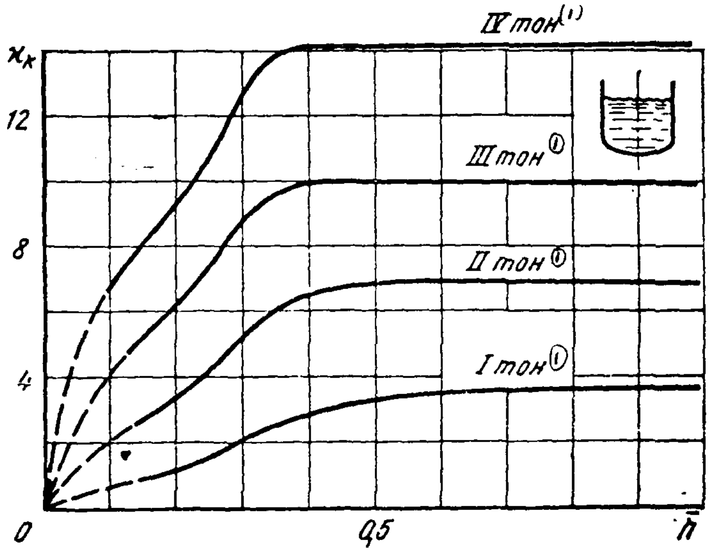

Fig. 3. Dependence of dimensionless natural frequencies of axisymmetric oscillations of liquid in cylindrical tank with combined bottom on relative level of filling.

Key: (1). ... tone.

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For the cylinder with the spherical bottom and coaxial inset within the geometry of external and internal generatrix it was assigned as follows:

$$\operatorname{tg}\alpha=\begin{cases}\dfrac{\bar{h}^2-1}{2} & 0,1\leqslant\bar{h}\leqslant 0,2;\\ -24,13(\bar{h}-0,3) & 0,2\leqslant\bar{h}\leqslant 0,3;\\ 0 & \bar{h}>0,3;\end{cases}\qquad(7)$$

$$\operatorname{tg}\alpha_1=\begin{cases}0 & 0,1\leqslant\bar{h}\leqslant 0,2;\\ -5,27(\bar{h}-0,2) & 0,2<\bar{h}\leqslant 0,48;\\ 5,27(\bar{h}-0,76) & 0,48<\bar{h}\leqslant 0,76;\\ 0 & \bar{h}>0,76.\end{cases}$$

With $\bar{h}>1$ solution for this region also asymptotically approaches solution for coaxial cylinders. The dependence of the frequencies of the first three tones on the level of filling is given in Fig. 4. Table 3 gives frequencies for the coaxial cylinders in the region with $\bar{h}=1$ in question.

b) skew-symmetric oscillations of liquid in spherical tank. The form of oscillations of free surface is assigned as follows:

$$f_k=\frac{\cos\varphi}{\sqrt{\pi a^2}}\sum_{n=1}^{\infty}C_{kn}\left(\frac{r}{a}\right)^{2n-1}.\qquad(8)$$

Fig. 5. depicts dependences of first three frequencies of skew-symmetric natural oscillations of liquid on level of filling.

Examples of solution of problem examined about natural oscillations of liquid i:. axisymmetric cavities by method of analytical continuation make it possible to positively consider possibilities of method for solution of such problems.

Table 2.

(1) Номер тона k	x_k	
	(2) Цилиндрический бак с плоским днищем	(3) Цилиндрический бак со сферическим днищем
I	3,83	3,820
II	7,01	7,015
III	10,17	10,17
IV	13,32	14,33

Key: (1). Number of tone k. (2). Cylindrical tank with flat/plane bottom. (3). Cylindrical tank with spherical bottom.

Table 3.

(1) Номер тона k	x_k	
	(2) Коаксиальные цилиндры с плоским днищем	(3) Коаксиальные обечайки сложной формы
I	5,32	5,33
II	10,42	10,43
III	16,12	15,35

Key: (1). Number of tone k. (2). Coaxial cylinders with flat/plane bottom. (3). Coaxial cowlings of complex form.

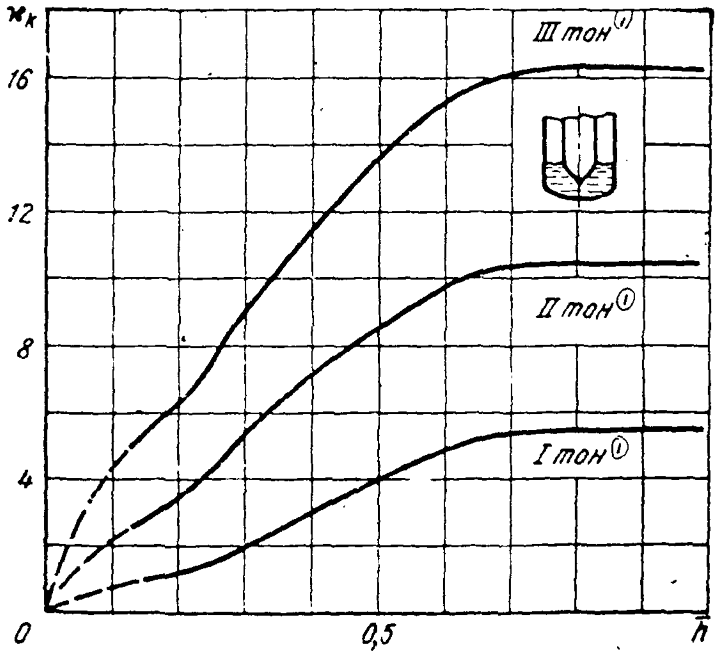

Fig. 4. Dependence of dimensionless natural frequencies of axisymmetric oscillations of liquid in tank with combined bottom and coaxial inset on relative level of filling.

Key: (1). ... tone.

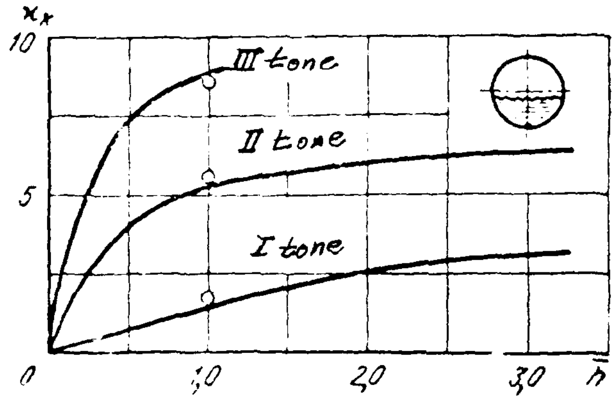

Fig. 5. Comparison of results of calculating frequencies of natural skew-symmetric oscillations of liquid in spherical tank with experiment - ○

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RAPIDLY-CONVERGING VARIATION ALGORITHM IN A PROBLEM ABOUT THE NATURAL OSCILLATIONS OF LIQUID IN THE VESSEL.

I. A. Lukovskiy, G. A. Shvets.

Variational method is used to solution of problem about free oscillations of liquid. Besides the harmonic polynomials, which traditionally are used during the solution of this problem, in the number of coordinate functions are included the functions with the properties, which reflect the surface character of gravity waves in the liquid. For their determination special boundary-value problem at eigenvalues is formulated. The efficiency of algorithm is illustrated on the solution about the oscillations of liquid in the cylindrical reservoir with horizontal generatrix and the conical vessel in the form of the inverted round cone. Is carried out the comparison of obtained data with the results of calculation only according to the harmonic polynomials.

To problem about natural oscillations of ideal incompressible fluid in vessel are devoted many research both in Soviet and in foreign literature. Large number of works is devoted to the methods of the construction of the actual solutions of the corresponding boundary-value problems, among which the widest acceptance obtained variational method. It proved to be very efficient for the solution of the problems about the oscillations of liquid in the vessels of axisymmetric form. However, because of the need for the solution of the problems about wave motions of liquid in the capacities/capacitances of complex geometric form it requires further improvement.

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This question is extremely urgent also in connection with the development of the methods of the study of the problems of nonlinear theory, for which the solutions of linear problems are chosen as the base. In the present work variational method is used in its traditional form. In the development of the basis of the ideas of work [1] system of coordinate functions in the method Ritz it is provided by the series/row of further properties (of type of the property of boundary layer), which leads to an improvement in the convergence of process and, in the final analysis, to the decrease of the dimensionality of algebraic systems.

1. Ritz's method. Selection of the system of coordinate functions. Research of the free oscillations of the ideal incompressible fluid in the vessel, as is known, is reduced to the solution of a certain problem at eigenvalues with the parameter under the boundary condition.

After representing velocity potential of liquid in the form

$$\Phi(x, y, z, t) = \varphi(x, y, z) \cos \sigma t, \tag{1}$$

from fundamental equations of linear wave theory we will obtain mentioned spectral problem in the following form:

$$\Delta\varphi = 0 \overset{(1)}{\text{B}}\ Q;\ \frac{\partial\varphi}{\partial v} = \varkappa\varphi \overset{(2)}{\text{Ha}}\ \Sigma_0;\ \ \frac{\partial\varphi}{\partial v} = 0 \overset{(2)}{\text{Ha}}\ S, \tag{2}$$

Key: (1). in. (2). on.

where Q - region, occupied with liquid; Σ_0 - undisturbed free surface of liquid; S - hydrophilic surface of cavity; $\kappa=\sigma^2/g$; g - free-fall acceleration on the Earth.

Basic properties of solutions of problem (2) are well studied. There is infinite consecutively/serially positive eigenvalues of this problem, and to each eigenvalue $\varkappa_n$ corresponds the finite number, generally speaking, the generalized solutions of problem φ_n.

Form of disturbed free surface is determined after determination of velocity potential in the form

$$f(y, z, t) = -\frac{1}{g} \frac{\partial\Phi(O, y, z, t)}{dt} \tag{3}$$

In conformity to problem at eigenvalues of (2) is formulated equivalent variational problem for functional

$$K(\varphi) = \frac{\int\limits_{Q} (\nabla\varphi)^2 dQ}{\int\limits_{\Sigma_0} \varphi^2 dS} \tag{4}$$

under further condition

$$\int_{\Sigma_0} \varphi dS = 0.$$

Minimum of functional (4) exists on class of continuous together with first-order derivatives of functions, and for its finding Ritz's method is used.

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The efficiency of the application of the latter depends on the successful selection of the adequate/approaching system of coordinate functions. The widest acceptance in this case received solid spherical harmonics and harmonic polynomials, for which were established/installed recursion formulas [3] important in the practical sense. This set of functions during its use in Ritz's method we is widened by the functions, which possess some analytical properties of the exact solution of problem [1]. In this plan/layout us, first of all, they will interest functions with the properties of the type of boundary layer, which reflect the clearly expressed surface character of the oscillations of liquid in the vessel.

Together with harmonic polynomials $W_k(x, y, z)$ let us consider solutions of equations of Laplace of type [1]

$$\varphi_i(x, y, z) = e^{k_i x} f_i(y, z), \qquad (5)$$

where $f_i(x, y)$ - complete on Σ_0 set of functions, which is solution of boundary-value problem

$$\Delta f + k^2 f = 0 \overset{(1)}{\text{на}} \ \Sigma_0; \qquad (6)$$

$$\frac{\partial f}{\partial n} + k\rho(s) f = 0 \overset{(1)}{\text{на}} \ l. \qquad (7)$$

Key: (1). on.

Here $\rho(s)$ - certain weight function; l - intersection Σ_0 and S; n - unit vector of external normal to l.

Boundary condition (7) follows from condition

$\frac{\partial \varphi}{\partial \nu} = 0$ on S. The solutions of the type (5) not only qualitatively reflect the surface character of the oscillations of liquid, but in a number of cases they bring also to good quantitative results of [2].

Solution $\varphi(x, y, z)$ let us represent now in the form

$$\varphi(x, y, z) = \sum_{k=1}^{n} a_k W_k(x, y, z) + \sum_{l=1}^{q} c_n \varphi_n(x, y, z), \qquad (8)$$

where a_k and c_n - unknown constants, to be ...rmined from system of Ritz

$$\sum_{j=1}^{N} b_j(\alpha_{ij} - \varkappa\beta_{ij}) = 0 \quad (j = 1,2,\ldots, N). \qquad (9)$$

By components of vector b in system (9) will be ordered in a specific manner constants a_k and c_n; α_{ij} and β_{ij} are determined by expressions

$$\alpha_{ij}=\int_Q (\nabla\psi_i,\ \nabla\psi_j)\,dQ;$$
$$\beta_{ij}=\int_{\Sigma_0} \psi_i\psi_j\,dS, \tag{10}$$

where ψ_i - sequence of functions, which consists of harmonic polynomials Ψ_i and functions φ_n (5).

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2. Numerical application of method of Ritz in some specific cases. Let us consider the case of the free oscillations of liquid in the cylindrical cavity with horizontal generatrix. The undisturbed free surface of liquid is rectangle and, therefore, boundary-value problem (6), (7) can be solved by the method of separation of variables. For example, for the forms of oscillations, skew-symmetric relative to y and symmetrical relative to z, we will obtain

$$f_{ni}(y,\ z)=\cos\frac{\pi n}{a}z\sin\sqrt{k_{ni}^2-\frac{\pi n}{a}}\,y, \tag{11}$$

moreover eigenvalues k_{ni} they are defined as the roots of the transcendental equation

$$k_{ni}\frac{h_1}{r_0}\sin\xi-\sqrt{k_{ni}^2-\frac{\pi n}{a}}\cos\xi=0, \tag{12}$$

where $h_1=h-R_0$; $r_0=\sqrt{R_0^2-h_1^2}$; R_0 - radius of cylinder; h - depth of filling of cavity.

Three-dimensional harmonic polynomials, obtained from system of solid spherical harmonics $C_n^m = b_{nm} R^n P_n^m (\cos\theta) \cos m\eta$ and $S_n^m = b_{mn}^m R^n P_n^m \times (\cos\theta) \sin m\eta$, take following form:

$$C_0^0 = 1;\ S_0^0 = 0;\ C_1^0 = x;\ C_1^1 = y;\ S_1^1 = z;\ C_2^0 = x^2 - \frac{1}{2}(y^2 + z^2);$$

$$C_2^1 = xy;\ C_2^2 = y^2 - z^2;\ S_2^1 = xz;\ S_2^2 = 2yz;\ C_3^0 = x^3 - \frac{3}{2}xy^2 - \frac{3}{2}xz^2;$$

$$C_3^1 = x^2y - \frac{1}{4}y(y^2 + z^2);\ S_3^1 = x^2z - \frac{1}{4}z(y^2 + z^2);\ C_3^2 = x(y^2 - z^2);\quad (13)$$

$$S_3^2 = 2xyz;\ C_3^3 = y(y^2 - 3z^2);\ S_3^3 = z(3y^2 - z^2).$$

Recursion relations, which make it possible to register functions and their derivatives for as much as desired large m and n [3], are obtained for them.

Table 1 shows convergence the first two values $\varkappa_{nj}$ $(j = 1, 2)$ of frequency parameter $\varkappa$ depending on number of harmonic polynomials (13) in expansion (8) at fixed value of q=3, radius of cylinder R_0=1, depth of filling h=1 and length of cylinder 2a=2.

Table 1.

n	$\varkappa_{n1}$	$\varkappa_{n2}$	$\varkappa^0_{n1}$	$\varkappa^0_{n2}$
3	1,3892	3,3795	1,5028	11,6237
4	1,3562	3,3795	1,3746	11,5993
5	1,3561	3,3795	1,3743	5,3202
6	1,3561	3,3665	1,3568	5,2927
7	1,3559	3,3665	1,3568	4,7702

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Two latter/last columns of this table $\varkappa^{\circ}_{nj}$ relate to the case of calculating the parameter κ, when potential φ is represented as expansion only in terms of the harmonic polynomials. Analogous results are represented in tables 2 and 3 for the values of parameters $h=1{,}5\,R_0, 2a=7{,}18\,R_0$ and j=1, 2, 3. In this case table 2 presents the results of calculation according to the algorithm, given in the present work (q=3), and in table 3 - the results of calculation with the use only of harmonic polynomials (q=0).

For case of cavities of axisymmetric form of solution of boundary-value problem (6), (7) in polar coordinate system they take form

$$f_{mn}(r,\ \eta)=J_m(k_{mn}r)^{\sin m\eta}_{\cos m\eta} \tag{14}$$

moreover eigenvalues k_{mn} are defined as roots of transcendental equation

$$\frac{dJ_m(\xi)}{d\xi}-\rho J_m(\xi)=0, \tag{15}$$

where $\xi=kr_0$; r_0 - radius of free surface.

System of harmonic polynomials, registered in cylindrical coordinate system, is biparametric set of functions $W_k^m(x, r)\sin m\eta$; $W_k^m(x, r)\cos m\eta$, for which are also established/installed simple recurrent correlations [3]. For case of m=1, for example, we have

$$W_1^1=r;\ W_2^1=xr;\ W_3^1=x^2r-\frac{1}{4}r^3;\ W_4^1=x^3r+\frac{3}{4}r^4,\ldots. \qquad (16)$$

Table 4 gives results of calculating first eigenvalues of boundary-value problem (2) for inverted cone with half-angle of $\theta_0=10°$ with value of $r_0=1$, $\rho=tg\theta_0$, q=3. Analogous data are cited in table 5 with q=2.

Results given above show that inclusion in number of coordinate of decision function of type (5), which reflect basic special features of behavior of liquid in vicinity of free surface, makes it possible to significantly influence convergence of Ritz's process not only during calculation of lowest, but also, in particular, highest frequencies and forms of oscillations of liquid. This relates first of all to the case of the substantially spatial problems, for which even partial separation of variables cannot be led. In this case also it is possible to substantial reduce the order of Ritz's system which implies the retention/maintaining the stability of calculating process and increasing the accuracy of calculations.

Table 2.

n	x_{n1}	x_{n2}	x_{n3}
3	2,0003	2,2187	2,7800
4	1,9077	2,2186	2,7800
5	1,9076	2,2186	2,7800
6	1,9076	2,1693	2,7791
7	1,9028	2,1693	2,7791

Table 3.

n	x^0_{n1}	x^0_{n2}	x^0_{n3}
3	2,8421	7,6975	437,0681
4	2,1584	7,6966	436,9401
5	2,1583	3,4926	59,5738
6	1,9599	3,4921	18,1121
7	1,9599	3,4140	18,0717

Table 4.

n	x_{n1}	x_{n2}	x_{n3}
1	1,6772	5,2470	8,4622
2	1,6748	5,2382	8,4546
3	1,6747	5,2322	8,4471
4	1,6746	5,2299	8,4419
5	1,6744	5,2297	8,4393

Table 5.

n	x_{n1}	x_{n2}	x_{n3}
6	1,6743	5,2296	37,8766
7	1,6743	5,2294	20,4115
8	1,6743	5,2293	18,9783
9	1,6743	5,2293	16,6370

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VARIATION PRINCIPLE OF M. A. LAVRENTYEV AND RT-ALGORITHM OF CONFORMAL MAPPING IN HYDRODYNAMICS AND DYNAMICS OF SOLID BODY WITH THE LIQUID.

B. I. Rabinovich, Yu. V. Tyurin.

Is described new numerical RT-algorithm of conformal mapping of arbitrary singly connected and double-bond regions with piecewise-smooth duct/contour on the average and annulus respectively. A RT-algorithm includes two recurrent procedures: internal (R-procedure) and external (T-procedure), at each step/pitch of which is used the variation principle of M. A. Lavrentyev. A RT-algorithm is realized in the form of program in the language FORTRAN-4.

Examples of solution of external and internal hydrodynamic problems are given, including problems, which are encountered in dynamics of solid body, which contains cavities with liquid.

Series of flat/plane boundary-value problems of mathematical physics with harmonic and biharmonic operators comparatively easily is solved, if initial region conformally can be mapped on the average (simply connected region) or annulus (doubly connected region).

This relates both to external and to internal problems of hydrodynamics of incompressible fluid, in particular, to problems of dynamics of liquid in mobile cylindrical cavities, which do not possess axial symmetry. Similar problems acquire ever larger rgency in the dynamics of different controlled objects, which have the sections of the complex configuration, which contain liquid.

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However, analytical formulas for mapping functions can be obtained only for regions of simple configuration (ellipse, elliptical ring, circular sector, etc.).

There is series/row of numerical methods of conformal mapping, whose bases are laid in work [3] see, for example, [10] and [11-13]). However, realization into the computer(s) of the corresponding algorithms, which reduce the problem to the problem of linear algebra, runs in the case of the regions of complex configuration, which require the assignment of the large number of points on the duct/contour, for the definite difficulties.

New possibilities in sense of creation of structural/design numerical algorithms of conformal mapping, which allow/assume efficient realization on computer(s), open/disclose variation principle of M. A. Lavrentyev [4, 5].

In work problems about conformal mapping of arbitrary quaver region on the average are examined in classical setting [3] and arbitrary doubly connected region with piecewise-smooth duct/contour to annulus. The corresponding exterior problems are led to those indicated with the help of the linear-fractional conversion.

Is presented multistage algorithm of conformal mapping (RT-algorithm), worked out by authors [8, 9], which includes two recurrent procedures: internal (R-procedure) and external (T-procedure). At each step/pitch is used M. A. Lavernt'yev's principle, to whom is dowry the form, convenient for the numerical realization on the computer(s).

RT-algorithm is realized in the form of program in language FORTRAN-4 and is checked on wide spectrum of flat/plane boundary-value problems.

As illustration of possibilities of RT-algorithm examples of solution of some external and internal hydrodynamic problems and problems of dynamics of liquid in tanks of flight vehicles are examined [1, 6-8].

1. Algorithm of precise conformal mapping of region, close to circle, on the average and inverse representation (R-procedure). Let in the plane complex variable z be assigned closed curve C, close to the unit circle Γ. To it correspond simply connected region $D_2(C)$ close to the circle and exterior $D_1(C)$. Proximity C to Γ is determined by the inequalities

$$|\delta(\varphi)| < \varepsilon; \quad |\delta'(\varphi)| < \varepsilon; \quad |\delta''(\varphi)| < \varepsilon; \qquad (1)$$
$$r = r(\varphi) = 1 - \delta(\varphi); \quad r(\varphi)|_\Gamma = 1 (0 \leqslant \varphi \leqslant \pi),$$

where $r(\varphi)$ - radius-vector of the point of duct/contour C; φ - vectorial angle; ϵ - small positive number; variation $\delta(\varphi) \gtrless 0$ is considered positive during the variation of duct/contour Γ in the direction of internal standard/normal.

Let w=f(z, C) - function, which reflects region D_2 of plane z to unit circle of plane w, moreover

$$w=f(z, C)|_{z=0}=0; \quad f'(z, C)|_{z=0}>0. \qquad (2)$$

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Using integral of Schwarz and law of mean [4], it is possible to give the variation principle of M. A. Lavrentyev [3, 4] to the following form:

Under the conditions (1) and (2) function.

$$w=f(z, C)=z[1+J(z)];$$
$$J(z)=\frac{1}{2\pi i}\oint_{\Gamma} u(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}, \qquad (3)$$

where $\zeta=e^{i\varphi}$; $u(\zeta)=\delta[\varphi(\zeta)]$ and duct/contour is bypassed so that the domain of definition z remains to the left, differs from the function, which realizes conformal mapping of region $D_2(C)$ onto the unit circle and $D_1(C)$ to its exterior, to the low, not lower than the second order relative to ϵ.

Let us introduce into examination function $\omega(\zeta)$

$$\omega(\zeta) = u(\zeta) + iv(\zeta) = \varepsilon\left(a_0 + \sum_{n=1}^{\infty} c_n \zeta^n\right);$$
$$c_n = a_n - ib_n;\ \zeta = e^{i\varphi};\ v(0) = v(\infty) = 0, \quad (4)$$

where εa_0, εa_n and εb_n $(n=1, 2, \ldots)$ - coefficients of expansion of function $\delta(\varphi)$, which is assumed to be single-valued, continuous and 2π-periodic, in Fourier series in segment $0 \le \varphi \le 2\pi$, and conjugated/combined with it $\overline{\omega(\zeta)}$.

Using Cauchy formula, it is possible to represent function J(z) (3) in another form:

$$J(z) = \begin{cases} \dfrac{1}{2\pi i} \oint\limits_{\Gamma} \dfrac{\omega(\zeta)\, d\zeta}{\zeta - z} = \varepsilon\left(a_0 + \sum\limits_{n=1}^{\infty} c_n z^n\right); & (5a) \\ \dfrac{1}{2\pi i} \oint\limits_{\Gamma} \dfrac{\overline{\omega(\zeta)}\, d\zeta}{\zeta - z} = \varepsilon\left(a_0 + \sum\limits_{n=1}^{\infty} \bar{c}_n z^{-n}\right) & (5b) \end{cases}$$

(a and b here and subsequently correspond to representation of interior onto unit circle or external to its exterior). The analytic function J(z) is continuous on the duct/contour Γ.

Using formulas (5a) and (5b), it is possible to give to mapping function w, entering (3), following form:

$$w = f(z, C) = \begin{cases} z\left[1 + \varepsilon\left(a_0 + \sum\limits_{n=1}^{\infty} c_n z^n\right)\right]; & (6a) \\ z\left[1 + \varepsilon\left(a_0 + \sum\limits_{n=1}^{\infty} \bar{c}_n z^{-n}\right)\right]. & (6b) \end{cases}$$

These expressions convenient for numerical realization, since they reduce entire problem to expansion of continuous 2π-periodic function to Fourier series.

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Function (6) differs from precise function, which translates duct/contour C of plane z into unit circle of plane w, to smalls of order not below ϵ^2.

Function $\Phi(w)$, which is in the first approximation of reverse/inverse for (6), is determined in accordance with M. A. Lavrentyev's principle by following formulas:

$$z = \Phi(w) = \begin{cases} w\left[1 - \varepsilon\left(a_0 + \sum_{n=1}^{\infty} c_n w^n\right)\right]; & (7a) \\ w\left[1 - \varepsilon\left(a_0 + \sum_{n=1}^{\infty} \bar{c}_n w^{-n}\right)\right]. & (7b) \end{cases}$$

This function also differs from precise function, reverse/inverse for f(z, C), to members of order not below ϵ^2.

However, there is possibility to construct on base of A. M. Lavrentyev's principle algorithm of representation of region, close to circle, on the average and inverse representation with any given accuracy.

Let us substitute into right side of (7a) as w values, which correspond to points of unit circle Γ. Inverse representation (7a) is translated Γ into duct/contour $C^{(1)}$, which differs from the initial duct/contour C by the smalls of order ϵ^2 or above.

Representation Γ on $C^{(1)}$ is precise. Problem now consists in constructing of the precise representation $C^{(1)}$ on Γ. We will use expression (6a), which let us represent in the form

$$z_1 = f_1(z, C^{(1)}) = z\left[1 + \varepsilon\left(a_0^{(1)} + \sum_{n=1}^{\infty} c_n^{(1)} z^n\right)\right], \tag{8}$$

where $\varepsilon a_0^{(1)}$ and $\varepsilon c_n^{(1)} = \varepsilon a_n^{(1)} - ib_n^{(1)}$ - Fourier coefficients the function $\delta^{(1)}(\varphi)$, which corresponds to duct/contour $C^{(1)}$.

Function (8) realizes representation of duct/contour $C^{(1)}$ on $C^{(2)}$, that differs from Γ by smalls of order is not below ϵ^2. After repeating the same operation m of times, we will obtain at the m step/pitch

$$z_m = f_m(z_{m-1}, C^{(m)}) = z_{m-1}\left[1 + \varepsilon\left(a_0^{(m)} + \sum_{n=1}^{\infty} c_n^m z_{m-1}^n\right)\right], \tag{9}$$

where coefficients $\varepsilon a_0^{(m)}$ and $\varepsilon c_n^{(m)}$ relate to function $\delta^m(\varphi)$, which corresponds to duct/contour $C^{(m)}$. Let us take for the distance from duct/contour $C^{(m)}$ to the unit circle Γ standard deviation Δ_{Rm}, where in view of the equality of Parseval

$$\Delta_{Rm}^2 = \frac{1}{2\pi}\int_0^{2\pi}[\delta^{(m)}(\varphi)]^2 d\varphi = \frac{\varepsilon^2}{2\pi}\left(a_0^{(m)2} + \sum_{n=1}^{\infty}|c_n^{(m)}|^2\right). \qquad (10)$$

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Relying on variation principle of M. A. Lavrentyev, it is ε_R possible to demonstrate that with any given small positive number with certain value of m=M will be fulfilled inequality $\varepsilon_{Rm} \leqslant \varepsilon_R$. sequence of mapping functions (9) descends according to norm Δ_{Rm}.

Analogously is constructed sequence of functions

$$z_m = f_m(z_{m-1}, C^{(m)}) = z_{m-1}\left[1 + \varepsilon\left(a_0^{(m)} + \sum_{n=1}^{\infty}\bar{c}_n^{(m)} z_{m-1}^{-n}\right)\right], \qquad (11)$$

corresponding to representation of exterior (expression (6b)). The set of functions f_m (9) or (11) (m=1, 2, ...) gives the precise in the sense indicated straight/direct representation w=F(z, $C^{(1)}$) closed domains (internal or external with respect to $C^{(1)}$), and functions $z=\Phi(w)$ (7) - precise inverse representations. We will subsequently call the appropriate recurrent procedure, described above, a R-procedure.

2. Recurrent algorithm of conformal mapping of arbitrary simply connected region on the average (T-procedure). Let in plane z be assigned arbitrary simply connected region D_2, limited by the piecewise-smooth duct/contour L_2. Is required to carry out conformal mapping D_2 region onto unit circle D_w of plane w with duct/contour Γ_w so that the given point O_1D_2 region would pass to the center of

this circle, and corresponding inverse representation.

Analogously is formed/shaped exterior problem (representation of region D_1 of plane z, external with respect to duct/contour L_1, onto exterior of unit circle of plane w and inverse representation). In this case the representation is determined with an accuracy to the arbitrary angle of rotation of duct/contour L_1 or L_2 around the axis, passing through point O, it is perpendicular to plane z.

Let us assume additionally that ducts/contours L_i are star with respect to point O (this assumption it is not fundamental and connected only with simplest version of construction of variation in majorant ducts/contours, described below).

Let us begin from exterior problem. Let us introduce auxiliary duct/contour in the form of the circle/circumference Γ_{10} of a radius ρ_{10} with the center in the beginning of coordinates O, inserted in duct/contour L_1 (in this case it is not excluded the presence in the circle/circumference Γ_{10} of common points with L_1).

We standardize variable z so as to obtain $\rho_{10}=1$ (Fig. 1a) let us designate $\delta_{10}(\varphi)$ variation in radius-vector of points of duct/contour L_1, which now we will consider positive, when it is directed in direction of external standard/normal, and let us determine as follows:

$$\delta_{10}(\varphi)=\varepsilon_{10}[r_{10}(\varphi)-1]; \ \varepsilon_{10}=\varepsilon_0\frac{\gamma_{10}}{\gamma_{10}+\varepsilon_0}; \qquad (12)$$
$$\gamma_{10}=[r_{10}(\varphi)]_{max}-1; \ 0\leqslant\varphi\leqslant 2\pi,$$

where $r_{10}(\varphi)\geqslant 1$ - radius-vector of arbitrary point of duct/contour L with beginning in wheelbarrow O; ϵ_0 - arbitrarily selective low parameter.

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The restrictions placed on duct/contour L_1, ensure uniqueness and continuity of function $\delta_{10}(\varphi)$, to which corresponds duct/contour C_{10}, close to the circle/circumference Γ_{10} (Fig. 1a). For the satisfaction to conditions (1) function $\delta_{10}(\varphi)$ can be subjected to further smoothing. Subsequently we will assume that this operation is carried out, and for all variations in condition (1) in question carried out.

We will use R-procedure, which let us register in new designations thus:

$$z=\Phi_{10}(z_1); \ z_1=F_{10}(C_{10}^{(1)}), \qquad (13)$$

by that based on expression (11), in which it is necessary to change sign before ϵ to opposite in accordance with (12). As a result we will obtain precise conformal mapping of duct/contour $C_{10}^{(1)}$, close to C_{10}, onto the unit circle of plane w and precise inverse representation (subscript it corresponds to the number of duct/contour). Duct/contour L_1 will pass by force (2.2) into the new duct/contour L_{11} with equation $z_1=F_{10}(\mathring{z}_{10}, C_{10}^{(1)})$, where $z=\mathring{z}_{10}(\varphi)$ - equation of duct/contour L_1.

Is put in L_{11} new auxiliary duct/contour Γ_{11} with center at point O_1, which is form O (Fig. 1~~B~~ b), which has perhaps common points with L_{11}, and we standardize variable z_1 so as to obtain unit radius in circle/circumference Γ_{11}.

Let us form variation $\delta_{11}(\varphi)$ radius-vector of points Γ_{11}, which satisfies conditions (1), analogous (12). to it will correspond duct/contour C_{11}, close to Γ_{11}.

Repeated application of R-procedure (13) makes it possible to construct precise representation of duct/contour $C_{11}^{(1)}$, close to C_{11}, onto exterior of unit circle of plane z_2 and corresponding inverse representation:

$$z_2=f_{11}(z_1, C_{11});\ z_1=\Phi_{11}(z_2);\ z_2=F_{11}(z_1, C_{11}^{(1)}). \qquad (14)$$

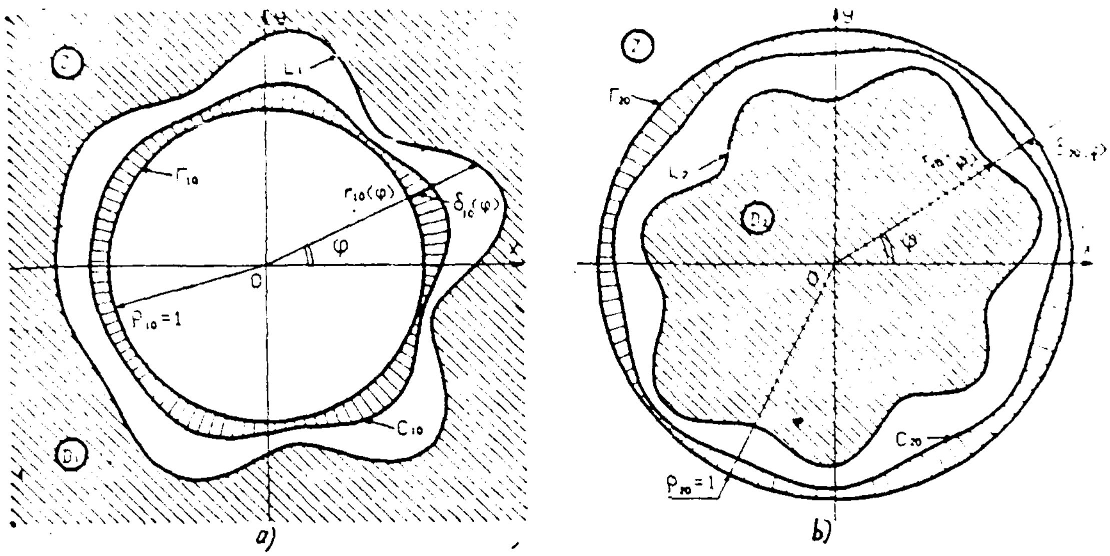

Fig. 1. Method of construction of variation for simply connected regions: a) exterior; b) interior.

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Set of enumerated operations forms the firsts of step/pitch of recurrent process

$$\begin{gathered} z_k = f_{1,k-1}(z_{k-1},\ C_{1,k-1}); \\ z_{k-1} = \Phi_{1,k-1}(z_k); \\ z_k = F_{1,k-1}\left(z_{k-1},\ C^{(1)}_{1,k-1}\right); \\ k=1,\ 2,\ldots;\ z_0 \equiv z;\ z_k = w \ \overset{(1)}{\text{при}}\ k = K_1, \end{gathered} \qquad (15)$$

Key: (1). with.

which leads to the sequence of the functions, which majorize duct/contour L_1. It is possible to show that with any given small positive number ε_T with the certain value of $k=K_1$ will be fulfilled inequality $\Delta_{Tk} \leqslant \varepsilon_T$, root-mean-square distance from duct/contour $C^{(1)}_{1,k}$ to the unit circle, i.e., occurs convergence according to norm Δ_{Tk}.

We will call described new recurrent procedure T-procedure.

Analogously is constructed T-procedure during solution internal of problem, i.e., representation D_2 region, internal with respect to boundary of L_2, onto unit circle. For its realization we will use circle/circumference Γ_{20} with the center at point O, which contains duct/contour L_2, so that Γ_{20} and L_2 can have common points, and we standardize z so that the radius Γ_{20} would become unit $\rho_{20}=1$ (see Fig 1b). Let us designate $\delta_{20}(\varphi)$ the variation in the radius-vector of the points of duct/contour Γ_{20}, positive during the variation in the direction of the internal standard/normal:

$$\delta_{20}(\varphi)=\varepsilon_{20}[1-r_{20}(\varphi)]\geqslant 0;\ \varepsilon_{20}=\varepsilon_0\frac{\gamma_{20}}{\gamma_{20}+\varepsilon_0};\quad (16)$$
$$\gamma_{20}=1-[r_{20}(\varphi)]_{\min};\ 0\leqslant\varphi\leqslant 2\pi,$$

where $r_{20}(\varphi)$ - the radius-vector of the arbitrary point of duct/contour L_2 with the beginning in center O of circle/circumference Γ_{20}.

Let us ascribe to function $\delta_{20}(\varphi)$ the same properties (1) that also $\delta_{10}(\varphi)$; then to it will correspond duct/contour C_{20}, close to Γ_{20}. Using a R-procedure (9), it is possible to construct a T-procedure for the internal problem according to the same diagram, as (16):

$$z_k=f_{2,k-1}(z_{k-1},\ C_{2,k-1});\ z_{k-1}=\Phi_{2,k-1}(z_k);\quad (17)$$
$$z_k=F_{2,k-1}(z_{k-1},\ C_{2,k-1});$$
$$k=1,\ 2,\ldots;\ z_0\equiv z;\ z_k=w\ \overset{(1)}{\text{при}}\ k=K_2.$$

Key: (1). with.

Duct/contour L_2 at first step/pitch of this process will pass into duct/contour with equation $z_2=F_{20}\,(\dot{z}_{20},\ C_{20}^{(1)})$, where $z=z_{20}{}^{\circ}(\varphi)$ - equation of duct/contour L_2.

It is possible to show that corresponding majorant sequence possesses the same property of convergence according to norm, that also generated by process of (15), i.e., with certain $k=K_2$ is fulfilled inequality $\Delta_{Tk}\leqslant\varepsilon_T$.

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By such form of function w(z) and z(w), generated by a T-procedure, realizing straight/direct and inverse transformations in both problems - internal and external, are obtained in the form of the recurrent sequences of Taylor series or Loran with the known coefficients.

3. Recurrent algorithm of conformal mapping of arbitrary doubly connected region onto annulus. Let us consider doubly connected region $D(C_1, C_2)$, limited by the closed curves, close to the concentric circles/circumferences Γ_1 and Γ_2 in the sense that are satisfied the conditions

$$\begin{gathered} |\delta_j(\varphi)| < \varepsilon h; \ |\delta_j'(\varphi)| < \varepsilon h; \ |\delta_j''(\varphi)| < \varepsilon h; \\ \delta_1(\varphi) = r_1(\varphi) - \rho_{10}; \ \delta_2(\varphi) = 1 - r_2(\varphi) \ (0 \leqslant \varphi \leqslant 2\pi); \\ r_1(\varphi)\,|\,_{r_1} = \rho_{10} < 1; \ r_2(\varphi)\,|\,_{r_2} = \rho_{20} = 1; \ h = 1 - \rho_{10}, \end{gathered} \tag{18}$$

where φ - vectorial angle; r_j - radius-vectors of the points of lines C_1, C_2, carried out from the overall center of circle Γ_1, Γ_2: ϵ - arbitrary small positive number.

Let $w=f(z, C_1, C_2)$ - function, which reflects region D of plane z to annulus of plane w with unit external radius, moreover

$$f(z, C_1, C_2)\,|_{z=0} = 0; \ f'(z, C_1, C_2)\,|_{z=0} > 0. \tag{19}$$

Variation $\delta_1(\varphi)$ we will, in contrast to (7), consider positive during variation Γ_1 in direction of external standard/normal; and $\delta_2(\varphi)$ - during variation Γ_2 in direction of internal standard/normal.

To variation principle of M. A. Lavrentyev it is possible in this case to give following form, analogous (3).

Under the conditions (18) function

$$w=f(z,\ C_1,\ C_2)=z[1-J_1(z)+J_2(z)];$$
$$J_j(z)=\frac{1}{2\pi i}\oint\limits_{\Gamma_j} u_j(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}, \qquad (20)$$

where $\zeta=e^{i\varphi};\ u_j(\zeta)=\delta_j[\varphi(\zeta)];\ j=1,\ 2$ and duct/contour $(\Gamma_1,\ \Gamma_2)$ is bypassed so that domain of definition z remains to the left, differs to the left, differs from function, that realizes conformal mapping of region $D(C_1,\ C_2)$ by annulus D_w, to smalls of second order relative to ϵ.

Mapping function w (8) can be represented by analogy with (6a, b) in the form

$$w=f(z,\ C_1,\ C_2)=z\left[1-\varepsilon\left(a_0+\sum_{n=1}^{\infty}c_{1n}z^{-n}\right)+\varepsilon\left(a_{20}+\sum_{n=1}^{\infty}c_{2n}z^{n}\right)\right], \qquad (21)$$

where $c_{jn}=a_{jn}-ib_{jn};\ \varepsilon a_{jn},\ \varepsilon a_{jn},\ \varepsilon b_{jn}$ - coefficients of expansion of functions $\delta_j(\varphi)$ in Fourier series in segment $0\le\varphi\le2\pi$.

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Let now in plane z be assigned region in the form of ring D, limited by two locked nonintersecting piecewise-smooth lines L_1 and

L_2. It is necessary to carry out conformal mapping D region onto the annulus of plane w so that the line L_2 would pass into unit circle Γ_{w2}, line L_1 - into circle/circumference Γ_{w1} of a smaller radius, and the given point O of plane z - to the center of these circles/circumferences, and inverse representation. The corresponding representations are determined as in p. 2, with an accuracy to the arbitrary angle of rotation D region relative to point O. This problem is reduced to the problem of the previous section, if we use function (6) and R- and T-procedures.

Let us place within duct/contour L_1 circle/circumference Γ_{10} of radius ρ_{10} and will standardize variable z so as to obtain $\rho_{10}=1$ (Fig. 2a). Let us assume in (3), (6) $\delta_2(\varphi)\equiv 0$, $a_{20}=0$, $c_{2n}\equiv 0$ and let us do the first step/pitch of external recurrent process exactly as in the case of simply connected region, according to diagram (13), beginning from the construction $\delta_{10}(\varphi)$ (see Fig. 2a). Difference will be only in the fact that upon transfer from plane z into plane z_1 will be changed not only line L_1, which will become L_{11} with equation $z_1=F_{10}\,(\overset{\circ}{z}_{10},\ C_{10}^{(1)})$ (where $z=\overset{\circ}{z}_{10}(\varphi)$ - equation of line L_1), but also line L_2. We will assume that the parameter ϵ_0 is selected sufficiently to low in comparison with the minimum width of ring D so that the line L_2 upon transfer in L_{21} in accordance with equation $z_1=F_{10}\,(\overset{\circ}{z}_{20},\ C_{10}^{(1)})$ (where $z=\overset{\circ}{z}_{20}(\varphi)$ - equation of duct/contour L_2) would be changed to the values of the higher order of smallness relative to ϵ_{10}, than L_1 upon transfer in L_{11}. For the realization of the second step/pitch let us lead around line L_{21} the circle/circumference Γ_{21} of a radius ρ_{21} with the center at point O_1 (form O in plane z_1) and will standardize the variable z_1 so as to obtain $\rho_{21}=1$ (Fig. 2b). Let us assume into (3) $\delta_1(\varphi)\equiv 0$ and respectively in (6) $a_{10}=0$; $\bar{c}_{1n}\equiv 0$, and it is realized the procedure, analogous (16), beginning from the construction of variation $\delta_{21}(\varphi)$ (see Fig. 2b).

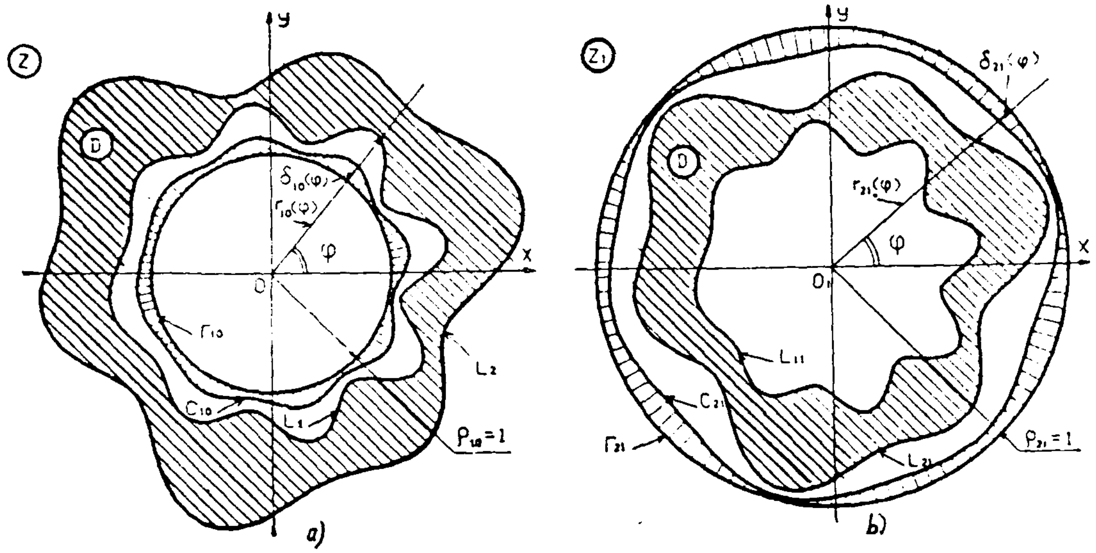

Fig. 2. Method of construction of variations for doubly connected regions: a) internal boundary; b) outer edge.

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It is assumed that line L_{11} upon transfer in L_{12} with equation $z_2=F_{21}(\dot{z}_{11}, C_{21}^{(1)})$ varies in view of selection ϵ_0; small of higher order relative to ϵ_{20}, than line L_{21} upon transfer in L_{22} with equation $z_2=F_{21}(\mathring{z}_{21}, C_{21}^{(1)})$. Further steps/pitches are constructed according to the same diagram. Here, thus, are realized two sequences of the functions, which majorize from two sides ring D, one of which is represented by Taylor series, and another with Laurent series. A question about its convergence is not trivial; the latter depends on the possibility to select the sufficiently low value ϵ_0, which does not lead to the loss of calculating stability due to a large required quantity of steps/pitches K for the completion of external recurrent process with the assigned accuracy, determined by parameter ε_T.

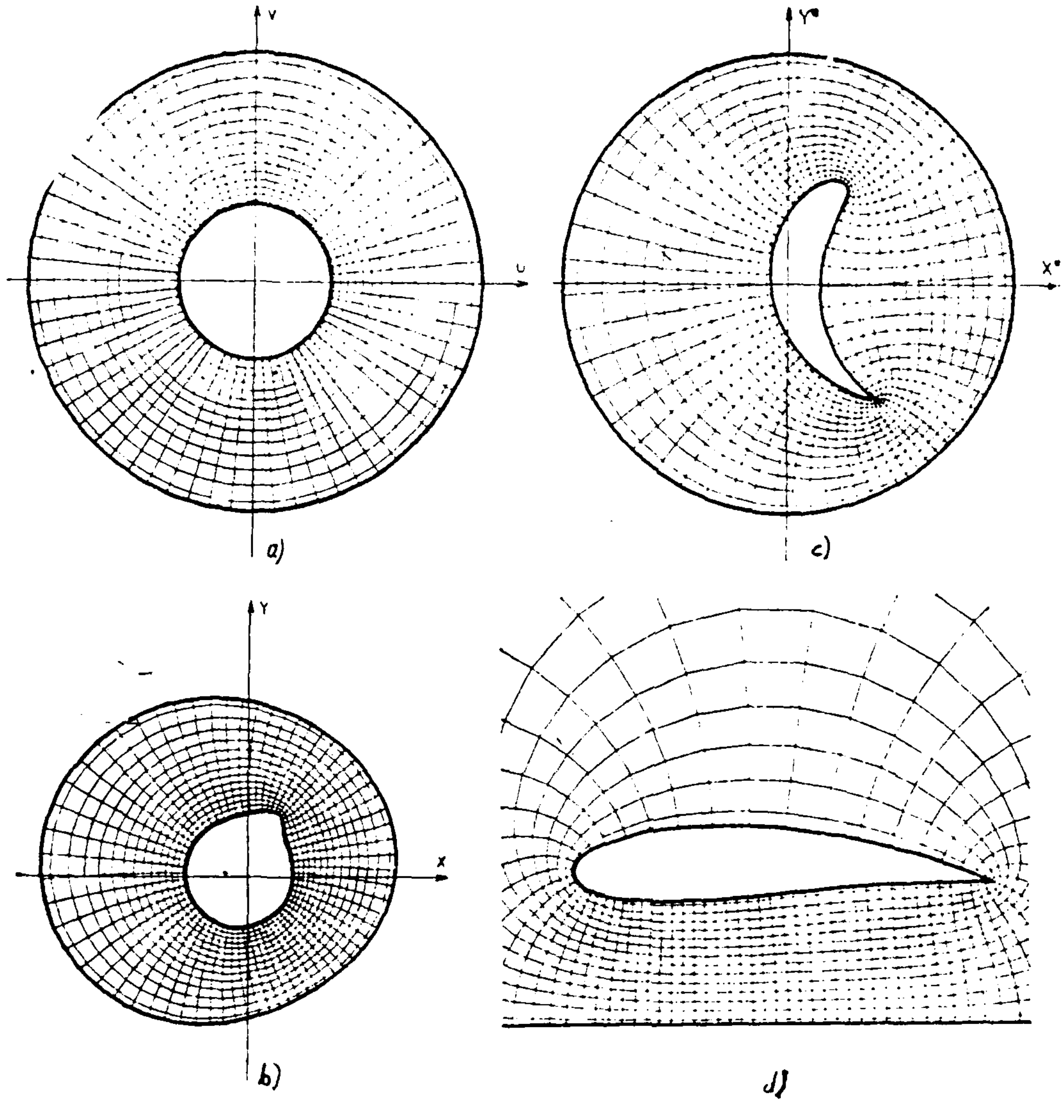

Fig. 3. Circulation "round wing" near earth/ground; stages of conformal mapping with the help of RT-algorithm of region cnto annulus: a) grid of polar coordinates in annulus, which is form of external with respect to profile/airfoil region; b) region before latter/last conversion of N. Ye. Zhukovskiy; c) region before reverse/inverse linear-fractional conversion; d) flow line and equipotential lines (grid, conformal-equivalent grid Fig. a).

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4. Some external and internal problems of hydrodynamics. The described recurrent algorithm was realized in the form of program in the language FORTRAN-4 for the cases of singly connected and doubly connected regions. In this case it was possible with the value of a certain complication of formula for the construction of a variation in the duct/contour of inscribed and circumscribed circles to remove/take the requirement of stellar regions.

For resolution in Fourier series was used program of rapid Fourier transform with use of 512 points of duct/contour and maximum number of members of series/row 129 (N=64). Parameters ε_R, ε_T, ε_0 were accepted during calculations by the following: $\varepsilon_R=\varepsilon_T=2\cdot10^{-5}$; $\varepsilon_0=0,2$. In this case the number of steps/pitches of internal cycle (R-procedure) did not exceed M=3, but external (T-procedure) K=50.

Examples of solution with the help of RT-algorithm of hydrodynamic problems, which are reduced to conformal mapping of fairly complicated regions on the average or annulus are given below.

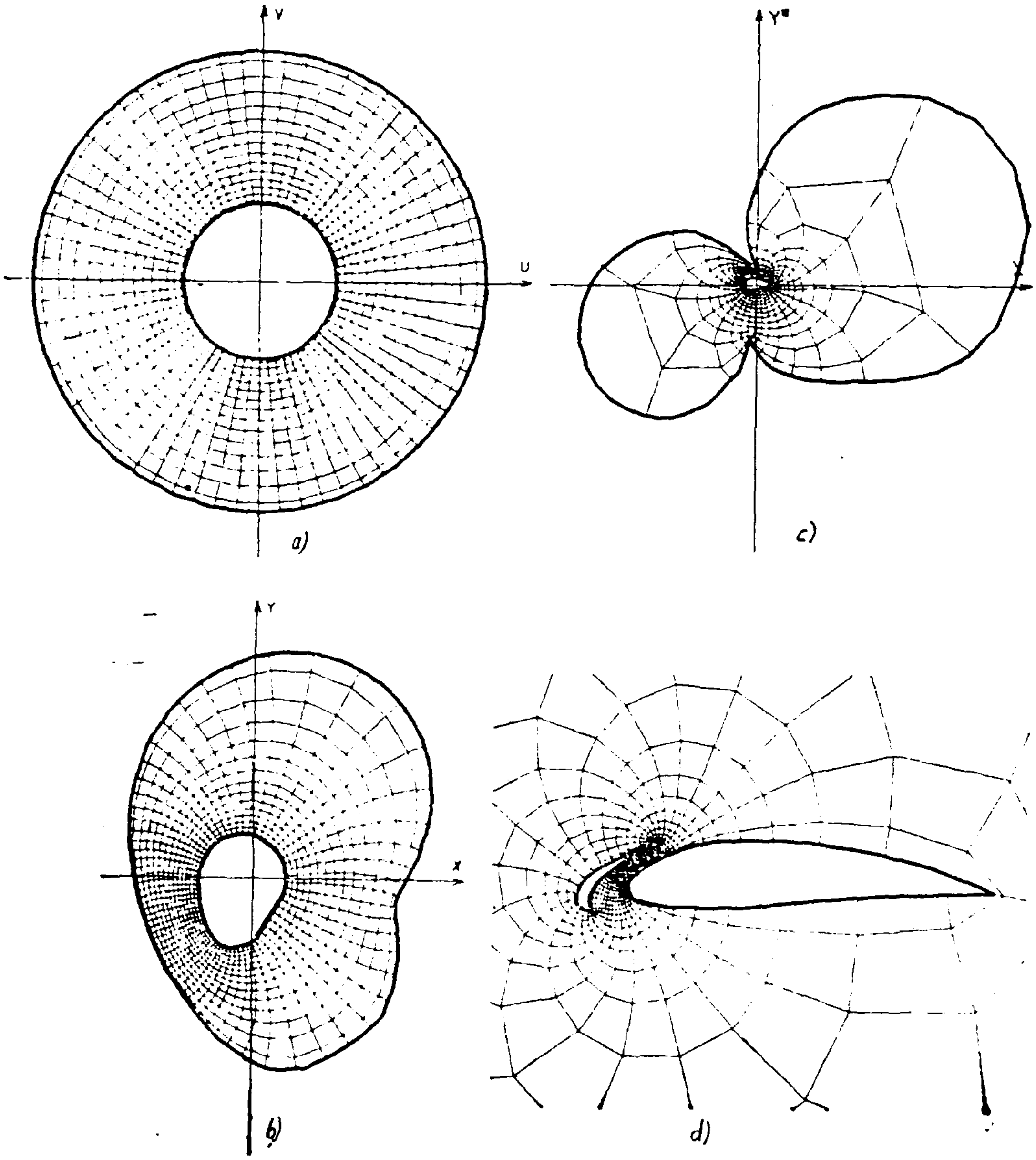

Fig. 4. Circulation "round wing" with slat (designation the same as in Fig. 3).

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Fig. 3, 4 present mapping onto annulus of regions, external with respect to wing near Earth and wing with slat, that gives one of possible flow patterns of corresponding profiles/airfoils of ideal fluid. Transition/transfer from the infinite region to the final was realized with the help of the linear-fractional conversion. Additionally, before beginning the work of a RT-algorithm, were used after conducting of the corresponding sections/cuts inverse transformations of N. Ye. Zhukovskiy, "straightening" flattened ducts/contours.

Fig. 3d and 4d shows final conformal-equivalent grids, elements of which are flow lines and equipotential lines during circulation flow around corresponding profiles/airfoils of flow of ideal incompressible fluid.

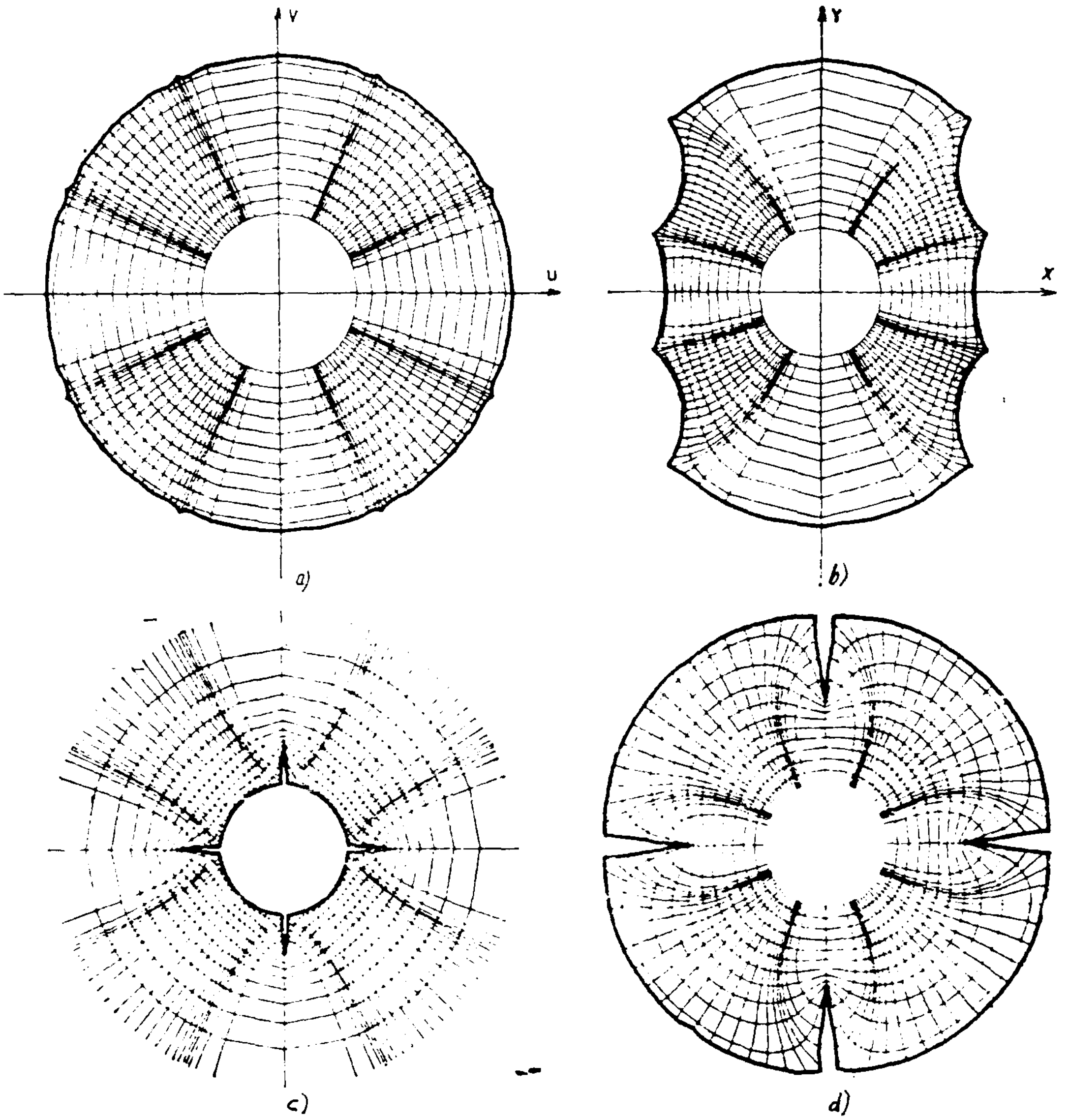

Fig. 5. Circulation flow around circular cylinder with edges/fins: a, b) the same as in Fig. 3a and 4a; c) flow line and equipotential lines in the case of external edges/fins; d) the same in the case of internal edges/fins.

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Fig. 5c d shows patterns of lines of current and equipotential lines during circulation flow around cylinder with external and internal radial edges/fins.

Fig. 6 depicts flow lines and equipotential lines, which correspond to motion of atmosphere of Jupiter in vicinity of "red spot" in presence of two point vortices/eddies [14].

5. Problem about motion of liquid in cylindrical cavities. Let us consider as an example doubly connected cylindrical cavity. In the case, when the free surface of liquid is perpendicular to the longitudinal axis of cylinder, the initial three-dimensional boundary-value problems, connected with the motion of liquid, are reduced by the method of separation of variables to the two-dimensional problems for S region, which is the cross section of the column of liquid [5]. The latter are solved by the method of the expansion of the unknown function in the Laurent series (heterogeneous problem) or by the method of Bubnov-Galerkin (uniform problem), if it is possible to conformally map doubly connected region S of the plane of complex variable z=x+iy to the annulus in plane w=u+iv.

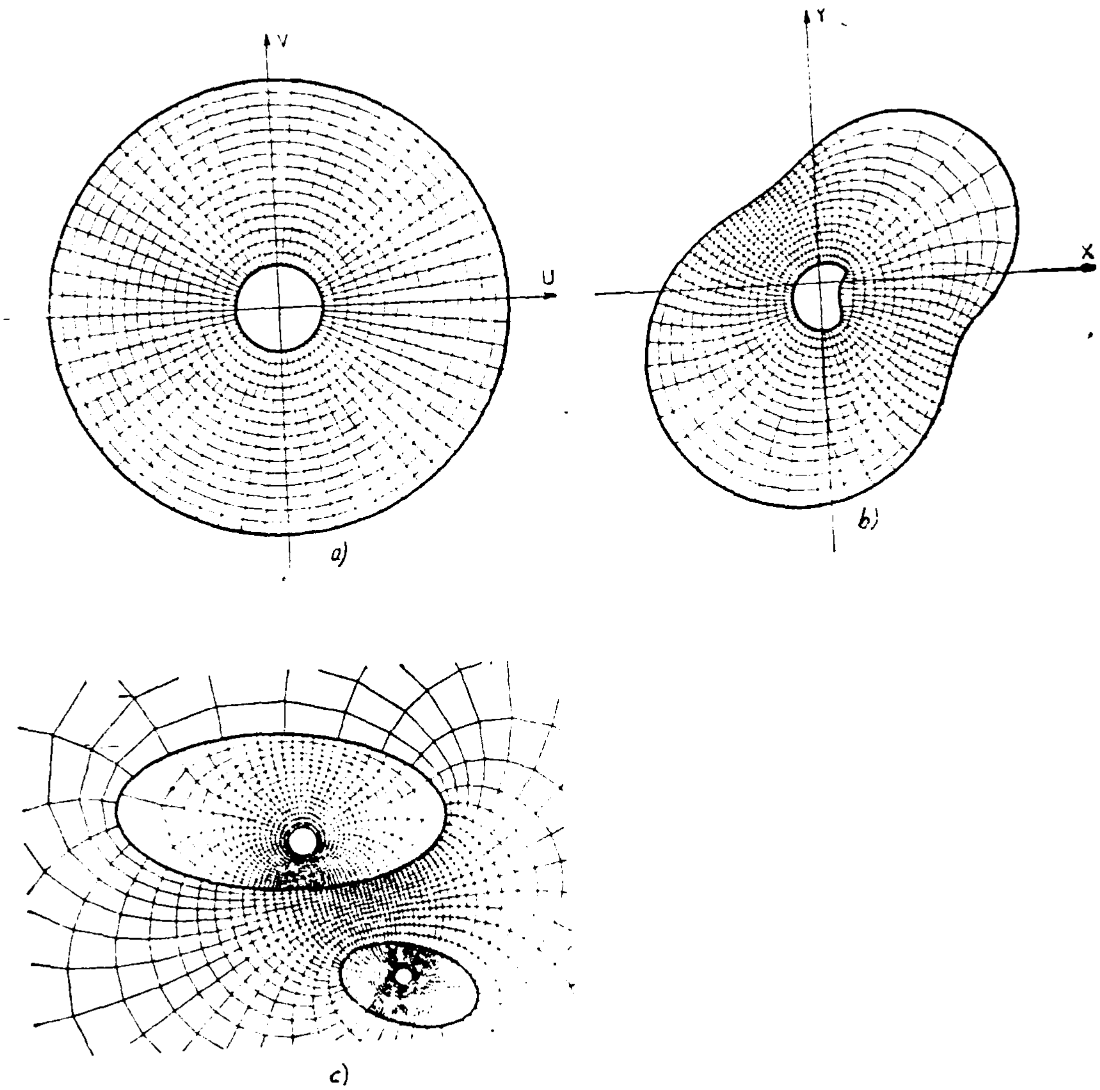

Fig. 6. Idealized diagram of flows in the atmosphere of Jupiter in vicinity of "red spot" (larger from closed domains), constructed on basis of photographs KA "Voyager" [14]: a) the same as in Fig. 3a and 4a; b) the same as in Fig. 3c and 4c; c) pattern of lines of current and equipotential lines (grid b, conformal-equivalent to grid Fig. a).

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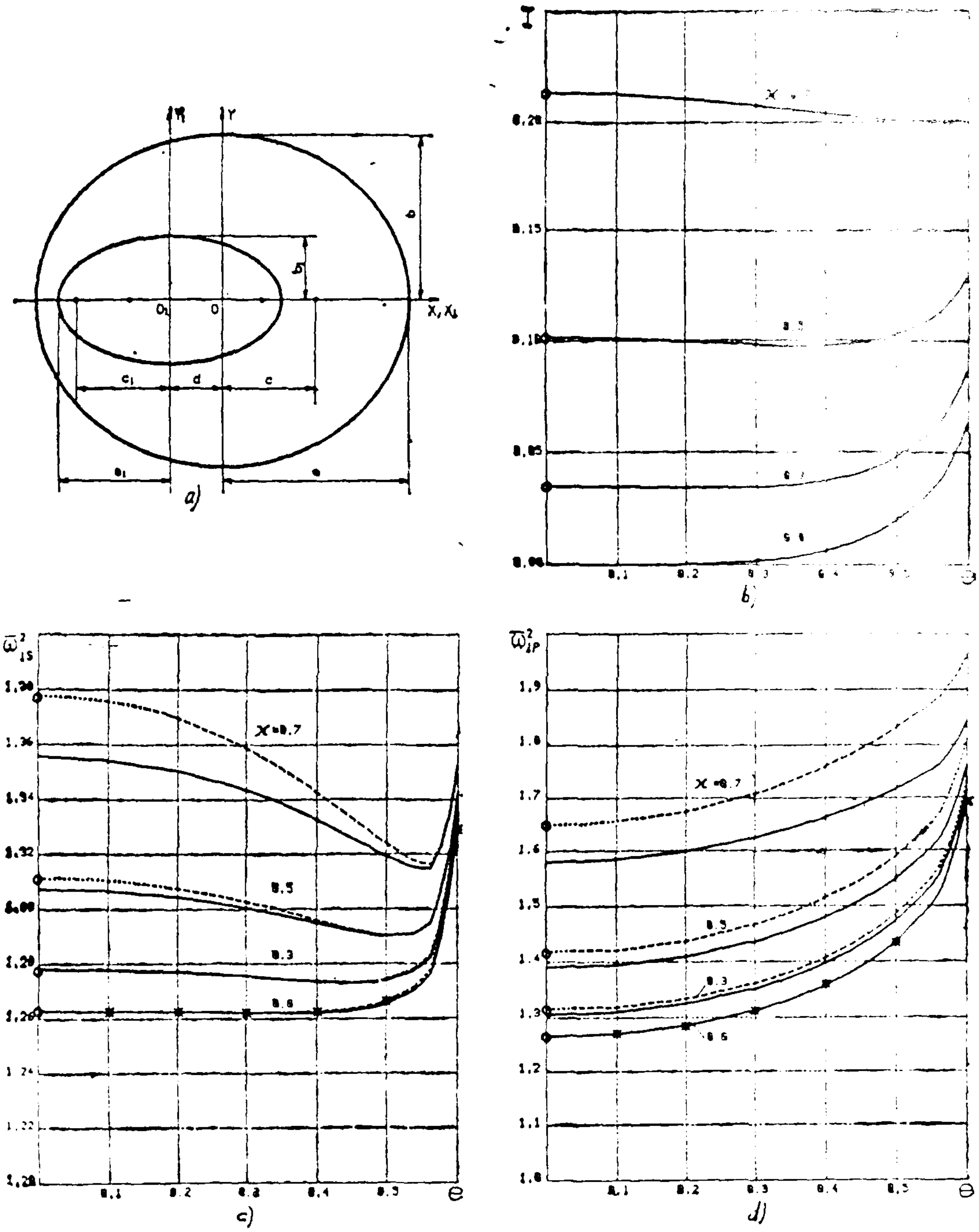

Fig. 7. Dimensionless parameters, which correspond to motion of liquid in section, formed by eccentric elliptical cylinders: a) cross section of column of liquid; b) dimensionless connected moment of

inertia of layer of liquid of unit thickness relative to longitudinal axis of jacket; c) square of dimensionless frequency of first tone of unsymmetric oscillations of infinitely deep liquid in plane, passing through minor axis of ellipse; d) the same in plane, passing through transverse; —— - calculation on basis of use of RT-algorithm by Bubnov-Galerkin method with four coordinate functions; - - - - the same with one function; ○. * - data from works [1] and [6].

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In the case of uniform problem is realized further representation of this annulus onto rectangle $S'(0\leqslant\xi\leqslant\xi_0;\ 0\leqslant\eta\leqslant2\pi)$ in plane $w=e^{\zeta}$, $\zeta=\xi+i\eta$ with the help of exponential function $w=e^{\zeta}$.

All parts of solution are described in works [1, 5, 6], and we on them do not stop.

Let us consider for example S region in the form of eccentric elliptical ring (Fig. 7c). As the dimensionless parameters let us introduce the following:

$$e=\frac{c}{a};\ \delta=\frac{a_1}{a};\ \varepsilon=\frac{d}{a};\ \varkappa=\frac{\varepsilon}{1-\delta}, \tag{22}$$

from which first three are independent variables (eccentricity e, the relationship/ratio of semimajor axes δ, the relative shift of the centers of internal and external ellipses ϵ). With $\epsilon=0$, S region passes into the elliptical ring, limited by the confocal ellipses: when e=0 - into the eccentric annulus, with $\epsilon=0$, e=0 is degenerated into the annulus.

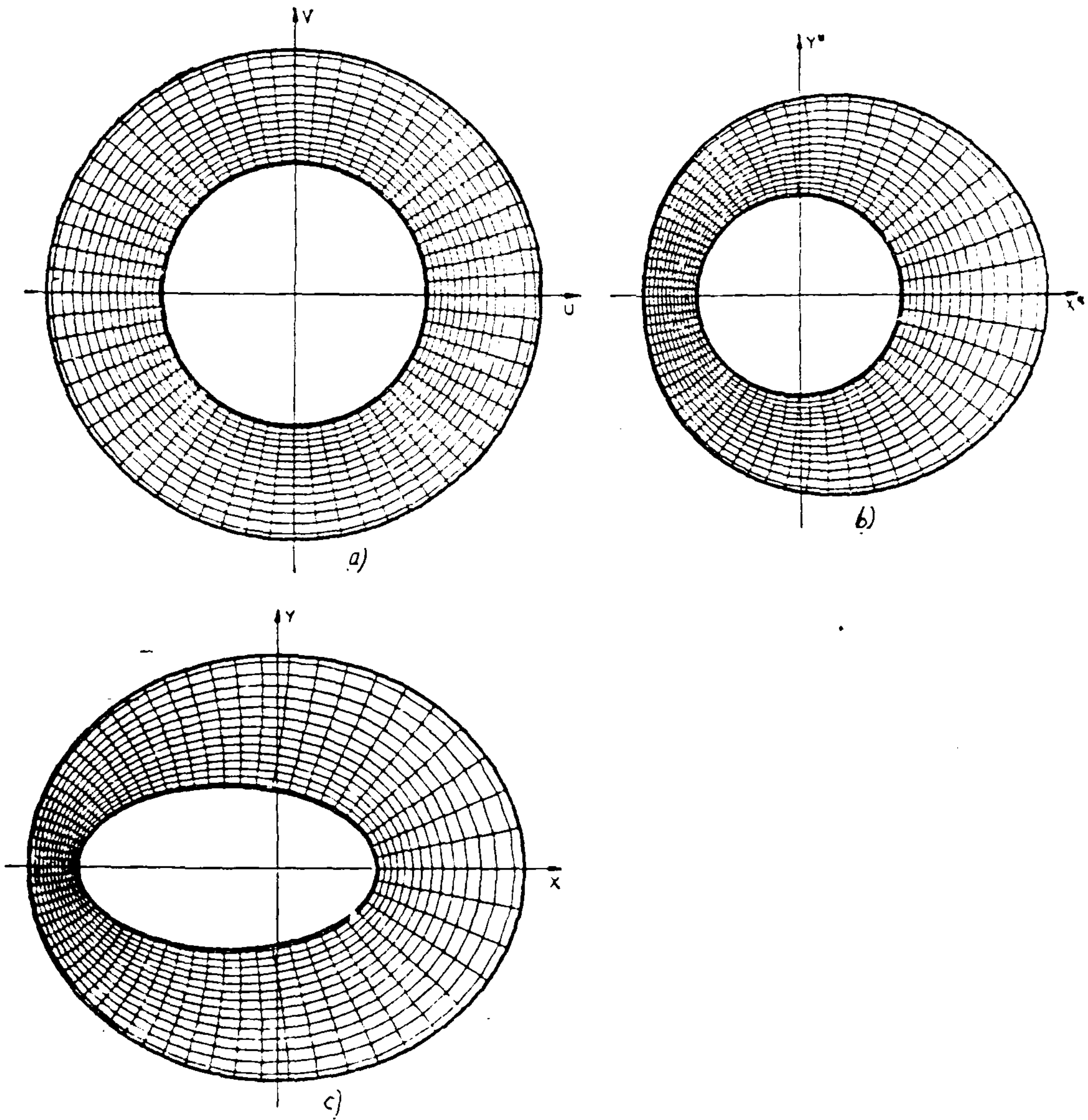

Fig. 8. Motion of liquid in section, formed by eccentric elliptical cylinders; stages of conformal mapping with the help of RT-algorithm of eccentric elliptical ring onto annulus (designation the same as in Fig. 3 and 4).

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Fig. 7b depicts values of dimensionless attachment/connection of moment of inertia of $\bar{I}$ layer of liquid of unit thickness relative to axis, passing through center of external ellipse. In Fig. 7c d - the value of the squares of the dimensionless frequencies of the first tone of the unsymmetric oscillations of infinitely deep liquid in the plane of the symmetry of section $(\bar{\omega}_{1p}^2)$ and in perpendicular plane $(\bar{\omega}_{1s}^2)$.

Following coordinate functions were used for calculation:

$$\gamma_{kp}=C_k^{\circ}+\cos(\mu-1)\bar{\xi}\cos\nu\eta;$$
$$\gamma_{ks}=\cos(\mu-1)\bar{\xi}\sin(2\nu-1)\eta\ (\mu,\ \nu=1,2,3,\ldots), \qquad (23)$$

where
$$C_k^{\circ}=-\frac{1}{S}\int_{S'}|z'(\zeta)|^2\cos(\mu-1)\bar{\xi}\cos\nu\eta\, d\xi\, d\eta;$$
$$S=\int_{S'}|z'(\zeta)|^2 d\xi\, d\eta;\ \bar{\xi}=\frac{\xi}{\xi_0}; \qquad (24)$$

moreover to each combination of indices μ, ν corresponds one value k.

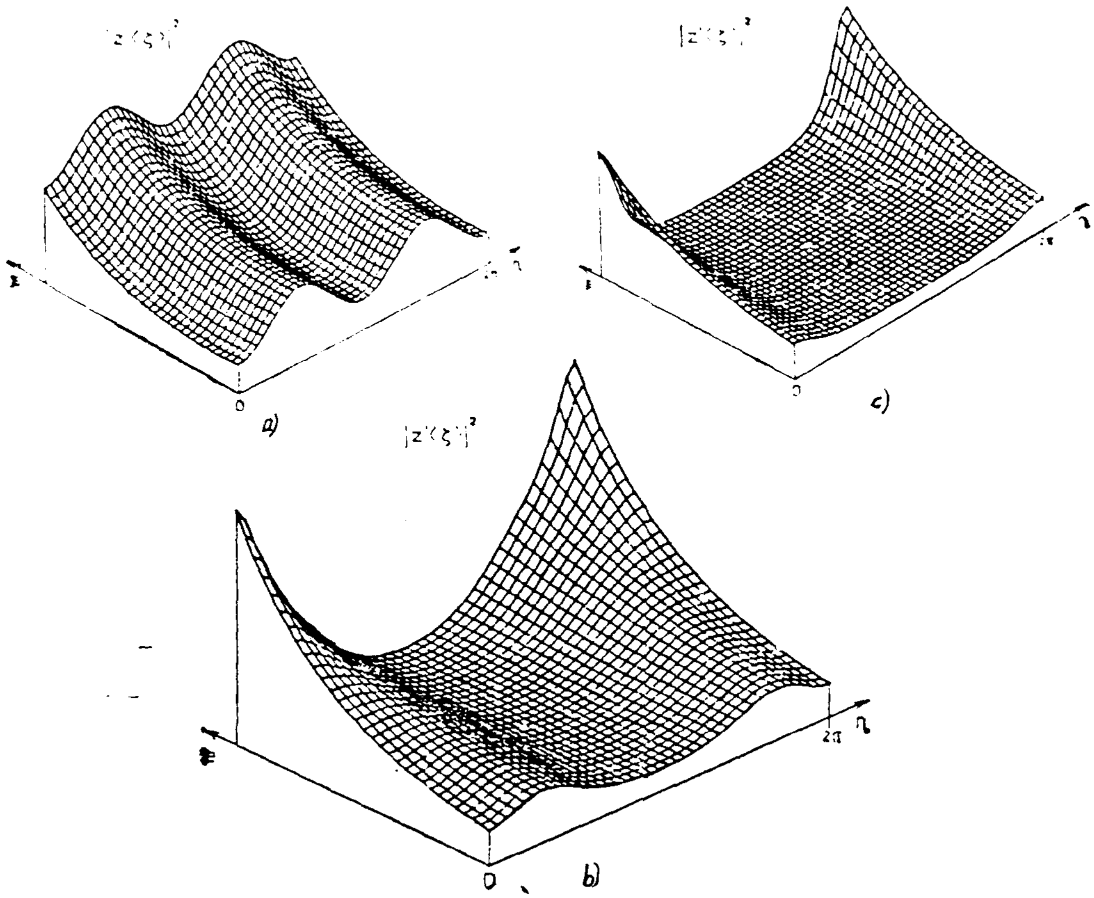

Fig. 9. Relief of jacobian of conversion $|z'(\zeta)|^2$ of initial region to rectangle of plane ζ above plane ζ: a) confocal elliptical ring; b) eccentric elliptical ring; c) eccentric annulus.

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As significant dimension is accepted value $l=(ab)^{1/2}$; parameter δ in all cases it is identical: $\delta=0.6$; maximum number of steps/pitches of T-procedure comprised k=10.

Together with values, obtained by method, described into [5], with use of RT-algorithm of conformal mapping, Fig. 7 gives appropriate values for confocal elliptical ring and eccentric annulus from works [6] and [1].

With critical value of e=0.6 internal ellipse is degenerated in line segment. In this case precise value $\bar{\omega}_{1p}^2$, which corresponds to $\kappa=0$, must not depend on the parameter κ. Actual difference $\bar{\omega}_{1p}^2$ with $\kappa \neq 0$ from $\bar{\omega}_{1p}^2$ with $\kappa=0$ (Fig. 7d), which increases with an increase κ, indicates the need for use during the calculation of the larger number of coordinate functions.

Fig. 8 depicts some stages of conformal mapping with the help of RT-algorithm of eccentric elliptical ring onto annulus, on Fig. 9 - relief of jacobian $|z'(\zeta)|^2$ of conversion $z=z(\zeta)$ above rectangle S' of plane $\zeta=\xi=i\eta$, to which is mapped initial doubly connected region S of plane $z=x+iy$.

Given examples give sufficient representation about great possibilities, which open/disclose RT-algorithm of conformal mapping for solving two-dimensional boundary-value problems, including problems of dynamics of liquid, partially filling cavity of solid body.

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OSCILLATIONS OF CYLINDRICAL CONTAINER WITH THE LIQUID AND THE ELASTIC RADIAL BAFFLES.

I. M. Mel'nikova, G. N. Mikishev.

Effect of elastic radial baffles on oscillations of cylindrical container, partially filled with liquid, is investigated. The approximate linearized equations, which describe the joint oscillations of vessel, the liquids also of the baffles, which are solved numerically by method of successive approximations, are given. It is shown that, beginning from a certain value, the relations of the partial frequencies of the system, the effect of the elasticity of baffles virtually are reduced to an increase in damping the oscillations of liquid. The dependence of equivalent attenuation factors on the relation of partial frequencies and air-gap clearance between the wall of vessel and the baffles is obtained.

Calculated results will agree well with results of carried out experiment.

Work [1] examines problem about oscillations of liquid in cylindrical container, which contains elastic radial baffles. The initial equations, which describe the joint oscillations of liquid and baffles, are obtained approximately with the use of a perturbation method [3, 4].

Here in the same setting is examined more general problem: about oscillations of cylindrical container with liquid and elastic radial baffles. Are given also some results of experimental research.

1. Let in field of mass forces, whose acceleration j, is located rigid vessel in the form of straight/direct circular cylinder with flat/plane bottom of radius r_0, partially filled with liquid of density ρ to level h and containing N of radial baffles in the form of flexible longitudinal bands of small width, stretched near wall at equidistance from each other. Mass of vessel with liquid M, the length of baffles l, width b $(b \ll l)$, thickness τ, the linear mass m°, the tensile stress of each band T. Baffles are completely immersed in liquid ($l \leq h$).

Vessel completes low progressive/forward oscillations under action of transverse force $P = P_0 \sin \omega t$. Let us compile the linearized equations of the joint oscillations of vessel, liquid and baffles. Most simply this to do by addition to the known equations for the vessel without the baffles (see [3]) equations of transverse vibrations of flexible bands and introduction to the right sides of the generalized forces, caused by baffles. During the compilation of equations we will be bounded to the account only of the fundamental tones of the oscillations of liquid and baffles. The appropriate generalized coordinates let us designate through s(t) and q(t). Let us connect moving coordinate system with the center of the bottom of vessel, after directing axis Ox opposite to vector j.

It is not difficult to obtain equations of transverse vibrations of flexible bands, identifying band with stretched string.

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Let us determine generalized forces approximately, after using the harmonically linearized dependence for the flow forces, which functions per unit of the length of rigid baffle [4]:

$$F_i = m\dot{v}_i + \Pi \rho b^{3/2} \omega^{1/2} v_{0i}^{1/2} v_i. \qquad (1)$$

Here

$m = \frac{\pi}{2} \rho b^2$ — linear apparent additional mass of baffle; v_i — normal to baffle component of velocity of liquid; i - number of baffle;

$$\Pi = 19{,}1\,(1 + 0{,}4\,\mathrm{e}^{0{,}138 x'/b}),\ x' = x - h. \qquad (2)$$

For normal to baffle component of velocity let us take

$$v_i = \dot{s}\,\frac{\partial \varphi}{\partial \nu_i} - q_i f_i, \qquad (3)$$

where φ - eigenfunction of uniform boundary-value problem about oscillations of liquid in cylinder without baffles;

$$\frac{\partial\varphi}{\partial\nu_i}=a\sin\theta_i,\ a=\frac{\mathrm{ch}\,(\xi_1 x/r_0)}{\xi_1\,\mathrm{sh}\,(\xi_1 h/r_0)};$$

θ_i — angle between plane of vibration and i baffle;

$f_i=f\sin\theta_i$, $f=\sin(\pi x/l)$ — natural mode of vibration of band.

Lowering intermediate unpackings/facings, let us register unknown linearized system of equations in the following form:

$$\begin{gathered}\ddot{u}+a_{us}\ddot{s}+\sum_{i=1}^{N}a_{uqi}\ddot{q}_i=\bar{P}_0\sin\omega t;\\ \ddot{s}+\varepsilon_s\dot{s}+\omega_s^2 s+a_{su}\ddot{u}+\sum_{i=1}^{N}a_{sqi}\ddot{q}_i+\sum_{i=1}^{N}\varepsilon_{sqi}\dot{q}_i=0;\\ \ddot{q}_i+\varepsilon_{qi}\dot{q}_i+\omega_q^2 q_i+a_{qiu}\ddot{u}+a_{qis}\ddot{s}+\varepsilon_{qis}\dot{s}=0\ (i=1,\ 2,\dots,\ N).\end{gathered}\tag{4}$$

Coefficients of equations are determined by following expressions:

$$\begin{gathered}\omega_s=\omega_s^{\circ}\sqrt{\frac{\mu}{\mu+\mu'}};\ \omega_q=\sqrt{\frac{T}{m^{\circ}+m}};\ a_{us}=\frac{\lambda}{M};\ a_{su}=\frac{\lambda}{\mu+\mu'};\\ a_{uqi}=\frac{2m^{\circ}l}{\pi M}\sin^2\theta_i;\ a_{qiu}=\frac{2m^{\circ}l}{\pi m'};\ a_{sqi}=-\frac{m}{\mu+\mu'}\int_0^l\alpha f\,dx\sin^2\theta_i;\\ a_{qis}=\frac{m}{m'}\int_0^l\alpha f\,dx;\ \varepsilon_s=\varepsilon_s^{\circ}+\frac{K}{\mu+\mu'}\sum_{i=1}^{N}|\sin\theta_i|^{5/2}\int_0^l\Pi\alpha^2\chi_{0i}^{1/2}dx;\\ \varepsilon_{qi}=\frac{K}{m'}\int_0^l\Pi f^2\chi_{0i}^{1/2}dx\,|\sin\theta_i|^{1/2};\ \varepsilon_{sqi}=-\frac{K}{\mu+\mu'}\int_0^l\Pi\alpha f\chi_{0i}^{1/2}\times\\ \times dx\,|\sin\theta_i|^{5/2};\end{gathered}\tag{5}$$

$$\varepsilon_{q i s}=-\frac{K}{m'}\int_{0}^{l}\Pi\alpha f\chi_{0i}^{1/2}dx\,|\sin\theta_i|^{1/2}.$$

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Here ω_s°, s_s°, λ, μ — hydrodynamic coefficients, which correspond to cylinder without baffles;

$$K=\left(\frac{2}{3\pi}\right)^{3/2}\rho\omega b^{-1/2};\ \mu'=\frac{Nm}{4\xi_1^2 sh^2(\xi_1 l/r_0)}\left[\frac{r_0}{2\xi_1}sh(2\xi_1 l/r_0)+l\right];$$

$$m'=\frac{l}{2}(m^{\circ}+m);\ \chi_{0i}=(\alpha^2 s_0^2+f^2q_{0i}^2-2\alpha f s_0 q_{0i}\cos\gamma_{sqi})^{1/2};$$

γ_{sq} — phase difference between generalized coordinates s and q_i.

2. System of equations (4) was solved numerically by method of successive approximations with use by computer(s). In this case the effect on the natural frequency of the baffles of the further chain forces, which appear during the oscillations, was taken into consideration.

Fig. 1 gives results of calculation of amplitude frequency characteristic of system according to coordinate u, in reference to P_0, obtained with different values of parameter $\varkappa=\omega_q/\omega_s$. Calculation is carried out with following initial data: M=56.6 kg; $P_0=0{,}313$ N; r_0=175 mm; l=365; b=58 and τ=0.2 mm; $\rho=10^3$ kg/m^3; h=l.

Case κ=100 virtually coincides with case of absolutely rigid baffles. The value of resonance peak with κ=100 is equal to 0.835 mm/N. With the decrease κ resonance peak is reduced. The frequencies of minimum and maximum of amplitude characteristic are changed weakly. With κ=1.34 becomes noticeable the second resonance peak, caused by the oscillations of baffles.

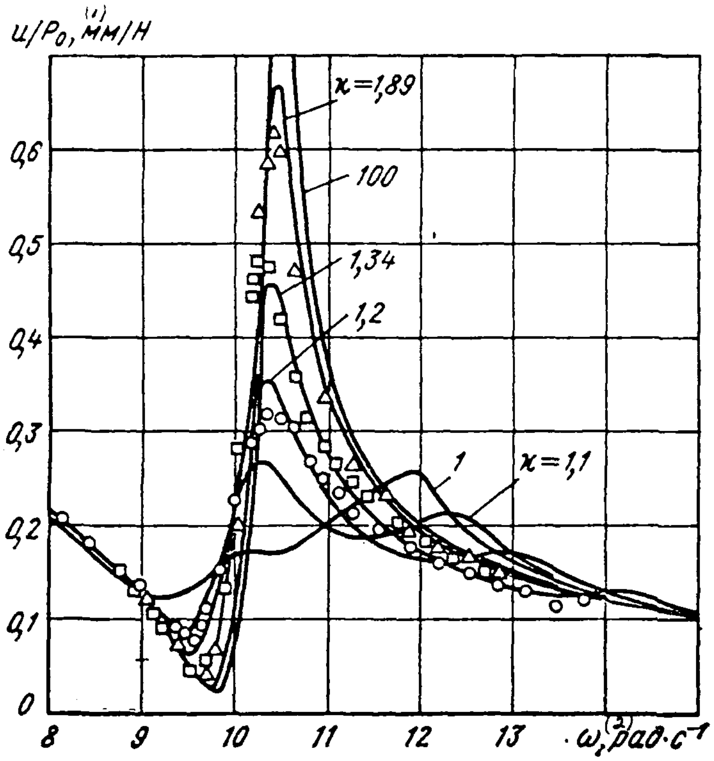

Fig. 1. Amplitude frequency system characteristics on coordinate u, in reference to P_0, with different values of parameter κ: —— - calculation; △□○ - experiment.

Key: (1). mm/N. (2). rad· s^{-1}.

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However, its value up to the value κ=1.2 considerably less than the first (basic) peak. As calculations showed, with an increase in the amplitude of external force the second peak virtually vanishes.

Thus, with values $\kappa \geq 1.2$ considered mechanical system behaves as system with two degrees of freedom (vessel - liquid). The decrease of the value of basic resonance peak is equivalent to an increase in damping the oscillations of liquid.

About this special feature of system testifies also phase response, which here is not given.

Major resonance continues to be reduced with further approximation/approach of parameter κ to 1, and amplitude characteristic is distorted by proximity of second resonance, which with κ=1 begins to prevail over the first.

When effect of elastic baffles in practice is reduced to decrease of resonance peak, amplitude characteristics can be used for determining equivalent damping coefficient of liquid from following formula:

$$\varepsilon_s^* = \frac{\omega_*^2 - \omega_0^2}{\omega^* \left[\left(u_0^* \omega_*^2 M / P_0 \right)^2 - 1 \right]^{1/2}}, \qquad (6)$$

where ω_0, ω^* - frequencies of minimum and maximum of amplitude characteristic; u_0^* -- value of resonance peak. This formula is obtained from the expression for the amplitude characteristics of linear system with two degrees of freedom when $\omega = \omega_*$ ($\omega_s \approx \omega_0$) [3].

Fig. 2 gives dependence of dimensionless equivalent attenuation factor $\delta_* = \pi \varepsilon_s^* / \omega_0$ on relative amplitude of oscillations $\bar{s}_0 = s_0 / r_0$.

In the case of elastic baffles equivalent attenuation factor is considerably higher than in the case of rigid baffles (κ=100); so, with κ=1.2 and $\bar{s}_0$=0.05, almost 3 times. This effect is caused by the fact that the elastic baffles, during the specific selection of their parameters, are peculiar dynamic damper [1, 2].

3. In order to be convinced of correctness of those obtained it is above results, and to also explain gap effect between wall of vessel and baffles, it was carried out experimental research.

While conducting of experimental research was used rigid cylindrical container with six radial baffles, prepared from aluminum alloy AMg6.

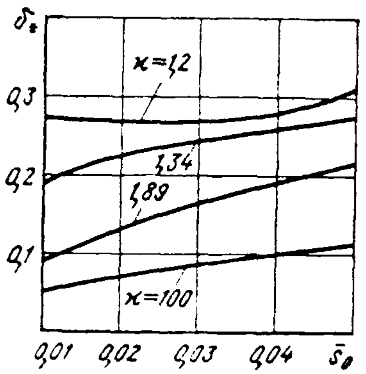

Fig. 2. Dependence of equivalent attenuation factor on amplitude of oscillations of liquid with different values of parameter κ.

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A radius of vessel and sizes/dimensions of baffles correspond to the calculated (see Section 2). Tension of each band was monitored in the process of experiment.

Frequency characteristics were determined on installation, which consists of rigid platform, air supports, electrodynamic vibration exciter and measuring system, which includes displacement pickup and low-frequency analyzer of oscillations. The masses of platform and vessel are equal to the masses, accepted in the calculations.

Experimental values of amplitude frequency characteristic, obtained with $P_0 = 0{,}313$ N and κ=1.89; 1.34 and 1.2, they are shown in Fig. 1. It is evident that they will agree well with calculation data, in spite of a comparatively large width of baffles. The experimental values of phase response also well will agree with the calculated.

Consequently, reference system of equations (4) with sufficient accuracy describes oscillations of body with cavity, partially filled with liquid and that containing elastic baffles.

Until now was examined case, when clearance between wall of vessel and baffles is absent. It is known that in the case of the rigid baffles, established/installed with a certain clearance, damping the oscillations of liquid can be substantially increased [3]. In particular, the radial baffles with a width of $r_0/3$, established/installed with the optimum clearance, attenuate of oscillations approximately 2 times more than without the clearance.

Research of gap effect on damping of oscillations in the case of elastic baffles was conducted with their different tension. The greatest increase in the equivalent attenuation factor δ_* was obtained with tension T=12.3 N, which in the absence of clearance corresponds to value κ=1.34. Fig. 3 for this case gives experimental dependence δ_* on the value of the relative clearance $\bar{\Delta}=\Delta/b$.

Attenuation factor δ_* attains maximum with $\bar{\Delta}\approx0.034$. It is important that with an increase in the amplitude of oscillations δ_* it grows. With amplitudes $\bar{s}_0=0{,}03...0{,}05$ δ_* it is approximately 6 times higher than for the rigid baffles without the clearance and 3 times higher than for the rigid baffles with the clearance. Thus, during the optimum combination of tension of baffles and clearance it is possible to increase substantially damping the oscillations of vessel with the liquid.

Together with progressive/forward oscillations of vessel angular oscillations were examined. And in this case the special features of system, caused by the elasticity of baffles, are retained.

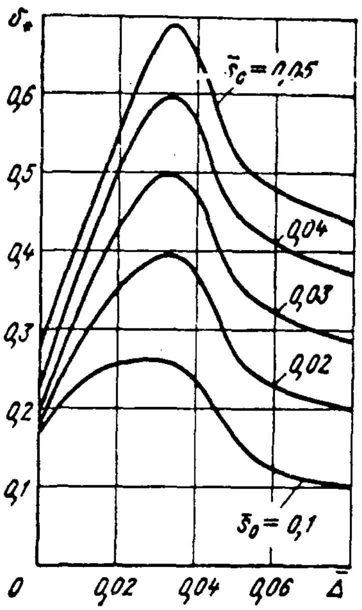

Fig. 3. Dependence of equivalent attenuation factor on air-gap clearance with different values of amplitude of oscillations of liquid.

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EFFECT OF SURFACE TENSION AND ANGLE OF WETTING ON THE OSCILLATIONS OF LIQUID IN THE VESSELS.

G. N. Mikishev, G. A. Churilov.

Results of experimental study of effect of surface tension and angle of wetting on oscillations of liquid in vessels, obtained for case of large numbers of Reynolds and Bond, are presented. The characteristics of the oscillations of liquid in the circular cylinder are in detail analyzed. The forces, which appear during the oscillations of liquid in the zone of meniscus, are determined. The simple methodology of the determination of attenuation factors, based on the use of empirical dependence for the dispersive forces, which function in the zone of meniscus, and the solutions of the corresponding problem about the oscillations of ideal fluid in the vessel is assumed. As examples are determined the attenuation factors for the fundamental tone of the oscillations of liquid in the circular cylinder with the flat/plane bottom and in the vessel, which has the form of rectangular prism.

To oscillations of liquid in vessels affect different factors, including surface tension and angle of wetting. With the large numbers of Reynolds and Bond the effect of these factors affects, mainly, damping of the oscillations of liquid. Attenuation factors can considerably differ from the coefficients, obtained within the framework of boundary-layer theory.

Theoretical determination of attenuation factors is associated with great difficulties. The experimental data about the coefficients are contained in works [6, 7, 10, 11, 12]. However, majority of them relates to a special case - a good wetting of walls. Furthermore, by the reliability of some data is caused doubt, since the angles of wetting were not determined.

Recently interest [1] again is exhibited to this question. This fact and dissatisfaction impelled the authors to conducting of experimental research for the purpose of the study of the effect of surface tension and angle of wetting on the characteristics of the oscillations of liquid in the vessels to the experimental data, available in the literature. The results of these research for the case of the large numbers of Reynolds and Bond are presented below.

1. Formulation of problem. The free linear oscillations of heavy low-viscosity liquid in the vessel by smooth walls with a sufficient accuracy are described by the following system of differential equations [7, 9]

$$\ddot{s}_n + 2\beta_n \dot{s}_n + \omega_n^2 s_n = 0 \quad (n = 1,\ 2 \ldots). \qquad (1)$$

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Here s_n — generalized coordinate, deflection of free surface of liquid at point of standardization; ω_n, β_n — natural frequency and attenuation factor; n - number of tone of oscillations. Natural frequencies and attenuation factors are determined from the formulas

$$\omega_n = \sqrt{j\varkappa_n};\ \beta_n = (\rho\sqrt{\nu\omega_n/2}/2\mu_n)\int_S (\nabla\varphi_n)^2 dS. \qquad (2)$$

Here ρ, ν - density and kinematic viscosity of liquid; j - acceleration of field of mass forces; $\varkappa_n, \varphi_n$ — eigenvalues and eigenfunctions of problem about oscillations of ideal fluid in vessel

$\mu_n = \rho\int_\Sigma \Psi_n^2 d\Sigma/\varkappa_n$ — generalized mass;

$\Psi = \left.\frac{\partial\varphi_n}{\partial x}\right|_\Sigma$ — form of oscillations of free surface; S - moistened surface of vessel; Σ - undisturbed surface of liquid.

With assigned form of vessel of formula (2) it is possible to represent in the form

$$\omega_n = \sqrt{j/l}\, f_n(h/l);\ \beta_n = \omega_n g_n(h/l)/\sqrt{\mathrm{Re}_n}, \qquad (3)$$

where l - significant dimension of vessel; h - depth of liquid; $\mathrm{Re}_n = \omega_n l^2/\nu$. Functions f_n and g_n completely are determined, if the solution of the problem about the oscillations of ideal fluid in the vessel is known.

Dimensionless parameter Re_n is ratio of inertial forces to viscous forces. They frequently call it Reynolds number. Assumption about the low viscosity of liquid is equivalent to condition $\mathrm{Re}_n \gg 1$. With satisfaction of this condition damping the oscillations of liquid is weak ($\beta_n \ll \omega_n$) and it is determined by the dissipation of vibrational energy in the thin wall boundary layer of vessel.

Objective parameters of oscillations of liquid, as show experiments [6, 11, 12], can differ significantly from characteristics, calculated by formulas (2), especially attenuation factors. A difference in the experimental characteristics from the calculated is caused by the action of the disregarded forces, which appear in the zone of meniscus during the oscillations of liquid.

In general case of forces, which function in zone of meniscus, they are nonlinear and depend on surface tension and wetting of walls of vessel. We will consider it their low in comparison with the inertial and gravitational forces. Then the nonlinear vibrations of liquid can be described in the first approximation, by equivalent linear system of equations, analogous system (1), with the only difference that the the coefficients of equations ω_n and β_n will be the functions of the amplitude of oscillations s_{0n}.

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Supplementing to that determining of parameter, which characterize oscillations of heavy low-viscosity liquid in vessel, surface tension σ and static angle of wetting (contact) α and using dimensionality method, let us represent equivalent natural frequencies and attenuation factors in the form of following generalized dependences:

$$\omega_n = \sqrt{j/l}\, f_n(B,\ \alpha,\ h/l,\ s_{0n}/l),$$
$$\beta_n = \omega_n g_n(\mathrm{Re}_n,\ B,\ \alpha,\ h/l,\ s_{0n}/l), \qquad (4)$$

where $B = \rho j l^2/\sigma$ — Bond number. With the assumption B>>1 done above.

Thus, problem about effect of surface tension and angle of wetting on oscillations of liquid in vessel is reduced to experimental determination of dependences (4).

2. Vessel in the form of circular cylinder. Let us establish for in with having been experimentally dependences (4) the fundamental tone of the oscillations of liquid the circular cylinder the flat/plane bottom, bounded in this case to the case of a deep liquid. For the characteristic linear dimension let us take a radius cylinder and r_0, while for the dimensionless attenuation factor - logarithmic decrement of oscillations $\delta_1 = 2\pi\beta_1/\omega_1$. Subsequently index n=1 for simplicity we will lower.

Description of experiments. Experiments were conducted with the use of six cylindrical containers by a radius of 38, 50, 100, 175, 375 and 1000 mm, prepared from the aluminum alloy AMg6. Vessels were filled with water or turpentine to level $h=2r_0$. Those realized in the experiments of the value of parameters Re, B and α lie/rest at ranges $2{,}2\cdot 10^4 \leqslant \mathrm{Re} \leqslant 3{,}6\cdot 10^6$; $194 \leqslant B \leqslant 1{,}33\cdot 10^5$ and $0 \leqslant \alpha \leqslant 108°$. The range of a change in the relative amplitude of oscillations was selected $0 \leqslant \bar{s}_0 \leqslant 0{,}1$. For amplitude s_0 is accepted the maximum deflection of the free surface of liquid on the wall.

Viscosity of water and turpentine was measured in process of tests by capillary tube viscometer, and values of density and surface tension were taken from handbooks according to physical properties of liquids. The static angles of wetting were determined by the method of the inclination/slope of plate by the adhered drop and by adhered bubble [4]. The zero angles of the wetting (more precise, close to zero) were obtained: for the turpentine - because of the complete wetting of the walls of vessels, prepared from alloy AMg6, for the water - by creation on the walls of microroughnesses [8, 10]. Virtually it was considered that the angle of wetting was equal to zero, if during the oscillations of the free surface of the wall of vessels they were covered with stable fluid film. Angle of α=108° is obtained with the plotting on the walls of a thin layer of hydrophobic lubricant and the filling of vessels with water. For the water this angle is maximum [4]. Intermediate angles are obtained just as zero angle for the water.

Natural frequencies and attenuation factors were determined by method of free oscillations on installations, which make it possible to measure total flow forces, which functions on vessel during oscillations of liquid.

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For the transition/transfer from the flow forces to the generalized coordinate was used relationship/ratio $P=\lambda\ddot{s}$, where $\lambda=\pi\rho r_0^3/\xi$. $\xi=1{,}841$ — - first nonzero root of equation $J_1'(\xi)=0$ [7].

Case of complete wetting ($\alpha=0$). At the zero angles of wetting, as experiments showed, natural frequency and logarithmic decrement do not depend on the amplitude of oscillations, i.e., the oscillations of liquid are linear.

Natural frequency does not depend also on Bond number. Virtually it coincides with the frequency for the heavy ideal fluid

$$\omega=\sqrt{\xi j/r_0}. \qquad (5)$$

Of this it is possible to be convinced from comparison of natural frequencies given in Fig. 1. To the case on the graph in question corresponds the calculated straight line $f=\sqrt{\xi}$ and the experimental points, obtained for the smallest cylinder (r_0=38 mm) with the filling with its turpentine (B=460).

Logarithmic decrement of oscillations is function of dimensionless parameters Re and B. The analysis of experimental data showed the possibility of the representation of logarithmic decrement in the form of the sum of two components, one of which depends only on parameter Re and, etc. - from B:

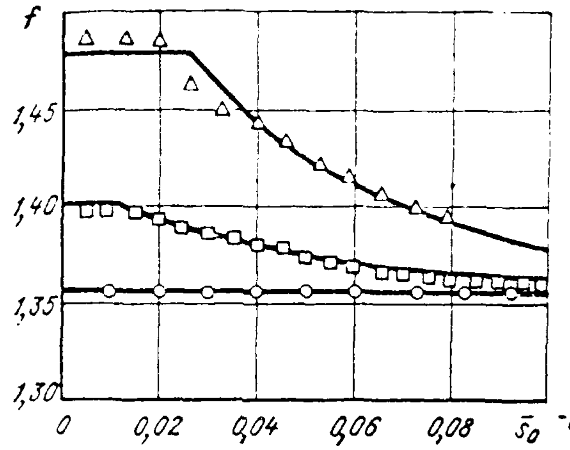
$$\delta=\delta_1(\mathrm{Re})+\delta_2(\mathrm{B}). \qquad (6)$$

Also data, obtained in work [12] for vessel in the form of rectangular prism, indicate this possibility.

Let us take for δ_1 dependence

$$\delta_1 = 4{,}08/\sqrt{Re}, \qquad (7)$$

obtained within the framework of boundary-layer theory for case of deep liquid, and let us determine component δ_2.



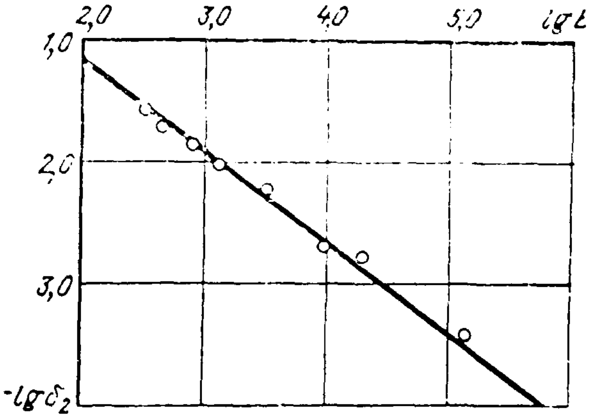


Fig. 1.

Fig. 2.

Fig. 1. Dependence of natural frequency on amplitude of oscillations with different values of number of Bond and angle of wetting:

○ – r_0=38 мм, B=460; □ – r_0=100 мм, B=1330, α=85°; △ – r_0=38 мм, B=194, α=108°

Fig. 2. Dependence of component of decrement δ_2 on Bond number with α=0: ○ – experimental points.

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Results of determining component δ_2 are shown in Fig. 2. Straight line is obtained by working/treatment of these points according to the method of least squares. The dependence

$$\delta_2 = 2{,}24/\sqrt[4]{B^3} \qquad (8)$$

answers it.

Thus, unknown logarithmic decrement takes following form:

$$\delta = 4{,}08\sqrt{Re} + 2{,}24\sqrt[4]{B^3}. \qquad (9)$$

First term into (9) is caused by dissipation of vibrational energy of liquid in wall boundary layer of cylinder, the second - by dissipation of energy in zone of meniscus. The results of calculations according to formula (9) showed that maximum disagreement with the experimental data are approximately 4%.

In series/row of works [6, 7, 10] for agreement with experiment theoretical dependences (3) formal correction, which increases attenuation factor 1.41 times, was introduced. The introduction of this correction, as it is not difficult to be convinced from formula (9), far from always can lead to the acceptable results. Comparison with the experimental data, examined above, showed that only for half of all cases the results are satisfactory (disagreement of less than 11%).

Case of partial wetting ($\alpha = 40...108°$). In the case of oscillation in question the liquids are nonlinear. The dependence of natural frequency and logarithmic decrement on the amplitude of oscillations testifies about this.

Natural frequency depends substantially on amplitude of oscillations only for small cylinders and large angles of wetting (see Fig. 1). Figure gives the experimental data, obtained for the cylinders with a radius of 100 and 38 mm.

Following empirical formula was obtained on the basis of experimental data for natural frequency:

$$f=\sqrt{\xi}\left[1+(5{,}3/B+2{,}4/\sqrt{B})\,e^{-25(\bar{s}_0-\bar{a})}\sin\alpha\right]^{1/2}, \qquad (10)$$

where $\bar{s}_0 \geqslant \bar{a}$; $\bar{a}=a/r_0$, $a\approx 1$ MM. With $\bar{s}_0<\bar{a}$ the frequency on the amplitude of oscillations does not depend and in the value coincides with the appropriate maximum value. The results of calculation according to formula (10) are shown in Fig. 1 by solid lines.

Maximum values of frequency at $\alpha=90°$ will agree well with calculation data of work [3], in which is examined problem about oscillations of ideal fluid in cylinder in the presence on its surface of elastic membrane/diaphragm, evenly stretched along duct/contour. For example, for number B=1330 disagreement in the frequencies composes only 0.6%.

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Known theoretical formula

$$f=\sqrt{\xi}\,[1+\xi^2/B]^{1/2},$$

obtained for case, when on displacement of line of contact of limitations it is not placed and $\alpha=90°$ (for example, see [2]), gives values of f, which differ little from $\sqrt{\xi}$.

Dependence of natural frequency on amplitude of oscillations is caused by interaction of line of contact with walls of cylinder. With the amplitudes $s_0<1$ mm the line of contact is completely engaged with the walls, natural frequency is maximum. With $s_0>1$ mm the line of contact begins to slip relative to walls, which leads to the decrease of natural frequency (in essence due to the decrease of the generalized "hardness" of system).

Let us consider now logarithmic decrement of oscillations. In contrast to the natural frequency logarithmic decrement strongly depends on the amplitude of oscillations, number of Bond and angle of wetting. As in the preceding case, let us represent it in the form of the sum of two components, one of which is caused by the dissipation of vibrational energy in the wall boundary layer of cylinder, and another - by dissipation of energy in the zone of the meniscus:

$$\delta=\delta_1(B)+\delta_2(B,\ \alpha,\ \bar{s}_0). \qquad (11)$$

Fig. 3 shows component of decrement δ_2, obtained for cylinders with different radii with different values of B and α depending on relative amplitude of oscillations.

With low amplitudes component δ_2 is close to zero. In this case logarithmic decrement it is determined by component δ_1. With an increase in the amplitudes, when line of contact begins to slip relative to walls, δ_2 sharply it grows, it reaches maximum values and then smoothly it is reduced. Maximum values δ_2 exceed the appropriate values δ_1 several times (for the cylinder with a radius of 175 mm - 7 times).

Dissipation of vibrational energy in zone of meniscus occurs as a result of complicated interaction of line of contact with walls of cylinder.

At angles $\alpha = 40...60°$ on walls of cylinder appears fluid film, which flows following stepping back liquid. The velocity of its runoff is lower than the velocity of liquid. Therefore line of contact lags behind the bulk of liquid.

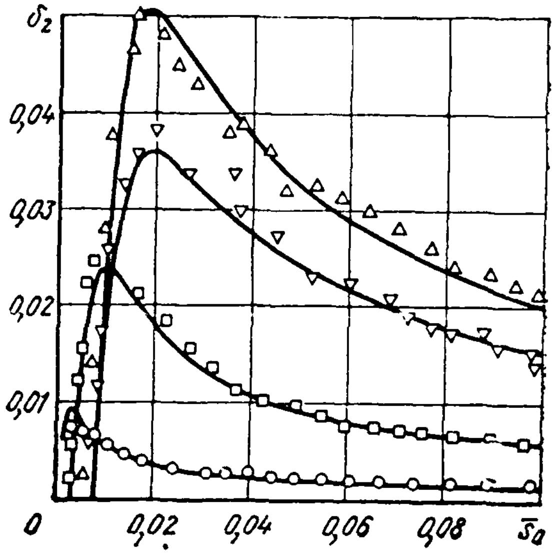

Fig. 3. Dependence of component of decrement δ_2 on amplitude of oscillations with different values of number

Bond and angle of wetting:

△ — $r_0=175$ мм, $B=4160$, $\alpha=108°$;
□ — $r_0=375$ мм, $B=1870$, $\alpha=108°$;
▽ — $r_0=1000$ мм, $B=4160$, $\alpha=75°$

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Abrupt change in the contact angle and formation of capillary waves on the free surface occurs upon the rendezvous of contact line with the heaving liquid. Liquid carries along after itself line of contact, surmounting in this case resisting force to its motion on the dry wall. With the increase of angle α the velocity of the runoff of film increases. The dissipation of vibrational energy grows.

At $\alpha=108°$ fluid film becomes unstable and it is displaced into drops, which are combined between themselves and jets they flow from walls, causing set of capillary waves on free surface. This mechanism of the dissipation of energy is characteristic for the large cylinders and the well developed amplitudes of the oscillations of liquid.

In contrast to previous case, to establish dependence $\delta_2(B, \alpha, \breve{s}_0)$ on the basis of obtained experimental data about decrements of oscillations did not succeed. For the establishment of the unknown dependence it is necessary to know the forces, which function in the zone of meniscus.

3. Determination of forces, which function in zone of meniscus. The flow forces, which function in the zone of meniscus, were determined for the rectangular plates, partially immersed in the liquid and accomplishing bouncing in their plane. Plates are prepared from alloy AMg6. The sizes/dimensions of the plates: length - 314 mm, height/altitude - 80 mm, thickness - is 0.65 mm. Submersion depth into the liquid - 40 mm. During the tests were used the same liquids and methods of changing the angles of wetting, as during the tests of cylinders.

To experienced/tested plate were assigned harmonic displacements. The unknown flow forces was determined from the relationship/ratio

$$F=F_0-F_1-F_2-F_3,$$

where F_0 - the composite force, which functions on the plate; F_1, F_2

and F_3 - inertial, dispersive and archimedian force. F_0 was measured directly by the sensor of force, to which was fastened the plate. Remaining forces were located by calculation with the use of the following dependences:

$$F_1 = -m_0 u_0 p^2 \sin pt + F_2 \operatorname{tg} pt;$$
$$F_2 = -\rho L \sqrt{2\nu p}\,(h_0 + u_0 \sin pt)\, u_0 p \cos pt;$$
$$F_3 = \rho j l d u_0 \sin pt,$$

where l, d - length and thickness of plate; $L=2(l+d)$ — length of line of contact; h_0 - submersion depth; m_0 - mass of plate and elements of its attachment; u_0, p - amplitude and the frequency.

Dependences for F_1 and F_2 are obtained on the basis of solution of problem about oscillations of flat surface in viscous fluid [5]. Experiments with the completeness by the immersed plate showed that the values of inertial and dispersive forces will agree well with appropriate computed values.

Fig. 4 shows dependence of F on displacement of plate $u = u_0 \sin pt$, obtained at angle of α=108°.

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With the low amplitudes of oscillations (u_0<1 mm) this dependence is linear. In this case the line of contact is engaged with the plate. With the amplitudes u_0>1 mm $F_{(u)}$ it is the hysteresis loops, whose area is equal to the dissipation of vibrational energy during the period. Judging by the form of hysteresis loops, dispersive force component F of the type of dry friction makes a basic contribution to the dissipation of vibrational energy. This indicate also the experimental data about the resisting forces to the motion of line of contact with the constant velocity (for example, see [8]).

With decrease of angle α hysteresis loops are deformed. Their area is reduced. In the limit (with $\alpha=0$) they become elliptical. In this case force F is degenerated in linear dispersive view of the type of viscous friction. However, directly it does not depend on the viscosity of liquid. Transition/transfer from the nonlinear frequency independent dispersive force to the linear force, proportional to velocity, occurs at angles of $\alpha<40°$.

Let us determine dispersive force, per unit of length of line of contact at angle $\alpha=0$. In the case in question, as already mentioned, dispersive force was proportional to the velocity

$$\bar{F}=\bar{b}\dot{u}. \tag{12}$$

Coefficient $\bar{b}$ is function ρ, σ and j. Since the number of independent units measurement is equal to three, then with an accuracy to constant factor it is possible to determine it on the basis of dimensional analysis

$$\bar{b}=c\sqrt[4]{\rho\sigma^3/j}. \tag{13}$$

Constant factor was determined experimentally by force measurement during oscillations of plate in turpentine. As a result was obtained value c=0.182.

Thus, unknown expression of dispersive force will be

$$F = 0,182\sqrt[4]{\rho\sigma^3/j}\,\dot{u}. \qquad (14)$$

In general case dispersive force is nonlinear. Let us determine the equivalent linear dispersive force, per unit of the length of line of contact.

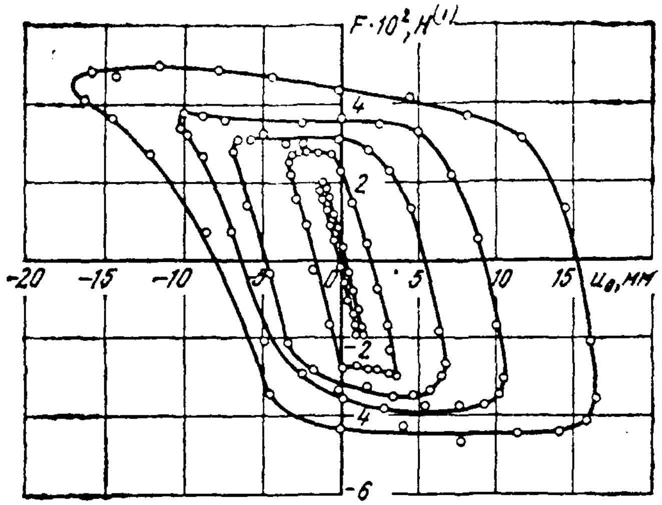

Fig. 4. Dependence of force F on displacement of plate $u=u_0 \sin pt$: ○ - experimental points.

Key: (1). N.

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In contrast to the previous case the unknown force does not depend on the frequency, at least for angles $\alpha=40...108°$,

$$\bar{F}=(\bar{b}_*/p)\,\dot{u}. \qquad (15)$$

Coefficient $\bar{b}_*$ is connected with dissipation of vibrational energy during period with relationship/ratio

$$\bar{b}_*=\Delta\bar{E}/\pi u_0^2, \qquad (16)$$

where $\Delta\bar{E}=\Delta E/L$.

Fig. 5 gives values $\Delta\bar{E}$ in dependence on amplitude of oscillations of plate at different angles α, obtained by determining area of hysteresis loops. Experimental points are approximated well by the dependence of the form

$$\Delta\overline{E}=A(u_0-a)^2+B(1-\cos\alpha)(u_0-a), \qquad (17)$$

where a - value of the amplitude of oscillations, with excess of which begins the slipping of line of contact relative to plate. For the angles of wetting in question with $a\approx 1$ MM. Curves 1, 2 and 3, given on the graph, are obtained with A=2.96 N/m², B=0.124 N/m.

Values A and B can depend only on ρ, σ and j.

Using dimensionality method, it is possible to express them in the following form:

$$A=0{,}111\sqrt{\rho\sigma j};\ B=1{,}69\sigma. \qquad (18)$$

Taking into account (18)

$$\Delta\overline{E}=0{,}111\sqrt{\rho\sigma j}(u_0-a)^2+1{,}69\sigma(1-\cos\alpha)(u_0-a). \qquad (19)$$

Obtained expression is correct for amplitudes of oscillations $u_0 \geqslant a=1$ mm. With amplitudes $u_0<a$ $\Delta\overline{E}$ it follows to assume it equal to zero.

Substituting (19) into (16), we will obtain

$$\bar{b}_*=0{,}0352\sqrt{\rho\sigma j}(1-a/u_0)^2+0{,}539\sigma(1-\cos\alpha)(1-a/u_0)/u_0. \qquad (20)$$

Dependence (15) and expression (20) completely determine equivalent dispersive force at angles $\alpha = 40...108°$.

During determination of forces, which function in zone of meniscus, it was assumed that they were directed perpendicularly to line of contact. Further experiments were conducted for checking the correctness of this assumption, the oscillations of plates along the line of contact in particular were investigated. As a result it was established that any forces, which function along the line of contact, it does not appear.

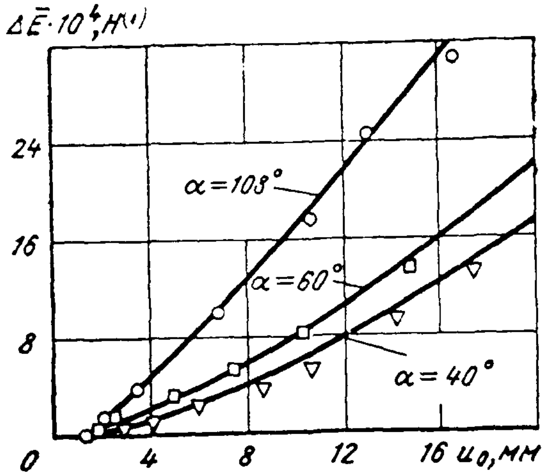

Fig. 5. Dependence of energy of dissipation in zone of meniscus on amplitude of oscillations of plate: ○, ▽, □ — experimental points.

Key: (1). N.

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4. Semi-empirical dependences for attenuation factors. As in Section 2, let us represent logarithmic decrement in the form of the sum of two components

$$\delta=\delta_1(\mathrm{Re},\ \bar{h})+\delta_2(\mathrm{B},\ \alpha,\ \bar{h},\ \bar{s}_0). \qquad (21)$$

Let us assume that solution of problem about oscillations of ideal fluid in vessel is known. Then the determination of the component of decrement δ_1 is reduced to the simple integration (see Section 1), and component δ_2 can be approximately found on the basis of experimental data for the dispersive forces, obtained in the previous section.

We will use relationship/ratio

$$\delta=\Delta E/2E, \qquad (22)$$

connecting logarithmic decrement with dissipation of vibrational energy during period and total energy of system. By defining ΔE as the work of the dispersive forces, which function in the zone of the meniscus

$$\Delta E=\pi\omega s_0^2\int\limits_L \bar{b}\Psi^2 dL,$$

and accepting as E maximum kinetic energy of system $0{,}5\,\mu\omega^2 s_0^2$, we will obtain

$$\delta_2=\frac{\pi}{\omega\mu}\int\limits_L \bar{b}\Psi^2 dL. \qquad (23)$$

This expression is correct for vessels, whose walls in area of duct/contour of free surface are vertical or close to vertical.

As examples let us determine logarithmic decrements for vessels in the form of circular cylinder with flat/plane bottom and rectangular prism.

Circular cylinder with flat/plane bottom. Component δ_1 takes following form [7]:

$$\delta_1=\frac{\pi}{\sqrt{2R}}\left[\frac{\xi^2+1}{\xi^2-1}+\frac{2\xi(1-\bar{h})}{\mathrm{sh}\,(2\xi\bar{h})}\right]. \qquad (24)$$

For determining component δ_2 from formula (23) it is necessary to know not only coefficient of $\bar{b}$, but also natural frequency ω, form of oscillations Ψ and generalized mass μ. From the solution of the problem about the oscillations of ideal fluid in cylinder [7] we have

$$\omega=\sqrt{\frac{\xi g}{r_0}\,\mathrm{th}\,\xi\bar{h}};\quad \Psi\mid_L=\sin\theta;\quad \mu=\frac{\rho r_0^3\pi(\xi^2-1)}{2\xi^3\,\mathrm{th}\,\xi\bar{h}}. \tag{25}$$

With α=0 coefficient $\bar{b}$ is determined by expression (13). Substituting (13) and (25) into (23), and also assuming/setting $dL=r_0d\theta$ and integrating, we find

$$\delta_2=2{,}2\sqrt[4]{\mathrm{th}\,\xi\bar{h}}/\sqrt{B^3}. \tag{26}$$

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For case of deep liquid obtained dependence virtually coincides with dependence (8). Disagreement in the coefficients composes only 1.8%.

When $\alpha=40...108°$ component of decrement δ_2 is located thus. In this case of $\bar{b}=\bar{b}_*/\omega$, where $\bar{b}_*$ is determined by expression (20), in which one should assume $u_0=s_1\sin\theta$. After integration we will obtain

$$\delta_2=\frac{0{,}1}{\sqrt{B}}\left(\pi-8\frac{a}{s_0}+2\pi\frac{a^2}{s_0^2}\right)+\frac{3{,}06}{B\bar{s}_0}(1-\cos\alpha)\left(2-\pi\frac{a}{s_0}\right), \tag{27}$$

where s_0 - amplitude of oscillations at the point $\theta=\frac{\pi}{2}$.

Results of calculations according to formula (27) are shown in Fig. 3 by solid lines. They satisfactorily will agree with the experimental data for all three cylinders.

Vessel in the form of rectangular prism. The component of decrement δ_1 is obtained into [12]

$$\delta_1 = \frac{\pi}{\sqrt{2\,\mathrm{Re}}}\left[(1+1/\bar{l}_1)+\frac{\pi(1-\bar{h})}{\mathrm{sh}\,\pi\bar{h}}\right]. \tag{28}$$

Here $\mathrm{Re} = \omega l^2/\nu$; $\bar{l}_1 = l_1/l$; $\bar{h} = h/l$; $2l$, $2l_1$ — length and the width of vessel.

component δ_2 it is not difficult to define analogously how this was done for cylinder. Using the solution of the problem about the oscillations of ideal fluid in rectangular vessel [7, 12], we find

$$\delta_2 = 1{,}46\,(1+1/2\bar{l}_1)\sqrt{\mathrm{th}\,(\pi\bar{h}/2)}/\sqrt[4]{\mathrm{B}^3}\ \overset{(1)}{\text{при}}\ \alpha = 0 \tag{29}$$

Key: (1). with.

and $$\delta_2 = \frac{0{,}22}{\sqrt{\mathrm{B}}}\left[(1+1/2\bar{l}_1)-\frac{2a}{s_0}(1+2/\pi\bar{l}_1)+\frac{a^2}{s_0^2}(1+1/\bar{l}_1)\right]+$$
$$+\frac{3{,}39}{\mathrm{B}\bar{s}_0}\left[(1+2/\pi\bar{l}_1)-\frac{a}{s_0}(1+1/\bar{l}_1)\right](1-\cos\alpha)\ \overset{(1)}{\text{при}}\ \alpha = 40\ldots108^\circ. \tag{30}$$

Key: (1). with.

Here $B = \rho j l^2/\sigma$; $\bar{s}_0 = s_0/l$; $\bar{s}_0 \geqslant a(1+\bar{l}_1)/(2/\pi+\bar{l}_1)$.

In work [12] for case $\alpha=0$ and $\bar{l}_1 \approx 4.61$, $\bar{h}=0.424$ is obtained empirical dependence

$$\delta_1 = 36,8/B. \qquad (31)$$

However, it much more badly will be coordinated with the experiment than semi-empirical dependence (29). The comparison of the results of calculations according to formula (31) with most reliable experimental data, obtained into [12] for the glass vessels with the filling with their water, showed that the disagreement is from 8.1 to 44%, whereas according to formula (29) - from 4.4 to 13%.

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DISTURBED MOTION OF THE NONROTATING FLIGHT VEHICLE WITH THE LIQUID IN THE TANKS. SIZABLE DISTURBANCES OF FREE SURFACE.

ON THE OSCILLATIONS OF LIQUID IN THE MOBILE CAVITIES.

G. S. Narimanov.

Work examines nonlinear slosh equations, which partially fills cavity in the form of straight/direct circular cylinder. It is assumed that the cavit. completes the steady harmonic oscillations in the plane, perpendicular to the longitudinal axis of cylinder. Investigated resonance excitation of the sizable oscillations of liquid with the assigned oscillatory motion of cavity.

On basis of theoretical analysis series/row of nonlinear effects is revealed (decrease of frequ--ncy of major resonance, difference in profile/airfoil of wave from that described by linear theory, etc.).

Obtained results are in complete agreement and by G. I. Mikishev's experiments.

Examination of problem about motion of liquid, which partially fills cavity of solid body, with assigned oscillatory motion of latter is indicated possibility of emergence of resonance oscillations of liquid. Naturally, in this case the theory, based on the prerequisite/premise about the smallness of oscillations, which reduces the problem of the study of motion to the solution of linear equations, cannot explain some special features of real motions.

G. N. Mikishev politely let results of his experimental works, dedicated to determination of possibility of applying linear equations for describing motion in region of resonance excitation of oscillations of liquid, to author. They discovered a change of the frequency of resonance oscillations into the dependence on relative value of the amplitude of the forcing oscillations, change in the profile/airfoil of resonance wave, limitedness of the amplitude of resonance oscillations.

By target of present article is theoretical analysis of phenomena on the basis of use indicated derived earlier than [1] general/common equations of motion of solid body, partially filled with liquid, that consider significance of motion of liquid.

1. General/common equations for case of cavity in the form of circular cylinder. We assume/set the motion of body by flat/plane, examining it in plane O*x*y*. Axis O*y* is vertical. The axis of the symmetry of cavity Oy is parallel O*x*y*, point O coincides with the center of gravity of system in the undisturbed state of the free surface of liquid.

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Liquid we assume/set by inviscid, and motion by its potential. The equation of the disturbed free surface can be represented in the form

$$z = y - c = \sum_{s=1}^{\infty} [a_s(t)\varphi_s + b_{0s}(t)\psi_{0s} + b_{2s}(t)\psi_{2s}]; \qquad (1)$$

$$\varphi_s = \sqrt{\frac{2}{\pi}} \frac{\xi_{1s}}{r_0\sqrt{\xi_{1s}^2 - 1}J_1(\xi_{1s})} J_1\left(\xi_{1s}\frac{r}{r_0}\right)\sin\alpha;$$

$$(s = 1,\ 2, \ldots;\ k = 0,2)$$

$$\psi_{ks} = \sqrt{\frac{2}{\pi}} \frac{\xi_{ks}}{r_0\sqrt{\xi_{ks}^2 - k^2}J_k(\xi_{ks})} J_k\left(\xi_{ks}\frac{r}{r_0}\right)\cos k\alpha.$$

J_1, J_k — function of Bessel of first order of corresponding order; r, a - polar coordinates in plane of normal axis of cavity (angle α it is counted off from normal to axis Ox); r_0 - radius of a circle; ξ_{1s}, ξ_{ks} — roots of equations

$$J_1'(\xi) = 0,\ J_k'(\xi) = 0.$$

All parameters of motion of solid body and liquid, with exception only of parameter a_1, we assume/set by such low that values of their products, squares and higher degrees can be disregarded/neglected. Relatively parameter a_1 we assume that the square of its value has the same order of smallness as the value of the remaining parameters.

Being based on this, in equations of motion we retain terms, whose values are of the order not less than third degree of value of order a_1.

We designate: x - bias/displacement of point O along axis O*x*, ϵ - angle between axes Oy and O*y*, ρ - density of liquid, m - total mass of system, J - moment of inertia of system in fixed position relative to body of undisturbed free surface of liquid, P - sum of external forces, which function on body, M - sum of external moments/torques.

Under assumptions of equation of motion of solid body indicated, which contains liquid, considering significance of motions by latter, can be represented by following infinite system of ordinary differential equations:

$$m\ddot{x}+\rho\sum_{s=1}^{\infty}\ddot{a}_s X_2^{(s)}=P; \qquad (2)$$

$$J\ddot{\varepsilon}+\rho\sum_{s=1}\left(a_s\Omega^{(s)}+a_s g X_2^{(s)}\right)+\rho\frac{d}{dt}\sum_{k=0,2}\sum_{s=1}^{\infty}\left(\dot{a}_1 b_{ks}\theta_{ks}+a_1\dot{b}_{ks}\theta^{(ks)}\right)+$$
$$+\rho\theta\frac{d}{dt}\left(\dot{a}_1 a_1^2\right)=M; \qquad (3)$$

$$\ddot{a}_n A_n+g a_n+\ddot{x}X_2^{(n)}+\ddot{\varepsilon}\Omega^{(n)}+\varepsilon g X_2^{(n)}+\frac{d}{dt}\sum_{k=0,2}\sum_{s=1}^{\infty}\left(\dot{a}_1 b_{ks}B_{1ks}^{(n)}+\right.$$
$$\left.+a_1\dot{b}_{ks}B_{sk1}^{(n)}\right)+B^{(n)}\frac{d}{dt}\left(\dot{a}_1 a_1^2\right)+C^{(n)}\dot{a}_1^2 a_1+\dot{a}_1\sum_{k=0,2}\sum_{s=1}^{\infty}\dot{b}_{ks}D_{ks}^{(n)}=0$$
$$(n=1,\ 2,\ldots); \qquad (4)$$

$$\ddot{b}_{ks}B_{ks}+g b_{ks}+B_1^{(ks)}\frac{d}{dt}\left(\dot{a}_1 a_1\right)+F^{(ks)}\dot{a}_1^2=0 \quad (k=0,\ 2,\ s=1,\ 2,\ldots). \qquad (5)$$

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Equations (2)...(5) can be solved by reduction on the basis of consecutive integration of final systems of equations, which are obtained by their (2)...(5) with finite and ever increasing values of n and s. In the first approximation, let us consider final system of equations with n=s=1. It is possible to expect that this approximation/approach will not require further refinement in the investigated problem. Actually, on the basis of linear theory for known [2], [3] that the study of the oscillations of liquid during the assigned harmonic oscillation of body with the frequency, close to the natural frequency of parameter a_1, can be carried out, disregarding the values of all parameters a_n $(n>1)$.

2. Resonance excitation of oscillations of liquid with assigned oscillatory motion of vessel. Let us consider the case of forced oscillations of liquid with the assigned oscillatory motion of the walls of cavity with the frequency, close to the frequency of the first form of the natural oscillations of liquid. Analogous problem, as it was mentioned above, she was solved by experimentally G. N. Mikishev.

We will assume that vessel with cylindrical cavity, partially filled with liquid, completes progressive/forward harmonic oscillations in direction of normal to axis of cylinder with frequency ω and amplitude of A.

Equations, which describe change in form of free surface of liquid, on the basis (4) and (5) with done above further simplifications will take following form:

$$\ddot{a}_1 A_1 g a_1 + \sum_{k=0,2}\left[\frac{d}{dt}\left(\dot{a}_1 b_{k1} B_{1k1}^{(1)} + a_1 \dot{b}_{k1} B_{k11}^{(1)}\right) + \dot{a}_1 \dot{b}_{k1} D_{k1}^{(1)}\right] +$$

$$+ B^{(1)}\frac{d}{dt}\left(\dot{a}_1 a_1^2\right) + C^{(1)}\dot{a}_1^2 a_1 = -A\omega^2 X_2^{(1)}\sin\omega t;$$

$$\ddot{b}_{k1} B_{k1} + g b_{k1} + B_1^{(k1)}\frac{d}{dt}\left(\dot{a}_1 a_1\right) + F^{(k1)}\dot{a}_1^2 = 0 \quad (k=0,2). \qquad (6)$$

We convert equations (6), after leading to dimensionless quantities entering in them parameters. Let us introduce dimensionless time $\tau=\omega_1$. We will designate differentiation on τ by primes. Further, we use the following designations of the dimensionless quantities of the parameters, entering equations (6):

$$\alpha_1 = \frac{a_1}{r_0^2};\quad \beta_k = \frac{b_{k1}}{r_0^2};\quad \omega_1^2 = \frac{g}{A_1};$$

$$m_k^2 = \frac{g}{B_{k1}\omega_1^2};\quad m = \frac{\omega}{\omega_1};\quad l_{1k} = \frac{B_{1k1}^{(1)} r_0^2}{A_1};$$

$$l_{k1} = \frac{B_{k11}^{(1)} r_0^2}{A_1};\quad d_k = \frac{D_{k1}^{(1)} r_0^2}{A_1};$$

$$l = \frac{B^{(1)} r_0^4}{A_1};\quad C = \frac{C^{(1)} r_0^4}{A_1};\quad l_k = \frac{B_1^{(k1)} r_0^2}{B_{k1}};$$

$$f_k = \frac{F^{(k1)} r_0^2}{B_{k1}};\quad a = \frac{A X_2^{(1)}}{A_1 r_0^2}.$$

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In these designations equations (6) are converted to form

$$\ddot{\alpha}_1+\alpha_1+\sum_{k=0,2}\left[\frac{d}{d\tau}(\alpha_1'\beta_k l_{1k}+\alpha_1\beta_k' l_{k1})+\alpha_1'\beta_k' d_k\right]+$$

$$+l\frac{d}{d\tau}(\alpha_1'\alpha_1^2)+C\alpha_1'^2\alpha_1=-m^2 a\sin m\tau; \tag{7}$$

$$\ddot{\beta}_k+m_k^2\beta_k+l_k\frac{d}{d\tau}(\alpha_1'\alpha_1)+f_k\alpha_1'^2=0 \quad (k=0,2). \tag{8}$$

As is known, steady forced oscillation of nonlinear system can be in the first approximation, (with an accuracy down to the terms, which possess frequency of change, equal to impressed frequency) it is represented in the form $\alpha_1=\gamma\sin(m\tau+\nu)$.

It is easy to see that for system, described by equations (7), (8), in which there is no damping, phase displacement ν will be equal to zero or π, i.e., it is considered by sign of value γ. Therefore the solution of these equations, which characterizes in the first approximation, steady forced oscillations, we will seek, on the basis of the expression

$$\alpha_1=\gamma\sin m\tau. \tag{9}$$

Substituting expression (9) into equation (8), let us lead latter to following form:

$$\beta_k^{\cdot\cdot}+m_k^2\beta_k=-\frac{1}{2}f_k\gamma^2m^2-\gamma^2m^2\left(l_k+\frac{1}{2}f_k\right)\cos 2m\tau. \qquad (10)$$

Steady-state oscillations of values β_k will be described by expressions

$$\beta_k=-\frac{f_k\gamma^2m^2}{2m_k^2}-\frac{(2l_k+f_k)\gamma^2m^2}{2(m^2-4m^2)}\cos 2m\tau. \qquad (11)$$

It follows from examination of formulas (7), (8) and (11) that accepted for construction of theory of nonlinear vibrations prerequisite/premise about comparability of order of magnitudes α_1^2 and β_k ceases to be accurate with sufficiently low values m, close to values $1/2m_k$, $1/4m_k$, $1/6m_k$....

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However, with these values of m generally there is no need in refinement of linear theory of low oscillations, according to which parameters b_{ki} are not excited with onset of oscillations of vessel or parameters a_i. The carried out refinement of the slosh equations due to the account of significance of the value of parameter a_1 or α_1 is substantial for describing the phenomena of oscillation only in the region of the resonance excitation, the parameter indicated, i.e., in region m=1. Keeping in mind the fact indicated, we will further conduct constructions under the condition

$$1/2m_k<0{,}8\leqslant m,$$

taking into account that at m>0.8 parameters b_{ks} can be accepted equal to zero, as it follows from the linear theory of the low oscillations of liquid.

For determining value γ let us substitute values β_k, expressed according to (11), into equation (7), in which α_1 is substituted by formula (9). After leading bringing similar terms, which contain factor sin $m\tau$ and after equating to zero given coefficient with this factor, we will obtain the equation, which describes the dependence of the amplitude of steady forced oscillations of the parameter α_1 on value m and values a (relative value of the amplitude of the oscillations of vessel) in the first approximation,

$$\gamma^3+p\gamma+q=0, \qquad (12)$$

$$p=\frac{4(1-m^2)}{Sm^2}; \quad q=\frac{4a}{S},$$

where $$S=m^2\sum_{k=0,2}\left[\frac{2l_{1k}f_k}{m_k^2}-\frac{(2l_k+f_k)(l_{1k}-2l_{k1}+2d_k)}{4m^2-m_k^2}\right]-3l.$$

Fig. 1 depicts curves γ/a, constructed on the basis of solution of equation (12) with three values of value a. The same figure depicts curve $\gamma_л/a$, which corresponds to the linear theory of the excitation of oscillations.

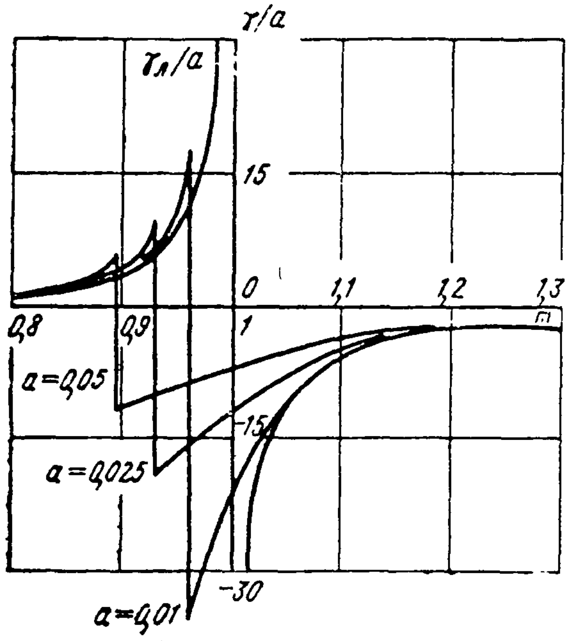

Fig. 1.

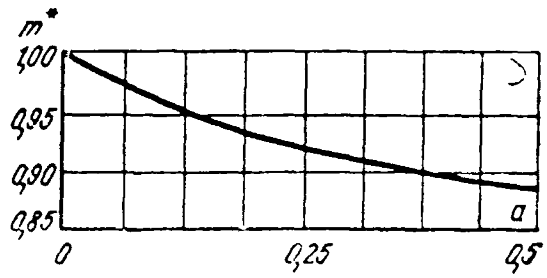

Fig. 2.

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If the value of the latter/last value, obtained on the basis of linear theory, does not depend on relative value of the amplitude of the exciting oscillations, then the account of nonlinear terms in the equations of motion stipulates the essential dependence of γ/a on value a. In this case γ/a, remaining the everywhere limited value, in proportion to the decrease of value a approaches curve $\gamma_{л}/a$, which corresponds to an increase in the field of the validity of the description of the phenomena of real oscillations with the help of the apparatus of linear theory.

At the same time derived equations, which consider nonlinear terms, are free from deficiencies in linear equations in direct vicinity m=1, in which $\gamma\pi/a$ suffers explosion.

As can be seen from graph, isobeen delirious in Fig. 1, account of nonlinearity in equations of motion, which stipulated dependence of frequency characteristics on value a, explains also another special feature of real oscillations, experimentally discovered by G. N. Mikishev, which consists in decrease of value of resonance frequency of excitation of oscillations of liquid in cylindrical tank with increase in a. The latter fact is illustrated by graph in Fig. 2.

Calculations of form of free surface of liquid during resonance excitation of oscillations, carried out on the basis of nonlinear equations given above, also showed very satisfactory description of real form of resonance wave.

On the basis of formulas (9) and (11), after determining value γ of (12) with resonance value of m=m*, calculated expression

$$\frac{z}{r_0} = \mp \alpha_1 r_0 \varphi_1 + \beta_0 r_0 \psi_{01} + \beta_2 r_0 \psi_{21}, \qquad (13)$$

where α_1, β_0, β_2 was taken with $\tau=\pi/2m^*$, and φ_1, ψ_{01}, ψ_{21} - with $\alpha=\pm\pi/2$, which corresponds to cross section of wave by plane, passing through axis of cylinder. The results of calculations are given in Fig. 3 and 4.

Depending on intensity of resonance phenomena, which, in turn, depends on value of amplitude a, is observed difference in profile/airfoil of wave (unbroken curves) from form, described by linear theory (dotted curves).

With a=0.025 ratio of height/altitude of protuberance of wave to depth of indentation is equal to 1.5, while with a=0.1 it already increases to 2.

Thus, nonlinear equations used make it possible to describe basic special features of real phenomena during resonance excitation of oscillations of liquid in mobile cavities.

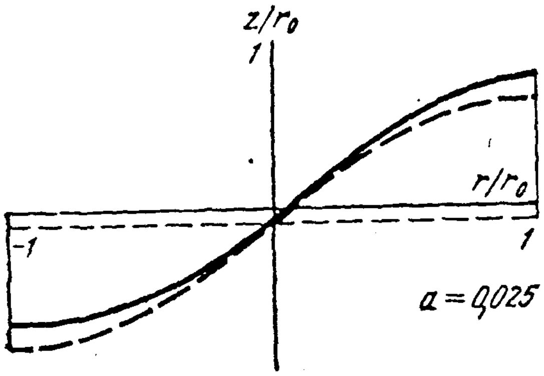

Fig. 3.

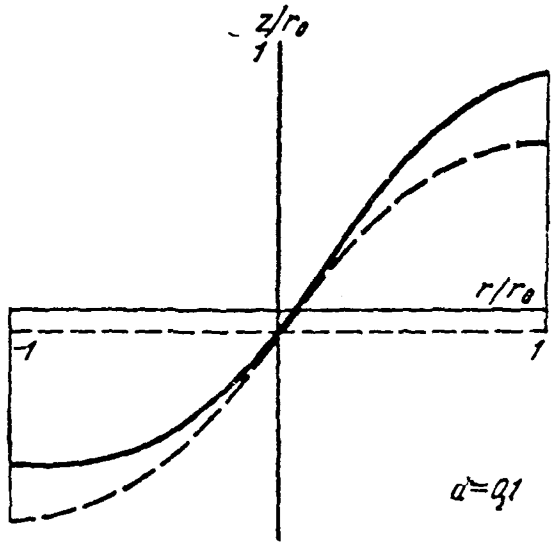

Fig. 4.

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Application of variation principle to the conclusion of the nonlinear equations of the disturbed motion of body - liquid system.

I. A. Lukovskiy.

Results of systematic application of variation principle of Ostrogradskiy to conclusion of nonlinear three-dimensional equations of motion of solid body with cavity, which contains liquid, are presented. Lagrange's function is selected in the form of two components, the first of which is the kinetic potential of solid body, and the second - integral of the pressure by the volume, occupied with liquid. A question about the selection of the form of the velocity potential of liquid and about the structural/design representation of its components is discussed. With some assumptions about the value of the parameters, which characterize the deflection of the free surface of liquid, is obtained the system of nonlinear equations of motion in the case of the cavities, formed by coaxial cylinders.

Basic work of G. S. Narimanov [9] marked beginning of development of nonlinear theory of motion of solid body with cavities, partially filled with ideal fluid. Under some assumptions relative to the parameters, which characterize the motion of mechanical system, they obtained nonlinear equations of motion, and is also proposed the method of calculation of their hydrodynamic coefficients in the case of the cavities of cylindrical form. Further development this method was obtained in works [3], [10], [13], and also in the number of research of the foreign authors. At the present time in nonlinear dynamics of bodies with the cavities, which contain liquid, wide acceptance were obtained also the methods, based on the application of variation principles of mechanics [2, 4-7, 11]. Below in general terms will be presented the conclusion/output of the nonlinear equations of the disturbed motion of bodies with the liquid filling with variational method. For the certainty the case of the cavity, which has cylindrical form is examined in the vicinity of free surface.

1. Basic assumptions. Let us connect with solid body the system of coordinates Oxyz, which moves together with the body relative to the absolute system of coordinates O_1XYZ, in which the field of mass forces has the potential function U. Subsequently this coordinate system is considered as inertial system.

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The axes of the absolute coordinate system are motionlessly connected with the Earth, moreover axis O_1Y is considered directed directly opposite to gravitational force, while axes O_1X and O_1Y are located in horizontal plane so that the coordinate system would be right. They coincide at the initial moment of the time of point O_1 and O, and axis Ox coincides in the direction with axis O_1Y. Axis Oz it is directed so that it at the initial moment would be parallel to axis O_1Z. Axis Oy will be directed in this case along axis O_1X to the opposite side. The directions of axes Ox, Oy, Oz relative to the absolute coordinate system are uniquely determined by three angles: pitch angle ϑ (angle between axis O_1X and plane OxZ), by yaw angle ψ (angle between axis Ox and plane OXY) and by attitude of roll γ (angle between axis Oy and intersection of planes Oyz and OXY). The cosines of the angles between the axes of absolute and body coordinate systems are given in Table 1.

Motion of solid body we will characterize by vector of forward velocity $\underline{v}_0$ of point O by vector of instantaneous angular velocity ω relative to point O. The kinematic equations, which establish/install connection/communication between the projections of angular velocity on the axis of body-fixed system with the angular parameters, which characterize the positions of body relative to the absolute coordinate system, take the following form:

$$\omega_1 = \dot{\gamma} - \dot{\vartheta} \sin \psi;$$
$$\omega_2 = \dot{\psi} \cos \gamma + \dot{\vartheta} \cos \psi \sin \gamma;$$
$$\omega_3 = -\dot{\psi} \sin \gamma + \dot{\vartheta} \cos \psi \cos \gamma.$$

Liquid, which fills cavity of body, is considered ideal and incompressible. If its initial motion is potential, then the same it will remain also at the subsequent moments of time. Let us represent the equation of the disturbed free surface in the form

$$\zeta(x, y, z, t) \equiv x - x_0 - h - f(y, z, t) = 0, \qquad (1)$$

where h - depth of liquid in the cavity; x_0 - coordinate of the bottom of cavity.

Table 1.

	O_1X_0	O_1Y_1	O_1Z_0
Ox	$\cos\vartheta\cos\psi$	$\sin\vartheta\cos\psi$	$-\sin\psi$
Oy	$\cos\vartheta\sin\psi\sin\gamma-$ $-\sin\vartheta\cos\gamma$	$\sin\vartheta\sin\psi\sin\gamma+$ $+\cos\vartheta\cos\gamma$	$\cos\psi\sin\gamma$
Oz	$\cos\vartheta\sin\psi\cos\vartheta+$ $+\sin\vartheta\sin\gamma$	$\sin\vartheta\sin\psi\cos\gamma-$ $-\cos\vartheta\sin\gamma$	$\cos\psi\cos\gamma$

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Velocity potential Φ(x, y, z, t), that describes absolute motion of liquid in moving coordinate system, is determined by solution of following nonlinear boundary-value problem [10]:

$$\Delta\Phi=0 \overset{(1)}{\text{в}}\ Q; \tag{2}$$

$$\frac{\partial\Phi}{\partial\nu}=(\mathbf{v}_0\boldsymbol{\nu})+(\omega(\mathbf{r}\times\boldsymbol{\nu}))\ \overset{(2)}{\text{на}}\ S; \tag{3}$$

$$\frac{\partial\Phi}{\partial\nu}=(\mathbf{v}_0\boldsymbol{\nu})+(\omega(\mathbf{r}\times\boldsymbol{\nu}))+u_\nu\ \overset{(2)}{\text{на}}\ \Sigma; \tag{4}$$

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}(\nabla\Phi)^2-(\nabla\Phi(\mathbf{v}_0+\omega\times\mathbf{r}))+U=0\ \overset{(2)}{\text{на}}\ \Sigma, \tag{5}$$

Key: (1). in. (2). on.

where Q - region, occupied with liquid; S and Σ - moistened surface of walls of cavity and disturbed free surface of liquid respectively; r - radius-vector of points of mechanical system relative to point O; u_ν - relative particle speed of free surface of liquid, determined by equation

$$u_v = \frac{f_t}{\sqrt{1 + f_y^2 + f_z^2}} = \frac{f_t}{N}; \quad N = |\nabla\zeta| = \sqrt{1 + (\nabla f)^2}. \tag{6}$$

Potential function U, as is known, is determined by relationship/ratio

$$U = -(\mathbf{g}\mathbf{r}) = -(\mathbf{g}(\mathbf{r}_0' + \mathbf{r}')), \tag{7}$$

where g - G-vector of gravity; r_0' - radius-vector of point O relative to O_1; r' - radius-vector of any point of system relative to O_1. For determining the pressure p in the liquid it is necessary to use Lagrange-Cauchy integral, which takes in fixed coordinate system Oxyz the form

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - (\nabla\Phi, (\mathbf{v}_0 + \omega \times \mathbf{r})) + U + \frac{p}{\rho} = 0, \tag{8}$$

where ρ - mass density of liquid.

Most general formulation of problem of dynamics of solid body with cavity, which contains liquid, assumes from known external forces applied to body definition of both motion of body itself and motion of liquid within its cavity, and also forces of interaction between body and liquid. The conclusion of the equations of motion of the mechanical system in question and the development of the methods of determining their hydrodynamic coefficients present the sufficiently complicated mathematical problem, whose solution in general form is difficult. However, during some limitations it is possible to obtain a comparatively complete system of nonlinear differential equations, which adequately describe the discovered experimentally complicated physical phenomena, which appear during the resonance interactions of solid body and liquid.

Subsequently we will consider that standard deviation of disturbed free surface is of the order of smallness ϵ.

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During the calculations we will disregard the values of order ϵ^2, i.e., we will retain values f, f^2 and f^3. Let us assume further that value ω^3 is of the order of smallness ϵ^2. Consequently, during the calculations must be taken into consideration besides values of the type ω^2, also products ωf, ωf^2, $f\omega^2$. The obtained below mathematical model is suitable for such states of motion of solid body, during which the free surface of liquid is not destroyed, and the amplitudes of oscillations appearing in this case do not exceed maximum.

2. Overall diagram of obtaining nonlinear equations of motion from variation principle of Ostrogradskiy. When solid body completes the assigned motion in the space, i.e., vectors v_c and ω are considered as the known functions of time, nonlinear boundary-value problem (2)...(5) follows from variation principle of [5]

$$\delta W = \delta \int_{t_1}^{t_2} L dt = 0, \quad (9)$$

where

$$L = \int_Q p dQ = -\rho \int_Q \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 - (\nabla \Phi, (\mathbf{v}_0 + \omega \times \mathbf{r})) + U \right] dQ. \quad (10)$$

Variation principle of Hamilton-Ostrogradskiy in the form of (9), (10) differs from others in terms of fact that all flow equations follow from it, including kinematic nonlinear free-surface conditions of (4). In the general case of moving the mechanical system it is impossible to find the analogous variation principle, for which it would be possible to formulate the appropriate problem of the calculus of variations.

More common point of view assumes use in the case of presence of potential forces and nonholonomic systems of variation principle of Ostrogradskiy in the form

$$\int_{t_1}^{t_2} (\delta L_1 + \delta L + \delta' A)\, dt = 0, \qquad (11)$$

where $L_1 = T - \Pi$ - kinematic potential of solid body; $\delta' A$ - elementary work of nonpotential forces, applied to solid body; L - Lagrange's function, determined by expression (10).

Let us represent now velocity potential of liquid in the form

$$\Phi = (\mathbf{v}_0, \mathbf{V}) + (\omega, \Omega) + \varphi, \qquad (12)$$

where in accordance with kinematic boundary conditions harmonic vector functions V and Ω and harmonic function φ satisfy following boundary conditions:

$$\left.\frac{\partial \mathbf{V}}{\partial \nu}\right|_{S+\Sigma}=\boldsymbol{\nu}; \quad \left.\frac{\partial \boldsymbol{\Omega}}{\partial \nu}\right|_{S+\Sigma}=\mathbf{r}\times\boldsymbol{\nu}; \tag{13}$$

$$\left.\frac{\partial \varphi}{\partial \nu}\right|_{S}=0; \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Sigma}=\frac{f_t}{\sqrt{1+f_y^2+f_z^2}}.$$

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Potential φ is in this case solution of nonlinear problem about wave motions of liquid in quiescent vessel. Vector function V with an accuracy to the arbitrary function of time is defined

$$\mathbf{V}=\mathbf{r}. \tag{14}$$

Let us assign form of disturbed free surface of liquid and potential φ in the form [4]

$$f(y, z, t)=\sum_i \beta_i(t) f_i(y, z); \tag{15}$$

$$\varphi(x, y, z, t)=\sum_n R_n(t) \varphi_n(x, y, z), \tag{16}$$

where $f_i(y, z)$ - complete orthogonal (upon inclusion in it of constant) set of functions, assigned on undisturbed free surface Σ_0; $\beta_i(t)$ - generalized Fourier coefficients, which play role of generalized coordinates and characterizing position free surfaces of liquid at the given instant; $\varphi_n(x, y, z)$ - system of harmonic functions, which are solutions of problem about low wave motions of liquid in vessel; $R_n(t)$ - unknown previously parameters, which characterize change of potential φ in time.

For potentials Ω_i it is possible also to introduce into examination of resolution of type (16) or to determine by their variational method, after formulating variational problems for functionals (i=1, 2, 3)

$$J(\Omega_i)=\int_Q (\nabla\Omega_i)^2 dQ-2\int_{S+\Sigma}\Omega_i(\mathbf{r}\times\nu)_i dS. \qquad (17)$$

As a result potentials Ω_i are defined as functions three-dimensional/space variables x, y, z parameters $\beta_i(t)$, characterizing position of free surface of liquid.

Thus, during selected division of velocity potential (12) and represen ions (15), (16) together with quasi-speeds $v_0(t)$ and $\omega(t)$ to determination are subject also unknown functions of time $\beta_i(t)$ and $R_n(t)$. For obtaining the equations of motion of system we will use variation principle of (11).

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After simple, but sufficient bulky conversions we will obtain the infinite system of the nonlinear ordinary differential equations of the following form:

$$\mathbf{M}[\overset{*}{\mathbf{v}}_0+\omega\times\mathbf{v}_0-\mathbf{g}+\dot{\omega}\times\mathbf{r}_c+\omega\times(\omega\times\mathbf{r}_c)]+$$
$$+m_1\overset{**}{\mathbf{r}}_{c_1}+2m_1(\omega\times\mathbf{r}_{c_1})=\mathbf{F}; \qquad (18)$$
$$(\mathbf{J},\dot{\omega})+(\mathbf{J}^1,\omega)+\omega\times(\mathbf{J},\omega)+M\mathbf{r}_c\times(\overset{*}{\mathbf{v}}_0+\omega\times\mathbf{v}_0-\mathbf{g})+$$

$$+\omega\times\overset{*}{\mathbf{l}}_{\omega}+\mathbf{l}_{\omega}-\mathbf{l}_{\omega t}-\omega\times\mathbf{l}_{\omega t}=\mathbf{M}^{\circ};\tag{19}$$

$$A_n-\sum_k A_{nk}R_k=0,\quad (n=1,2,\ldots);\tag{20}$$

$$\sum_n R_n\frac{\partial A_n}{\partial\beta_i}+\frac{1}{2}\sum_n\sum_k\frac{\partial A_{nk}}{\partial\beta_i}R_nR_k+\left(\dot{\omega}\left(\frac{\partial\mathbf{l}\omega}{\partial\beta_l}-\frac{\partial\mathbf{l}\omega t}{\partial\beta_i}\right)\right)+$$
$$+\left(\omega\left[\frac{\partial\mathbf{l}\omega t}{\partial\beta i}-\left(\frac{\partial\mathbf{l}\omega t}{\partial\beta i}\right)^*\right]\right)+\left((\overset{*}{\mathbf{v}}_0+\omega\times\mathbf{v}_0-\mathbf{g})\ \frac{\partial\mathbf{l}}{\partial\beta i}\right)-$$
$$-\frac{1}{2}\left(\omega\ \left(\frac{\partial\mathbf{J}^1}{\partial\beta i}\ \ \omega\right)=0\quad (i=1,2,\ldots),\tag{21}$$

where $A_n=\rho\int\limits_Q\varphi_n dQ;\quad A_{nk}=A_{kn}=\rho\int\limits_Q(\nabla\varphi_n,\ \nabla\varphi_k)\,dQ;$

$$\mathbf{l}_{\omega}=\rho\int\limits_Q\Omega dQ;\ \ \mathbf{l}_{\omega t}=\rho\int\limits_Q\frac{\partial\Omega}{\partial t}dQ;\ \ \mathbf{l}=\rho\int\limits_Q\mathbf{r}dQ;\ \ J^1_{ij}=\rho\int\limits_Q(\nabla\Omega_i,\ \nabla\Omega_j)\,dQ;\tag{22}$$

m_1 - mass of liquid; M - mass of entire system; $\mathbf{r}_c$ - radius-vector of the center of mass of system; $\mathbf{r}_{c_1}$ - radius-vector of the center of mass of liquid; F - main vector of all effective forces, applied to solid body; M° - main moment of these forces relative to point O; J - tensor of the inertia of mechanical system, which consists of the tensor of the inertia of solid body J° and certain tensor of second order J^1, called the tensor of the inertia of liquid. In equations (19)...(21) asterisks designated the vectors, whose projections on the axis of system Oxyz are equal to derivatives of the projections on them of the corresponding vectors. The right sides of the equations of forces (18) and moments/torques (19) are determined by the character of the decided problem. For example, among the forces, which function on the flight vehicle, are distinguished the engine thrust, the aerodynamic forces, control forces, Coriolis forces, connected with the relative particle motion within the revolving housing of apparatus, and forces, caused by the displacements of the center of mass of system relative to housing.

Nonlinear equations (20) and (21) reflect mobility of liquid in cavity of body, moreover system of equations (20) is result of satisfaction of kinematic free-surface conditions of liquid. As a result of its linearity relative to parameters $R_k(t)$ it can be permitted relative to these parameters, after which from equations (21) is obtained the infinite system of the nonlinear ordinary differential equations of the second order only relative to generalized coordinates $\beta_i(t)$.

For practical purposes only finite-dimensional analogs of system of equations can be used (18)...(21).

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One of the efficient versions of this system is obtained during the introduced higher limitations by the amplitudes of the oscillations of the free surface of liquid and the angular velocity of the motion of body in the case of the cavity, formed by straight/direct circular cylinders [6].

Assuming that basic nonlinear phenomena in controlled system occur into vicinity of major resonance of free surface of liquid [9], form of free surface $f(\xi, \eta, t)$ and velocity potential $\varphi(x, \xi, \eta, t)$

in cylindrical system $Ox\xi\eta$ let us structurally assign in the form

$$f(\xi, \eta, t) = p_0(t) f_{01}(\xi) + [r_1(t)\sin\eta + p_1(t)\cos\eta] f_{11}(\xi) + + [r_2(t)\sin 2\eta + p_2(t)\cos 2\eta] f_{21}(\xi); \quad (23)$$

$$\varphi(x, \xi, \eta, t) = P_0(t)\psi_{01}(x, \xi) + [R_1(t)\sin\eta + P_1(t)\cos\eta]\psi_{11}(x, \xi) + + [R_2(t)\sin 2\eta + P_2(t)\cos 2\eta]\psi_{21}(x, \xi), \quad (24)$$

where

$$f_{mn}(\xi) = Y_m(k_{mn}\xi) = \frac{J_m(k_{mn}\xi) N'_m(\zeta_{mn}) - N_m(k_{mn}\xi) J'_m(\zeta_{mn})}{J_m(\zeta_{mn}) N'_m(\zeta_{mn}) - N_m(\zeta_{mn}) J'_m(\zeta_{mn})}; \quad (25)$$

$$\psi_{mn} = \frac{\operatorname{ch} k_{mn}(x - x_0)}{\operatorname{ch} k_{mn} h} Y_m(k_{mn}\xi); \quad (26)$$

$J_m(k_{mn}\xi)$ and $N_m(k_{mn}\xi)$ - function of Bessel and Neumann of m order; $\zeta_{mn} = k_{mn}R_1$ - roots of transcendental equation

$$J'_m(\delta\zeta) N'_m(\zeta) - N'_m(\delta\zeta) J'_m(\zeta) = 0; \quad (27)$$

$\delta = R_0/R_1$; R_0 and R_1 - radii of internal and jackets respectively. Retaining in equations (18)...(21) the small third-order quantity relative to parameters $r_1(t)$ and $p_1(t)$, which characterize the skew-symmetric oscillations of the free surface of liquid, and counting parameters $p_0(t)$, $r_2(t)$ and $p_2(t)$ by the values of order r^2_1 and p^2_1, let us write out the scalar analog of system of equations (18)...(21).

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Equations of forces in projections on axis of body coordinate system with center of inertia G of system body - liquid with hardened in undisturbed state liquid are represented in the following form:

$$Mw_x+\lambda_{1x}(\dot{r}_1r_1+\dot{p}_1p)^{\cdot}+\lambda[2(\omega_y\dot{r}_1-\omega_z\dot{p}_1)+$$
$$+r_1(\dot{\omega}_y+\omega_x\omega_z)-p_1(\dot{\omega}_z-\omega_x\omega_y)]=F_x; \tag{28}$$
$$Mw_y+\lambda[\ddot{p}-2\omega_x\dot{r}_1-r_1(\dot{\omega}_x-\omega_y\omega_z)-p_1(\omega_x^2+\omega_z^2)]+$$
$$+\lambda_{1x}\left[\omega_z(r_1^2+p_1^2)^{\cdot}+\frac{1}{2}\dot{\omega}_z(r_1^2+p_1^2)\right]=F_y; \tag{29}$$
$$Mw_z+\lambda[\ddot{r}_1+2\omega_x\dot{p}_1+p_1(\dot{\omega}_x+\omega_y\omega_z)-r_1(\omega_x^2+\omega_y^2)]-$$
$$-\lambda_{1x}\left[\omega_y(r_1^2+p_1^2)^{\cdot}+\frac{1}{2}\dot{\omega}_y(r_1^2+p_1^2)\right]=F_z, \tag{30}$$

where through w_x, w_y, w_z are designated projections of apparent acceleration $\mathbf{w}_ж=\dot{\mathbf{v}}_G+\boldsymbol{\omega}\times\mathbf{v}_G-\mathbf{g}$ on axis of body coordinate system. Momental equations relative to the principal axes of inertia solid bodies at point G will take the form

$$J_{11}^0\dot{\omega}_x+(J_{33}^0-J_{22}^0)\,\omega_y\omega_z=M_x+M_{xж}+M_{xж}^{\omega}; \tag{31}$$
$$(J_{22}^0+G_{22}^0)\,\dot{\omega}_y+(J_{11}^0-J_{33}^0-G_{22}^0)\,\omega_x\omega_z=M_y+M'_{yж}+M_{yж}^{\omega}; \tag{32}$$
$$(J_{33}^0+G_{22}^0)\,\dot{\omega}_z-(J_{11}^0-J_{22}^0-G_{22}^0)\,\omega_x\omega_y=M_z+M_{zж}+M_{zж}^{\omega}, \tag{33}$$

where moments/torques $M_{xж}$, $M_{xж}^{\omega}$, $M_{yж}^{\omega}$, $M_{yж}$, $M_{zж}$, $M_{zж}^{\omega}$, which appear as a result of the mobility of liquid, are determined by the expressions

$$M_{xж}=\mu_1(r_1\ddot{p}-\ddot{r}p_1);$$
$$M_{xж}^{\omega}=\lambda_0(\dot{\omega}_yp_1+\dot{\omega}_zr_1+\omega_x\omega_yr_1-\omega_x\omega_zp_1)-\lambda(p_1w_z-r_1w_y)-$$
$$-\mu_1\left[\dot{\omega}_x(r_1^2+p_1^2)+\omega_x(r_1^2+p_1^2)^{\cdot}\right];$$
$$M_{yж}=\lambda_0\ddot{r}_1-c_1(\dot{r}_1r_1^2+r_1p_1\dot{p}_1)^{\cdot}-c_2(r_1p_1\dot{p}_1-\dot{r}_1p_1^2)^{\cdot}-c_3(\dot{r}_1p_0)^{\cdot}-$$
$$-c_4(\dot{p}r_2-r_1p)^{\cdot}-c_5(r_1\dot{p}_0)^{\cdot}-c_6(p_1\dot{r}_2-r_1\dot{p}_2)^{\cdot};$$
$$M_{yж}^{\omega}=\lambda_0[2\omega_x\dot{p}_1+p_1(\dot{\omega}_x+\omega_y\omega_z)+r_1(\omega_z^2-\omega_x^2)]-G_{22}^1(p_0\omega_y)^{\cdot}-$$
$$-G_{22}^2(p_2\omega_y+r_2\omega_z)^{\cdot}-G_{22}^3(r_1^2\omega_y-r_1p_1\omega_z)^{\cdot}-G_{22}^4(p_1^2\omega_y+r_1p_1\omega_z)^{\cdot}+$$

$$+\mu_1(r_1\dot{p}_1-\dot{r}_1 p)\omega_z-\lambda r_1 w_x+\frac{1}{2}\lambda_{1x}(r_1^2+p_1^2)w_z; \quad (34)$$

$$M_{z ж}=-\ddot{\lambda}_0 p_1+c_1(\dot{p}_1 p_1^2+r_1 p_1\dot{r}_1)^{\cdot}-c_2(\dot{p}_1 r_1^2-r_1 p_1\dot{r}_1)^{\cdot}+c_3(\dot{p}_1 p_0)^{\cdot}+$$
$$+c_4(\dot{p}_1 p_2+\dot{r}_1 r_2)^{\cdot}+c_5(p_1\dot{p}_0)^{\cdot}+c_6(p_1\dot{p}_2+r_1\dot{r}_2)^{\cdot};$$

$$M_{z ж}^{\omega}=\lambda_0\left[2\dot{r}_1\omega_x+r_1(\dot{\omega}_x-\omega_y\omega_z)+p_1(\omega_x^2-\omega_y^2)\right]-G_{22}^{1}(p_0\omega_z)^{\cdot}+$$
$$+G_{22}^2(p_2\omega_z-r_2\omega_y)^{\cdot}-G_{22}^3(p_1^2\omega_z-r_1 p_1\omega_y)^{\cdot}-G_{22}^4(r_1^2\omega_z+r_1 p_1\omega_y)^{\cdot}-$$
$$-\mu_1(r_1\dot{p}_1-\dot{r}_1 p_1)\omega_y+\lambda p_1 w_x-\frac{1}{2}\lambda_{1x}(r_1^2+p_1^2)w_y.$$

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System of equations relative to generalized coordinates $r_i(t)$ and $p_i(t)$ takes the following form:

$$\mu_1(\ddot{r}_1+\varkappa_{11}w_x r_1)+L_1(r_i,\ p_i)+L_1^{\omega}(r_i,\ p_i,\ \omega)+\lambda w_z-\lambda_0\dot{\omega}_y=0;$$
$$\mu_1(\ddot{p}_1+\varkappa_{11}w_x p_1)+L_2(r_i,\ p_i)+L_2^{\omega}(r_i,\ p_i,\ \omega)+\lambda w_y+\lambda_0\dot{\omega}_z=0;$$
$$\mu_0(\ddot{p}_0+\varkappa_{01}w_x p_0)+L_3(r_i,\ p_i)+L_3^{\omega}(r_i,\ p_i,\ \omega)=0; \quad (35)$$
$$\mu_2(\ddot{r}_2+\varkappa_{21}w_x r_2)+L_4(r_i,\ p_i)+L_4^{\omega}(r_i,\ p_i,\ \omega)=0;$$
$$\mu_2(\ddot{p}_2+\varkappa_{21}w_x p_2)+L_5(r_i,\ p_i)+L_5^{\omega}(r_i,\ p,\ \omega)=0$$

where

$$L_1(r_i,\ p_i)=d_1 r_1(r_1\dot{r}_1+p_1\dot{p}_1)^{\cdot}+d_2(p_1\ddot{r}_1+2p_1\dot{r}\dot{p}_1-r_1 p_1\ddot{p}_1-2r_1\dot{p}_1^2)-$$
$$-d_3(p_2\dot{r}_1-r_2\dot{p}_1)^{\cdot}+d_4(r_1\ddot{p}_2-p_1\ddot{r}_2)+d_5(p_0\dot{r}_1)^{\cdot}+d_6 r_1\ddot{p}_0;$$

$$L_1^{\omega}(r_i,\ p_i,\ \omega)=c_1(r_1^2\dot{\omega}_y-r_1 p_1\dot{\omega}_z-r_1\dot{p}_1\omega_z-p_1\dot{p}_1\omega_y)-c_2(p_1^2\dot{\omega}_y+r_1 p_1\dot{\omega}_z+$$
$$+3r_1\dot{p}_1\omega_z+3p_1\dot{p}_1\omega_y)+c_3(p_0\omega_y)^{\cdot}-c_4(p_2\omega_y+r_2\omega_z)^{\cdot}-c_5\omega_y\ddot{p}_0+$$
$$+c_6(\omega_y\dot{p}_2+\omega_z\dot{r}_2)+\mu_1(p_1\dot{\omega}_x+2\omega_x\dot{p}_1-r_1\omega_x^2)+\lambda_0\omega_x\omega_z+$$
$$+G_{22}^3\omega_y(\omega_z p_1-\omega_y r_1)-G_{22}^4(\omega_y p_1+\omega_z r_1)\omega_z;$$

$$L_2(r_i,\ p_i)=d_1 p_1\,(r_1\dot r_1+p_1\dot p_1)^{\cdot}+d_2\,(r_1^2\ddot p_1-r_1 p_1\ddot r_1+2r_1\dot r_1\dot p-2p_1\dot r_1^2)+$$
$$+d_3\,(p_2\dot p_1+r_2\dot r_1)^{\cdot}-d_4\,(p_1\ddot p_2+r_1\ddot r_2)+d_5\,(p_6\dot p_1)^{\cdot}+d_6 p_1\ddot p_0;$$
$$L_2^{\omega}(r_i,\ p_i,\ \omega)=-c_1\,(p_1^2\dot\omega_z-r_1p_1\dot\omega_y-r_1\omega_z\dot r_1-p_1\omega_y\dot r_1)+c_2\,(r_1^2\dot\omega_z+$$
$$+r_1p_1\dot\omega_y+3r_1\dot r_1\omega_z+3p_1\dot r_1\omega_y)^{\cdot}-c_3(p_0\omega_z)^{\cdot}-c_4\,(p_2\omega_z-r_2\omega_y)^{\cdot}+c_5\dot p_0\omega_z-$$
$$-c_6\,(\dot r_2\omega_y-p_2\omega_z)-\mu_1\,(r_1\dot\omega_x+2\dot r_1\omega_x+\omega_x^2p_1)+\lambda_0\omega_x\omega_y+G_{22}^3\,(r_1\omega_y\omega_z-$$
$$-\omega_z^2p_1)-G_{22}^4\,(\omega_y\omega_z r_1+\omega_y^2 p_1);$$
$$L_3(r_i,\ p_i)=d_6\,(r_1\ddot r_1+p_1\ddot p)+d_8\,(\dot r_1^2+\dot p_1^2);$$
$$L_3^{\omega}(r_i,\ p_i,\ \omega)=c_3\,(\omega_z\dot p_1-\omega_y\dot r_1-p_1\omega_x\omega_y-r_1\omega_x\omega_z)+c_7\omega_x\,(r_1\dot p_1-\dot r_1p_1)+$$
$$+c_5\,(\omega_y r_1-\omega_z p_1)^{\cdot}-\frac{1}{2}G_{22}'\,(\omega_y^2+\omega_z^2); \qquad (36)$$
$$\bar L_4(r_i,\ p_i)=-d_4\,(\ddot r_1p_1+p\ddot r_1)-2d_7\dot r_1\dot p_1;$$
$$L_4^{\omega}(r_i,\ p_i,\ \omega)=-c_4\,(\omega_y\dot p_1-\omega_z\dot r_1)+c_6\,(\omega_y p_1-\omega_z r_1)^{\cdot}+c_8\,(\dot\omega_x r_1^2+\dot\omega_x p_1^2)+$$
$$+c_{10}\omega_x\,(\dot r_1r_1+p_1\dot p_1)+c_9\,(\dot\omega_x p_2+2\omega_x\dot p_2)-G_{12}^3\omega_x\,(\omega_z p_1+\omega_y r_1)-$$
$$-G_{22}^2\omega_y\omega_z=0;$$
$$L_5(r_i,\ p_i)=d_4\,(\ddot r_1r_1-\ddot p_1p_1)+d_7\,(\dot r^2-\dot p^2);$$
$$L_5^{\omega}(r_i,\ p_i,\ \omega)=c_4\,(\dot r_1\omega_y+\dot p_1\omega_z)-c_6\,(\omega_y r_1+\omega_z p_1)^{\cdot}+2c_8\dot\omega_x r_1p_1+$$
$$+c_{10}\omega_x\,(r_1p_1)^{\cdot}-c_9\,(\dot\omega_x r_2+2\omega_x\dot r_2)+G_{12}^3\omega_x\,(\omega_z r_1-\omega_y p_1)+$$
$$+\frac{1}{2}G_{22}^2\,(\omega_z^2-\omega_y^2).$$

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Hydrodynamic coefficients of equations of motion by (28)...(33), (35) are determined by some quadratures from cylindrical functions. Their expressions are given in work [6], and numerical values - into

[7].

System of equations (28)...(33), (35) is most complete of known systems, which consider interaction of various forms of oscillations of liquid in vicinity of major resonance. The solution on their basis of some particular problems of dynamics and stability of motion of mechanical systems of the type in question showed that the basic physical phenomena are described with the high degree of accuracy both with the quantitative and from qualitative side [11].

Equations given above are generalization of G. S. Narimanov's equations, given in works [9], [10]. They serve as basis for obtaining the equations of the disturbed motion of system body - liquid, suitable for the analysis of stability of its motion when hypothesis about the smallness of all parameters of motion is not applicable. This relates, in particular, to that case, when in the process of motion occurs the resonance excitation of any of the forms of the oscillations of liquid, that leads to the complicated three-dimensional/space free surface motions with the finite amplitude even with the low linear or angular displacements of solid body.

3. Simplification in general/common nonlinear equations of disturbed motion. In conclusion we will obtain the system of the nonlinear equations of the disturbed motion, being based on the system of the hypotheses, utilized usually in the dynamics of flight vehicles. We convert the given above nonlinear system of equations

(28)...(33), (35), after selecting as the undisturbed motion of system its programmed motion, which is determined by the simplified equations of motion of the material point, which coincides with the center of mass of system [1], [12]. The free surface of liquid in the undisturbed motion is close to the plane, perpendicular to the longitudinal axis, and parameters r_i, p_i are considered equal to zero.

In programmed motion yaw angles and bank are usually equal to zero. We will also consider that in the undisturbed motion

$$\psi=0, \ \gamma=0. \qquad (37)$$

During conclusion/output of differential equations of disturbed motion of system let us agree to consider as low: a) disturbance/perturbation of basic parameters, which characterize motion of solid body; b) disturbance/perturbation of mass Δm and moments of inertia of solid body; c) disturbances/perturbations, connected with work of engine; d) time derivative of pitch angles, yaws also of bank, and also angle of attack and its time derivative.

In view of latter/last assumption low values will be also projections of speed v_G on axis Gy and Gz and derivatives of them on time. We will consider the parameters, which characterize the motion of liquid, finite quantities. Let us designate the disturbed values of the basic parameters of motion by prime.

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Then the disturbance/perturbation of the basic parameters can be defined as a difference in their disturbed and quiescent values;

$$\begin{aligned}
&\vartheta(t)=\vartheta'(t)-\vartheta^0(t);\quad \psi(t)=\psi'(t)-\psi^0(t)\equiv\psi'(t);\\
&\gamma(t)=\gamma'(t)-\gamma^0(t)\equiv\gamma'(t);\quad \mathbf{v}_G(t)=\mathbf{v}'_G(t)-\mathbf{v}^0_G(t);\\
&r_i(t)=r_i^{\bullet}(t)-r_i^0(t)\equiv r'_i(t);\quad p_i(t)=p'_i(t)-p_i^0(t)\equiv p'_i(t).
\end{aligned}\tag{38}$$

Basic connections/communications between true and undisturbed values are established/installed taking into account fact that body coordinate system at moment of time t in accordance with programmed and proper motion they occupy different position Gxyz and $G'x'y'z'$ respectively. Transition/transfer from the system of axes in the proper motion to the system of the axes of the undisturbed motion is realized by a matrix of transition/transfer [1]

$$M=\begin{bmatrix} 1 & \vartheta & -\psi \\ -\vartheta & 1 & \gamma \\ \psi & -\gamma & 1 \end{bmatrix}. \tag{39}$$

With an accuracy to small second-order quantities occur following relationships/ratios:

$$\omega'_{x'}=\dot{\gamma};\quad \omega'_{y'}=\dot{\psi};\quad \omega'_{z'}=\dot{\vartheta}^0+\dot{\vartheta}; \tag{40}$$

$$v'_{G_{x'}}=v^0_{G_x}\vartheta+v_x;\quad v'_{G_{y'}}=-v^0_{G_x}\vartheta+v^0_{G_y}+v_y;\quad v'_{G_{z'}}=v^0_{G_x}\psi+v^0_{G_z}+v_z; \tag{41}$$

$$g_{x'}=g_x+g_y\vartheta;\quad g_{y'}=-g_x\vartheta+g_y;\quad g_{z'}=g_x\psi-g_y\gamma+g_z. \tag{42}$$

Converting system of nonlinear equations (28)...(33), (35) taking into account limitations introduced above, we will obtain equations of disturbed motion of object in the following form:

$$M\ddot{x}_{G'}+a\dot{x}_{G'}-mg_y\vartheta+\lambda_{1x}(\dot{r}_1r_1+\dot{p}_1p_1)^{\cdot}+\lambda(2\dot{r}_1\dot{\psi}-\dot{\vartheta}p_1+r_1\ddot{\psi}-p_1\ddot{\vartheta}-$$

$$-\ddot{\vartheta}^0\dot{p}_1-\dot{\vartheta}^0p_1)=c_{x\delta}\delta_x+\Delta F_x; \qquad (43)$$

$$M\ddot{y}_{G'}+b\dot{y}_{G'}+\nu_y\dot{\vartheta}-(P+bv_{G_x})\vartheta+\lambda(\ddot{p}_1-2\dot{\gamma}\dot{r}_1-r_1\ddot{\gamma})=c_{y\delta}\delta_\vartheta+\Delta F_y;$$

$$M\ddot{z}_{G_1}+b\dot{z}_{G_1}+\nu_z\dot{\psi}+(P+dv_{G_x})\psi+mg_y\gamma+\lambda(\ddot{r}_1+2\dot{\gamma}\dot{p}_1+p_1\ddot{\gamma})=$$

$$=c_{z\delta}\delta_\psi+\Delta F_z.$$

$$J_{11}^0\ddot{\gamma}+\mu_x\dot{\gamma}-\lambda_0(p_1\ddot{\psi}+r_1\ddot{\vartheta}^0+r_1\ddot{\vartheta})-\mu_1(r_1\ddot{p}_1-\ddot{r}_1p_1)+\lambda p_1(\dot{v}_{G_z}-\dot{v}_{G_x}\psi+$$

$$+\ddot{z}_{G'}-g_x\psi+g_y\gamma-g_z)-\lambda r_1(\dot{v}_{G_y}-\dot{v}_{G_x}\vartheta+\ddot{y}_{G'}+\dot{\vartheta}^0v_{G_x}+g_x\vartheta-g_y)+$$

$$+c_{\gamma\delta}\delta_\gamma+\Delta M_x;$$

$$(J_{22}^0+G_{22}^0)\ddot{\psi}+\mu_y\dot{\psi}-x_Fb\dot{z}_{G'}-x_Fbv_{G_x}\psi-\lambda_0(2\dot{\gamma}\dot{p}_1+p_1\ddot{\gamma})-\lambda r_1(\dot{v}_{G_x}-g_x+$$

$$+\ddot{x}_{G'}-g_y\vartheta)-M_{y\,ж}=c_{\psi\delta}\delta_\psi+\Delta M_y;$$

$$(J_{33}^0+G_{22}^0)\ddot{\vartheta}+\mu_z\dot{\vartheta}+x_Fb\dot{y}_{G'}-x_Fbv_{Gx}\vartheta-\lambda_0(2\dot{r}_1\dot{\gamma}+r_1\ddot{\gamma})-\lambda p_1(\dot{v}_{Gx}-$$

$$-g_x+\ddot{x}_{G'}-g_y\vartheta)-M_{z\,ж}=c_{\vartheta\delta}\delta_\vartheta+\Delta M_z, \qquad (44)$$

$$\mu_1(\ddot{r}_1+\omega_1^2r_1)+\mu_1(p_1\ddot{\psi}+2\dot{\gamma}\dot{p}_1)-\lambda_0\ddot{\psi}+\lambda(\dot{v}_{Gx}-g_x)\psi+\lambda\ddot{z}_{G'}+$$

$$+\lambda g_y\gamma+L_1(r_i,\ p_i)=0;$$

$$\mu_1(\ddot{p}_1+\omega_1^2p_1)-\mu_1(r_1\ddot{\gamma}+2\dot{r}_1\dot{\gamma})+\lambda_0\ddot{\vartheta}-\lambda(\dot{v}_{Gx}-g_x)\vartheta+$$

$$+\lambda\ddot{y}_{G'}+L_2(r_i,\ p_i)=0;$$

$$\mu_2(\ddot{p}_2+\omega_2^2p_2)+c_3\dot{p}_1(\dot{\vartheta}^0+\dot{\vartheta})-c_3\dot{r}_1\dot{\psi}+c_5(\dot{\psi}r_1-\dot{\vartheta}^0p_1-\dot{\vartheta}p_1)^{\cdot}+ \qquad (45)$$

$$+L_3(r_i,\ p_i)=0;$$

$$\mu_2(\ddot{r}_2+\omega_2^2r_2)-c_1(\dot{\psi}\dot{p}_1-\dot{\vartheta}^0\dot{r}_1-\dot{\vartheta}\dot{r}_1)+c_6(\dot{\psi}p_1-\dot{\vartheta}^0r_1-\dot{\vartheta}r_1)^{\cdot}+$$

$$+L_4(r_i,\ p_i)=0;$$

$$\mu_2(\ddot{p}_2+\omega_2^2p_2)+c_4(\dot{r}_1\dot{\psi}+p_1\dot{\vartheta}^0+\dot{p}_1\dot{\vartheta})-c_6(\dot{\psi}r_1+\dot{\vartheta}^0p_1+\dot{\vartheta}p_1)^{\cdot}+$$

$$+L_5(r_i,\ p_i)=0.$$

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Here $x_{G'}$, $y_{G'}$, $z_{G'}$ are designated coordinates of center of inertia of object in disturbed motion in system of coordinates Gxyz. In the right sides of equations (43) and (44) they cost the projection of the disturbed forces and moments/torques in the direction of the axes of the system of coordinates Gxyz. The parameters of motion $x_{G'}$, $y_{G'}$, $z_{G'}$, ϑ, ψ, γ, r_i and p_i, if is known the position of body axes in the undisturbed motion, completely define the configuration of system at the moment of time t and they can be considered as its generalized coordinates.

When deriving the equations of disturbed motion of forces of (43) and momental equations (44) we have used known analogous equations of disturbed motion of object without taking into account liquid filling [1]. The possibilities of further simplification in the system of equations of the disturbed motion given above can be revealed during the more careful analysis of the kinematic and dynamic properties of concrete/specific mechanical systems.

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RESEARCH OF TRANSIENT PROCESSES DURING THE LARGE DISTURBANCES/PERTURBATIONS OF THE FREE SURFACE OF LIQUID IN THE LOCKED SECTION.

I. B. Bogoryad, I. A. Druzhinin, S. V. Chakhlov.

Application of point-by-point method to calculation of large three-dimensional/space displacements of ideal incompressible fluid with free surface in rigid section is examined. The a posteriori evaluations of the accuracy of the obtained results are proposed. The effect of the method of the application of method on its stability is analyzed. Are given the results of the calculations of the displacements of liquid in the mobile sections of various forms, including under the conditions of weightlessness and in the presence of fluid flow rate from the section.

In nonlinear dynamics of solid body, which contains liquid masses with free surface, great mathematical difficulties appear during solution of corresponding hydrodynamic problems. The conformity of dynamic diagram to real process is determined, as a rule, by the accuracy of the description of the motion of liquid and solution of these problems.

G. S. Narimanov [5] for the first time constructed nonlinear dynamic flow chart of solid body with liquid, which has free surface, and was proposed asymptotic method of solving boundary-value problems of hydrodynamics. In this case as the low parameter the maximum deflection of free surface from the surface of liquid is chosen. G. S. Narimanov's method for the almost thirty years receives further development and continuation in the work of many authors [3, 6, 8, etc.] and it is actually the basic "working" method.

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At the same time G. S. Narimanov's method as any other, constructed during asymptotic approximation/approach, sets limitations by an order of values of unknown functions and their derivatives. The propagation of G. S. Narimanov's method in the case of capillary liquid and sections of complex geometric form is connected with the great mathematical difficulties.

Attempts at solution of nonlinear boundary-value problems of hydrodynamics by variational methods [4, 7, etc.] did not widen in comparison with G. S. Narimanov's method boundary of application of nonlinear theory. This is explained in the basic fact that in these works the exceptionally/exclusively Eulerian approach to the description of the motion of liquid is used.

In connection with this point-by-point method of solution of nonlinear problems of dynamics of liquid with free surface in locked section [1, 2, 9], which combines Eulerian and lagrangian approaches and using sampling process was proposed and realized. In this article point-by-point method applies to the case of mobile cavities and spatial motions of liquid taking into account its outflow from the cavity.

1. Point-by-point method. The boundary-value problem about the irrotational motion of the ideal incompressible capillary fluid in the section, which accomplishes the assigned motion, takes the form

$$\Delta\Phi(z, \rho, \beta, t) = 0 \ \overset{(1)}{\text{в}}\ Q;$$

$$\frac{\partial\Phi}{\partial\nu} = (v_\rho + \zeta\omega_\beta)\zeta' - (v_z - r\omega_\beta)r' + u_S \ \overset{(2)}{\text{на}}\ S; \tag{2}$$

$$\frac{d\rho}{dt} = \frac{\partial\Phi}{\partial\rho} - v_\rho - z\omega_\beta; \quad \frac{dz}{dt} = \frac{\partial\Phi}{\partial z} - v_z + \rho\omega_\rho; \tag{3}$$

$$\rho\frac{d\beta}{dt} = \frac{1}{\rho}\frac{\partial\Phi}{\partial\beta} - v_\beta - \rho\omega_z + z\omega_\beta;$$

$$\frac{d\Phi}{dt} = \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{\mathrm{Fr}}(g_x\rho\cos\beta + g_y\rho\sin\beta + g_z z) + \frac{2H}{\mathrm{We}} \ \overset{(2)}{\text{на}}\ \Sigma; \tag{4}$$

$$\theta = \mathrm{const} \ \overset{(2)}{\text{на}}\ L; \tag{5}$$

$$\Phi(z, \rho, \beta, 0) = \Phi^0; \quad \Sigma(0) = \Sigma^0. \tag{6}$$

Key: (1). in. (2). on.

It is assumed that section has shape of surface of rotation, generatrix to which is assigned by equations

$$\rho = r(l), \ z = \zeta(l), \tag{7}$$

where l - arc length of generatrix.

Here Φ - potential of absolute velocity of liquid in system of coordinates (z, ρ, β connected with section - cylindrical or x, y, z - Cartesian); Q - volume of liquid, limited by free Σ and moistened S by surfaces; ν - unit vector of external normal to $S+\Sigma$, $\mathbf{v}(t)=\{v_z, v_\rho, v_\beta\}$, $\omega(t)=\{\omega_z, \omega_\rho, \omega_\beta\}$ - known vectors of linear and angular velocities of section; $\mathbf{g}=\{g_x, g_y, g_z\}$ - G-vector of potential field of mass forces; H - mean curvature Σ; u_S - normal component of fluid emission rate from section through intake opening; L, θ - line and angle of wetting. Problem is registered in dimensionless variables, Fr, We - numbers of Froude and Veber.

Is introduced replacement of continuous time $t\in[0, T]$ by discrete/digital so that $0\leqslant t=t^n=n\tau\leqslant T;\ n=0, 1, \ldots, T/\tau$.

In terms of values $\Phi^n=\Phi\{z, \rho, \beta, t^n\}$, $(z, \rho, \beta)\in\Sigma^n$ with the help of difference analogues of boundary conditions (3), (4) and condition (5) known for certain moment of time t^n are determined form of free surface Σ^{n+1} and potential on it Φ^{n+1} at moment of time t^{n+1}.

For determination of potential Φ^{n+1} in volume of liquid, limited when $t=t^{n+1}$ by surfaces Σ^{n+1} and S^{n+1}, is solved boundary-value problem

$$\Delta F^{n+1}=0 \overset{(1)}{\text{B}} Q;$$

$$\frac{\partial F^{n+1}}{\partial \nu}=\left(v_\rho^{n+1}+\zeta\omega_\beta^{n+1}\right)\zeta'-\left(v_z^{n+1}-r\omega_\beta^{n+1}\right)r'+u_S \overset{(2)}{\text{Ha}} S^{n+1}; \qquad (8)$$

$$F^{n+1}=\Phi^{n+1} \overset{(2)}{\text{Ha}} \Sigma^{n+1}.$$

Key: (1). in. (2). on.

Solution of problem (8), which let us name base, makes it possible to find Σ^{n-2} and $\Phi^{n-2}[(z, \rho, \beta) \in \Sigma^{n-2}]$. Further the cycle of calculations is repeated.

2. Numerical realization of point-by-point method. In contrast to [1], where variational method of Ritz is used, here the solution of boundary-value problem (8) is carried out by the methods of least squares and collocation. In the latter case to S, Σ and the lines of the three-phase contact L are introduced respectively the set of the control points

$$\begin{aligned} S_j &= (\zeta(l_j),\ r(l_j),\ \beta_j), \quad j = 1,2,\dots,J; \\ \Sigma_i &= (z_i,\ \rho_i,\ \beta_i), \quad i = 1,2,\dots,I; \\ L_k &= (\zeta(l_k),\ r(l_k),\ \beta_k), \quad k = 1,2,\dots,K \end{aligned} \tag{9}$$

and bounding surface is approximated by linear triangular elements with the apexes/vertexes at the control points.

During axisymmetric flows of liquid instead of (9) control points are introduced only on forming free surface

$\Sigma_i = (\rho_i, z_i)$, $i = 1, 2, \dots, I-1$; $\Sigma_I = (r(l_I), \zeta(l_I))$ and most generatrix is approximated by interpolation parametric cubic spline with parametrization along total chord length, which combine control points. In this case accurately is satisfied condition (5) and is computed mean curvature H of surface Σ.

Solution (8) is sought/found in the form

$$F^{n+1}=\sum_{m=0}^{M} C_m^{n+1}\varphi_m(z,\rho,\beta)+\Psi^{n+1}(z,\rho,\beta), \qquad (10)$$

where φ_m - harmonic polynomials, and function Ψ^{n+1} harmonic in Q satisfies condition

$$\frac{\partial\Psi^{n+1}}{\partial\nu}=u_S \overset{(1)}{\text{на}} S^{n+1}.$$

Key: (1). on.

Coefficients C_m^{n+1} are determined from condition of minimum of functional, constructed according to method of least squares

$$J_1^{n+1}=\int_{S^{n+1}}\left(\sum_{m=0}^{M} C_m^{n+1}\frac{\partial\varphi_m}{\partial\nu}+\frac{\partial\Psi^{n+1}}{\partial\nu}-\frac{\partial\Phi^{n+1}}{\partial\nu}\right)^2 dS+$$

$$+\gamma\int_{\Sigma^{n+1}}\left(\sum_{m=0}^{M} C_m^{n+1}\varphi_m+\Psi^{n+1}-\Phi^{n+1}\right)^2 dS, \qquad (11)$$

or according to method of collocation

$$J_2^{n+1}=\delta_1+\gamma\delta_2,$$

$$\delta_1=\sum_{p=1}^{P}\left[\sum_{m=0}^{M} C_m^{n+1}\frac{\partial\varphi_m(S_p)}{\partial\nu}+\frac{\partial\Psi^{n+1}(S_p)}{\partial\nu}+\frac{\partial\Phi^{n+1}(S_p)}{\partial\nu}\right]^2; \qquad (12)$$

$$\delta_2=\sum_{p=1}^{P}\left[\sum_{m=0}^{M} C_m^{n+1}\varphi_m(\Sigma_p)+\Psi^{n+1}(\Sigma_p)-\Phi^{n+1}(\Sigma_p)\right]^2. \qquad (13)$$

Here γ - numerical parameter, Σ_p, S_p - units of collocation on

Σ_{n-1} and S^{n-1} respectively.

Parameter l into (7) is calculated from difference equation

$$\frac{dl}{dt}=\left(\frac{\partial\Phi}{\partial\rho}-v_\rho-\zeta\omega_3\right)\frac{dr}{dl}+\left(\frac{\partial\Phi}{\partial z}-v_z+r\omega_\rho\right)\frac{d\zeta}{dl} \qquad (14)$$

according to diagram predictor - corrector. Difference analogues of equations (3), (4) are calculated according to the same diagram.

3. Control/check of accuracy and organization of calculations. An error in the solution of problem by point-by-point method is determined by an error in the solution of base problem (8) at each step/pitch and by errors in the integration for the time of boundary conditions (3), (4). In turn, an error in solution (8) depends on errors linear algebraic equations initial data in the system relatively C_m^{n+1}, errors in the solution of this system of equations, M number and approximating properties of functions φ_m. Complete error analysis is very complex. Therefore we will be bounded to the examination of the quantitative criteria of the accuracy of solution (8) and role of the parameter γ utilized in the calculations of (11), (12) in control of the value of the resulting error.

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As the criteria indicated are selected the a posteriori evaluations of discrepancies in satisfaction of boundary conditions to Σ and S, which were calculated by the formulas

$$\varepsilon_1=\frac{\sqrt{\delta_1}}{P},\quad \varepsilon_2=\frac{\sqrt{\delta_2}}{P}, \qquad (15)$$

where δ_1, δ_2 they are determined from formulas (13), in which S_p and Σ_p - points, evenly distributed on S and Σ. In the case of applying the functional (12) these points should not be placed in the units of collocation.

Criteria ϵ_1 and ϵ_2 make sense of measure of absolute errors in satisfaction of boundary conditions on S and Σ.

Fig. 1 gives characteristic graph/diagrams of dependences $\epsilon_1(\gamma)$, $\epsilon_2(\gamma)$.

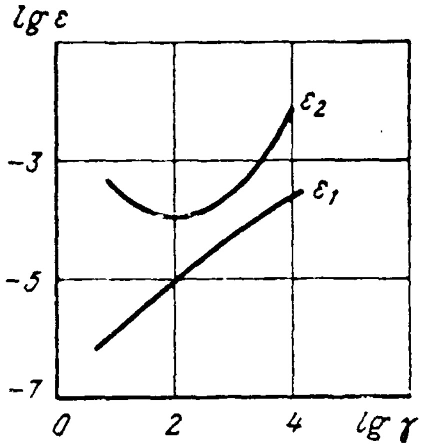

Fig. 1.

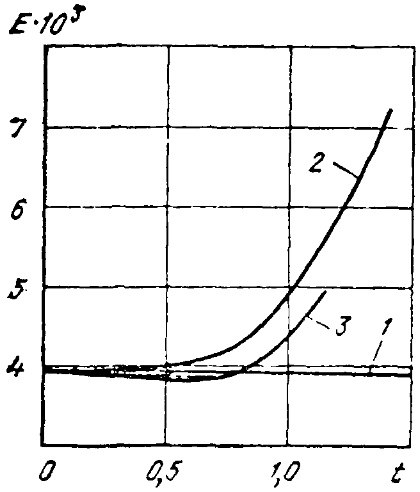

Fig. 2.

Fig. 1. Dependence of absolute errors in satisfaction of boundary conditions on S and Σ on numerical parameter γ.

Fig. 2. Change in time of total energy of motion of liquid depending on diagram of integration for time: 1 - diagram predictor - corrector (τ=0.02); 2, 3 - Euler's diagram with τ=0.02 and 0.01.

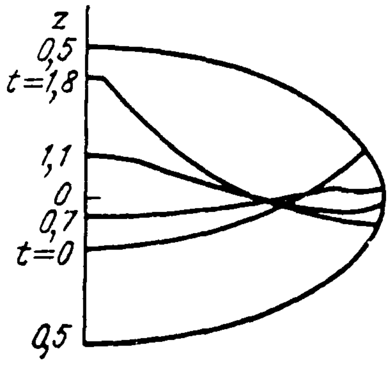

Fig. 3.

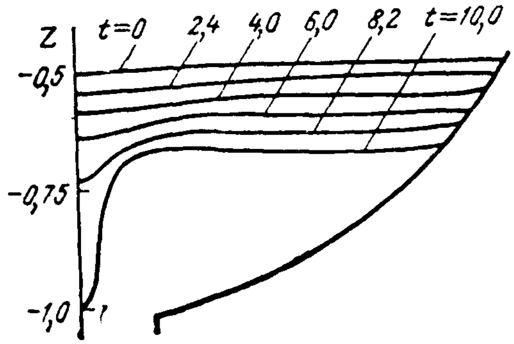

Fig. 4.

Fig. 3. Motion of liquid in section of ellipsoidal form.

Fig. 4. Displacement of liquid, exhausted from spherical section without waves on free surface.

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It is evident that there is an optimum value γ, to which correspond the smallest errors in the solution in the assigned level of the accuracy of intermediate calculations (integration, the solution of linear system of equations, etc.). The presence of optimum is explained by dependence on γ the contribution to the matrix elements of the system of linear equations, which introduce integrals (sums) Σ and S and by change in this case the numbers of conditionality of matrix.

At each step/pitch on time calculation ϵ_1 and ϵ_2 is used for assignment of value γ at following time step, i.e., criteria (15) make it possible not only to consider accuracy of obtained solution, but also it is correct to organize numerical process.

Accuracy in the integration for time of difference analogues of equations (3), (4) and (14) a posteriori is evaluated according to accuracy of accomplishing of law of conservation of total energy of liquid. By calculation it is established, just as into [10], where it is examined the motion of liquid in the cylinder, that the deflection of the instantaneous value of total energy from the initial is extremely sensitive to errors in the method of integrating the boundary conditions. This is important for organizing the calculations fact, since due to errors in the discretization/digitization, generally speaking, it is not possible to expect the constancy of total energy, and therefore it is expedient to track the sufficiently lasting tendencies in its changes.

As illustration of results of control of numerical process Fig. 2 gives change in time of total energy of axisymmetric motion of liquid, which is developed in spherical section from initial state $\Phi^0 = 0$ $(\rho,\ z = -0{,}59 + 0{,}24\rho^2) \in \Sigma^0$. It is evident that the first diagram is numerically stable (value of total energy it remains constant in margins of error in the solution of base problem), the second - it is unstable (value of total energy rapidly it grows).

It is interesting that instability of calculation (in evaluations according to energy) is developed against the background of constant/invariable within limits of working accuracy of volume of liquid mass. Computed value of volume is completely determined by the accuracy of the solution of base problem; therefore it can serve as the criterion of this accuracy. Calculating experiment shows that if the base problem for the certain step/pitch n is solved insufficiently accurately, then at the following steps/pitches the value of volume rapidly "is swung".

4. Numerical results. Let us consider examples of the calculation of the motions of liquid within the sections of various forms. We will assume that at the initial moment of time the liquid is found in state of rest relative to the moistened surface, and then either section is set in motion or instantly varies the field of mass

forces.

Motion of liquid, which half fills it cut off ellipsoidal form, when $g/Fr=(0, 0, -1)$, $We=\infty$ it is represented in Fig. 3. It is evident that nonlinear interaction of different tones of oscillations leads to the appearance of fine/small waves on the free surface of liquid near the wall of section.

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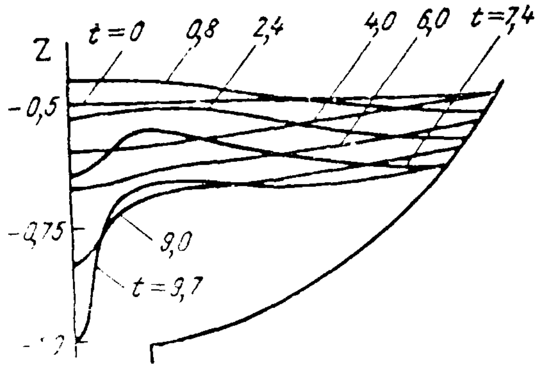


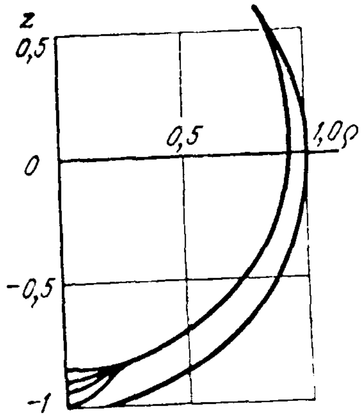
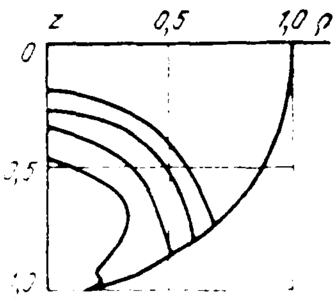


Fig. 5.

Fig. 6.

Fig. 5. Formation of funnel/hopper in presence of waves on free surface.

Fig. 6. Formation of funnel/hopper under conditions of weightlessness ($\theta=5°$).



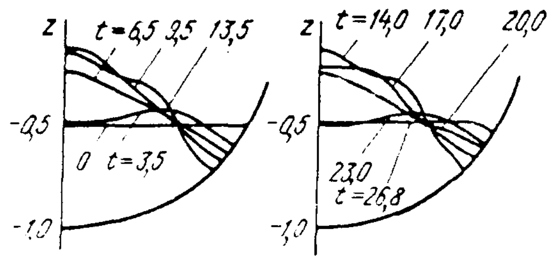


Fig. 7.

Fig. 8.

Fig. 7. Formation of funnel/hopper under conditions of weightlessness ($\theta = 120°$)

Fig. 8. Change in form of free surface of liquid, which instantly loses weight ($\theta = 90°$)

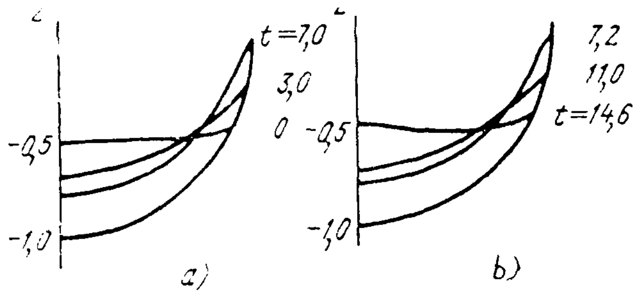


Fig. 9. Change in form of free surface of liquid, which instantly loses weight [illegible]

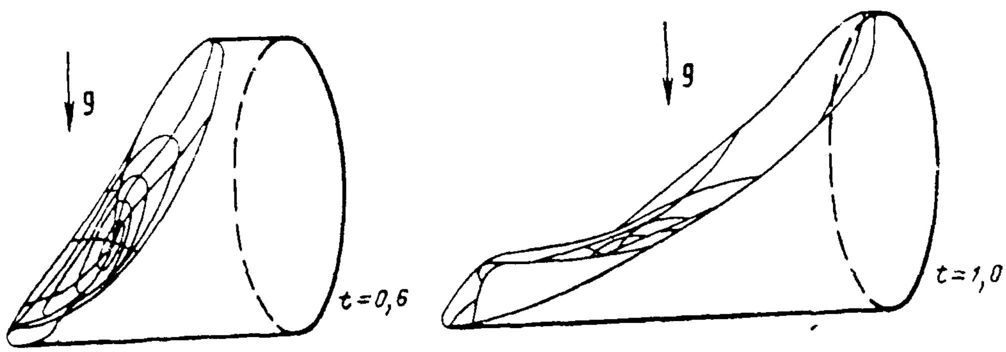


Fig. 10. Motion of liquid in cylindrical section with horizontal generatrix.

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Further is shown formation of axisymmetric funnel/hopper on free surface of liquid, exhausted from spherical section in the absence of waves on free surface (Fig. 4) and in their presence (Fig. 5). It turned out that when $g/Fr=(0, 0, -1)$; $We=\infty$; $u_s=0{,}5$ the oscillations of liquid on the first axisymmetric tone contribute to the more rapid formation of funnel/hopper.

Numerical experiment established that with suction of liquid from spherical section under conditions of weightlessness volume of liquid, which remains in section at moment of entry of gas into intake opening, depends substantially on angle of wetting. The results of calculations with the filling of section to level $z=-0.5$ $(Fr=\infty, We=10^3, u_s=0.5)$ are given in Fig. 6, 7. There is an optimum angle of the wetting, for which in the given ones expenditure/consumption and other determining parameters the remainder/residue of liquid is minimum.

Behavior of liquid, which fills spherical section to 0.156 of entire volume and that instantly losing weight shown in Fig. 8 for $Fr=\infty$, $We=10^3$, $\theta=90°$ and 9 for $Fr=\infty$, $We=10^2$, $\theta=30°$. It is evident that the motion of liquid, shown in Fig. 8, is accompanied by complicated interaction of fine/small waves and qualitatively it differs from smooth and regular capillary waves by Fig. 9.

Fig. 10 shows change of form of liquid volume in straight/direct circular cylinder, filled at depth, equal to radius after direction of acceleration of field of mass forces instantly was changed on 90°, i.e., section proved to be lying/horizontal "on side". With an instantaneous change in the direction of the acceleration of the field of mass forces on 31.3° in plane zy and during the rotation of section with $\omega_x=1$, $\omega_y=\omega_z=0$ around the axis, passing through the center of free surface, the form of liquid volume is shown in Fig. 11. It is well noticeable that during the rotation of section the free surface of liquid longer retains flat/plane form.

Let in spherical section of unit radius be located volume of liquid in the form of spherical segment by height/altitude 0.1. At first liquid segment is deflected from direction g at angle of 45°. Fig. 12 shows the evolution of the form of this segment in the course of time under the action of gravitational force.

If at initial moment of time it cut off with liquid it begins to quickly move to the right with acceleration of

$\frac{dv_y}{dt} = 0{,}5g_z$ (this equivalent to instantaneous change in direction of acceleration of field of mass forces at angle of 31.3°), then form of liquid volumes within sections in the form of cone and straight/direct circular cylinder with hemispherical cover/cap will be such as it is depicted in Fig. 13 and 14 respectively.

Fig. 15 shows form of liquid volume, which is located in cylindrical section with hemispherical cover/cap, under the same external influences, as in Fig. 11. The comparison of Fig. 11 and 15 shows that the motion of the center section of the free surface during the rotation of section weakly depends on its form.

Point-by-point method makes it possible to calculate transient processes in liquid and during other, more complicated motions of section.

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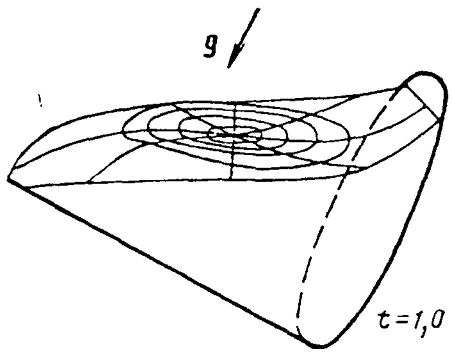

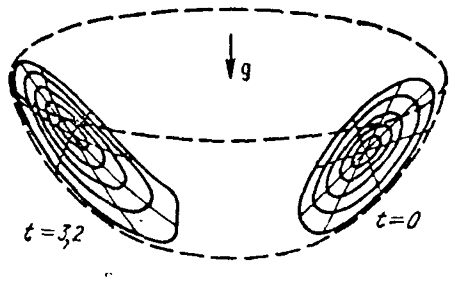

Fig. 11. Fig. 12.

Fig. 11. Characteristic forms of volume of liquid in cylindrical section, which accomplishes progressive/forward and rotations.

Fig. 12. Characteristic forms of volume of liquid of low depth in spherical section.

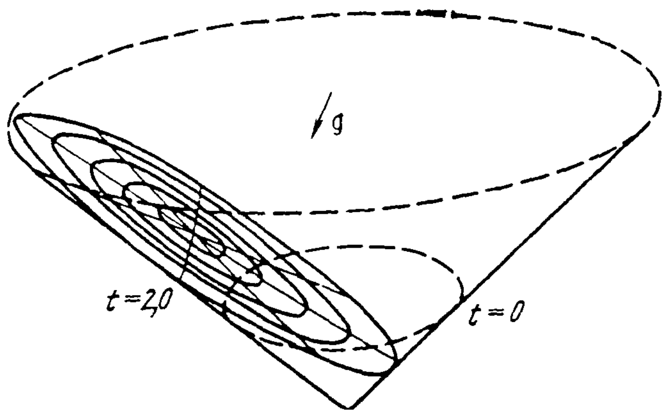

Fig. 13. Form of liquid volume in quickly moving/driving conical section.

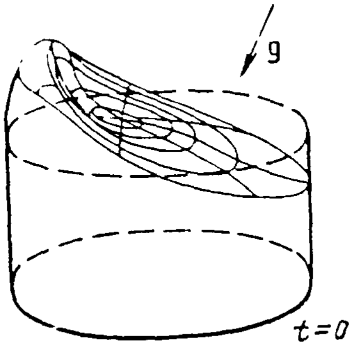

Fig. 14.

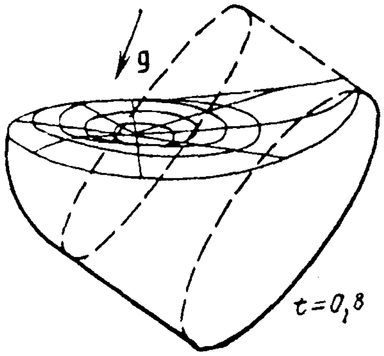

Fig. 15.

Fig. 14. Form of volume of liquid in quickly moving/driving cylindrical section with spherical cover/cap.

Fig. 15. Displacement of liquid in cylindrical section with spherical cover/cap, which accomplishes progressive/forward and rotations.

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About the motion of solid body with the liquid, whose free surface is closed with the nonlinearly deformed shell.

V. A. Trotsenko.

Are derived equations of disturbed motion of solid body with cavity, wholly filled with liquid, part of surface of which is axisymmetrically deformed by hydrostatic pressure dome-shaped shell of highly elastic material. Investigated methods of solving the corresponding boundary-value problems, connected with the determination of the hydroelastic coefficients of the obtained equations. Are given some results of the calculations of the free oscillations of mechanical system liquid - shell.

As one of means of limitation of mobility of liquid in axisymmetric vessels structurally/design can be used preliminarily deformed soft shell of rotation, coaxially attached on walls of vessel and closing free surface of liquid.

In the present work setting and solution of nonlinear problem about determination of state of equilibrium of shell, prepared from highly elastic material and which is located under hydrostatic pressure is given. According to G. S. Narimanov's method in the linear setting are derived the equations of the disturbed motion of mechanical system body - liquid - shell and is proposed the approximation method of determining the hydroelastic coefficients of those obtained equation. The results of the calculations of the frequency characteristics of the oscillations in coupled circuit of liquid and deformed shell are given.

1. It is known that under conditions of weightlessness reorientation of liquid in capacity/capacitance in comparison with its usual position under conditions of gravitation is possible. Furthermore, under these conditions under the influence of weak disturbances/perturbations the presence of free surface unavoidably leads to the disturbance of the continuity of liquid, and the latter is converted into the blistered mass, it falls into the series/row of the large and small drops, distributed throughout entire volume of vessel. In connection with this in the engineering practice are used different kind the pressurization diaphragms, whose basic purpose - the stationary isolation/evolution of gas from the liquid taking into account the possibility of drain or servicing with the latter. The devices/equipment, prepared on the basis of the synthetic rubbers, which possess the high coefficients of elongation/aspect ratio and ultimate tensile strengths, are most promising of such type of devices/equipment. The principle of the work of pressurization diaphragms from the elastomers can be clarified based on the following example. Let the initially flat/plane free surface of liquid, which is located in the axisymmetric vessel, be closed with the circular membrane/diaphragm from the material in question rigidly attached on the walls of vessel. Then a change of the volume of liquid in the

vessel will be accompanied by the ultimate strains of the membrane/diaphragm, which forms in this case a certain axisymmetric surface. In the more general case the diaphragm can be prepared in the form of arbitrary shell of the rotation of variable/alternating thickness closed in the pole.

Analysis of behavior of liquid in cavities with pressurization devices/equipment from elastomers and account of its effect on dynamics of body it is connected with solution of whole complex of problems. The task of the definition of the elastic deformation of diaphragm, which during the defined limitations to fluid emission rate from the cavity can be considered as the task about the ultimate strains of the highly elastic shells of revolution under the effect of hydrostatic load, is one of them. This task relates to the number physically and of the geometrically nonlinear tasks of the theory of soft shells.

Let diaphragm in unloaded state be thin shell of constant thickness h_0, whose median surface is obtained by rotation of plane curve to angle 2π relative to axis Ox_0, which lies at its plane. Let us assume that the shell is attached on the parallel of radius R_0 on the walls of the cavity of rotation and the form of its meridian is assigned in the form

$$x_0 = x_0(s);\ r_0 = r_0(s), \qquad (1)$$

where s - arc length of meridian, calculated off the pole of shell.

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We will examine shells, for which in vicinity s=0 function $r_0(s)$ is represented in the form

$$r_0(s) = s + c_1 s^3 + c_2 s^5 + \dots \tag{2}$$

This class of shells includes, in particular, shells, which have form of spherical canopy, paraboloid, ellipsoid and hyperboloid of rotation.

For describing geometry of deformed shell let us introduce cylindrical coordinate system $Ox\eta r$ with beginning in center of attached duct/contour of shell and axis Ox, directed opposite to gradient g of field of mass forces. Since the axisymmetric load is applied to the shell, the main directions of deformation at point $P(x, \eta, r)$ will coincide with the meridians, the parallels and the standards/normals of deformed median surface of shell. The main degrees of elongations/aspect ratios in these directions let us designate through λ_1, λ_2 and λ_3 respectively. Assuming that the shell is prepared from the incompressible isotropic material, we will obtain

$$\lambda_1 = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dr}{ds}\right)^2}; \quad \lambda_2 = \frac{r}{r_0}; \quad \lambda_3 = \frac{1}{\lambda_1 \lambda_2}. \tag{3}$$

Forces of median surface of deformed shell in its biaxial stressed state are determined from formulas [1]

$$T_i = 2h_0\lambda_3(\lambda_i^2 - \lambda_3^2)\left(\frac{\partial W}{\partial I_1} + \lambda_{3-i}^2 \frac{\partial W}{\partial I_2}\right), \quad (i = 1, 2). \tag{4}$$

Here W - elastic potential, which is function of invariant of deformation I_1 and I_2

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2; \quad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}. \tag{5}$$

Conditions of equilibrium infinitesimal element of deformed surface of shell make it possible to obtain equations

$$\begin{aligned} &\frac{dT_1}{ds} + \frac{1}{r}\frac{dr}{ds}(T_1 - T_2) = 0; \\ &\frac{T_1}{R_1} + \frac{T_2}{R_2} = Q(x), \end{aligned} \tag{6}$$

where $Q(x) = C + Dx$; $C = P^* - P^0$; $D = -\rho\gamma g$; ρ - density of liquid; γ - load factor; P^* - constant component of fluid pressure, P^0 - pressure from side of gas to shell; R_1 R_2 - main radii of curvature of shell in final state. The constant C depends on the volume of liquid.

Thus, equations of equilibrium (6) taking into account of given relationships/ratios and conditions of attachment of shell are boundary-value problem for system of two nonlinear differential second order equations relative to functions x(s) and r(s).

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For solution of formulated problem we will use variational method, it is sufficient to efficient during solution of similar tasks [8]. On the basis of the principle of the stability of potential energy of shell, the solution of initial problem can be reduced to finding of the extremum of the functional

$$I=\int_0^{s_0}\left[2h_0Wr_0(s)+Q(x)r^2\frac{dx}{ds}\right]ds \tag{7}$$

on the class of the functions, which satisfy the conditions

$$\left.\frac{dx}{ds}\right|_{s=0}=x(s_0)=r(0)=0;\ r(s_0)=R_0. \tag{8}$$

Following Ritz's method, solutions for functions x(s) and r(s) let us represent in the form

$$\begin{aligned} x(s)&=x_0(s)+\sum_{k=1}^{p}x_ku_k(s);\\ r(s)&=r_0(s)+\sum_{k=1}^{p}x_{k+p}v_k(s). \end{aligned} \tag{9}$$

Coordinate sets of functions $\{u_k(s)\}$ and $\{v_k(s)\}$ let us select as follows

$$u_k(s)=(s^2-s_0^2)\,s^{2k-2};\ v_k(s)=su_k(s). \tag{10}$$

Taking into account stability condition of functional (7), constants x_k, that form by itself 2 p-component vector x, we find from solution of nonlinear algebraic system

$$\mathbf{f}(x)=0. \tag{11}$$

In this case components of vector function f take form

$$f_i = \int_0^{s_0} \left\{ 2h_0 U(\lambda_1, \lambda_2) \frac{1}{\lambda_1} \frac{dx}{ds} \frac{du_i}{ds} - Q(x) \lambda_2 \frac{dr}{ds} u_i \right\} r_0 ds;$$

$$f_{i+p} = \int_0^{s_0} \left\{ 2h_0 \left[U(\lambda_1, \lambda_2) \frac{1}{\lambda_1} \frac{dr}{ds} \frac{dv_i}{ds} + U(\lambda_1, \lambda_2) \frac{v_i}{r_0} \right] + \right.$$

$$\left. + Q(x) \lambda_2 \frac{dx}{ds} v_i \right\} r_0 ds; \qquad (12)$$

$$U(\lambda_1, \lambda_2) = \left(\lambda_1 - \frac{1}{\lambda_1^3 \lambda_2^2} \right) \frac{\partial W}{\partial I_1} + \left(\lambda_1 \lambda_2^2 - \frac{1}{\lambda_1^3} \right) \frac{\partial W}{\partial I_2} \; (i = \overline{1, p}).$$

For solving system of nonlinear algebraic equations (11) is sufficient quantity of iterative methods.

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Converging of them is most rapidly Newton's method, according to whom the approximations/approaches are computed according to the diagram

$$\mathrm{X}^{(k+1)} = \mathrm{X}^{(k)} - \mathrm{H}^{-1}(\mathrm{X}^{(k)}) f(\mathrm{X}^{(k)}). \qquad (13)$$

where H(x) - Jacobi matrix set of functions $f_1, f_2, \ldots, f_{2p}$ relative to variables $x_1, x_2, \ldots, x_{2p}$.

During construction of initial approximation/approach in iterative process of (13) is used method of continuing solutions by parameter of load Q, and in the case of presence of singular solutions of equations (11) - method of replacing parameters of continuation.

Subsequently in calculation formulas was used most widespread elastic potential $W(I_1, I_2)$ in the form of Mooney

$$W(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3). \tag{14}$$

Here C_1 and C_2 - physical constants.

Carried out numerous calculations show that method of solution of initial problem proposed above proves to be sufficiently efficient from point of view of accuracy of calculations and expenditures of machine time.

As example let us give results of calculating strained and stressed state of shell, which has in initial state form of circular membrane/diaphragm with following geometric and physical characteristics:

$$R_0 = 1\ \text{м};\ h_0 = 2\cdot10^{-3}\ \text{м};\ C_1 = 93{,}19{,}5\cdot10^{-4}\ \overset{(1)}{\text{Н}}/\text{м}^2;$$
$$C_2 = 17{,}168\cdot10^{-4}\ \overset{(1)}{\text{Н}}/\text{м}^2. \tag{15}$$

Key: (1). N/m^2.

Fig. 1 depicts profiles/airfoils and force of median surface of deformed shell with different values of dimensionless parameter of load C (transition/transfer to dimensionless quantities is realized according to formulas [8]). As can be seen from figure, with the low values of C the large part of the shell is in stressed state close to the uniform. In proportion to its increase, which is accompanied by an increase of the efforts/forces in median surface of shell, this region is reduced.

2. Let us consider solid body, which has cavity of rotation with liquid. Let us assume that region Q, occupied with liquid, is limited by the rigid walls of cavity S and by the axisymmetrically deformed shell Σ.

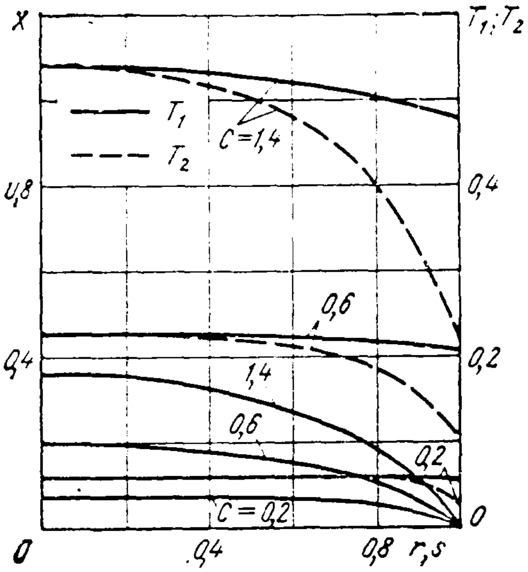

Fig. 1. Internal efforts/forces and profiles/airfoils of deformed shell with different values of parameter of load C.

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Let us introduce the system of coordinates $O^*x^*y^*z^*$ with axis O^*x^*, which coincides with the axis of the symmetry of cavity. Let us take the assigned motion of the system of coordinates $O^*x^*y^*z^*$ for that not disturbed. In the description of the disturbed motion we will use the hypothesis of the smallness of the parameters of motion. Let us introduce the system of coordinates Oxyz connected with the body and it will place it so that during the undisturbed motion of solid body this coordinate system would coincide with system $O^*x^*y^*z^*$. The unit vectors of axes Ox, Oy and Oz we will designate through i_1, i_2 and i_3. The disturbed motion of body we will characterize by the vector of low displacement u_0 of point O relative to O^* and by the vector of the low rotation θ of the system of coordinates Oxyz relatively $O^*x^*y^*z^*$. For describing the motion of liquid in the cavity used the displacement potential χ, for which in the case in question must be satisfied the conditions

$$\Delta\chi = 0 \ {}^{(1)}\text{в}\ Q,$$

$$\left.\frac{\partial\chi}{\partial\nu}\right|_S = (\mathbf{u}_0, \boldsymbol{\nu}) + \theta(\mathbf{R}\times\boldsymbol{\nu}); \quad \left.\frac{\partial\chi}{\partial\nu}\right|_\Sigma = (\mathbf{u}_0, \boldsymbol{\nu}) + \theta(\mathbf{R}\times\boldsymbol{\nu}) + w, \qquad (16)$$

Key: (1). in.

where R - radius-vector of the arbitrary point of surface Σ+S relative to point O; ν - unit vector of external normal to Q region; w - sagging/deflection of shell in the direction of external normal to the surface $\bar{\Sigma}$.

It is analogous with [5], [6], [7], displacement potential is represented in the form

$$\chi = (\mathbf{u}_0, \boldsymbol{\Phi}) + (\theta, \boldsymbol{\Psi}) + \varphi. \qquad (17)$$

Then functions Φ, Ψ and φ must be harmonic functions in Q region and satisfy following boundary conditions

$$\begin{aligned} &\left.\frac{\partial\boldsymbol{\Phi}}{\partial\nu}\right|_{S+\Sigma} = \boldsymbol{\nu}; \quad \left.\frac{\partial\boldsymbol{\Psi}}{\partial\nu}\right|_{S+\Sigma} = \mathbf{R}\times\boldsymbol{\nu}, \\ &\left.\frac{\partial\varphi}{\partial\nu}\right|_{S} = 0; \quad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\Sigma} = w. \end{aligned} \qquad (18)$$

Furthermore, to function w must be superimposed further limitation

$$\int_{\Sigma} w\,d\Sigma = 0, \qquad (19)$$

escaping/ensuing from condition of retaining/maintaining volume of liquid.

Vector function Φ with an accuracy to arbitrary constant is determined by formulas $\Phi_1 = x$; $\Phi_2 = y$; $\Phi_3 = z$, vector function Ψ subsequently we will assume known.

Displacements of shell we will characterize by vector u with components u, v and w respectively in directions of meridian, parallel and normal of median surface of shell.

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Then the disturbed motion of shell will be described by the following system of partial differential equations:

$$\begin{aligned} &L_{11}(u) + L_{12}(v) + L_{13}(w) = r\lambda_1 \delta Q_1; \\ &L_{21}(u) + L_{22}(v) + L_{23}(w) = r\lambda_1 \delta Q_2; \\ &L_{31}(u) + L_{32}(v) + L_{33}(w) = r\lambda_1 [\delta Q_3 + P]. \end{aligned} \qquad (20)$$

In this case functions u(s, η, t), v(s, η, t) and w(s, η, t) must satisfy boundary conditions of rigid attachment on duct/contour l

$$u\,|_l = v\,|_l = w\,|_l = 0 \qquad (21)$$

and limitedness of deformations in pole of shell.

Here L_{ij} - well-known differential operators those preliminarily deformed by hydrostatic load of dome-shaped different shells of rotation [9]. Under $\delta Q_i\,(i=\overline{1,\,3})$ are understood the loads, caused by the inertia of material of the shell

$$\delta Q_1 = -\rho_0 h\,\frac{\partial^2 u}{\partial t};\ \delta Q_2 = -\rho_0 h\,\frac{\partial^2 v}{\partial t^2};\ \delta Q_3 = -\rho_0 h\,\frac{\partial^2 w}{\partial t^2}, \tag{22}$$

where ρ_0 and h - material density and the variable/alternating thickness of the deformed shell.

Pressure from side of liquid

$$P = -\rho g \delta x_\theta - \rho g \delta x_\Sigma - \rho\,\frac{\partial^2 \chi}{\partial t^2} \tag{23}$$

will function besides these loads on shell during its disturbed motion.

First two terms in formula (23) determine additional hydrostatic pressures on shell, caused by rotation of system of coordinates Oxyz relatively $O^*x^*y^*z^*$ to angle θ and by deformation of surface Σ. In this case with an accuracy down to the terms first-order of the smallness

$$\delta x_\theta = \theta_2 z - \theta_3 y;\ \delta x_\Sigma = w\cos\alpha - u\sin\alpha, \tag{24}$$

where α - angle between the standard/normal of surface Σ and axis Ox.

Latter/last term into (23) registered on the basis of Lagrange-Cauchy integral determines with an accuracy to arbitrary function of time dynamic pressure of liquid on shell.

Let us introduce into examination operator of Neumann H, who to values of function f, assigned on surface Σ, places in conformity function φ, determined in region Q and being solution of boundary-value problem

$$\Delta\varphi = 0 \overset{(1)}{\text{B}} Q; \ \frac{\partial\varphi}{\partial\nu}\Big|_{\Sigma} = f; \ \frac{\partial\varphi}{\partial\nu}\Big|_{S} = 0. \tag{25}$$

Key: (1). in.

Let us register this conformity in the form

$$\varphi = Hf. \tag{26}$$

Operator H is integral operator, kernel of which is Green's function for task (25).

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Then disturbed motion of shell will be described by operational equation:

$$L\mathbf{u} = -r\lambda_1 h\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{X}. \tag{27}$$

Here L - matrix operator, generated by differential equations (20) and determined on the set of functions, which satisfies conditions (19) and (21).

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ \overset{*}{L}_{31} & L_{32} & \overset{*}{L}_{33} \end{bmatrix},$$

moreover $\overset{*}{L}_{31}$ and $\overset{*}{L}_{33}$ - changed differential expressions L_{31} and L_{33} due to second term in formula (23). Vector X is determined in this case as follows:

$$\mathbf{x}=\left\{0,\ 0,\ r\lambda_1\left[-\rho g(\theta_2 z-\theta_3 y)-\rho(\ddot{\mathbf{u}}_0,\mathbf{\Phi})-\rho(\ddot{\theta},\mathbf{\Psi})-\rho H\frac{\partial^2 w}{\partial t^2}\right]\right\}'. \quad (28)$$

Prime in curly brace indicates operation of transposition.

Let us consider task at eigenvalues, which describes free oscillations of mechanical system liquid - shell in fixed vessel

$$L\,\mathbf{u}_j-\omega_j^2 B\,\mathbf{u}_j=0;\ B=\operatorname{diag}[r\lambda_1 h\rho_0,\ r\lambda_1 h\rho_0,\ r\lambda_1 h\rho_0,\ r\lambda_1(h\rho_0+\rho H)]. \quad (29)$$

Boundary-value problem (29) has discrete spectrum, and all its eigenvalues are positive. The set of eigenfunctions in a sense possesses completeness and satisfies the conditions of the orthogonality

$$\int_\Sigma (L\,\mathbf{u}_i,\mathbf{u}_j)\,ds\,d\eta=0;\ \int_\Sigma (B\,\mathbf{u}_i,\mathbf{u}_j)\,ds\,d\eta=0,\ (i\neq j). \quad (30)$$

Let us represent displacements of median surface of shell u(s, η, t) during its disturbed motion in the form of generalized series along system of vector functions $\mathbf{u}_j(s,\eta)$ of task (29)

$$\mathbf{u}(s,\ \eta,\ t)=\sum_{j=1}^{\infty} s_j(t)\,u_j(s,\ \eta). \quad (31)$$

Equations of disturbed motion of mechanical system body - liquid - shell, that establish/install differential linkage between vectors u₀ θ and generalized coefficients $s_j(t)$, it is possible to obtain, after registering equations of motion of body, taking into account in this case power and moment effects from side of liquid to body. Let us find the third equation after the substitution of expression (31) in equation (27) and the subsequent application of Bubnov-Galerkin procedure.

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Assuming that the axes of the introduced coordinate system are the principal and central axes of system body - hardened liquid, we convert the equations of the disturbed motion to the form

$$(m_0+m)\ddot{\mathbf{u}}_0+\sum_{n=1}^{\infty}\ddot{s}_n\beta_n=\mathbf{P}';$$

$$(I_0+I)\ddot{\theta}+\sum_{n=1}^{\infty}\ddot{s}_n\beta_{0n}+g\sum_{n=1}^{\infty}s_n[\beta_n\times\mathbf{i}_1]=\mathbf{M}'; \tag{32}$$

$$\mu_n(\ddot{s}_n+\omega_n^2 s_n)+(\mathbf{u}_0,\beta_n)+(\theta,\beta_{0n})+g(\beta_{n3}\theta_2-\beta_{n2}\theta_3)=0,\quad (n=1,\ 2,\ldots),$$

where

$$\mu_n=\int_{\Sigma}[\rho_0 h(u_n^2+v_n^2+w_n^2)+\rho H w_n w_n]\,d\Sigma;\ \beta_n=\rho\int_{\Sigma}\Phi w_n d\Sigma;$$

$$\beta_{0n}=\rho\int_{\Sigma}\Psi w_n d\Sigma;\ I_{ii}=\rho\int_{S+\Sigma}\Psi_i\frac{\partial\Psi}{\partial\nu}ds;\ \beta_n=\{\beta_{n1},\ \beta_{n2},\ \beta_{n3}\};$$

m_0, I_0 - mass and tensor of the inertia of solid body; m - mass of liquid; I - tensor of the inertia of liquid with nonzero components I_{ii}; $\mathbf{P}'$ and M' - main vector and main moment with respect to point O of other external forces, which function on the body during its disturbed motion.

3. Further perturbation analysis of mechanical system in question is possible only after determination of hydrodynamic coefficients of equations (32), which is conjugated/combined with determination of solutions of corresponding boundary-value problems. For determining of function Ψ, liquid in the cavity describing motion, during the low rotations of body to the angle θ it is possible to use the methods, presented, for example, in work [4]. As far as finding the free oscillations of liquid and shell in the quiescent cavity is concerned, solution of the corresponding problem of hydroelasticity after its reducing to the system of integrodifferential equations (29) can be replaced with the solution of variational problem for the functional

$$J=\int_{\Sigma}(L\mathbf{u},\ \mathbf{u})\,ds\,d\eta \Big/ \int_{\Sigma}(B\mathbf{u},\ \mathbf{u})\,ds\,d\eta. \qquad (33)$$

In this case minimum value of functional (33) and function, on which it is realized, coincides in accordance with eigenvalues and eigenfunctions of task (29).

For finding minimum of functional (33) we will use Ritz's method.

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Examining the skew-symmetric oscillations of liquid and shell in plane Oxy, let us represent the components of vector u in the form of the expansions

$$u=\cos\eta\sum_{k=1}^{q}y_k u_k(s);\ v=\sin\eta\sum_{k=1}^{q}y_{k+q}v_k(s);$$
$$w=\cos\eta\sum_{k=1}^{q}y_{k+2q}w_k(s). \qquad (34)$$

After isolation of angular coordinate end-point s=0 is regular singular point for operator L. Therefore, taking into account the established/installed earlier asymptotic behavior of the unknown solutions with the approach to the pole of shell, coordinate sets of functions $\{u_k(s)\}$, $\{v_k(s)\}$ and $\{w_k(s)\}$ let us select in the following form [9]:

$$u_k(s)=(s^2-s_0^2)\,s^{2(k-1)};\ v_k(s)=u_k(s);\ w_k(s)=(s^2-s_0^2)\,s^{2k-1}, \qquad (35)$$

after assuming in this case in expansions (34) $y_{q+1}=-y_1$.

As a result let us reduce task of determining (3q-1)-component vector $y=\{y_1, y_2, ..., y_q, y_{q+2}, ..., y_{3q}\}$ to solution of generalized algebraic problem at eigenvalues

$$(\mathbf{A}-\omega^2\mathbf{B})\,\mathbf{y}=0. \qquad (36)$$

For determining matrix elements B knowledge of functions $\varphi_k^{(1)}=H^{(1)}w_k$, which are solutions of heterogeneous boundary-value problems in region G of meridian cut of cavity

$$\frac{\partial}{\partial x}\left(r\frac{\partial\varphi_k^{(1)}}{\partial x}\right)+\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_k^{(1)}}{\partial r}\right)-\frac{1}{r}\varphi_k^{(1)}=0; \quad (37)$$

$$\left.\frac{\partial\varphi_k^{(1)}}{\partial\nu}\right|_{L_0}=w_k;\ \left.\frac{\partial\varphi_k^{(1)}}{\partial\nu}\right|_{L_1}=0,\quad (k=\overline{1,\ q}),$$

where L_0 and L_1 - parts of boundary of the region G, which lie respectively on surface of shell Σ and hydrophilic surface of cavity S respectively, is necessary.

Formulated problems (37) are solved on basis of variational method, worked out for solution of analogous problems in work [4], if we select for function $\varphi^{(1)}{}_k$ in the form of sequence of linear combinations of coordinate functions

$$\varphi_k^{(1)}=\sum_{m=1}^{n}z_m^{(k)}\zeta_m(x,\ r). \quad (38)$$

Sets of functions $\{\zeta_m(x,\ r)\}$, which ensure convergence of sequence (38), are constructed in accordance with recommendations, proposed in works [3], [4].

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Let us give some results of dynamic response computation of free oscillations of liquid and preliminarily deformed shell for cavity in the form of straight/direct circular cylinder. It was assumed that shell in the unstressed state it is the circular membrane/diaphragm with characteristics (15), attached at a distance $H/R_0=0{,}5$ from the bottom of vessel. Table 1 gives the convergence of the first three values of dimensionless frequency parameter $\varkappa_n=R_0{}^2\rho_0\omega^2{}_n/(2C_1)$

depending on the number of approximations/approaches q in expansions (34) with the fixed/recorded number of functions into (38) (m=10). The dimensionless physical parameters of the mechanical system in question relied by the equal to

$$C=1,\ 4;\ D=-2{,}631579;\ \Gamma=\frac{C_2}{C_1}=0{,}184210;\ \frac{\rho}{\rho_0}=1;\ \frac{R_0}{h_0}=500. \tag{39}$$

Inertia of material of shell in tangential direction in this case was not considered. On the other hand, the effect of the number of approximations/approaches m in expansions (38) into the determination of frequency parameter $\varkappa_n$ is represented in Table 2. Analogous convergence is observed also during the determination of the forms of oscillations. The obtained results testify about a sufficient efficiency of the proposed approach to the solution of the problem during the determination of its lowest eigenvalues and eigenfunctions in question corresponding to them.

Fig. 2 gives graphic dependence of first two values of dimensionless parameter $\omega\sqrt{R_0/g}$ from value of increase in volume of liquid in vessel $\overline{\Delta V}=\Delta V/(\pi R_0^3)$ when $\delta=R_0/h_0=200;500.$ Dotted lines plotted the first four dimensionless natural vibration frequencies of liquid in the cylindrical container with the free surface of liquid.

Table 1.

q	1	2	3	4	5	6
$\varkappa_1$	0,038495	0,036532	0,035860	0,035844	0,035843	0,035843
$\varkappa_2$	—	0,27951	0,23283	0,23097	0,23096	0,2386
$\varkappa_3$		—	1,09715	0,74836	0,72226	0,72189

Table 2.

n	4	6	8	10	12
$\varkappa_1$	0,036024	0,035851	0,035847	0,035843	0,035843
$\varkappa_2$	0,42981	0,23944	0,23097	0,23096	0,23091
$\varkappa_3$	—	—	0,81385	0,72226	0,72180

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With ΔV, that vanishes, the natural frequencies of mechanical system shell - liquid approach the appropriate natural vibration frequencies of liquid in the vessel with free surface of [2]. An increase in the parameter ΔV, which is accompanied by an increase in the stressed state of the shell (see Fig. 1), leads to a considerable increase in the natural frequencies of the hydroelastic system in comparison with the natural vibration frequencies of liquid in the equivalent vessel with the free surface Σ. Consequently, the structural equipment in question can be also used for the capacities/capacitances, which are located in the strong gravitational fields as the means of a substantial change frequently of the natural oscillations of liquid in the direction of their increase.

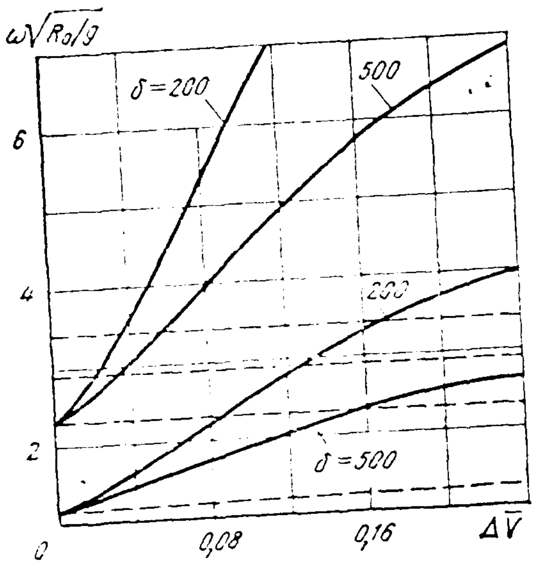

Fig. 2. Dependences of natural frequencies on value of increase in volume of liquid in vessel with different relative thicknesses of shell.

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SELF-EXCITATION OF THE LOW-FREQUENCY OSCILLATIONS OF LIQUID DURING THE HIGH-FREQUENCY OSCILLATIONS OF VESSEL.

B. L. Venediktov, R. A. Shibanov.

Are presented and analyzed results of experimental analyses of special features of behavior of liquid during excitation of high-frequency (to 50-70 Hz) oscillations of tank, leading to fountain effect drops from free surface of liquid and onset of oscillations of liquid and tank with frequency of 5-20 times lower than frequency of vibrations of forcing force. Is determined the dependence of these phenomena on frequency and amplitude of the oscillations of the longitudinal exciting force and transverse vibrations of tank, and also on some parameters of tank. Are revealed some special features, which make it possible to make more precise nature of the phenomenon being investigated and its effect on the oscillations of tank itself and elastic constructions/designs, component part of which it is.

Nonlinear effects, which appear during oscillations of liquid in tanks of flight vehicles, can substantially affect their dynamic characteristics and must be considered with stabilization and controllability of flight, work of fuel-supplying systems, dynamic structural strength of flight vehicles with liquid propellant [3, 4].

Nonlinear effects during oscillations of liquid in tanks can be described within the framework of following three classes of phenomena:

1. The effects, which appear, mainly, as a result of the special features of tank geometry (due to the edges/fins, the grids, perforated/punched diaphragms, etc.), which are exhibited even with the sufficiently low amplitudes of the oscillations of exciting forces and biases/displacements of tank.

2. Effects, caused, in essence, with large amplitudes of oscillations of exciting forces and biases/displacements of tank. The nonlinear effects of this class (limitedness of resonance amplitudes, the asymmetry of the profile/airfoil of wave, etc.), observed in the region of major resonance, appear with the amplitudes of the oscillations of liquid, which exceed 0.25 of radius of tank [2].

3. Effects, connected with implication of other substantially (in comparison with linear oscillations) forms of behavior of liquid, that appear with instability of oscillations of liquid during interaction with oscillations of shell of tank. With this type phenomena are connected:

the excitation of the subharmonic oscillations of liquid, whose frequency is 2 times lower than the frequency of perturbing force, during the longitudinal (normal to the free surface of liquid) oscillations of tank due to the instability, which appear in this case the parametric variations of liquid. Linear theories do not make it possible to determine the amplitude of the oscillations of liquid, which appear during the longitudinal excitation of tank;

the rotation of the free surface of liquid during transverse vibrations of tank with the large amplitude with the frequency, close to the first tone of the oscillations of liquid;

the fountain effect of drops from the free surface of liquid, the introduction of the bubbles of air into the liquid during the sufficiently intense high-frequency oscillations of tank;

the onset of low-frequency oscillations of liquid during the high-frequency oscillations of tank, whose frequencies differ into dozens of times.

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The known results of experimental research of this phenomenon will be coordinated with the assumption, independently advanced by M. S. Galkin in 1960 and Yarimovich [4] that the main factor, which supports the low-frequency oscillations of liquid, are the impacts/shocks of the drops, which fall to the free surface of the liquids, formation and flight time of which is synchronized with the low-frequency oscillations of the surface of liquid.

Excitation of lowest tones of natural oscillations of free surface of liquid during high-frequency oscillations of tank was discovered during experimental studies of longitudinal vibrations of rigid vessels Yarimovich and by Kahn [4] it is independent during transverse high- frequency oscillations of vessels - by B. L. Venediktov in 1960 and during longitudinal vibrations of free tank with elastic bottom - R. A. Shibanov in 1965. However, the satisfactory theory of the emergence of this phenomenon is absent, and it needs further research.

In this work results of experimental studies of phenomenon of radio-frequency drive of low-frequency oscillations of liquid and process of spray formation connected with it are presented and are analyzed.

1. Description of experiment. The bouncing of the tank with the water, suspended/hung from the shock cords, with the help of the directed eccentric radiator and the spring energizer is excited. The natural frequency of oscillation of tank on the suspension is located in the range of frequencies from 1 to 4.5 Hz. Tank can be moved only in the vertical direction.

Tank is cylinder with flat elastic bottom, whose inner diameter is equal to 300 mm. Cylindrical part is done from organic glass a thickness of 3 mm, bottom - from the steel plate with a thickness of 0.2 mm.

Are varied: frequency of vibrations of exciting force ω from 1 to 50 Hz; depth of filling of tank with water H $(H/R=0{,}02\ldots1{,}73$, where R - inside radius of tank); amplitude of oscillations of exciting force, created by radiator, is equal to $P_0\omega^2/\omega_0^2$, where $\omega_0=33{,}33$ Hz, and P_0 vary within the range of 15 to 175 N. Bias/displacement and acceleration of the end/face of the cylindrical part of the tank are measured.

2. Characteristic types of behavior of liquid. Characteristic forms of oscillations and type of the behavior of the free surface of liquid, discovered during these tests, they are shown on a series of the photographs Fig. 1.

During excitation of oscillations of tank in the range of frequencies from 4.44 to 20 Hz well visually observed oscillations of free surface of liquid appear.

In the range of frequencies from 5 to 10 Hz harmonic axisymmetric oscillations of surface (see Fig. 1B) usually are excited.

With increase in frequency and amplitude of oscillations of tank axisymmetric oscillations are destroyed, appears zone of unstable oscillations of liquid, where whimsical forms of oscillations with large amplitude (see Fig. 1c) frequently appear.

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The frequency of the vibrations, during which appears the instability, grows with the reduction of the amplitude of the oscillations of tank. With $H/R=0{,}67$ and $85\leqslant P_0\leqslant 175$ N the instability begins in the range of frequencies from 7 to 10 Hz, while under force of P_0=49 of N at the frequency of 15-16 Hz, with P_0=15 N at the frequency of 36-37 Hz.

Steady subharmonic oscillations of free surface of liquid with frequency of from 2.23 to 4.1 Hz are observed, frequency of vibrations of exciting force in this case is 2-2.1 times higher.

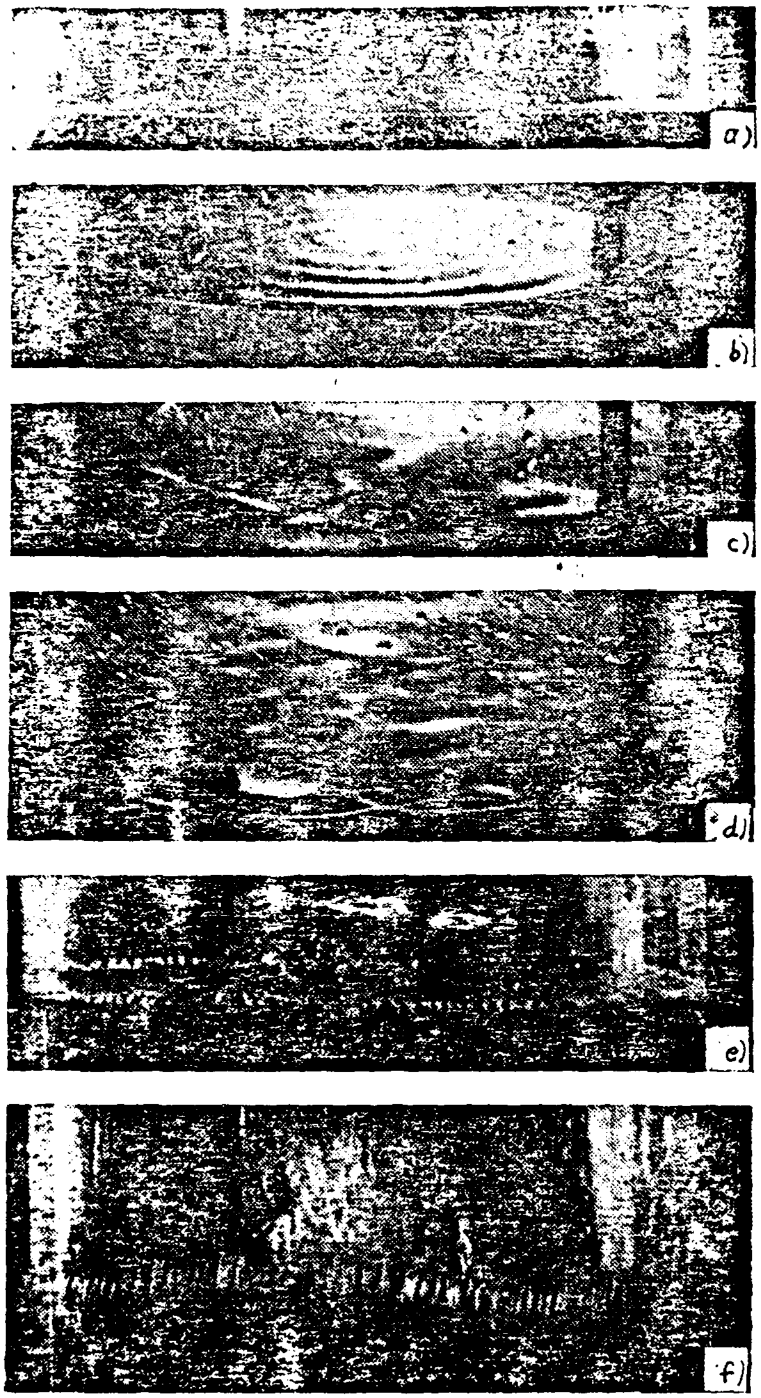

Fig. 1. Characteristic cases of oscillations of free surface of liquid during longitudinal vibrations of tank with different frequency and amplitude.

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The frequencies of the vibrations of free surface to 1-9% are lower than appropriate theoretical values of the natural frequencies of oscillation of the liquid

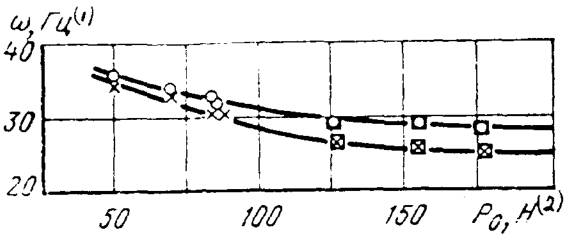
$$\Omega_{ij}=\sqrt{\frac{g}{R}\xi_{ij}\,\mathrm{th}\,\xi_{ij}\frac{H}{R}}, \qquad (1)$$

where g - gravitational acceleration; ξ_{ij} - roots of derivative of the Bessel function of the i first-order order $J'_i(\xi_{ij})=0$.

Observing low-frequency tones correspond to roots ξ_{02}, ξ_{04}, ξ_{21}, ξ_{22}, ξ_{23}, ξ_{51} and have natural frequencies, close ones to natural frequency of oscillation of tank during suspension or are 2 times lower than it. The smallest frequency, at which is observed the manifestation of the instability of the surface of liquid (7 Hz), two times exceeded the frequency of the vibrations of tank during the suspension.

Lowest axisymmetric tone (natural frequency ≤2.5 Hz) in this case is not excited. This tone (2.5 Hz) is recorded (see Fig. 1a) during the free oscillations of tank with a frequency of 5 Hz (logarithmic decrement of oscillations 0.02), which appear after the excitation of tone during spray formation in the process of the high-frequency oscillations of tank and subsequent removal/taking of exciting force.

Excitation of subharmonic oscillations and emergence of instability of oscillations of free surface of liquid can be explained with the help of theory of parametric variations, described during longitudinal harmonic excitation by equations of Mathieu [1, 5]. Let us note that the dynamic characteristics of tank substantially are reflected in the sizes/dimensions of the zone of the instability of the subharmonic oscillations of the liquid: both due to the increase in the amplitude of the harmonic oscillations of tank at the resonance frequency of excitation and due to reduction in the amplitude of the oscillations of tank, necessary for the parametric excitation of the subharmonic oscillations of liquid in the case of the proximity of subharmonic to the natural frequency of oscillation of tank.



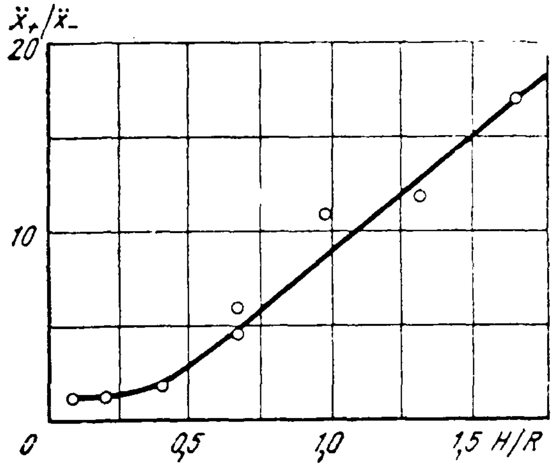


Fig. 2. Fig. 3.

Fig. 2. Dependence of minimum frequency of vibrations of longitudinal forcing force, during which occurs fountain effect of liquid, from its amplitude P_0 with increase (○) and decrease (×) of frequency (□ - self-excitation of low-frequency oscillations; $H/R=0.67$)

Key: (1). Hz.

Fig. 3. Dependence of ratio of amplitude of oscillations of longitudinal acceleration of tank with fountain effect of liquid ($\ddot{x}_+$) to appropriate amplitude without fountain effect ($\ddot{x}_-$) on depth of filling of tank with liquid (excitation it corresponds to boundary of drop-forming P_0=126 N).

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With increase in frequency of rotation of rotor of radiator instability, where oscillations of liquid depend substantially on random factors (first of all, initial disturbances), vanishes, and on free surface of liquid appear steady subharmonic oscillations with frequency of $\omega/2$ with large number of waves in radial and tangential directions (see Fig. 1d). The corresponding to them natural frequencies and the forms of the oscillations of liquid weakly depend on the geometry of cavity and are determined in by the fundamental principles of capillary and gravitational forces.

Number of waves on surface grows with increase in frequency of vibrations of tank. Evaluations showed that the mean radius of the crater of one wave approximately can be considered with the help of the equality

$$\Omega^2 = \frac{g}{r}\xi_{01}\left(1 + \frac{\xi_{01}^2}{\rho g r^2}\sigma\right),$$

of the corresponding to expression for the frequency first axisymmetric tone of the natural oscillations of liquid in a deep vessel of radius r taking into account surface tension σ [5]. Here Ω - frequency of the vibrations of liquid; ρ - density of liquid; $\xi_{01} = 3{,}83$.

3. Drop-forming. An increase in the frequency of the rotation of the rotor of radiator in higher than the certain limit leads to the destruction of the crests/peaks of capillary-gravitational waves; to the formation it is high (in comparison with the amplitude of wave oscillations) of the gushing columns of water and to the breakaway of the drops of liquid from the crests/peaks (see Fig. 1e). The greater the amplitude P_0 of the oscillations of the force, created by radiator, the lower the minimum frequency of the vibrations of tank,

during which occurs the formation of drops (Fig. 2). If we reduce the frequency of the rotation of radiator after the emergence of the process of drop-forming, ther drop-forming is retained to the lower frequencies and the amplitudes of the oscillations of exciting force.

The greater amplitude of oscillations of exciting force, the greater quantity of drops is formed and the higher trajectory of their flight. The greater the frequency of the vibrations of tank, the less the diameter of drops. The maximum/overall diameter of the gushing drops is equal to 5...8 mm, which is approximately 20...30% of wavelength of the corresponding oscillations of surface with a frequency of 20 Hz. Finer/smaller drops have the high initial velocity and heave above. In the majority of the cases the drop-forming occurs only in the center section of the surface of the liquid (see Fig. 1e). With the decrease of the depth of filling of tank the surface of liquid, included by drop-forming, increases. The majority of drops (is especially highly gushing) flies away from the center of tank to the periphery.

With emergence of drop-forming amplitude of oscillations of tank sharply grows (Fig. 3). The greater the depth of the filling of tank with liquid, the more the amplitude increases.

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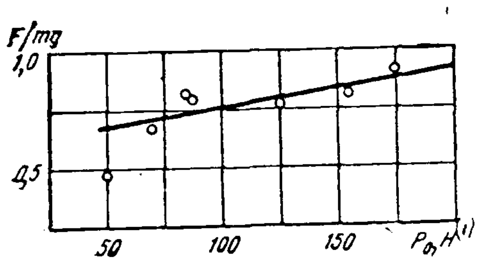
With $H/R>1$ the amplitude of the oscillations of tank grows due to the drop-forming 10...20 times.

Minimally necessary for excitation of process of drop-forming of amplitude of oscillations of composite force, which functions on liquid, compose 0.5...10 from weight of liquid (Fig. 4, 5). With an increase in the depth of the filling of tank with liquid with $H/R>0.4$ the necessary amplitudes of forces are reduced.

Drop-forming is retained in sufficiently wide frequency band of forced oscillations (Fig. 6). The greater the amplitude of the oscillations of exciting force, the wider this range.



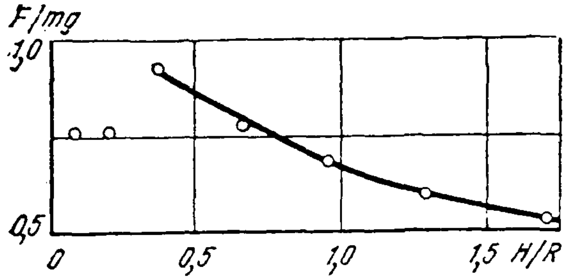


Fig. 4. Fig. 5.

Fig. 4. Dependence of amplitude of oscillations of total longitudinal force of F, which functions on liquid on boundary of drop-forming, in reference to its weight mg, on amplitude of oscillations of that forcing P_0 ($H/R=0,67$).

Key: (1). N.

Fig. 5. Dependence of amplitude of oscillations of total longitudinal force of F, which functions on liquid on boundary of drop-forming, in reference to its weight mg, on depth of filling of tank with liquid (P_0=126 N).

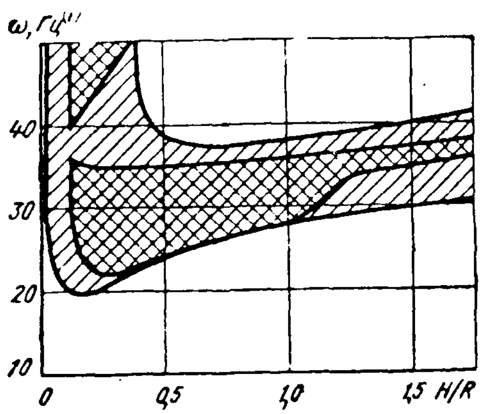


Fig. 6.

Fig. 7.

Fig. 6. Frequency regions of fluctuations of forcing force and depths of filling of tank with liquid, with which occurred drop-forming and self-excitation of low-frequency oscillations (P_0=126 N).

Key: (1). Hz.

Fig. 7. Excited with drop-forming oscillations of free surface of liquid with frequency of 1st axisymmetric tone of 2.5 Hz (ω=30 Hz).

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4. Self-excitation of low-frequency oscillations. During the excitation of the longitudinal vibrations of tank with the frequency, equal or several that exceeding the minimum frequency of the vibrations, which cause the fountain effect of liquid (if the amplitude of exciting force is sufficiently great), appear the stable oscillations of the surface of liquid and tank itself with the low frequency and the large amplitude (Fig. 7, 8). Their frequency is 5...20 times lower than the frequency of the vibrations of exciting force.

Excitation of low-frequency oscillations of liquid and tank is observed over a wide range of frequencies of induced high-frequency vibrations of tank (see Fig. 6). With the low depths in the investigated frequency region of forced oscillations (to 50 Hz) there are two frequency bands, at which appear low-frequency oscillations. The value of minimum frequency for the second range is 1.9-2.2 times more than for the first. To the harmonic oscillations of tank with the frequency of exciting force in the 2nd range are placed the commensurate in the amplitude subharmonic oscillations with the frequency two times of less. In all observed cases the necessary condition of the onset of low-frequency oscillations during the radio-frequency drive is the fountain effect of liquid, which usually (but not always) it is accompanied by drop-forming.

During excitation of high-frequency longitudinal vibrations of tank appeared low-frequency oscillations of different forms with frequency from 1.4 to 5 Hz. Form and their frequency depends on the hardness of suspension and depth of filling of tank, on initial conditions and amplitude of exciting force, etc. The excitation of the 1st and 2nd axisymmetric, 1st and 2nd transverse tones of the oscillations of liquid in the tank and the oscillations of tank with the frequency, equal to the natural frequency of oscillation of tank during the suspension, is discovered.

1st axisymmetric tone of oscillations of liquid (see Fig. 7) most frequently is excited with frequency, which was varying in accordance with depth of filling of tank from 1.55 to 2.5 Hz. Frequency corresponded to formula (1). In this case the tank oscillated with the double frequency.

Greatest amplitudes of oscillations of tank are noted during oscillations with frequency of from 3 to 4 Hz, coinciding with natural frequencies of oscillation of tank during suspension (see Fig. 8). The free surface of liquid in this case oscillates weakly and drops it is formed (especially with the larger depths of the filling of tank with liquid) much less than during the excitation of the liquid tones of oscillations.

Oscillograms of bias/displacement and acceleration of tank with emergence of such oscillations, which were conceived in process of transition/transfer of boundary of drop-forming with slow increase in frequency of rotation of radiator, are represented in Fig. 9.

Process of fountain effect of drops from surface of liquid is changed synchronously with excited low-frequency oscillations of liquid.

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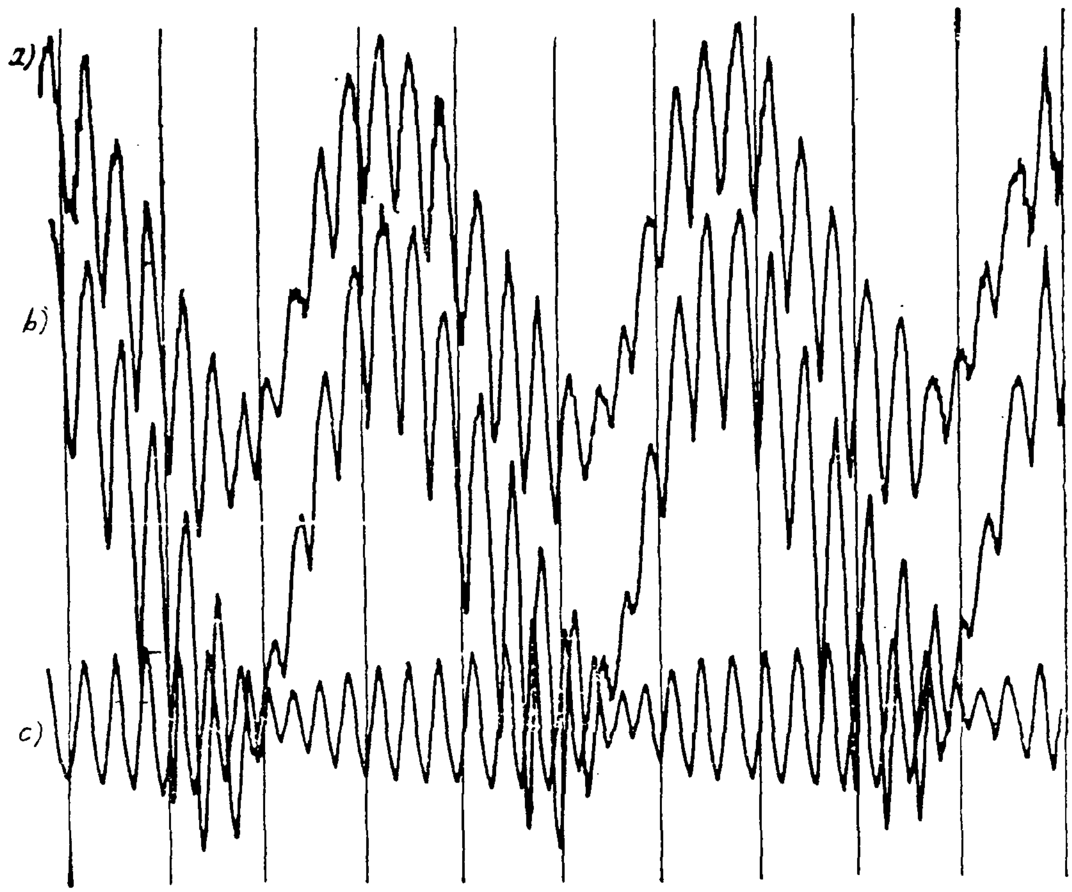

Fig. 8. Oscillograms of oscillations of longitudinal displacement (a, b) and acceleration (c) of tank during emergent in process of drop-forming steady-state oscillations of tank with frequency of 3 Hz (ω=33 Hz).

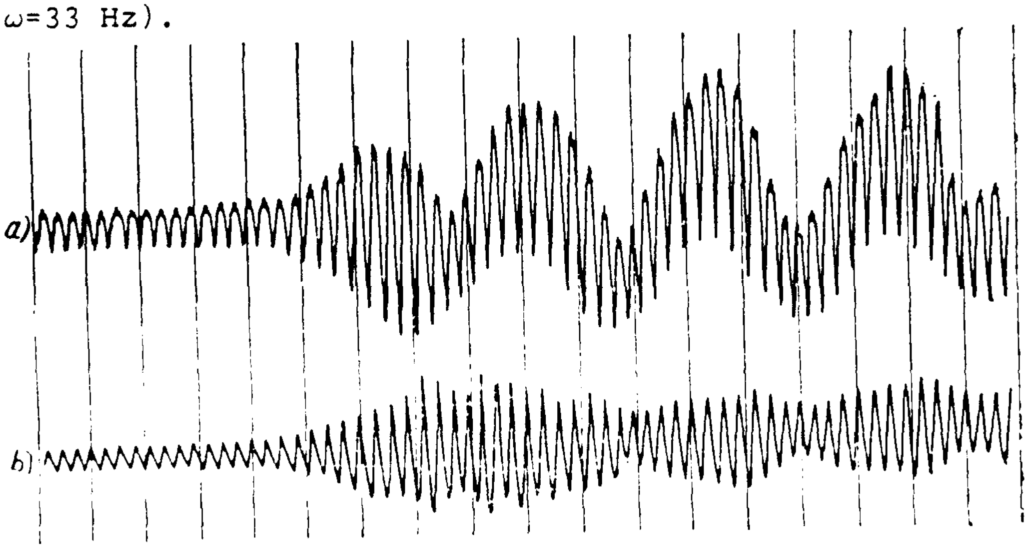

Fig. 9.

Fig. 9. Oscillograms of oscillations of longitudinal displacement (a) and acceleration (b) of tank with emergence of drop-forming and oscillations of tank with frequency of 3 Hz (ω=36 Hz).

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Drops during the excitation of the low-frequency oscillations of the surface of liquid are concentrated near the antinodes, although the center is always characterized by the most intense drop-forming, which apparently, is caused by the more intense oscillations of the center section of the bottom.

Three stages of low-frequency oscillations of surface of liquid during radio-frequency drive of longitudinal vibrations of tank are shown on series of photographs (Fig. 10). When the center section of the surface is located below, fountain effect does not occur (see Fig. 10a), with its approximation/approach to the upper position (see Fig. 10b) begins fountain effect. Fountain effect reaches maximum at the maximum altitude of the crest of the wave (see Fig. 10c).

Synchronously with low-frequency oscillations amplitude modulation of high-frequency oscillations of tank occurs. Strongest modulation appears during the excitation of the suspension tone (see Fig. 8, 9). The amplitude of high-frequency oscillations reaches maximum in the periods of maximum drop-forming.

Bottom with drop-forming oscillates with large amplitude. An increase in the hardness of bottom raises minimally necessary for the drop-forming and excitations of low-frequency frequency variations and the amplitude of exciting force.

5. Polyharmonic excitation. For the study of the effect of the induced high-frequency oscillations of tank with the liquid on its induced low-frequency oscillations the simultaneous excitation of high-frequency (by eccentric radiator) and low-frequency (by spring energizer) oscillations is produced. It turned out that the amplitude of the induced low-frequency oscillations of tank with the fountain effect of drops considerably (to 2.5 times) grows in comparison with the oscillations of tank in the absence of drop-forming (Fig. 11, 12). The maximum increase of the amplitude of the oscillations of tank due to the drop-forming occurs at the frequencies, close to the natural frequency of oscillation of tank during the suspension. Specifically, with this frequency oscillate during the excitation of the high-frequency longitudinal vibrations of tank by one radiator.

6. Transverse vibrations. Research showed that the low-frequency oscillations of liquid can appear, also, during the transverse high-frequency oscillations of the vessels, moved in the plane of the free surface of liquid. Research was conducted with the cylindrical containers from organic glass with a diameter of 60 mm, 80 mm, 100 mm and 300 mm, which were filled with water from $H/R=0.46$ to

H/R=3.3. Wall thickness was 4-6 mm. Frequency ω and amplitude A of the induced harmonic oscillations of the vessel, established/installed on the platform of vibration table in the vertical position, was varied in the ranges: ω=0...70 Hz, $A=0...1{,}25$ мм.

During oscillations of vessels with appropriate low frequency and by low amplitude are excited stable oscillations of free surface of liquid on first skew-symmetric tone. The frequencies coincide with theoretical values of (1). With an increase in the amplitude of the oscillations of vessel at the frequency, close to the frequency of the 1st skew-symmetric tone, the instability appears and the slow rotation of nodal curve appears [2, 4].

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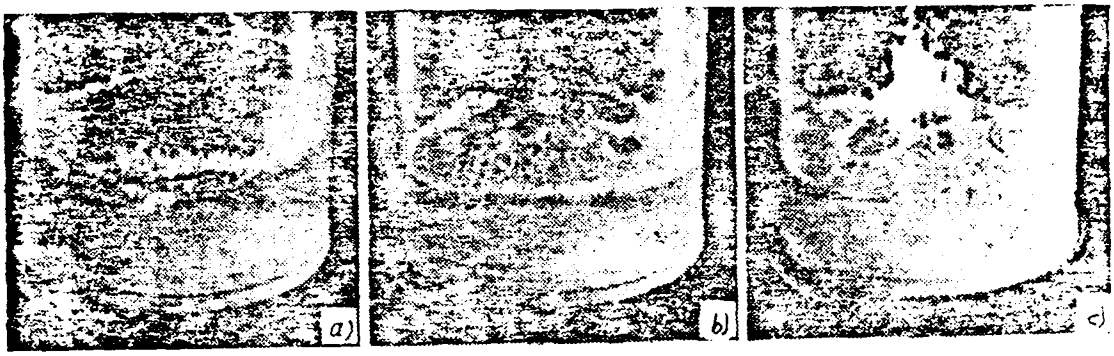

Fig. 10. Three stages of low-frequency oscillations of liquid and process of drop-forming, that arose during radio-frequency longitudinal drive of tank.

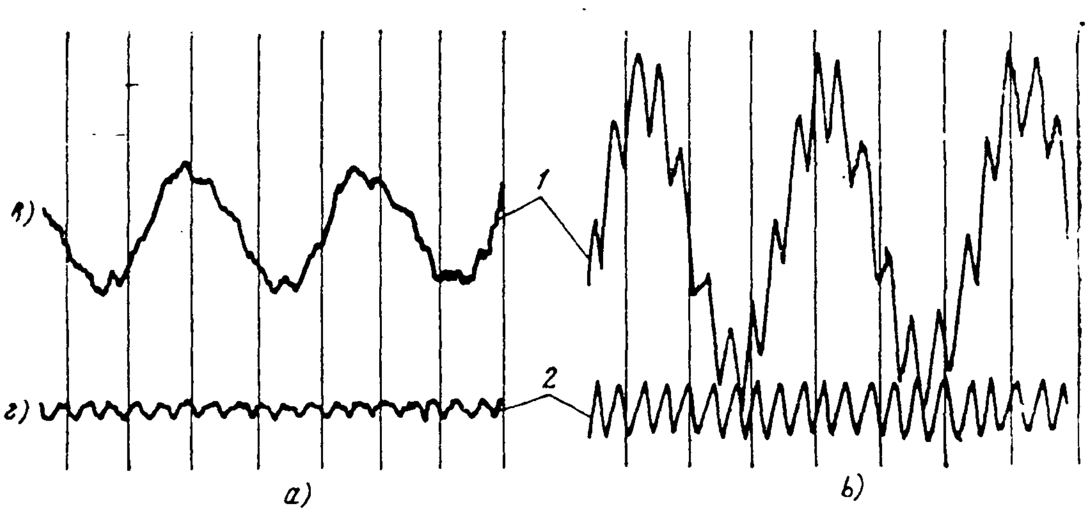

Fig. 11. Oscillograms of longitudinal vibrations of bias/displacement (1) and acceleration (2) of tank during polyharmonic excitation (ω_1=3.4 Hz, ω_2=25,5 ... 27,5 Hz) without drop-forming (a) and with drop-forming (b).

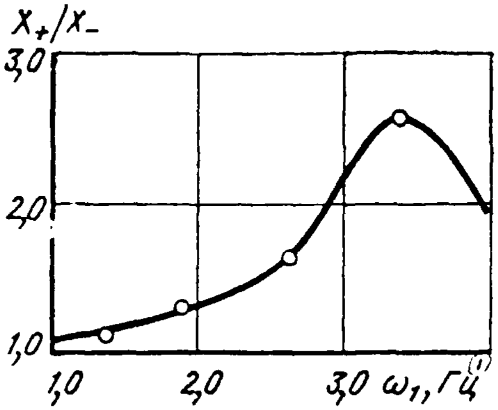

Fig. 12. Dependence of ratio of amplitude of longitudinal low-frequency oscillations of tank during polyharmonic excitation with drop-forming to their amplitude (fountain effect it is absent), from frequency of excitable low-frequency harmonic.

Key: (1). Hz.

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Further increase in frequency leads with sufficiently large amplitudes of oscillations to decomposition of free surface of liquid with formation of sprays (Fig. 13). Sprays are formed in the larger degree in the walls of tank, perpendicular to plane of vibration.

Oscillations of free surface of liquid with frequency of 1st skew-symmetric tone appear with specific values of frequency and amplitude of oscillations of vessel; moreover position of nodal curve of wave is unstable (Fig. 14). The frequency of the vibrations of vessel in this case exceeded the frequency of the vibrations of the free surface of liquid 8-20 times.

Boundary of emergence of generatable with drop-forming 1st skew-symmetric tone of oscillations of liquid in region of those determining this phenomenon of parameters of external excitation: amplitude of oscillations of transverse acceleration $\ddot{y}$ of vessel, in reference to gravitational acceleration, and frequency of vibrations of vessel it is represented in Fig. 15.

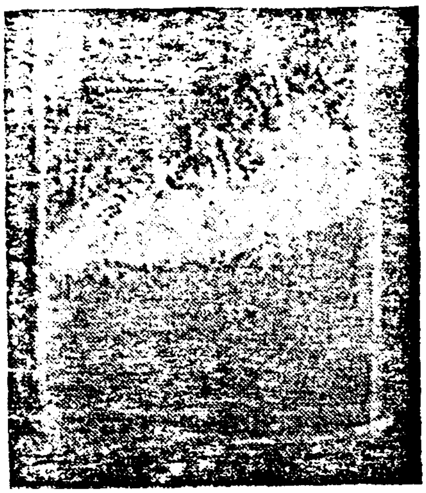

Fig. 13.

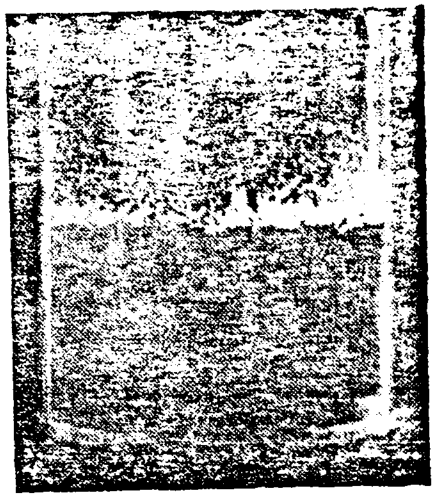

Fig. 14.

Fig. 13. Fountain effect of drops of liquid during transverse vibrations of vessel.

Fig. 14. Excited with drop-forming oscillations of free surface of liquid with frequency of 1st skew-symmetric tone. Transverse excitation.

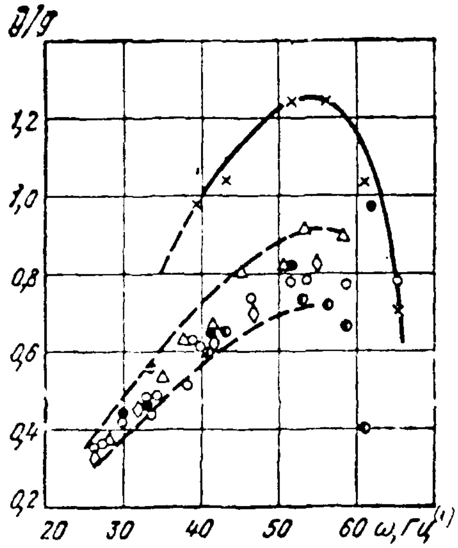

Fig. 15. Dependence of amplitude of oscillations of transverse

acceleration of vessel $\ddot{y}$, in reference to gravitational acceleration, with which in process of drop-forming appeared low-frequency oscillations of liquid, from frequency of induced transverse vibrations of vessel:

× – $H/R=0,5$; △ – $H/R=1,0$; ◐ – $H/R=1,5$;
○ – $H/R=2,0$; ◇ – $H/R=2,5$; ● – $H/R=3$

Key: (1). Hz.

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Amplitude of the acceleration of tank necessary for the excitation of these oscillations of liquid reaches maximum in 50-60 Hz areas and substantially it is raised with the decrease of the depth of filling of tank by water with H/R<1.

7. Conclusion. The obtained results will agree with the results of analogous research in [4, 7] and the works generalized in them. Let us conduct short sums, accentuating attention in the new results.

Onset of self-excited low-frequency (in the range of frequencies of lowest tones of natural oscillations of free surface of liquid) oscillations of tank with liquid occurs during sufficiently intense nonlinear high-frequency oscillations of surface of liquid, which lead to decomposition of resultant capillary-gravitational waves with fountain effect of streams of liquid and drops, which have different value and initial velocity. The appearing upon the decomposition of capillary-gravitational interference waves of the surface of liquid, apparently, can be described as random process. In work [6] it is theoretically proved that with the sufficiently high value of the spectral density of the random vertical excitation of vessel the surface of liquid becomes unstable for the lowest tones of oscillations, remaining stable for the high. At the adequate/approaching external radio-frequency drive of the oscillations of tank the emergent low-frequency oscillations can stably be supported due to the synchronously changing process of the fountain effect of the liquid, when the flight time of drops (or the gushing jets) reaches 1/4...1/2 periods of low-frequency oscillations of the surface of liquid.

With specific values of amplitude and frequency of induced longitudinal vibrations of tank with liquid stable oscillations of tank with large amplitude and low frequency, which composes 1/20...1/5 from frequency of excitation and which coincides with natural frequency of structural tone of oscillations of tank (among other things of that caused by characteristics of entire construction/design, component part of which it is tank) can appear or with frequency, two times of exceeding first axisymmetric tone of natural oscillations of free surface liquid.

Due to drop-forming amplitude of high-frequency induced longitudinal vibrations of tank on sizable (in comparison with radius of tank) level of filling with liquid can strongly increase without change in amplitude and frequency of vibrations of forcing force. An increase in the amplitude to 20 times is recorded. Drop-forming can substantially increase the amplitude of the induced low-frequency oscillations of tank with the liquid without a change in the level of the polyharmonic excitation of the low-frequency and causing drop-forming high-frequency oscillations of tank. An increase in the amplitude of low-frequency oscillations to 2.5 times is recorded.

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Conducted investigations showed that emergence of process of drop-forming occurs with satisfaction of following conditions: amplitude of oscillations of total longitudinal force, which functions on liquid from side of tank, reaches value of 0.5-1.0 from weight of liquid;

radius of craters, which are parametrically excited on surface of liquid of capillary-gravitational waves corresponded to condition

$$\mathrm{Bo} = \rho g r^2/\sigma < 20 \ldots 30,$$

where Bo - Bond number, which characterizes relationship/ratio between gravitational forces and forces of surface tension σ.

By these conditions, apparently, are determined necessary for drop-forming amplitude and frequency of vibrations of exciting force.

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DISTURBED MOTION OF THE REVOLVING FLIGHT VEHICLE WITH THE LIQUID IN THE TANKS.

On the motion of the symmetrical gyroscope, whose cavity is partially filled with liquid.

G. S. Narimanov.

Equations of disturbed motion of rapidly spinning gyroscope with cylindrical cavity, partially filled with ideal fluid, are derived. It is assumed that the longitudinal axis of cavity coincides with the axis of the high-spin motion of gyroscope, and the depth of liquid - is low, so that during the undisturbed stationary rotation of gyroscope the free surface of liquid and wall cavities are coaxial cylinders with close radii. This makes it possible to use the concept of long waves theory.

Are derived equations of slight disturbances of stationary rotation of symmetrical gyroscope, whose cylindrical cavity is partially filled with liquid. The obtained equations can be used for the analysis of stability of stationary rotation and for the evaluation of the effect on it of the values of different parameters of system.

1. Formulation of problem. We assume that the axis of the symmetry of gyroscope is simultaneously the rotational axis of the cavity, which is straight/direct circular cylinder. During the stationary rotation the gyroscope and the included in the cavity liquid revolve as single solid body around the axis of symmetry, which in this case is placed vertically.

We will assume difference in disturbed motion of gyroscope and liquid from stationary rotation low, which allow/assume possibility of linearization of equations for values of variations in parameters of disturbed motion.

It will place fixed coordinate system $Ox^*y^*z^*$ so that axis Oz^* would be vertical, and point O coincided with fixed point of gyroscope. Besides this system let us introduce into the examination two additional moving coordinate systems, one of which Oxyz (axis Oz it coincides with the axis of symmetry) it is rigidly connected with the gyroscope, another Ox_0y_0z is partially connected.

Transformation of system of coordinates $Ox^*y^*z^*$ into Ox_0y_0z is realized with the help of two rotations: one rotation around axis Oy^* clockwise to angle δ_2 and second rotation around axis Ox_0 counterclockwise to angle δ_1. The system of coordinates Ox_0y_0z is converted into system Oxyz by one rotation clockwise around axis Oz to the angle δ_3.

Let i_0, j_0, k and i, j, k - unit vectors of moving coordinate systems, respectively; r_0 - radius of cylinder of lateral surface of cavity; H - height/altitude of cylindrical cavity, ω - angular velocity vector of disturbed motion of gyroscope, ω - angular velocity of stationary rotation of gyroscope, r, φ, z - cylindrical coordinates in system Oxyz.

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In description of motion of liquid we disregard gravitational field in comparison with centrifugal-force field, which corresponds to sufficiently high value of value ω. We assume/set the equation of free surface in the form

$$r_1 - r = \varkappa(\varphi, z, t), \qquad (1)$$

where r_1 - radius of the free surface, which has cylindrical form (during stationary rotation $h = r_0 - r_1$).

Assuming/setting value h sufficient low in comparison with length circle/circumference and height/altitude of cylindrical surface of cavity, we will disregard dependence of parameter values of relative motion of liquid from value r. In further unpackings/facings we will lower index during designation r, accepting as it always mean radius $r = (r_0 + r_1)/2$.

As radial component relative velocity of liquid v let us also take its average value

$$v_r = -\frac{1}{2}\frac{\partial \varkappa}{\partial t}. \qquad (2)$$

Let us designate through l displacement vector of particles of liquid in relative motion

$$\frac{\partial \mathbf{l}}{\partial t} = \mathbf{v}. \qquad (3)$$

Equation of continuity in these designations can be represented in the form

$$\varkappa = -\frac{\partial}{\partial z}(hl_z) - \frac{h}{r}\frac{\partial l_\varphi}{\partial \varphi}. \qquad (4)$$

2. Expression of moment of momentum of system. The moment of momentum of system G is represented in the form

$$\mathbf{G} = \mathbf{G}_T + \mathbf{G}_0, \qquad (5)$$

where $\mathbf{G}_T$ — moment of momentum of the instantly hardened system; G_0 - the moment of momentum of relative motion of liquid.

Expression $\mathbf{G}_T$ with an accuracy down to the terms first-order of smallness can be represented in the form

$$G_T = A\omega_{x_0}\mathbf{i}_0 + A\omega_{y_0}\mathbf{j}_0 + C\omega_z\mathbf{k} - I^0_{xz}\omega_z\mathbf{i}_0 - I^0_{yz}\omega_z\mathbf{j}_0. \qquad (6)$$

Here A, C - respectively transverse and pitching moments of inertia of hardened system during stationary rotation;

$$\omega_{x_0}=-\dot{\delta}_1;\ \omega_{y_0}=\dot{\delta}_2;\ \omega_z=\dot{\delta}_3; \tag{7}$$

I^0_{xz}, I^0_{yz} — centrifugal moments/torques of hardened system in system of coordinates Ox_0y_0z; ω_{x_0}, ω_{y_0}, ω_z — component of velocity ω in this coordinate system;

$$I^0_{xz}=\rho r^2\int\limits_{z_1}^{z_2}\int\limits_0^{2\pi}\varkappa z\cos(\varphi+\delta_3)\,dzd\varphi;$$
$$I^0_{yz}=\rho r^2\int\limits_{z_1}^{z_2}\int\limits_0^{2\pi}\varkappa z\sin(\varphi+\delta_3)\,dzd\varphi, \tag{8}$$

where ρ - density of liquid; z_1 - distance from point O to lower bottom of cavity; $z_2=z_1+H$.

Expression of moment of momentum of relative motion of liquid takes form

$$\mathbf{G}_0=\rho hr\left\{\mathbf{i}_0\int\limits_{z_1}^{z_2}\int\limits_0^{2\pi}[rv_z\sin(\varphi+\delta_3)-zv_\varphi\cos(\varphi+\delta_3)]\,dzd\varphi-\right.$$
$$\left.-\mathbf{j}_0\int\limits_{z_1}^{z_2}\int\limits_0^{2\pi}[rv_z\cos(\varphi+\delta_3)+zv_\varphi\sin(\varphi+\delta_3)]\,dzd\varphi-\mathbf{k}\int\limits_{z_1}^{z_2}\int\limits_0^{2\pi}rv_\varphi dzd\varphi\right\}. \tag{9}$$

Using formulas (5)...(9), let us compose expression of components of total quantity of moment of momentum in system of coordinates Ox_0y_0z:

$$G_{x_0} = -A\dot{\delta}_1 - \rho r^2\omega_z \int_{z_1}^{z_2}\int_0^{2\pi} \varkappa z \cos(\varphi + \delta_3)\, dz d\varphi + \rho h r \int_{z_1}^{z_2}\int_0^{2\pi} [r v_z \sin(\varphi + \delta_3) - \\ - z v_\varphi \cos(\varphi + \delta_3)]\, dz d\varphi; \quad (10)$$

$$G_{y_0} = A\dot{\delta}_2 - \rho r^2\omega_z \int_{z_1}^{z_2}\int_0^{2\pi} \varkappa z \sin z(\varphi + \delta_3)\, dz d\varphi - \rho h r \int_{z_1}^{z_2}\int_0^{2\pi} [r v_z \cos(\varphi + \delta_3) + \\ + z v_\varphi \sin(\varphi + \delta_3)]\, dz d\varphi; \quad (11)$$

$$G_z = C\omega_z + \rho r^2 h \int_{z_1}^{z_2}\int_0^{2\pi} v_\varphi dz d\varphi. \quad (12)$$

3. Equations of relative motion of liquid. We assume/set the liquid of inviscid. The Euler equation for relative motion of liquid will take the form

$$\frac{\partial \mathbf{v}}{\partial t} + 2\,\omega \times \mathbf{v} + \frac{d\,\omega}{dt} \times \mathbf{R} + \omega \times (\omega \times \mathbf{R}) = -\frac{1}{\rho}\operatorname{grad} p, \quad (13)$$

where R - radius-vector relative to point O; p - pressure of liquid.

Taking into account condition (2), let us drop/omit equation, which determines change in component of (13), after replacing with its static equation of motion in accordance with basic hypothesis of long waves theory.

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Remaining equations will take the form

$$\frac{\partial v_\varphi}{\partial t} - \omega_z \frac{\partial \varkappa}{\partial t} - (\dot{\omega}_x \cos\varphi + \dot{\omega}_y \sin\varphi)\, z + \dot{\omega}_z r + \omega_z z(\omega_y \cos\varphi - \omega_x \sin\varphi) = \\ = -\frac{1}{r\rho}\frac{\partial p}{\partial \varphi}; \quad (14)$$

$$\frac{\partial v_z}{\partial t} + r(\dot{\omega}_x \sin\varphi - \dot{\omega}_y \cos\varphi) + \omega_z r(\omega_x \cos\varphi + \omega_y \sin\varphi) = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \quad (15)$$

Static equation of pressure under assumptions accepted will take form

$$p - p_0 = \rho\omega_z^2 r(h + \varkappa). \quad (16)$$

Differentiating right and left of part of equality (16) on φ and on z and after replacing $\varkappa$ according to (4), we will obtain expression of components of pressure gradient through components of displacement vector of particles of liquid:

$$\begin{aligned} \frac{\partial p}{\partial\varphi} &= -\rho\omega_z^2 h\left(r\frac{\partial^2 l_z}{\partial z\partial\varphi} + \frac{\partial^2 l_\varphi}{\partial\varphi^2}\right); \\ \frac{\partial p}{\partial z} &= -\rho\omega_z^2 h\left(r\frac{\partial^2 l_z}{\partial z^2} + \frac{\partial^2 l_\varphi}{\partial\varphi\partial z}\right). \end{aligned} \quad (17)$$

Let us show that in the case of absence of component of moment of external forces along axis Oz value $\dot{\omega}_z$ is equal to zero, and, that means angular rate of rotation of gyroscope relative to axis Oz will be in entire time of motion equal to angular velocity of stationary rotation. For the proof let us register the momental equation of momentum relative to axis Oz, using (12):

$$C\dot{\omega}_z + \rho r^2 h\int_{z_1}^{z_2}\int_0^{2\pi}\dot{v}_\varphi\, dz\, d\varphi = 0. \quad (18)$$

On the basis (14) and boundary conditions (21), given below, it is easy to obtain equality

$$\int_{z_1}^{z_2}\int_0^{2\pi} \dot{v}_\varphi dz d\varphi = 0.$$

Hence follows validity of expressed affirmation. We will further assume the moment of external forces relative to axis Oz the equal to zero.

Equations (14) and (15) on the basis (17) and upon consideration (3), (4) and (7) can be finally converted to following form:

$$\frac{\partial^2 l_\varphi}{\partial t^2} + \omega h \frac{\partial^2 l_z}{\partial t \partial z} + \omega \frac{h}{r} \frac{\partial^2 l_\varphi}{\partial t \partial \varphi} - \omega^2 \frac{h}{r} \frac{\partial^2 l_\varphi}{\partial \varphi^2} = \omega^2 h \frac{\partial^2 l_z}{\partial \varphi \partial z} +$$
$$+ z\ddot{\delta}_2 \sin(\varphi + \delta_3) - z\ddot{\delta}_1 \cos(\varphi + \delta_3); \qquad (19)$$

$$\frac{\partial^2 l_z}{\partial t^2} - \omega^2 hr \frac{\partial^2 l_z}{\partial z^2} = \omega^2 h \frac{\partial^2 l_\varphi}{\partial z \partial \varphi} + r\ddot{\delta}_1 \sin(\varphi + \delta_3) + r\ddot{\delta}_2 \cos(\varphi + \delta_3) +$$
$$+ 2\omega r\dot{\delta}_1 \cos(\varphi + \delta_3) - 2\omega r\dot{\delta}_2 \sin(\varphi + \delta_3). \qquad (20)$$

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Let us register boundary conditions

$$\left[\frac{\partial l_z}{\partial t}\right]_{z=z_1} = \left[\frac{\partial l_z}{\partial t}\right]_{z=z_2} = 0. \qquad (21)$$

4. Compilation of equations of motion of system. Let us register the expression of the components of the moment of

gravitational force along axes Ox_0 and Oy_0:

$$M_{x_0} = -D\delta_1; \quad M_{y_0} = D\delta_2, \tag{22}$$

where value D depends on the mass of system and distance of the center of its masses during the stationary rotation from point O. Let us introduce complex variable

$$\delta = \delta_2 + i\delta_1. \tag{23}$$

Momental equation of momentum relative to complex variable δ on the basis of expressions (10), (11) and (22) takes following form:

$$A\ddot{\delta} - iC\omega\dot{\delta} - D\delta + \rho \frac{d}{dt}\left\{ir^2\omega \int_{z_1}^{z_2}\int_0^{2\pi} \varkappa z\, e^{i(\varphi+\omega t)}\, dz\, d\varphi - \right.$$

$$\left. - hr \int_{z_1}^{z_2}\int_0^{2\pi} (rv_z - izv_\varphi)\, e^{i(\varphi+\omega t)}\, dz\, d\varphi\right\} = 0. \tag{24}$$

Components of displacement vector of particles of liquid can be in general form represented as follows:

$$l_z = r \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [a_{nm}(t) \cos m\varphi + b_{nm}(t) \sin m\varphi] \sin \frac{\pi n}{H}(z - z_1); \tag{25}$$

$$l_\varphi = r \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [d_{nm}(t) \cos m\varphi + c_{nm}(t) \sin m\varphi] \cos \frac{\pi n}{H}(z - z_1).$$

Expressions indicated can be substantially simplified, after excluding from examination those parameters a_{nm}, b_{nm}, c_{nm}, d_{nm}, whose change does not depend on motion of gyroscope, and which, in turn, do not affect motion of latter.

After substituting (25) into equations (24), (19) and (20), we see that such parameters include those, whose indices satisfy conditions m≠1 or n - even number (n≠0).

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Keeping in mind not to consider these parameters, let us drop/omit index m and we will assume that index n takes only odd values or n=0; formulas (25) in this case are converted to form

$$l_z = r \sum_{n=1,3,\ldots} [a_n(t)\cos\varphi + b_n(t)\sin\varphi] \sin\frac{\pi n}{H}(z - z_1);$$

$$l_\varphi = r \sum_{n=1,3,\ldots} [d_n(t)\cos\varphi + c_n(t)\sin\varphi] \cos\frac{\pi n}{H}(z - z_1) + \qquad (26)$$

$$+ r(d\cos\varphi + c\sin\varphi).$$

Let us substitute expressions (4.5) into equations (3.7) and (3.8). Let us multiply the left side of each of these equations consecutively/serially to the coefficient in parameters a_n, b_n, d_n, c_n of (26) and will integrate the obtained products on φ in the range from 0 to 2π and on z range from z_1 to z_2.

As a result we will obtain following system of equations:

$$\ddot{a}_n + \pi^2 n^2 \omega^2 \frac{hr}{H^2} a_n = -\pi n \omega^2 \frac{h}{H} c_n + \frac{4}{\pi n} (\ddot{\delta}_1 \sin \omega t + \ddot{\delta}_2 \cos \omega t +$$

$$+ 2\omega\dot{\delta}_1 \cos \omega t - 2\omega\dot{\delta}_2 \sin \omega t) ;$$

$$\ddot{b}_n + \pi^2 n^2 \omega^2 \frac{hr}{H^2} b_n = \pi n \omega^2 \frac{h}{H} d_n + \frac{4}{\pi n} (\ddot{\delta}_1 \cos \omega t - \ddot{\delta}_2 \sin \omega t -$$

$$- 2\omega\dot{\delta}_1 \sin \omega t - 2\omega\dot{\delta}_2 \cos \omega t) ; \qquad (27)$$

$$\ddot{d}_n + \omega^2 \frac{h}{r} d_n + \omega \frac{h}{r} \dot{c}_n = \pi n \omega^2 \frac{h}{H} b_n - \pi n \omega \frac{h}{H} \dot{a}_n +$$

$$+ \frac{4H}{\pi^2 n^2 r} (\ddot{\delta}_1 \cos \omega t - \ddot{\delta}_2 \sin \omega t) ;$$

$$\ddot{c}_n + \omega^2 \frac{h}{r} c_n - \omega \frac{h}{r} \dot{d}_n = -\pi n \omega^2 \frac{h}{H} a_n - \pi n \omega \frac{h}{H} \dot{b}_n -$$

$$- \frac{4H}{\pi^2 n^2 r} (\ddot{\delta}_1 \sin \omega t + \ddot{\delta}_2 \cos \omega t);$$

$$\ddot{d} + \omega^2 \frac{h}{r} d + \omega \frac{h}{r} \dot{c} = \frac{z_1 + z_2}{2r} (\ddot{\delta}_2 \sin \omega t - \ddot{\delta}_1 \cos \omega t) ;$$

$$\ddot{c} + \omega^2 \frac{h}{r} c - \omega \frac{h}{r} \dot{d} = \frac{z_1 + z_2}{2r} (\ddot{\delta}_1 \sin \omega t + \ddot{\delta}_2 \cos \omega t) .$$

We convert equations (27), after introducing complex quantities

$$\alpha_n = (a_n + ib_n) e^{i\omega t}; \ \beta_n = (d_n + ic_n) e^{i\omega t};$$
$$\beta = (d + ic) e^{i\omega t}.$$

Through the same parameters is expressed variable κ, which is connected with l_z and l_φ according to (4), and values v_z and v_φ, determined on (3). Let us introduce the following designations: $m = 2\pi r h H \rho$ — the mass of the liquid, which is contained in the cavity; k=h/r - charge/weight ratio of cavity with liquid; $\lambda = r/H$ — half of reciprocal value of the elongation/aspect ratio of cavity; C_1=mr² - moment of the inertia of liquid mass; $\mu = (z_1 + z_2)/2r$.

Lowering intermediate unpackings/facings, let us register finally joint system of equations, which describes disturbed motion of gyroscope and liquid containing in its cavity:

$$
\begin{gathered}
A\ddot{\delta} - iC\omega\dot{\delta} - D\delta - \frac{C_1}{\pi} \sum_{n=1,3,\dots} \frac{1}{n} \left(\ddot{\alpha}_n - 2i\omega\dot{\alpha}_n + \frac{i}{\pi n \lambda} \ddot{\beta}_n \right) + \frac{i}{2} C_1 \mu \ddot{\beta} = 0; \\
\ddot{\alpha}_n - 2i\omega\dot{\alpha}_n + (\pi^2 n^2 \lambda^2 k - 1)\omega^2 \alpha_n = i\pi n k \lambda \omega^2 \beta_n + \frac{4}{\pi n} (\ddot{\delta} - 2i\omega\dot{\delta}); \\
\ddot{\beta}_n - i\omega(2+k)\dot{\beta}_n - \omega^2 \beta_n = -\pi n k \lambda \omega \dot{\alpha}_n - \frac{4i}{\pi^2 n^2 \lambda} \ddot{\delta} \; (n = 1, 3 \dots); \qquad (28) \\
\ddot{\beta} - i\omega(2+k)\dot{\beta} - \omega^2 \beta = i\mu\ddot{\delta}.
\end{gathered}
$$

Equations (28) are infinite system of ordinary linear differential equations with constant coefficients.

Their solution can be constructed by method of reduction, by finding solutions of final systems of equations, obtained of (28) with limited and consecutively/serially increasing numbers n.

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The stability of rotation in the string of axisymmetric solid bodies with the cavities, filled with liquid.

A. Yu. Ishlinskiy, M. L. Gorbachuk, M. Ye. Temchenko.

Stability of motion of axially symmetrical body with liquid filling, suspended/hung from string and which revolves with constant angular velocity ω around stationary position of dynamic equilibrium is investigated. String is assumed to be inertia-free, nonductile and nontwisted. The cavity, filled with liquid, has a shape of body of revolution, the axis of symmetry of which coincides with the axis of the symmetry of solid body. It is assumed also that in the steady motion liquid and solid body form single whole. Are examined two special cases; cavity in the form of straight/direct circular cylinder and in the form of ellipsoid of revolution.

Present article to a certain extent is close to scientific thematics of Georgiy Stepanovich Narimanov. To the authors of article infinitely it's a pity, that already it does not exist among us.

In work [9] S. L. Sobolev examined task about stability of motion of gyroscope with axisymmetric cavity, wholly filled with ideal incompressible fluid. In this case it was assumed that in the steady motion the mechanical system solid body - liquid revolves as single solid body. This task led to the analysis of the structure of the spectrum of the self-adjoint operator in the space with the indefinite metric, as a result of which it was possible to isolate the zones of stability. As it proved to be, these zones depend substantially on the form of cavity. In work [5] it was shown that the task considerably is simplified, if the motion of the liquid filling cavity is related to the coordinate system, rigidly connected with the body of gyroscope. This allowed in the case of ellipsoidal and cylindrical cavities to solve it, after using only the method of separation of variables.

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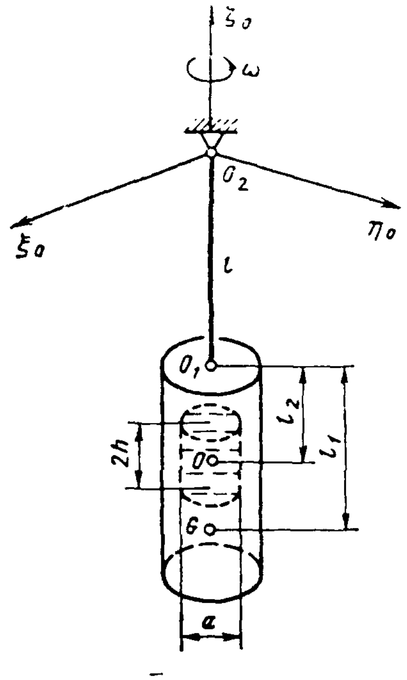
Methodology of conclusion/output of equations of motion [5] is used in this work during solution of more complex problem about stability of motion of axially symmetrical body suspended/hung from string, within which there is cavity in the form of body of revolution with axis of symmetry, which coincides with axis of symmetry of solid body. Is assumed that the string is inertia-free, nonductile and nontwisted, i.e., it is examined only as geometric constraint; in the steady motion liquid and solid body form seemingly single solid whole. In this case it is accepted that the string and the axis of the symmetry of body lie/rest on the vertical straight line. The posed problem is solved with the help of the theory of the self-adjoint operators in the space with the indefinite metric, developed in [3]. The method of study is close to the method, used into [9]. However, in contrast to [9], where the indefinite metric has not more than one negative square, both one and two negative squares here can appear. Is shown that under specific conditions, assigned on the form of cavity, depending on the coefficients of system of equations, to the corresponding task in question, interval $(0, \infty)$ of a change in the angular velocity ω can be decomposed into two or three intervals: $(0, \infty)=(0, \omega_1)\cup(\omega_1, \infty)$ or $(0, \infty)=(0, \omega_1)\cup(\omega_1, \omega_2)\cup(\omega_2, \infty)$, in which a quantity of negative squares of the indefinite metric is constantly equal to 0.1 or 0, 1, 2 respectively. If $\omega\in(0, \omega_1)$, then the solution of system is stable. In all remaining cases of change ω the stability of system depends on the form of cavity.

Cavities in the form of straight/direct circular cylinder and ellipsoid of revolution are examined as examples. In the first case in as conveniently distant interval (ω', ω'') $(\omega'>\omega_1)$ there is a countless set ω, with which the system will not be stable.



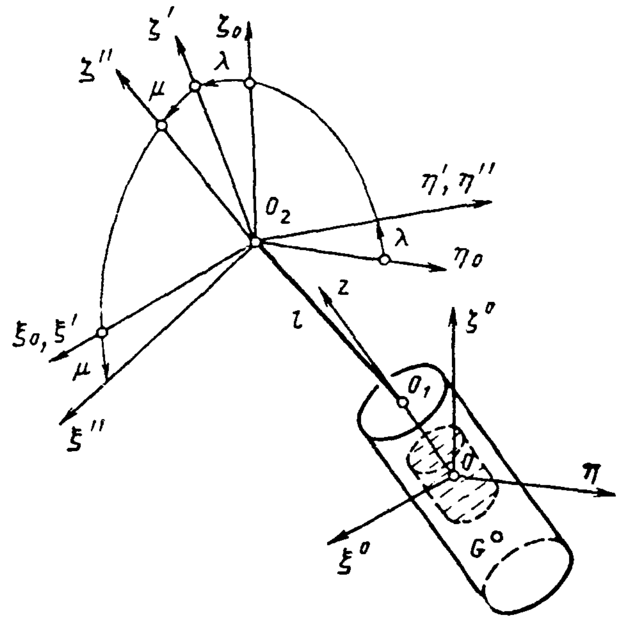


Fig. 1.

Fig. 2.

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In the second case, passing to the ellipsoidal coordinates and using the connected polynomials of Legendre, with the help of the separation of variables and graph-analytic reception/procedure from [4, 10] it is shown that there are two intervals $(\omega_1^*, \omega_1^{**})$ $(\omega_1^* > \omega_1)$ and $(\omega_2^*, \omega_2^{**})$ $(\omega_2^* > \omega_1^{**})$ the changes ω, in which the motion of system is unstable; with ω, that lie out of these intervals, motion is stable.

1. Let us derive differential equations of disturbed motion of solid body, using equations of Lagrange of II order. Let us introduce fixed coordinate system $\xi_0\eta_0\zeta_0$ with the center at suspension point of string to fixed base O_2 (Fig. 1). Axis ζ_0 it is directed vertically

upward, axes ξ_0 and η_0 it will place in the horizontal plane. In the center of cavity - to point O - let us place the reference point of the progressively/forwardly moving system of coordinates $\xi^\circ\eta^\circ\zeta_0$ (Fig. 2), whose axes are respectively parallel to axes ξ_0, η_0, ζ_0. At the same point let us place the reference point of the system of coordinates xyz (Fig. 3), rigidly connected with solid body so that z axis is directed along the axis of the symmetry of body upward, and x and y axes are located in the plane, perpendicular to the axis of symmetry. Let us determine the position of string in the space by angles μ and λ; μ - angle between the straight line, directed along the string, and by its projection on the plane η_0, ζ_0, λ - angle between the projection indicated and the axis ζ_0.

Let us determine position of solid body in space by three angles of Euler-Krylov α, β and φ.

It is assumed that in case on body in question they function force of gravity m_1g and force p* of hydrodynamic pressure of liquid on wall of cavity. The latter are reduced to the main vector F and the main moment/torque M_0 relative to the center of cavity O.

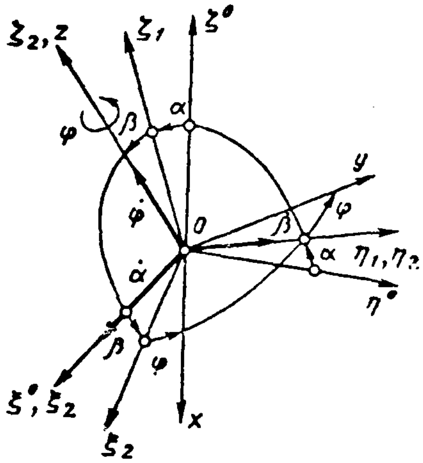

Fig. 3.

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Equations of motion of solid body in the case of small angles α, β, λ, μ and angular velocities $\dot\alpha$, $\dot\beta$, $\dot\lambda$, $\dot\mu$ take form [2]

$$(A_1+m_1l_1^2)\ddot\alpha+C_1\dot\varphi\dot\beta+C_1\ddot\beta\varphi+m_1l_1l\ddot\lambda=M_x\cos\varphi-M_y\sin\varphi+M_z\beta+$$
$$+l_2(F_x\sin\varphi+F_y\cos\varphi)-m_1l_1g\alpha;$$
$$(A_1+m_1l_1^2)-C_1\dot\alpha\dot\varphi+m_1l_1l\ddot\mu=M_x\sin\varphi+M_y\cos\varphi-$$
$$-l_2(F_x\cos\varphi-F_y\sin\varphi)-m_1gl_1\beta; \qquad (1)$$
$$C_1\ddot\varphi=0;$$
$$m_1(l\ddot\lambda+l_1\ddot\alpha)=F_x\sin\varphi+F_y\cos\varphi+(\lambda-\alpha)F_z-m_1g\lambda;$$
$$m_1(l\ddot\mu+l_1\ddot\beta)=F_y\sin\varphi-F_x\cos\varphi+(\mu-\beta)F_z-m_1g\mu,$$

where A_1, C_1 - main central moments of inertia of solid body: m_1 - its mass; l_1 - distance from center of mass of body to attachment point to it of string (see Fig. 1); g - acceleration of gravity ; F_x, F_y, F_z and M_x, M_y, M_z — projection on axis of coordinates x, y, z (main vector and main moment of forces of pressure of liquid on solid body); they are derived/concluded analogously how this is done into [6].

Projections F_x, F_y, F_z and M_x, M_y, M_z we determine from known formulas of hydrodynamics

$$F_x=\int_\tau \frac{\partial p^*}{\partial x}\,d\tau;\ F_y=\int_\tau \frac{\partial p^*}{\partial y}\,d\tau;\ F_z=\int_\tau \frac{\partial p^*}{\partial z}\,d\tau; \tag{2}$$

$$M_x=\int_\tau \left(y\frac{\partial p^*}{\partial z}-z\frac{\partial p^*}{\partial y}\right)d\tau;\ M_y=\int_\tau \left(z\frac{\partial p^*}{\partial x}-x\frac{\partial p^*}{\partial z}\right)d\tau, \tag{3}$$

in which integration it is conducted throughout entire volume of cavity τ.

For determining hydrodynamic pressure p*(x, y, z, t) we will use flow equations in moving coordinate system [6]

$$\frac{d\mathbf{u}}{dt}=\mathbf{P}-\frac{1}{\rho}\operatorname{div}\mathbf{p}^*-\mathbf{w}_e-2(\boldsymbol{\omega}\times\mathbf{u}). \tag{4}$$

Here u - vector of relative speed of particle motion of liquid in cavity of solid body; ρ - specific density of liquid; P - mass forces; $\mathbf{w}_e$ — vector of translational acceleration of particles of liquid, determined by relationship/ratio

$$\mathbf{w}_e=\mathbf{w}^0+\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}), \tag{5}$$

where r(x, y, z) -- radius-vector of particle of liquid, w° - vector of absolute acceleration of point O.

Projections of acceleration w° on axis of coordinates x, y, z with an accuracy to smalls of second order relative to derived angles α, β, μ and λ can be presented in the form

$$w_x^0 = -(l\ddot{\mu} + l_2\ddot{\beta})\cos\varphi + (l\ddot{\lambda} + l_2\ddot{\alpha})\sin\varphi;$$
$$w_y^0 = (l\ddot{\mu} + l_2\ddot{\beta})\sin\varphi + (l\ddot{\lambda} + l_2\ddot{\alpha})\cos\varphi; \qquad (6)$$
$$w_z^0 = 0,$$

where l_2 - distance from point O_1 of suspension of solid body to string to center of cavity O (see Fig. 1).

With the same degree of accuracy it is possible to take taking into account third equation (1) that projections of angular velocity vector ω on axis x, y, z will be following:

$$\omega_x = \dot{\alpha}\cos\varphi + \dot{\beta}\sin\varphi,\ \omega_y = -\dot{\alpha}\sin\varphi + \dot{\beta}\cos\varphi,\ \omega_z = \dot{\varphi} = \omega = \text{const}. \qquad (7)$$

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Taking into account recently obtained equations (5)-(7), flow equations (4) in projections on axis x, y, z let us represent in

$$\frac{\partial u_x}{\partial t} - 2\omega u_y = -\frac{1}{\rho}\frac{\partial p_1}{\partial x} - 2z\dot{\omega}_y;$$
$$\frac{\partial u_y}{\partial t} + 2\omega u_x = -\frac{1}{\rho}\frac{\partial p_1}{\partial y} + 2z\dot{\omega}_x; \qquad (8)$$
$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial z},$$

where

$$p_1 = p^* - \rho\left[(l\ddot{\mu} + l_2\ddot{\beta})\cos\varphi - (l\ddot{\lambda} + l_2\ddot{\alpha})\sin\varphi\right]x +$$
$$+\rho\left[(l\ddot{\mu} + l_2\ddot{\beta})\sin\varphi + (l\ddot{\lambda} + l_2\ddot{\alpha})\cos\varphi\right]y - \frac{\rho}{2}(x^2 + y^2)\omega^2 +$$
$$+\rho\omega z(x\omega_x + y\omega_y) - \rho z(x\dot{\omega}_y - y\dot{\omega}_x) - \rho(xg_x + yg_y + zg_z). \qquad (9)$$

According to the condition of incompressibility, function u_x, u_y, u_z, the entering system (8), must satisfy the equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \qquad (10)$$

and the boundary condition

$$u_x \cos nx + u_y \cos ny + u_z \cos nz = 0. \qquad (11)$$

Latter means that projection of relative particle speed of liquid, which is touched with boundary of cavity, on normal n to it is equal to zero.

2. Let us turn now to equations (1). Let us introduce the complex-valued functions of real variable $\zeta^* = \alpha + i\beta$; $z^* = \lambda + i\mu$. Then

$$(A_1 + m_1 l_1^2)\ddot{\zeta}^* - iC_1\omega\dot{\zeta}^* + m_1 l_1 g\zeta^* + m_1 l l_1 \ddot{z}^* = (M_x + iM_y)e^{i\omega t} -$$
$$-il_2(F_x + iF_y)e^{i\omega t}; \qquad (12)$$
$$m_1 l_1 \ddot{\zeta}^* + m_1 l \ddot{z}^* + m_1 g z^* = -i(F_x + iF_y)e^{i\omega t} + F_z(z^* - \zeta^*).$$

Let us determine expressions $F_x + iF_y$; F_z, $M_x + iM_y$, which stand in right sides of recently obtained equations.

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Taking into account relationships/ratios (2), (3), (9), we obtain

$$F_x+iF_y=-iM_z\left(l_2\ddot{\zeta}^*+l\ddot{z}^*+g\zeta^*\right)e^{-i\omega t}+\int_\tau\left(\frac{\partial p_1}{\partial x}+i\frac{\partial p_1}{\partial y}\right)d\tau;$$

$$F_z=-m_2g+\int_\tau\frac{\partial p_1}{\partial z}\,d\tau; \qquad (13)$$

$$M_x+iM_y=-i\int_\tau(x+iy)\frac{\partial p_1}{\partial z}\,d\tau+i\int_\tau z\left(\frac{\partial p_1}{\partial x}+i\frac{\partial p_1}{\partial y}\right)d\tau-$$

$$-\left(\ddot{\zeta}^*-2i\omega\dot{\zeta}^*\right)(C_2-A_2)\,e^{-i\omega t}.$$

Let us note that in latter/last equalities through m_2 is designated mass of liquid in cavity of bodies, and are also used relationships/ratios

$$\int_\tau\rho x d\tau=\int_\tau\rho y d\tau=\int_\tau\rho z d\tau=0;$$

$$\int_\tau\rho xy d\tau=\int_\tau\rho xz d\tau=\int_\tau\rho yz d\tau=0;$$

$$\int_\tau\rho\left(z^2-y^2\right)d\tau=\int_\tau\rho\left(z^2-x^2\right)d\tau=A_2-C_2,$$

in which A_2 and C_2 - respectively equatorial and axial moments of inertia of liquid relative to axes x, y, z.

Bearing in mind that $\zeta^*=\zeta e^{i\omega t}$, and assuming/setting $z^*=\eta e^{i\omega t}$, let us register set of equations (12) in matrix form

$$\mathbf{A}\begin{bmatrix}\ddot{\zeta}\\ \ddot{\eta}\end{bmatrix}+i\omega\mathbf{B}\begin{bmatrix}\dot{\zeta}\\ \dot{\eta}\end{bmatrix}+\omega^2\mathbf{C}\begin{bmatrix}\zeta\\ \eta\end{bmatrix}+\begin{bmatrix}\frac{i}{2}\int\limits_S p\nu\,dS\\ 0\end{bmatrix}=0, \qquad (14)$$

where

$$\mathbf{A}=\begin{pmatrix}a_{11}a_{12}\\ a_{12}a_{22}\end{pmatrix};\ \mathbf{B}=\begin{pmatrix}b_{11}b_{12}\\ b_{12}b_{22}\end{pmatrix};\ \mathbf{C}=\begin{pmatrix}c_{11}c_{12}\\ c_{12}c_{22}\end{pmatrix};$$

$$a_{11}=A_1+L;\ L=m_1l_1^2+m_2l_2^2;\ a_{12}=lK_1;\ K_1=m_1l_1+m_2l_2;$$

$$a_{22}=l^2m;\ m=m_1+m_2;\ b_{11}=2A_1-C_1+2L;\ b_{12}=2lK_1;$$

$$b_{22}=2l^2m;\ c_{11}=D-L+\frac{K_1}{\omega^2}g;\ D=C_1-A_1+C_2-A_2;$$

$$c_{12}=-lK_1;\ c_{22}=-l^2m+\frac{m}{\omega^2}lg;\ \nu=z(\cos nx+i\cos ny)-$$

$$-(x+iy)\cos nz;$$

$$\frac{1}{\rho}p=\frac{1}{\rho}p_1+\frac{\ddot{\zeta}-\ddot{\bar{\zeta}}}{2i}xy+\frac{\ddot{\zeta}+\ddot{\bar{\zeta}}}{2}yz;$$

S - surface of cavity.

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3. As shown into [2], on the assumption that S satisfies conditions

$$\int\limits_\tau\frac{\partial p_1}{\partial z}d\tau=\int\limits_\tau\frac{\partial p_1}{\partial y}d\tau=\int\limits_\tau\frac{\partial p_1}{\partial x}d\tau=0$$

(this it always occurs, for example, for ellipsoid of revolution and circular cylinder), system (8), (10), (11), (14) can be registered in canonical form

$$\frac{du(t)}{dt}=iH_\omega u(t),$$

where u(t) - vector function with values in certain Hilbert space; H_ω — bounded operator in it. Therefore the Cauchy problem for system

(8), (10), (11), (14) is always solved.

It proves to be that operator $H_\omega - J_\omega$— self-adjoined (i.e. $J_\omega H_\omega = H_\omega^* J_\omega$), in this case gap/interval [0, ∞) of change ω can be represented as $[0, \infty) = [0, \omega_1) \cup [\omega_1, \infty)$ or $[0, \infty) = [0, \omega_1) \cup [\omega_1, \omega_2) \cup [\omega_2, \infty)$, where ω_1, ω_2 they are such, that operator J_ω positively determined when $\omega \in [0, \omega_1)$ when $\omega \in (\omega_1, \infty)$ in first case and $\omega \in (\omega_1, \omega_2)$ in second case J_ω— indefinite operator with one negative square, when $\omega \in (\omega_2, \infty)$ J_ω it has two negative squares. This indicates the stability of the system when $\omega \in [0, \omega_1)$; in question however, as far as the remaining cases, are concerned, here stability does not always have the place (they can appear the solutions, which grow on ∞ both exponentially, and exponentially) - this depends on the form of cavity.

Let us note that division [0, ∞) to three subintervals and appearances in connection with this in J_ω two negative squares characteristically precisely in presence of string (compare with [9]). The numbers ω_1 and ω_2 are defined as the roots of the equation

$$\det C = [-(\gamma + K_1^2)\, l\omega^4 + g(\gamma - mlK_1)\omega^2 + K_1 mg^2]\, l\omega^4 = 0,$$

where $\gamma = m(D - L)$. If $\gamma + K_1{}^2 > 0$, then det C becomes zero only when

$\omega_1 = \sqrt{\frac{g(\gamma - mlK_1) + \sqrt{\Delta}}{2l(\gamma + K_1^2)}}$; $\Delta = g^2(\gamma + mlK_1)^2 + 4lK_1^3 mg^2$; if $\gamma + K_1{}^2 = 0$, then $\omega_1 = \sqrt{mg/(ml + K_1)}$. The cases indicated correspond to division [0, ∞) to two subintervals. But if $\gamma + K_1{}^2 < 0$, then

$[0, \infty) = [0, \omega_1) \cup [\omega_1 \omega_2) \cup [\omega_1, \infty)$, where

$$\omega_1 = \sqrt{[g(\gamma - mlK_1) + \sqrt{\Delta}]/2l(\gamma + K_1^2)},$$
$$\omega_2 = \sqrt{[g(\gamma - mlK_1) - \sqrt{\Delta}]/2l(\gamma + K_1^2)}.$$

As already mentioned, when $\omega \in [0, \omega_1)$ system (8), (10), (11), (14) can have those exponentially growing for ∞ solutions. Such solutions exist when and only when operator H_ω has insubstantial eigenvalues. Them there can be not more than four (two complexly conjugated/combined pairs), since J_ω has not more than two negative squares [3].

4. Let us show that if cavity of solid body is straight/direct circular cylinder, then there is countless set ω, for which operator H_ω has at least one pair of conjugated/combined with each other insubstantial eigenvalues.

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In fact, task at eigenvalues $H\omega f = \lambda_1 f$ for operator H_ω brings, as shown into [2], to equation

$$(A^* + C_2 - A_2)\lambda_1^2 - (C_1 + 2C_2 - 2A_2)\omega\lambda_1 - K_1 g - \frac{K_1^2}{m}\frac{\lambda_1^4}{(\lambda_1^2 - g/l)} = $$
$$= \lambda_1(\lambda_1 - \omega)\tilde{D}(\lambda_1), \qquad (15)$$

where

$$\tilde{D}(\lambda_1) = \frac{128\rho a^2 h^3}{\pi^3(\omega + \lambda_1)} \sum_{k=0}^{\infty} \frac{[4\omega^2 - 3(\lambda_1 - \omega^2)] J_1(ka) + ka(\lambda_1 - \omega)^2 J_0(ka)}{(2l+1)^4 [ka(\lambda_1 - \omega) J_0(ka) - (\lambda_1 + \omega) J_1(ka)]},$$

$A^{*}=A_1+m_1 l_1{}^2+m_2 l_2{}^2$, $J_1(ka)$ and $J_0(ka)$ - function of Bessel of first order, a, h - radius and height/altitude of cylinder.

Let us establish existence of countless set ω, for which equation (15) has at least one insubstantial root.

Let us designate $\lambda_1 - \omega = 2\omega g$ and let us divide both parts (15) to ω^2. Then

$$(A^{*}+C_2-A_2)(2q+1)^2-(C_1+2C_2-2A_2)(2q+1)-\frac{K_1 g}{\omega^2}-$$
$$-\frac{K_1^2}{m}\cdot\frac{(2q+1)^4}{(2q+1)^2-g/\omega^2 l}=D(q), \qquad (16)$$

where

$$D(q)=\frac{2q(2q+1)\cdot 128\rho a^2 h^3}{\pi^3(q+1)}\sum_{k=0}^{\infty}\frac{kaq^2 J_0(ka)-(3q^2-1)J_1(ka)}{(2l+1)^2[kaqJ_0(ka)-(q+1)J_1(ka)]}.$$

Let us assume further

$$D_N(q)=\frac{2q(2q+1)\cdot 128\rho a^2 h^3}{\pi^3(q+1)}\sum_{k=0}^{N}\frac{kaq^2 J_0(ka)-(3q^2-1)J_1(ka)}{(2l+1)^2[kaqJ_0(ka)-(q+1)J_1(ka)}.$$

Sequence of analytic functions $D_N(q)$ evenly approaches D(q) within any region, which does not contain segment [-1, 1) (see [9]). Therefore the insubstantial roots of equation (16) with sufficiently large N are close to the insubstantial roots of the equation

$$(A^* + C_2 - A_2)(2q+1)^2 - (C_1 + 2C_2 - 2A_2)(2q+1) - \frac{K_1 g}{\omega^2} -$$

$$-\frac{K_1^2}{m}\,\frac{(2q+1)^4}{(2q+1)^2 - g/\omega^2 l} = D_N(q). \qquad (17)$$

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Since

$$\frac{(2q+1)^4}{(2q+1)^2 - g/\omega^2 l} = (2q+1)^2\Big[1 + \frac{g}{(2q+1)^2\omega^2 l} +$$

$$+ \frac{g^2}{(2q+1)^4\omega^4 l^2}\,\frac{1}{1 - g/(2q+1)^2\omega^2 l}\Big],$$

that left side (17) is represented in the form

$$\left(A^* + C_2 - A_2 - \frac{K_1^2}{m}\right)(2q+1)^2 - (C_1 + 2C_2 - 2A_2)(2q+1) -$$

$$-\left(K_1 g + \frac{K_1^2 g}{ml}\right)\frac{1}{\omega^2} - Q(q),$$

where

$$Q(q) = \frac{K_1^2}{m}\,\frac{g^2}{l^2\omega^4[(2q+1)^2 - g/\omega^2 l]}.$$

When

$\omega \geqslant \max \left\{\omega_1,\ 4\sqrt{\frac{g}{l}}\right\}$ and $-\frac{1}{4} \leqslant q < 1$

$$|Q(q)| \leqslant \frac{16K_1^2 g^2}{3m\omega^4 l^2}. \qquad (18)$$

As shown in [9], $D_N(q)$ has countless set of simple poles with unique accumulation point in zero, moreover for negative q_0^- and positive q_0^+ poles deduction $\mathrm{Res}\, D_N(q_0) < 0\ (>0).$ This speaks, that curve about the poles takes the form, indicated in Fig. 4. In interval (-3/4, -1/4) $D_N(q)$ has the finite number of poles.

Let us consider first equation

$$\Pi(q)=\left(A^{*}+C_2-A_2-\frac{K_1}{m}\right)(2q+1)^2-(C_1+2C_2-2A_2)(2q+1)-$$
$$-\left(K_1 g+\frac{K_1^2 g}{ml}\right)\frac{1}{\omega^2}=D_N(q). \qquad (19)$$

For research of insubstantial roots of (19) with given one a/h let us construct curve $D_N(q)$. Real roots (19) are located in the points of intersection of parabola $\Pi(q)$ with curve $D_N(q)$.

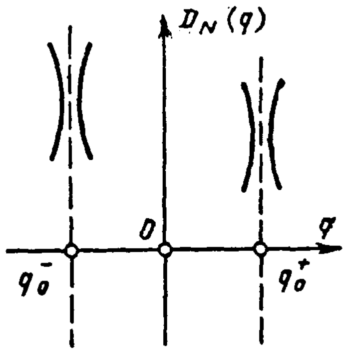

Fig. 4.

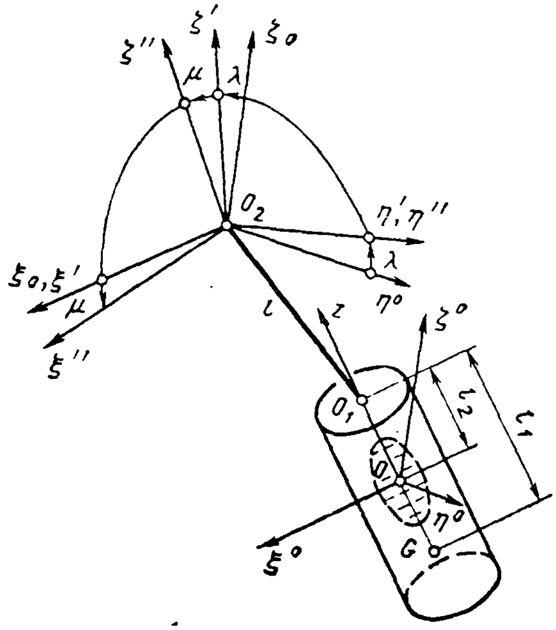

Fig. 5.

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Since in the low section of change q parabola Π(q) is close to the straight line, the equation (19) has insubstantial roots when parabola passes between two branches $D_N(q)$. Moving parabola Π(q) so that the position of its axis would remain constant/invariable, it is possible to find all dangerous gaps/intervals of change ω from the value of the segment, intercepted/detached by parabola to straight line q=-1/2, equal to $(K_1 g + K_1{}^2 g/ml)/\omega^2$. With these ω the parabola falls between branches $D_N(q)$, which it has countless set. Therefore dangerous gaps/intervals there will be also infinitely much.

As a result of the fact that in interval $(-3/4, -1/4)$ $D_N(q)$ has finite number of branches, and out of this interval (see inequalities (18)] with sufficiently large $\omega\Pi(q)+Q(q)$ it differs little from $\Pi(q)$, then for equation (17) will also exist infinite set of gaps/intervals ω in which it will have insubstantial roots, moreover these gaps/intervals, with exception of finite number, differ little from appropriate gaps/intervals for equation (19).

5. We investigate now stability of motion of body on the assumption that in question cavity, filled with liquid, has form of ellipsoid of revolution (Fig. 5).

We will seek solution of system of equations (8) in this case in the form

$$\begin{aligned} p_1 &= \tilde{p}(x,\ y,\ z)\,e^{i(\lambda_1-\omega)t};\ u_x=\tilde{u}(x,\ y,\ z)\,e^{i(\lambda_1-\omega)t};\\ u_y &= \tilde{v}(x,\ y,\ z)\,e^{i(\lambda_1-\omega)t};\ u_z=\tilde{w}(x,\ y,\ z)\,e^{i(\lambda_1-\omega)t}. \end{aligned} \tag{20}$$

After transformations, analogous to those carried out into [5], taking into account of equations (10), (11) and designations

$$\dot{\omega}_x=\Omega_x\,e^{i(\lambda_1-\omega)t};\ \dot{\omega}_y=\Omega_y\,e^{i(\lambda_1-\omega)t} \tag{21}$$

for determining function $\tilde{p}(x, y, h)$ we will obtain equation of Laplace

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial \tilde{z}^2}\right)\tilde{p}=0, \tag{22}$$

where

$$\tilde{z}=z\varkappa;\ \varkappa=(\lambda_1-\omega)\left[(\lambda_1-\omega)^2-4\omega^2\right]^{-1/2}. \tag{23}$$

For determination of solution of equation (22) let us switch over to ellipsoidal coordinates μ, ξ_*, ψ, which are connected with x, y, z by relationships/ratios [1, 8]

$$x = k(1-\mu^2)^{1/2}(\xi_*^2-1)^{1/2}\cos\psi,\quad y = k(1-\mu^2)^{1/2}(\xi_*^2-1)^{1/2}\sin\psi,$$
$$\tilde{z} = k\mu\xi_*, \tag{24}$$

in this case ellipsoidal coordinates let us select so that equation of surface of cavity

$$\frac{x^2+y^2}{a^2}+\frac{\tilde{z}^2}{\varkappa^2c^2}=1 \tag{25}$$

would belong to confocal family. Let, for example, $\xi_* = \xi_0$.

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Then the arbitrary value k in relationships/ratios (24) and constant ξ_0 should be taken by the equal to

$$k = c\varkappa e;\ \xi_0 = e^{-1};\ k = a(\xi_0^2-1)^{1/2}, \tag{26}$$

where $e = (c^2\varkappa^2 - a^2)^{1/2}(c\varkappa)^{-1}$ — eccentricity of ellipsoid of revolution with semi-axes a and $c\kappa$. The solution of equation (22) now can be represented in the form [1]

$$\tilde{p} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n^m(\xi_*)P_n^m(\mu)\left[A_n^m\cos m\psi + B_n^m\sin m\psi\right], \tag{27}$$

where $P_n{}^m$ — connected polynomials of Legendre; $A_n{}^m, B_n{}^m$ — arbitrary constants.

Values $A_n{}^m, B_n{}^m$ let us determine from boundary condition (11), which with the help of equalities (8), (20), (21), (23)...(27) is converted to form

$$\frac{c^2\varkappa^2}{\rho i(\lambda_1-\omega)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1-\mu^2e^2}{\xi_*^2-\mu^2}\xi_*(\xi_*^2-1)\frac{dP_n^m(\xi_*)}{d\xi_*}P_n^m(\mu)(A_n^m\cos m\psi+$$

$$+B_n^m\sin m\psi)+\frac{2\omega c^2}{\rho[4\omega^2-(\lambda_1-\omega)^2]}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}mP_n^m(\xi_*)P_n^m(\mu)\times$$

$$\times(-A_n^m\sin m\psi+B_n^m\cos m\psi)=\frac{c^2k^2}{9\varkappa[4\omega^2-(\lambda_1-\omega)^2]}\{[4\omega\Omega_x-$$

$$-2i(\lambda_1-\omega)\Omega_y]\cos\psi+[4\omega\Omega_y+2i(\lambda_1-\omega)\Omega_x]\sin\psi\}P_2^1(\xi_*)P_2^1(\mu).$$

Substituting them into expression (27) and taking into account first relationship/ratio (20), and also equality (9), let us find function p*, which determines pressure of liquid on wall of cavity. Using further formulas (2), (3), (9), (13) and (21), and then integrating by the volume of cavity τ, after appropriate unpackings/facings let us arrive at the relationships/ratios

$$M_x+iM_y=\frac{4}{15}\pi\rho a^2c(c^2-a^2)\left[\frac{2c^2i(i\ddot{\zeta}^*+\omega\dot{\zeta}^*)(\lambda_1-\omega)}{(c^2+a^2)(\lambda_1-\omega)+2a^2\omega}+\ddot{\zeta}^*-\right.$$

$$\left.-2i\omega\dot{\zeta}\right]e^{-i\omega t};\;\; F_x+iF_y=-im_2(l_2\ddot{\zeta}^*+l\ddot{z}^*+g\zeta^*)e^{-i\omega t};\;\; F_z=-m_2g.$$

Let us substitute recently obtained expressions into system of equations (12) and we will seek its solution on the basis of relationships/ratios (20) in the form

$$\zeta^*=\tilde{\zeta}^*e^{i\lambda_1 t};\quad z^*=\tilde{z}^*e^{i\lambda_1 t}.$$

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Characteristic equation, which corresponds to system in question, is reduced to the form

$$f_1(\lambda, \omega) = (A^0 + k_*\eta_*)\lambda^5 - \omega[C_1 + (A^0 + k_*)\eta_*]\lambda^4 + [-K - \\ -(A^* + k_*\eta_*)g/l + C_1\omega^2\eta_* + k_*\omega^2(1-\eta_*)]\lambda^3 + [K\eta_* + C_1 g/l + \\ + (A^* + k_*)\eta_* g/l]\omega\lambda^2 + [K + k_*\omega^2(1-\eta_*) - C_1\omega^2\eta_*]\lambda g/l - K\omega\eta_* g/l = \\ = 0. \qquad (28)$$

In this equation besides the designations, which were being already mentioned earlier,

$$A^* = A_1 + m_1 l_1^2 + m_2 l_2^2; \quad \eta_* = \frac{c^2 - a^2}{c^2 + a^2}; \quad k_* = \frac{4}{15}\pi\rho a^2 c(c^2 - a^2);$$

$$z_C = K_1/m; \quad A^0 = A^* - z_C K_1; \quad K = gK_1.$$

It is obvious that criterion of stability of motion in situation in question is matter of roots of characteristic equation (28). For their research we will use the graph-analytic reception/procedure, described in [4, 10]. Let us note, first of all, that according to the rule of Descartes [7], the number of positive roots of equation (28) with any values of the parameter ω cannot be more than three, but negative - it is more than two. It is possible to show that this equation with $\omega>0$ always has two and only two negative roots. Actually, it is easy to ascertain that

$$f_1(-\infty, 0) < 0; \quad f_1\left(-\sqrt{g/l}, \omega\right) = (g^2/l^2)(A^* - A^0)\left(\omega\eta_* + \sqrt{g/l}\right) > 0.$$

Consequently, in the intervals $-\infty < \lambda < -\sqrt{g/l}$ and $-\sqrt{g/l} < \lambda < 0$ is located through one negative root.

determination of character of three remaining roots of equation (28) is realized by means of construction and corresponding study of plotted function

$$\omega=\omega(\lambda)=\frac{q(\lambda)\pm\sqrt{q^2(\lambda)-4p(\lambda)r(\lambda)}}{2p(\lambda)}, \tag{29}$$

in which

$$p(\lambda)=\lambda(\lambda^2-g/l)\varepsilon\eta_*^2;\quad \varepsilon=\eta_*^{-2}[C_1\eta_*+k(1-\eta_*)]>0;$$
$$\varkappa=k\eta_*^{-2}(1-\eta_*)(1-\eta_*^2);$$
$$q(\lambda)=\eta_*\{(A^0+k\eta_*)\lambda^4-[K+(A^*+k\eta_*)g/l]\lambda^2+Kg/l+$$
$$+\lambda^2(\lambda^2-g/l)(\varepsilon-\varkappa);$$
$$r(\lambda)=\lambda\{(A^0+k\eta_*)\lambda^4-[K+g^0(A^*+k\eta_*)]\lambda^2+g^0K\}.$$

Analysis of this graph gives possibility for each fixed/recorded parameter ω to determine, everything roots of equation (28) will be real, and therefore will be motion of solid body stable or not.

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In that range of values λ, where the discriminant

$$\Delta(\lambda)=4p(\lambda)r(\lambda)-q^2(\lambda)=-\eta_*^2[a_1\lambda^4-(K+g/la_2)\lambda^2+g/lK]\times$$
$$\times[b_1\lambda^4-(K+b_2g/l)\lambda^2+Kg/l],$$

where $a_1=A^0+k\eta-(\sqrt{\varepsilon}-\sqrt{\varkappa})^2$; $b_1=A^0+k\eta-(\sqrt{\varepsilon}+\sqrt{\varkappa})^2$; $a_2=A^*+$

$$+k\eta-(\sqrt{\varepsilon}-\sqrt{\varkappa})^2;\quad b_2=A^*+k\eta-(\sqrt{\varepsilon}+\sqrt{\varkappa})^2$$

it is positive, equation (28) will have a pair of the complexly conjugate roots, and therefore the motion of the body being investigated will be unstable. For the same values λ, where $\Delta(\lambda)<0$ it is stable.

For determining sign Δ(λ) let us consider polynomials

$$F_1(\Omega)=a_1\Omega^2-(K+a_2gP/l)\Omega+KgP/l;$$
$$F_2(\Omega)=b_1\Omega^2-(K+b_2gP/l)\Omega+KgP/l. \quad (30)$$

It is not difficult to show that their roots are always real, signs of these roots depending on signs of coefficients a_1 and b_1. If $b_1>0$ and $a_1>0$ (that in the case in question it occurs), then all zeros of the polynomial (30) are positive. In this case one root of each of the polynomials is less than g/l, and another - is more than this value. Fig. 6 depicts plotted function (29). In it through $\omega_1{}^*$, $\omega_2{}^*$, $\omega_1{}^{**}$, $\omega_2{}^{**}$ are designated the critical values of the angular rate of rotation of the bodies, which, according to [10], are determined from the relationships/ratios

$$\omega_i^*=\frac{\sqrt{\varepsilon}-\sqrt{\varkappa}}{\eta_*\sqrt{\varepsilon}}\lambda_i^*;\quad \omega_i^{**}=\frac{\sqrt{\varepsilon}+\sqrt{\varkappa}}{\eta_*\sqrt{\varepsilon}}\lambda_i^{**}\quad (i=1,2),$$

where

$$\lambda_i^*=\sqrt{\Omega_i^*};\quad \lambda_i^{**}=\sqrt{\Omega_i^{**}}\quad (i=1,2)$$

Ω_1^*, Ω_2^*, Ω_1^{**}, Ω_2^{**} — the roots respectively of first and second polynomials (30).

Analysis of plotted function (29) shows that with values of angular velocity, which are changed within limits $\omega_1^* < \omega < \omega_1^{**}$ and $\omega_2^* < \omega < \omega_2^{**}$, motion of body being investigated is unstable. The motion is stable out of these limits of a change in the angular velocity ω.

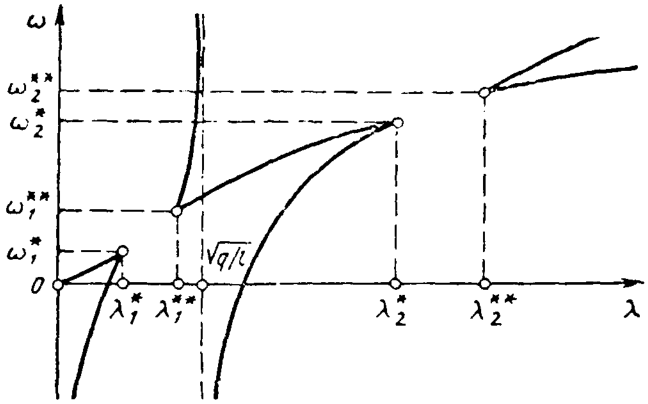

Fig. 6.

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Dynamics of the fast-turning flight vehicle with the eccentrically arranged sections, partially filled with liquid.

L. V. Dokuchayev, V. L. Yezdakov.

Is examined motion of body with cylindrical cavity, which has radial diaphragms and by partially filled liquid. For the compilation of equations of motion is used the setting by G. S. Narimanov the task about the nonlinear irrotational motion of liquid in the field of mass forces. It is assumed that the external forces and moments/torques are low in comparison with the centrifugal forces, so that entire mass of liquid is forced against lateral surface, and free surface completes oscillations relative to the cylindrical undisturbed surface. The analytical solutions of boundary-value problems are given and are determined hydrodynamic coefficients taking into account three tones. Are analyzed the necessary stability conditions of the stationary rotation, which upon consideration of the dissipation of energy are sufficient.

Develop theorying of nonlinear motion of body with liquid, G. S. Narimanov in his first works [1, 2] considered two cases of steady state of liquid object. In one case the body is located in the field of the mass forces, parallel to vertical axis so that the free surface of liquid in the undisturbed state is horizontal. Compiling the nonlinear equations of the wave oscillations of liquid, taking into account the members of the third order of smallness, G. S. Narimanov could explain the phenomenon of the "circular" wave observed in the experiment, when the inclined plane of the free surface of liquid begins to complete rotation relative to vertical axis, while the exciting forces are directed perpendicularly to this axis. In other case the body is located in the radial field of mass forces, as, for instance, the liquidfilled gyroscope, whose centrifugal forces considerably exceed gravitational forces. On the flight vehicle, which accomplishes rotational motion with the engines off, the radial field of mass forces appears virtually at any velocity of torsion. Using long wave theory, G. S. Narimanov derives the equations of rotation of body with a circular thin layer of liquid. Is investigated below the stability of rotational motion LA with the cavities, situated on a certain removal/distance symmetrically relative to the axis of torsion. On the single-connected volume of liquid functions the radial field of mass forces.

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For the compilation of equations of motion is used setting of G. S. Narimanov's first task.

1. Let us introduce connected with solid housing system of coordinates OXYZ, moving relative to inertial space with forward velocity of V_0 and angular velocity ω. In the undisturbed motion KA

it is twisted with an angular velocity of ω_0 around axis OX, which coincides with axis O_1Z_1 of the system of coordinates $O_1X_1Y_1Z_1$, arranged/located so that axis O_1X_1 lies/rests on the intersection of two planes of the symmetry of the cavity (see Fig. 1). The free surface of an incompressible fluid Σ completes oscillations relative to cylindrical surface Σ_0. At the initial moment of time the flow of ideal fluid is assumed to be irrotational, therefore, and subsequently the flow will be also irrotational, i.e., there is a velocity potential Φ, which satisfies the equation of Laplace and the condition of nonpassage on the moistened surface of cavity.

Let set of functions $f_i(y, z)$ be complete on surface Σ_0. The coefficients of expansion in the generalized series of Fourier in these functions of the deflection of free surface from the surface Σ_0 are designated through $\beta_i(t)$ (they have a dimensionality of length). The velocity potential and the hydrodynamic coefficients, determined below, are decomposed/expanded according to the degrees of the low parameter, such it is the deflection of the free surface of f, moreover only terms are retained, which in the equations of motion will give terms not higher than the 1st order of smallness. It is assumed that the vector of the apparent acceleration

$\mathbf{j}=\frac{d\mathbf{V}_0}{dt}+\boldsymbol{\omega}\times\mathbf{V}_0-\mathbf{g}$ ($\mathbf{g}$ — free-fall acceleration), the vector of angular acceleration ω, projection of angular velocity on the transverse axes ω_2, ω_3, generalized coordinates $\beta_i(t)$ are the values of the 1st order of smallness, and the velocity of torsion ω_0 -

zeroth-order quantity of smallness. Velocity potential Φ we will seek in the following form:

$$\Phi=\mathbf{V}_0\mathbf{r}+\omega\left(\Omega_0+\sum \beta_i\Omega_i+\sum_i\sum_j \beta_i\beta_j\Omega_{ij}\right)+\sum_i \dot{\beta}_i A_i.$$

Here harmonic functions $A_i, \Omega_0, \Omega_i, \Omega_{ij}$ satisfy boundary-value problems

$$\Delta A_i=0;\ \left.\frac{\partial A_i}{\partial \nu}\right|_S=0;\ \left.\frac{\partial A_i}{\partial \nu}\right|_{\Sigma_0}=f_i; \qquad (1)$$

$$\Delta \Omega_0=0;\ \left.\frac{\partial \Omega_0}{\partial \nu}\right|_{S+\Sigma_0}=\mathbf{r}\times\boldsymbol{\nu}; \qquad (2)$$

$$\Delta \boldsymbol{\Omega}_i=0;\ \left.\frac{\partial \Omega_i}{\partial \nu}\right|_S=0;\ \left.\frac{\partial \Omega_i}{\partial \nu}\right|_{\Sigma_0}=-(\mathbf{r}\times\nabla f_i)+\nabla(f_i\nabla\Omega_0); \qquad (3)$$

$$\Delta \boldsymbol{\Omega}_{ij}=0;\ \left.\frac{\partial \Omega_{ij}}{\partial \nu}\right|=0;\ \left.\frac{\partial \Omega_{ij}}{\partial \nu}\right|_{\Sigma_0}=$$

$$=-\frac{1}{2}\left[\nabla(f_j\nabla\boldsymbol{\Omega}_i+f_i\nabla\boldsymbol{\Omega}_j+\nabla\left(f_if_j\nabla\frac{\partial\Omega_0}{\partial\nu}\right)-\boldsymbol{\nu}\times\nabla(f_if_j)\right]. \qquad (4)$$

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Operator ∇ indicates two-dimensional gradient over surface Σ_0. The numerical methods of solving the first and second boundary-value problems, which relate to the linear theory of the wave oscillations of liquid, are presented in work [4]. Let us consider two special cases, with which it is possible to obtain analytical expressions for the coefficients of nonlinear equations of motion.

Let cavity take form of circular cylindrical sector (Fig. 1) with planes of symmetry $O_1X_1Z_1$, $O_1X_1Y_1$ and aperture angle of sector 2α. External radius of sector R_0, internal δR_0, the height/altitude of cylinder $2\bar{h}R_0$. Let us introduce cylindrical coordinate system z, r, η. Then the solution of boundary-value problem (1) can be sought by the method of separation of the variables

$$A_i(z, r, \eta) = M_{sk}(r) H_s(\eta) Z_k(z), \tag{5}$$

where

$$M_{sk}(r) = \frac{1}{\beta} \frac{I_m(\beta r) K_m'(\beta) - K_m(\beta r) I_m'(\beta)}{I_m'(\beta r) K_m'(\beta) - K_m'(\beta\delta) I_m'(\beta)};$$

$$m = \frac{\pi s}{2\alpha}; \quad \beta = \frac{\pi k}{2\bar{h}} \ (k, s = 0, 1 \ldots); \quad H_s = \cos m(\eta + \alpha);$$

$$Z_k = \cos \beta (z + h), \ h = \bar{h} R_0. \tag{6}$$

Functions M_{sk} — linear combinations of functions of Bessel and Neumann, which satisfy conditions

$$M_{sk}'(r)|_{r=\delta} = 1; \quad M_{sk}'(r)|_{r=1} = 0. \tag{7}$$

Solution of boundary-value problem (2) for projections functions $\Omega_0(\Omega_0^1, \Omega_0^2, \Omega_0^3)$, called Zhukovskiy's potentials, for case of circular sector are obtained in work [4]. We will consider during the solution of boundary-value problems (3) and (4) that are excited only the first three their own forms of the free surface:

$$f_1 = H_1(\eta); \quad f_2 = Z_1(z); \quad f_3 = H_1(\eta) Z_1(z). \tag{8}$$

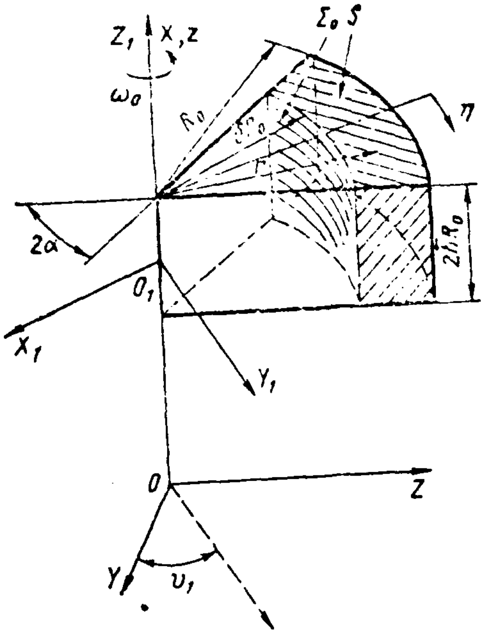

Fig. 1.

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Substituting in boundary conditions on Σ_0 in boundary-value problem (3) of expression for Ω_0^j $(j=1, 2, 3)$ from (4) and expressions (8) we obtain expansion

$$\frac{\partial \Omega_i^j(\eta, z)}{\partial \nu} = -(r \times \nabla f_i)_j + \nabla (f_i \nabla \Omega_0^j) = \sum_{s,k=0,1\ldots}\sum K_{sk}^{ij} H_s Z_k \quad (i, j=1,2,3), \tag{9}$$

where

$$K_{sk}^{ij} = \frac{\delta}{a\bar{h} b(s)\, b(i)} \int_{-a}^{a} \int_{-h}^{h} \frac{\partial \Omega_i^j(\eta, z)}{\partial \nu} H_s Z_k \, d\eta \, dz;$$

$$b(s) = \begin{cases} 2 \text{ при } s=0, \\ 1 \text{ при } s \neq 0. \end{cases}$$

Key: (1). with.

Hence taking into account boundary conditions (7) we will obtain

$$\Omega_i^j=\sum\sum_{s,k=0,1\ldots} K_{sk}^{ij} M_{sk} H_s Z_k. \tag{10}$$

Determination of functions Ω_{ij} is not compulsory, since into coefficients of equations of motion of system will enter only boundary values of these functions on free surface Σ_0. Moreover in view of the symmetry of cavity zero are only potentials with the diagonal indices i=j. We will obtain these boundary conditions, after substituting into the boundary condition formula (7), (8) and (10)

$$\frac{\partial \Omega_{ii}^j(\eta, z)}{\partial \nu}\Bigg|_{r=\delta}=-f_i(\nu\times\nabla f_i)_j+\nabla f_i\nabla\Omega_i^j-f_i\frac{\partial^2\Omega_i^j}{\partial\nu^2}+$$

$$+f_i\nabla f_i\frac{\partial}{\partial\nu}(\nabla\Omega_0^j)-\frac{f_i^2}{2}\frac{\partial^3\Omega_0^j}{\partial\nu^3}. \tag{11}$$

Let us compute kinetic energy of body with liquid by analogy with [3], using obtained resolution of potential Φ.

$$T=\frac{M}{2}V_0^2+\mathbf{L}(\mathbf{V}_0\times\omega)+\frac{1}{2}\omega J\omega+\mathbf{V}_0\dot{\mathbf{L}}+\sum_i\omega\mathbf{R}_i\dot{\beta}_i+\sum_i\sum_j\mu_{ij}\dot{\beta}_i\dot{\beta}_j. \tag{12}$$

Let us give expressions of hydrodynamic coefficients, entering (12). Static torque L and tensor of inertia J take the form

$$\mathbf{L}=\mathbf{L}^0+\mathbf{L}^1;\quad J=J^0+\sum_i(J^i\beta_i+J^{ii}\beta_i^2).$$

Here L° and J° - statistical moment/torque and tensor of inertia of solid body with undisturbed liquid

$$\mathbf{L}^1 = \mathbf{i}_y \lambda_2 \beta_1 + \mathbf{i}_z \lambda_3 \beta_2; \quad \lambda_2 = -\frac{16\rho \bar{h} a^2 \cos aR_0^3 \delta 2}{4a^2 - \pi^2}; \quad \lambda_3 = -\frac{16\rho \bar{\delta} h^2 a R_0^3}{\pi^2}; \quad (13)$$

$\mathbf{i}_y$, $\mathbf{i}_z$ — unit vectors of system $O_1X_1Y_1Z_1$.

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$$J^i = \rho R_0^4 \int_{\Sigma_0} \left\{ \Omega_0, \frac{\partial \Omega_i}{\partial \nu} + (\mathbf{r} \times \nu), (\mathbf{r} \times \nu) f_i - \left[(\mathbf{r} \times \nu) \frac{f_i}{\delta} + \mathbf{r} \times \nabla f_i \right], \Omega_0 \right\} ds; (14)$$

$$J^{ii} = \rho R_0^3 \int_{\Sigma_0} \left\{ \Omega_0 \frac{\partial \Omega_{ii}}{\partial \nu} + (\mathbf{r} \times \nu), \left(\frac{\partial \Omega_i}{\partial \nu} f_i + \frac{f_i^2}{2} \frac{\partial^2 \Omega_0}{\partial \nu^2} \right) - \right.$$
$$- \left[(\mathbf{r} \times \nu) \frac{f_i}{\delta} + \mathbf{r} \times \nabla f_i \right], \left(\Omega_i + \frac{\partial \Omega_0}{\partial \nu} f_i \right) +$$
$$\left. + \frac{1}{2} \left[(\mathbf{r} \times \nabla (f_i^2) \frac{1}{\delta} - \nu \times \nabla (f_i^2) \right], \Omega_0 \right\} ds. \quad (15)$$

In tensor (13) nonzero and sizable they are only on two components, which correspond to 2nd and 3rd tones: $J_{31}^2 = J_{13}^2$ and $J_{23}^3 = J_{32}^3$. In (14) it is necessary to consider only J_{33}^{11}, J_{33}^{22}, J_{33}^{33}. Expressions for coefficient R_i $(i = 1, 2, 3)$ take the form

$$R_1 = \mathbf{i}_z \lambda_{01}^z; \quad R_2 = \mathbf{i}_y \lambda_{02}^y + \mathbf{i}_z \lambda_{23}^z \beta_3; \quad R_3 = \mathbf{i}_x \lambda_{03}^x + \mathbf{i}_z \lambda_{32}^z \beta_2, \quad (16)$$

where

$$\lambda_{01}^z = \rho R_0^4 \int_{\Sigma_0} \Omega_0^z f_1 ds; \quad \lambda_{02}^y = \rho R_0^4 \int_{\Sigma_0} \Omega_0^y f_2 ds;$$

$$\lambda_{03}^z = \rho R_0^4 \int_{\Sigma_0} \Omega_0^x f_3 ds; \quad \lambda_{23}^z = 2R_0^3 \rho \delta \bar{h} a K_{01}^{33}; \quad \lambda_{32}^z = \rho R_0^3 \delta \bar{h} a K_{11}^{33}.$$

In coefficients μ_{ij} nonzero are only μ_{ii} $(i=1, 2, 3)$

$$\mu_{11}=2R_0^3\rho\alpha\bar{h}M_{10}(\delta);\quad \mu_{22}=2R_0^3\rho\delta\bar{h}\bar{\alpha}M_{01}(\delta);\quad \mu_{33}=R_0^3\rho\delta\bar{\alpha}\bar{h}M_{11}(\delta). \quad (17)$$

Let us take into account dissipation of energy in cavity, after introducing dispersive function $\varepsilon(\dot{\beta}_1^2+\dot{\beta}_2^2+\dot{\beta}_3^2)/2$.

2. Let us consider case of cavity, formed by cylindrical surface and two planes, perpendicular to rotational axis. From this cavity with the help of k of radial baffles are formed k of the sectors (see Fig. 1), with the half-angle α=π/k. Cavities contain identical amount of liquid in the cavities. Instead of 3k the generalized coordinates, which describe the motion of liquid in the cavities, let us introduce 9 generalized coordinates, which completely determine the effect of the mobility of liquid on the motion of the body

$$T_1=\frac{1}{k}\sum_i \beta_1^i;\quad T_2=\frac{1}{k}\sum_i \beta_1^i\cos\vartheta_i;\quad T_3=\frac{1}{k}\sum_i \beta_1^i\sin\vartheta_i;$$

$$S_1=\frac{1}{k}\sum_i \beta_2^i;\quad S_2=\frac{1}{k}\sum_i \beta_2^i\cos\vartheta_i;\quad S_3=\frac{1}{k}\sum_i \beta_2^i\sin\vartheta_i;$$

$$P_1=\frac{1}{k}\sum_i \beta_3^i;\quad P_2=\frac{1}{k}\sum_i \beta_3^i\cos\vartheta_i;\quad P_3=\frac{1}{k}\sum_i \beta_3^i\sin\vartheta_i.$$

After substituting obtained expressions for coefficients into (12) and using Euler-Lagrange equations, we will obtain equations of forces and moments/torques, and using equations of Lagrange of 2nd order - equation for generalized coordinates.

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Equations of forces we use in order to get rid of the apparent accelerations in the remaining equations, after which we will obtain system from 12 equations: 3rd momental equations and 9 equations for the generalized coordinates. This system of 12 equations is divided/marked off into three independent systems. The first system consists of the 2nd equations

$$\left(\mu_{22}-\frac{k(\lambda_3)^2}{M}\right)\ddot{S}_1-\omega_0^2J_{33}^{22}S_1+(\lambda_{23}^z-\lambda_{32}^z)\,\omega_0\dot{P}_1-\varepsilon\dot{S}_1=0,$$
$$\mu_{33}\ddot{P}_1-\omega_0^2J_{33}^{33}P_1+(\lambda_{32}^z-\lambda_{23}^z)\,\omega_0\dot{S}_1-\varepsilon\dot{P}_1=0. \quad (18)$$

The second system also consists of the 2nd equations

$$J_{11}^0\dot{\omega}_1+k\lambda_{01}^z\ddot{T}_1=0,\quad \mu_{11}\ddot{T}_1+\lambda_{01}^z\dot{\omega}_1-\omega_0J_{33}^{11}T_1-\varepsilon\dot{T}_1=0. \quad (19)$$

The third system consists of by the eighth of the equations

$$J_{22}^0\dot{\omega}_2+(J_{11}^0-J_{33}^0)\,\omega_0\omega_3+k\,\{J_{13}^2\omega_0\dot{S}_3+\lambda_{03}^x\ddot{P}_3-\omega_0^2J_{23}^3P_3-\omega_0\lambda_{02}^y\dot{S}_3+$$
$$+\omega_0J_{23}^3\dot{P}_2+\lambda_{02}^y\ddot{S}_2+\omega_0^2J_{13}^2S_2+a\lambda_2\,(-\omega_0\dot{T}_2-\ddot{T}_3+\omega_0^2T_3)\}=0,$$
$$J_{33}^0\dot{\omega}_3+(J_{22}^0-J_{33}^0)\,\omega_0\omega_2+k\,\{-J_{13}^2\omega_0\dot{S}_2-\lambda_{03}^x\ddot{P}_2+J_{23}^3\omega_0^2P_2+\lambda_{02}^y\omega_0\dot{S}_2+$$
$$+J_{23}^3\omega_0\dot{P}_3+\lambda_{02}^y\ddot{S}_3-J_{13}^2\omega_0^2S_3+d\lambda_2\,(-\omega_0\dot{T}_3+\ddot{T}_2-\omega_0^2T_2)\}=0,$$
$$\mu_{11}\ddot{T}_2-\omega_0^2J_{33}^{11}T_2+\frac{k(\lambda_2)^2}{2M}(\omega_0\dot{T}_3-\ddot{T}_2+\omega_0^2T_2)+\frac{d\lambda_2}{2}(\dot{\omega}_3+\omega_0\omega_2)=0, \quad (20)$$
$$\mu_{11}\ddot{T}_3-\omega_0^2J_{33}^{11}T_3+\frac{k(\lambda_2)^2}{2M}(-\dot{T}_2\omega_0-\ddot{T}_3+\omega_0^2T_3)+\frac{d\lambda_2}{2}(-\dot{\omega}_2+\omega_0\omega_3)=0$$

$$\mu_{22}\ddot{S_2} + \frac{\lambda_{02}^{y}}{2}\dot{\omega_2} + \lambda_{23}^{z}\omega_0\dot{P_2} + J_{13}^{2}\frac{\omega_0}{2}\omega_3 - J_{33}^{22}\omega_0^2 S_2 - \lambda_2\omega_2\dot{P_2} = 0,$$

$$\mu_{22}\ddot{S_3} + \lambda_{02}^{y}\frac{\dot{\omega_3}}{2} + \lambda_{23}^{z}\omega_0\dot{P_3} - J_{13}^{2}\frac{\omega_0}{2}\omega_2 - J_{33}^{22}\omega_0^2 S_3 - \lambda_2\omega_0\dot{P_3} = 0,$$

$$\mu_{33}\ddot{P_2} - \lambda_{03}^{x}\frac{\dot{\omega_3}}{2} + \lambda_{32}^{z}\omega_0\dot{S_2} - J_{23}^{3}\frac{\omega_0}{2}\omega_2 - J_{33}^{33}\omega_0^2 P_2 - \lambda_{23}\omega_0\dot{S_2} = 0,$$

$$\mu_{33}\ddot{P_3} - \lambda_{03}^{x}\frac{\dot{\omega_2}}{2} + \lambda_{32}^{z}\omega_0\dot{S_3} - J_{23}^{3}\frac{\omega_0}{2}\omega_3 - J_{33}^{33}\omega_0^2 P_3 - \lambda_{23}^{z}\omega_0\dot{S_3} = 0.$$

Equations of motion (18), (19), (20) are comprised in system of coordinates OXYZ, connected with center of gravity of undisturbed system, and hydrodynamic coefficients of (13)...(17) are determined in connected with cavity system $O_1X_1Y_1Z_1$, displaced up to distance of d along axis ox (see Fig. 1).

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3. We investigate systems (18), (19), (20) to asymptotic stability. It is possible to show [5] which for the stability is sufficient so that the absolute term in each characteristic polynomial of systems (18), (19), (20) would be more than zero. The first system is stable, if $J_{33}^{22}>0$ and $J_{33}^{33}>0$, which always occurs. The solution of system (19) is unstable with respect to the cyclic variable ω_1. The stability of the trivial solution of system (20) is ensured with accomplishing of following inequalities:

$$J_{11}^{0} - J_{33}^{0} + \frac{kM(\lambda_2 d)^2}{2MJ_{33}^{11} - k(\lambda_2)^2} + \frac{k\left(J_{13}^{2}\right)^2}{2J_{33}^{22}} + \frac{k\left(J_{23}^{3}\right)^2}{2J_{33}^{33}} > 0;$$

$$J_{11}^{0} - J_{22}^{0} + \frac{kM(\lambda_2 d)^2}{2MJ_{33}^{11} - k(\lambda_2)^2} + \frac{k\left(J_{13}^{2}\right)^2}{2J_{33}^{22}} + \frac{k\left(J_{23}^{3}\right)^2}{2J_{33}^{33}} > 0. \qquad (21)$$

Let us consider another case, which admits analytical solution. Let the body contain the cylindrical cavities, situated on the spot of sectors so that the rotational axis of the i cavity coincides with axis O_iX_i - by the intersection of the planes of the symmetry of the i sector cavity, and the rotational axis of the 1st cylindrical cavity coincides, correspondingly, with the intersection of the planes of the symmetry of the i sector. Functioning by analogy with the previous case, we will obtain expression for the kinetic energy of liquid. On the strength of the fact that the cavity is axisymmetric, some hydrodynamic coefficients, which correspond to sector cavity, will become zero. The third tone in connection with this is not excited and is not proven to be effect on the motion of body. If we make the assumption that the cavity is located sufficiently far from the rotational axis OX, then free surface Σ_0 can be considered flat/plane. A quantity of hydrodynamic coefficients in this case even more will be shortened also instead of (21) we will obtain the following stability condition:

$$\begin{gathered}
J_{11}^{0}-J_{33}^{0}+\frac{kM(\lambda_2 d)^2}{2MJ_{33}^{11}-k(\lambda_2)^2}+\frac{k\left(J_{13}^{2}\right)^2}{2J_{33}^{22}}>0;\\
J_{11}^{0}-J_{22}^{0}+\frac{kM(\lambda_2 d)^2}{2MJ_{33}^{11}-k(\lambda_2)^2}+\frac{k\left(J_{13}^{2}\right)^2}{2J_{33}^{22}}>0.
\end{gathered}\tag{22}$$

Is simple mechanical analog for body, which contains cylindrical cavities with liquid, this body with simple pendulums attached to it. The structure of the equations of motion of body with the liquid will be the same as the structure of the equations of certain equivalent solid body of mass M, with the tensor of inertia J°, to which at the points, which lie on the rotational axes of cavities, are attached the pendulums (masses m, lengths l), which are located in the undisturbed position at a distance of c from the rotational axis OX, moreover:

$$m=\frac{\lambda_{01}^2}{\mu_{22}}; \quad l=\frac{\mu_{22}}{N^2}; \quad c=\frac{\lambda_{01}^z}{\lambda_2}.$$

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In new designations stability condition (22) is converted into following:

$$\begin{aligned} &J_{11}^0-J_{22}^0-\frac{kmd^2l}{2M(r-[c+l]F)+kml}-\frac{km(r-c)^2l}{r-cG}>0;\\ &J_{11}^0-J_{33}^0-\frac{kmd^2l}{2M(r-[c+l]F)+kml}-\frac{km(r-c)^2l}{r-cG}>0, \end{aligned} \tag{23}$$

where

$F=\dfrac{J_{33}^{22}l}{\mu_{22}(c+l)}$; $G=\dfrac{J_{33}^{11}l}{\mu_{22}c}$; $r-$ distance from OX to bottom.

If F and G are equal to one, then complete coincidence of stability conditions of body with liquid and bodies with pendulums occurs.

Evidently from (23), as affects disturbed free surface motion of liquid stability of stationary rotation of body. An increase in the radius of the free surface of liquid, and also distance d from the center of mass of system to the plane, in which are located the cavities, worsens/impairs stability.

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EXPERIMENTAL ANALYSIS OF STABILITY OF ROTATION OF BODIES WITH THE LIQUID FILLING.

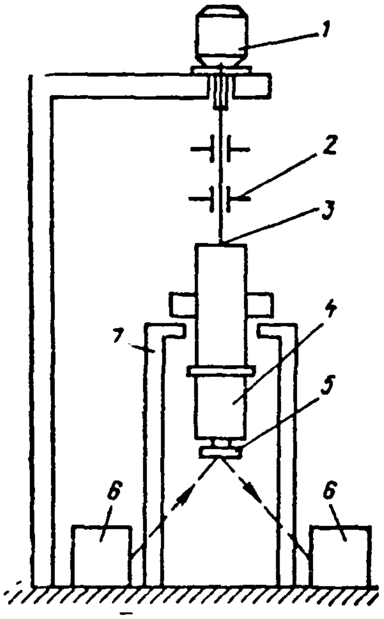
V. T. Desyatov.

Method of experimental study and experimental installation are described, analysis of results of experimental research is carried out. It is shown that to the stability of rotation of bodies with the cylindrical form the cavity mainly affects the volume of the poured liquid, value of the ratios of the length of cavity to its diameter and axial moment inertia of body up to the axial moment/torque. The loss of stability of body sets in with the defined filling of cavity with liquid and can occur both with the partial and with full-stroke admission.

In 50's G. S. Narimanov paid considerable attention to research of dynamics of solid body, whose cavity is partially filled with liquid [2, 3]. Into the circle of its scientific interests entered also questions of the study of the stability of rotation of bodies with the liquid filling. On the initiative of G. S. Narimanov and under his management in these years by the author of article was carried out the cycle of experimental works on the analysis of stability of rotation of the bodies with the liquid filling, whose some results are presented in the data to article.

1. For imparting to bodies (models) rotation during experimental research was used string drive, proposed by S. V. Malashenko, who permitted implemention of rotation of investigated model around principal axis of inertia with high angular velocity (to 10000 r/min) [1].

Study of stability of rotation of models was conducted on installation, which consists of two parts: worker and recording. Test section (Fig. 1) consists of electric motor 1, to shaft of which is fastened thin string 3 with diameter of 0.1-0.3 mm. By lower end/lead the string is combined with the model 4 being investigated. For the extinguishing of transverse vibrations of string damper 2 was used. For the purpose of the preservation of the incidence/drop in the model with the break of string on recorder is used detector 7. On the shaft of motor is attached the permanent magnet, which revolves in the induction coil. The ends/leads of the induction coil are connected to the frequency meter, the number of revolutions of the shaft of motor is determined in the frequency of the aimed in the induction current coil.



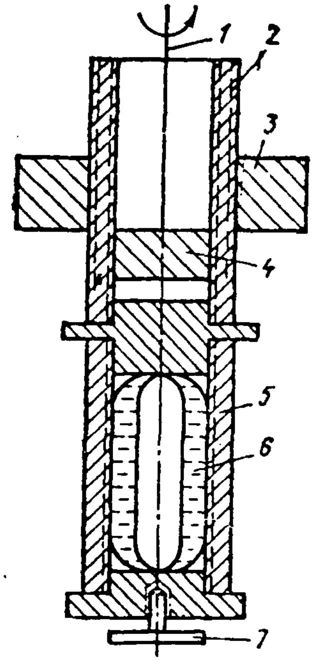


Fig. 1. Fig. 2.

Fig. 1. Schematic diagram of experimental installation: 1 - electric motor; 2 - damper; 3 - string; 4 - experimental model; 5 - mirror; 6 - recorder; 7 - detector.

Fig. 2. Schematic of experimental model: 1 - string; 2 - cylinder; 3 - ring; 4 - nut; 5 - experimental capacity/capacitance; 6 - liquid; 7 - mirror.

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2. Models (Fig. 2), which were being used during research, consisted of several parts, which gave possibility to change moments of inertia of models, their relation with set of different parts, sizes/dimensions and form of poured cavity over wide limits. A change in axial moment inertia of model was achieved by the displacement of

ring 3 over cylinder 2. A change in the axial moment/torque was produced by the replacement of this ring. By the replacement of capacity/capacitance 5 or by setting different in the thickness packing was achieved a change in the sizes/dimensions of the poured cavity. The attachment of string, as a rule, was realized in the area of the center of mass of model on nut 4. On the end/face of model was placed flat/plane mirror with 7 diameter of 8 mm. The construction of the base of mirror, is such, that there was a possibility of regulating the angle of the inclination/slope of the plane of mirror relative to the end/face of model. For conducting the visual observation of the behavior of liquid capacities/capacitances 5, prepared from organic glass, were used. A change in the form of the cavity of body was realized by installation into the cylindrical cavity of the shaped ogival or conical insets. The parameters of insets were changed over wide limits. The parameters of models with the cylindrical form of cavity were varied in the limits of the values of the ratio of the length of cavity to its diameter $L/D=0{,}5\ldots4{,}0$ and in the limits of the values of the ratios of axial moment inertia to axial $A/C=3\ldots3{,}5$. The measurement of amount of liquid, poured into the cavity of models, and the metering of the necessary amount of liquid were realized with the error not more than 1%.

3. Recording part made it possible to carry out recordings of oscillations of revolving model, which correspond to deviations of axis of body from position of stable equilibrium in the range of $\alpha=0\ldots7°$. Instrument is carried out according to the diagram, shown in Fig. 3.

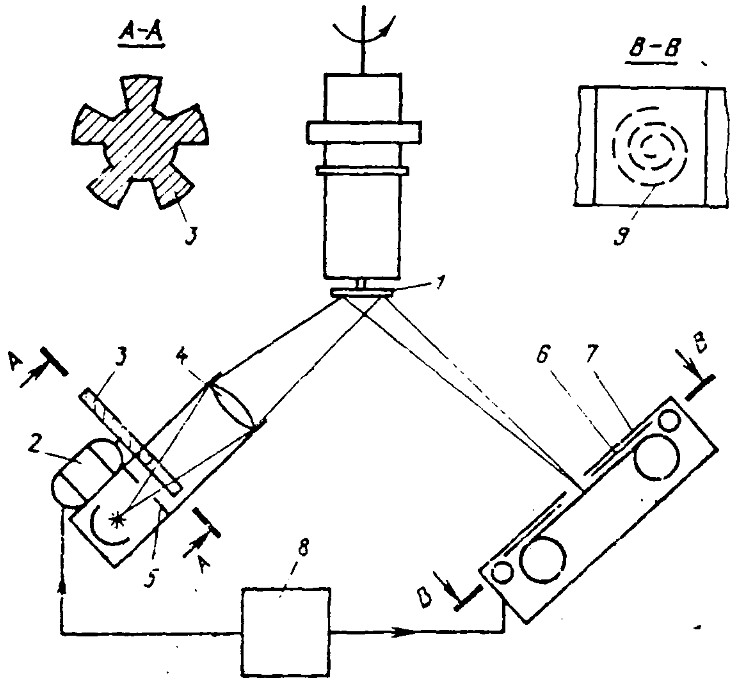

Fig. 3. Schematic of recorder: 1 - mirror; 2 - electric motor; 3 - disk-interrupter; 4 - lens; 5 - diaphragm; 6 - gate/shutter; 7 - photographic paper; 8 - synchronizing unit; 9 - photograph.

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The light beams, emitted by tube and directed by mirror, consecutively/serially pass through the capacitor/condenser, diaphragm 5 with the opening/aperture 0.06 mm and lens 4. On the path of the inswept beam of light, which emerges from the objective, flat/plane mirror 1, fastened/strengthened to the end/face of model, is placed. The reflected beam of light/world falls on photographic paper 7, included in the cassette. It is possible to attain by the regulation of the distances of the optical arms of instrument information of the beam of light, reflected from the mirror of model, to the photographic paper into the point.

Disk 3, rotated by electric motor 2 with a velocity of 60 r/min, is placed between diaphragm and lens. The leading drum of cassette revolves by the step-type electric motor, which is synchronized with disk 3. Disk has five cutouts in the form of the sectors of the strictly defined sizes/dimensions. This device/equipment was the peculiar camera shutter, which makes it possible to obtain the light dotted line of specific sizes/dimensions of 9 on the photographic paper through the equal time intervals. Thus, knowing the number of revolutions of disk, the sizes/dimensions of cutouts, measuring the length of dotted line and the diameter of the broken circle, obtained on the photographic paper, it is possible to compute angular velocity align the precessions of model and the rate of the increase of amplitude oscillation.

Work on installation was conducted as follows. The model, suspended/hung to the shaft of motor, gradually was accelerated/dispersed to the velocity, at which were conducted the experiments. Oscillations of model appeared in this case were damped. On reaching/achievement of given speed the damping ceased. If in this case the axis of model remained in the vertical position, then light beam was projected on the fixed shield into the image of focus. Least deviation of the axis of model from the vertical position led to the bias/displacement of "light spot".

Oscillations of model relative to center of mass it led to description with "light spot" of trace on photographic paper in the form of circle of specific diameter. While conducting of photographing the motion of "light spot" were switched on the motors, which revolved disk and leading drum of cassette. The synchronization of these motors was realized in such a way that the image of the motion of "light spot" was obtained with the necessary delay through the equal time intervals.

Experiments showed that negative effect of liquid on stability of rotation of models is exhibited in the form of noticeable oscillations of body relative to center of mass, identified by us with rapid gyro precession. The loss of stability of model is characterized by a gradual increase in the amplitude of these oscillations. For small angles of oscillations their amplitude with the loss of stability of rotation calculation according to the law, which is expressed by the known exponential function of time $s = \dot{s}e^{k\omega t}$. In the conducted research the dimensionless parameter "k" was considered the measure of the intensity of the loss of stability of model. The velocity of the rapid precession of model was determined by the results of the photorecording, obtained on the instrument with the relative error not more than 3...5%.

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4. Preliminary research was conducted on models with cavities of cylindrical form, whose characteristics were represented in Table 1.

Water was working fluid. Models were suspended to the string in the area of the center of mass.

Model No 1, whose cavity is completely filled with liquid, during rotation proved to be such unstable which with great difficulty was possible to untwist it to 4000 r/min. After the cessation/discontinuation of damping the model almost instantly lost the stability of rotation. With the decrease of a quantity of poured liquid the intensity of loss of stability was reduced also with the filling of cavity to 60% rotation of model it became stable. The same model without the liquid filling revolved stably in the wide interval of number of revolutions ($n>2000$ r/min) even under the influence on it sufficient strong external moments/torques.

Models No 2 and 3 in this series of experiments untwisted to velocities, which correspond to n=5000 r/min. In this case model No 2 lost the stability of rotation with the fillings of the corresponding to range of charge/weight ratios $B=V/V_n=0{,}31\ldots0{,}33$. By charge/weight ratio is understood the ratio of the volume of the poured liquid to entire volume of cavity. The greatest intensity of loss of stability was observed with B=0.32, a change in the filling in all to 1% led to sharp reduction in the intensity of the phenomenon of instability. And without it the model revolved stably with all other fillings. Model No 3 lost the stability of rotation already in other range of charge/weight ratios $B=0{,}57\ldots0{,}62$. The character of the stability of model No 3 is analogous to the character of the loss of stability of models No 1 and 2.

5. For purpose of determination of effect parameter of string and places of its attachment to model were carried out research on the same models, but with change in characteristics of string (material, length and diameter of its cross section), place of its attachment (value of inverted and righting moments) and value of angular velocity.

Results of research showed that investigated factors do not change very fact of loss of stability and do not displace center of range of charge/weight ratios, with which occurs loss of stability. The factors indicated can have an effect on the intensity of loss of stability and on the width range B.

Table 1.

(1) № модели	(2) Длина полости L'', мм	(3) Диаметр полости D, мм	(4) Осевой момент инерции $C \cdot 10^{-4}$ кг·м²	(5) Экваториальный момент инерции $A \cdot 10^{-4}$ кг·м²	(6) Масса модели, г
1	30	30	0,95	11,1	290
2	60	30	1,0	15	320
3	70,5	30	1,1	23	370

Key: (1). No of model. (2). Length of cavity L", mm. (3). Diameter of cavity D, mm. (4). Axial moment of inertia $C \cdot 10^{-4}$ kg·m². (5). Axial moment inertia $A \cdot 10^{-4}$ kg·m². (6). Mass of model, g.

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For example, an increase in the number of revolutions of model from n=3000 to $n = 10\,000$ r/min leads to an increase in the intensity of the process of loss of stability. In the process of these experiments the views of accuracy and convenience in conducting the experiment of the value of the investigated parameters most acceptable from the point were selected. The diameter of the cross section of steel string must be not more than 0.25 mm, distance from the attachment point of string to the model to the latter/last limiter is not less than 250 mm, the attachment point of string with the model should be placed in the area of the center of mass of model, the number of revolutions of model must be not less than 2500-3000.

Thus, liquid filling exerts a substantial influence on stability of rotation of bodies. Loss of stability can occur both with full-stroke admission and with the partial. In this case the width of the range of charge/weight ratios, in limits of which occurs the loss of stability, and the coordinate of its center B_{II} depend mainly on the characteristics of the revolving body, its cavity and liquid.

6. Further systematic studies of effect of sizes/dimensions of cavity and moments of inertia of body on arrangement of centers of unstable regions B_{II} and width of these regions made it possible to establish appropriate graphic dependences, represented in Fig. 4. During the experiments all models untwisted to the angular velocities n=5000 r/min. The diameter of the cavities of models was equal to 30 mm. Water served as working fluid. Processing the results of experiments was conducted in the criterial form.

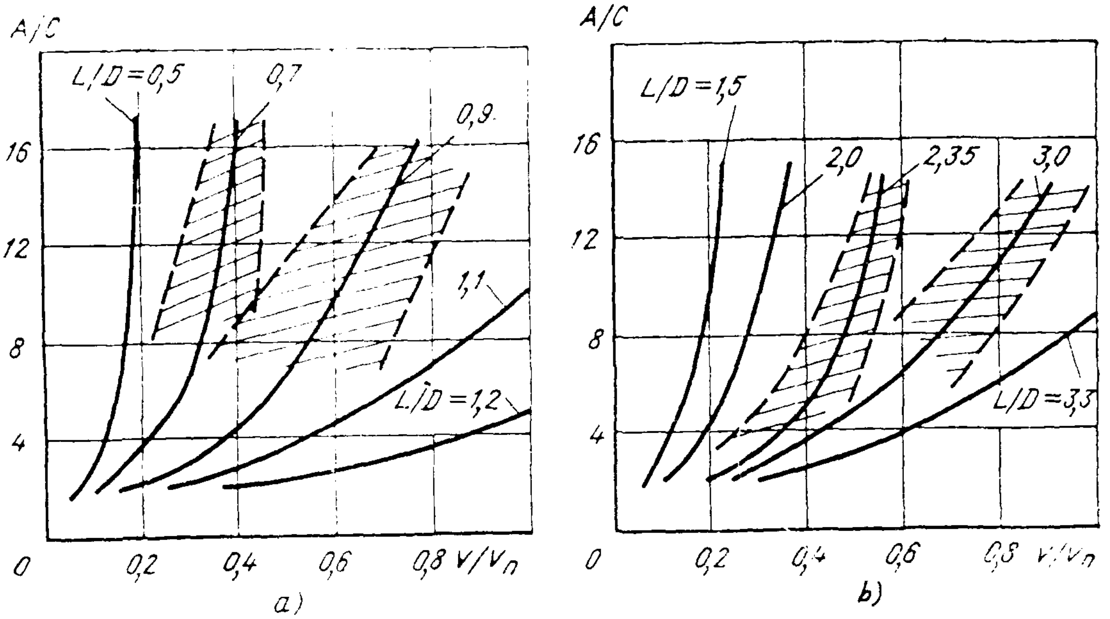

Fig. 4. Dependence of charge/weight ratios of cylindrical cavity with liquid, which correspond to instability, on relation of moments of inertia of body and relation of significant dimensions of cavity.

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It follows from examination of obtained dependences that with increase in value of ratio of length of cavity to its diameter L/D in two groups of change in values $L/D=0,1...1,2$ and $L/D=1,3...3,3$ in constant/invariable ratio of axial moment inertia of body to axial A/C center of unstable region is moved in direction of high values up to 1.0, i.e., full-stroke admission. In this case the width of unstable regions $B_{\text{н}}$ also grows with an increase of value of L/D within the limits of each group L/D. With an increase in value of A/C and $L/D=\text{const}$ the center of unstable regions is displaced in the direction of the larger values, in this case the width of the range of instability also is reduced. The intensity of the phenomenon of instability, which corresponds to the center of the coefficient domain of filling, in which is observed the instability with an increase in value of L/D, grows and is reduced with an increase in value A/C, i.e., "follows" a change in the width of unstable region. The intensity of the phenomenon of instability, which corresponds to the first group of values L/D, several times more than for the second group of value L/D.

Results of experimental studies of arrangement of centers of coefficient domains of filling, with which occurs loss of stability of rotation of bodies, depending on change in values of L/D and A/C can be approximated for low filling (B<0.5) with following dependence

$$B_{н}=a\left[\frac{L}{D}\left(1-\frac{C}{A}\right)\right]^{2},$$

where a - coefficient, for first group L/D $a=0{,}81$; for second group L/D $a=0{,}27$.

For purpose of study of viscosity effect of liquid on stability of rotation of bodies with liquid filling were used different liquids, whose characteristics were given in Table 2. The results of research showed that with numbers $Re<10^5$ $Re=\omega D^2/2\nu>10^5$ (ω — the angular velocity of body) the intensity of the phenomenon of instability is weakened/attenuated with decrease of Re, and the center of unstable region somewhat is displaced in the direction of the high values B.

This bias/displacement can be approximated by following dependence:

$$\Delta B_{н} = 30/Re^{0,6}.$$

The viscosity effect of liquid with numbers $Re>10^5$ on the stability of rotation of bodies with the liquid filling is not discovered.

Table 2.

(1) Наименование жидкости	(2) Кинематический коэффициент вязкости $\nu \cdot 10^{-6}$ $м^2/с$	(3) Плотность $\rho \cdot 10^3$ $кг/м^3$
(4) Вода	1,0	1,0
(5) Раствор глицерина в воде	2,7 12,3 34	1,1 1,15 1,18
(6) Бензин	0,76	0,75

Key: (1). Designation of liquid. (2). Kinematic modulus of viscosity $\nu \cdot 10^{-6}$ m^2/s. (3). Density $\rho \cdot 10^3$ kg/m^3. (4). Water. (5). Solution/opening of glycerin in water. (6). Gasoline.

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