

SECURITY CLASSIFICATION OF THIS PAGE

## ORT DOCUMENTATION PAGE

1a. <b>AD-A214 102</b>		1b. RESTRICTIVE MARKINGS	
2a.		3. DISTRIBUTION/AVAILABILITY OF REPORT	
2b.		Approved for public release; distribution unlimited.	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) <b>ARO 24409.3-MA</b>	
6a. NAME OF PERFORMING ORGANIZATION University of Houston Dept. of Mathematics	6b. OFFICE SYMBOL (if applicable)	7a. NAME OF MONITORING ORGANIZATION U. S. Army Research Office	
6c. ADDRESS (City, State, and ZIP Code) 4800 Calhoun Houston, Texas 77204-3476		7b. ADDRESS (City, State, and ZIP Code) P. O. Box 12211 Research Triangle Park, NC 27709-2211	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION U. S. Army Research Office	8b. OFFICE SYMBOL (if applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER <b>DAAL03-86-K-0138</b>	
8c. ADDRESS (City, State, and ZIP Code) P. O. Box 12211 Research Triangle Park, NC 27709-2211		10. SOURCE OF FUNDING NUMBERS	
		PROGRAM ELEMENT NO.	PROJECT NO.
		TASK NO.	WORK UNIT ACCESSION NO.
11. TITLE (Include Security Classification) Numerical Methods for Parabolic Equations and Inequalities in Very High Dimensions.			
12. PERSONAL AUTHOR(S) R. Glowinski			
13a. TYPE OF REPORT Final	13b. TIME COVERED FROM 5/86 TO 5/89	14. DATE OF REPORT (Year, Month, Day) 89/10/5	15. PAGE COUNT 20
16. SUPPLEMENTARY NOTATION The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.			
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB-GROUP	
		Computational Mathematics	
19. ABSTRACT (Continue on reverse if necessary and identify by block number)  See back of page.			
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT <input type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS		21. ABSTRACT SECURITY CLASSIFICATION Unclassified	
22a. NAME OF RESPONSIBLE INDIVIDUAL		22b. TELEPHONE (Include Area Code)	22c. OFFICE SYMBOL

## 19. Abstract

The research supported by this grant was essentially concerned with the numerical solution of the Zakai equation from nonlinear filtering. The Zakai equation is a linear stochastic equation of parabolic type and its solution provides a nonnormalized probability density from which we can reconstruct the solution of a dynamical system governed by stochastic ordinary differential equations when the observation is itself noisy. The Zakai equation presents some unique features in the sense that

- (i) It has practical applications;
- (ii) It is a highly advective, advection-diffusion linear parabolic equation;
- (iii) The space dimension is very large (of the order of ten and more for practical applications);
- (iv) It is a stochastic equation.

Each of the difficulties listed in (ii), (iii), (iv) leads to very challenging problems when it goes to the numerical solution, but their combination provides a formidable numerical problem which from our point of view requires computer power beyond the possibility of the existing hardwares and softwares.

In view of developing know now we have been investigating systematic methods for solving the Zakai equation, which have proved reliable for solving problems with space dimension six at most (this number will increase with the progress of computers). The methodology that we have been studying relies on operator splitting methods in which one decouples, via time discretization, the stochastic part and the deterministic part of the Zakai equation. Using this approach we obtain subproblems which have either closed form solution (the stochastic part) or can be solved (the deterministic part; modulo the difficulty of the dimension) by efficient methods for traditional elliptic and parabolic problems. The same methodology can be extended to the parabolic inequalities of obstacle type originating from impulse control.

Concerning the space approximation itself it can be achieved by finite difference or finite element methods, with a specific treatment of the first order terms for which the first order upwinding methods investigated by some authors are by far too dissipative. We have been therefore investigating second order upwinding methods which are by far much less dissipative (in the detailed final report we also comment on third order upwinding methods).

FINAL REPORT  
CONCERNING  
GRANT NO. DAAL03-86-K-0138

Numerical Methods for Parabolic Equations and  
Inequalities in Very High Dimensions



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DTIC TAB	<input checked="" type="checkbox"/>
Unannounced	<input type="checkbox"/>
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FINAL REPORT CONCERNING GRANT DAAL03-86-K-0138

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1. Introduction. Motivation

The main goal of this projects was to investigate the numerical solution of *parabolic equations in very high dimension*, the principal motivation being the solution of the so-called *Zakai equation* from nonlinear filtering. As it will be seen in Section 2, below, this equation is of the *advection-diffusion* type and therefore is closely related to other equations of this type such as the Navier-Stokes equations from Fluid Dynamics; since these equations are also of interest for the Department of Defense some aspects of their numerical solution, byproducts of the present investigation will be considered in the present document.

The content of this report is the following:

In Section 2, we describe the Zakai equation and its origin. In Section 3, we consider the solution of time dependent problems by operator splitting methods, and show in Sections 4, 5, 6 how these methods apply to the Zakai equation, to parabolic inequalities of obstacle type and to Navier-Stokes equations.

In Section 7 we comment on the solution of advection-diffusion problems by upwinding, particle and characteristics methods. In Section 8 we go back to the Zakai equation and its stochastic dynamical system origin and comment about the feasibility of the Zakai equation approach for the analysis of such systems.

2. The Zakai equation.

We consider a *signal process* given by

$$(2.1) \quad dx_t = b(t, x_t)dt + \sigma(t, x_t)dw_t.$$

Here  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  and  $x_0$  has a given probability density  $p_0(x)$ . The vector  $w_t = \{w_t^1, \dots, w_t^d\}$  is a  $d$ -dimensional Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, p^*)$ . Also,  $b(t, x) = \{b_i(t, x)\}_{i=1}^d$  and  $\sigma(t, x) = (\sigma_{ij}(t, x))$ ,  $i, j = 1, \dots, d$ , have components defined on  $[0, T] \times \mathbb{R}^d$  which are bounded and smooth. We suppose now that the observation is given by

$$(2.2) \quad dy_t = h(t, x_t) + g(t)dw_t + \bar{g}(t)d\bar{w}_t.$$

Here,  $y \in \mathbb{R}^m$ ,  $y_0 = 0$  and  $\bar{w}_t = \{\bar{w}_t^j\}_{j=1}^m$  is a  $m$  dimensional Brownian motion defined on  $(\Omega, \mathcal{F}, p^*)$  and is independent of  $w$ . The function  $h(t, x) = \{h_j(t, x)\}_{j=1}^m$  has essentially the same smoothness and boundedness properties than  $b$  and  $\sigma$ . We shall suppose that matrices  $g(t) = (g_{ij}(t))$ ,  $i=1, \dots, m$ ,  $j=1, \dots, d$ , and  $\bar{g}(t) = (\bar{g}_{kl}(t))$ ,  $k, l=1, \dots, m$  are continuous on  $[0, T]$ . We suppose that

$$(2.3) \quad g(t)g(t)^T + \bar{g}(t)\bar{g}(t)^T = I.$$

Condition (2.3) is a normalization hypothesis which can always be satisfied by re-scaling the observation process.

Denoting by  $g_j$  the  $j$ -th row of  $g$  we define the vector  $c_j$  by

$$c_j(t) = \sigma(t) g_j^T(t),$$

and then  $\bar{h}_j$  by

$$\bar{h}_j(t) = h_j(t) - \text{div } c_j(t), \quad j=1, \dots, d.$$

Define now an operator  $H_j$ , mapping  $H^1(\mathbb{R}^d)$  into  $L^2(\mathbb{R}^d)$ , by

$$H_j(t) u(t) = \bar{h}_j(t) u(t) - (C_j(t), \nabla u(t)),$$

for  $u \in H^1(\mathbb{R}^d)$ .

The elliptic operator  $L^*$  is the formal adjoint of the generator of (2.1), i.e.

$$L^*(t) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(t) - \sum_{i=1}^d \frac{\partial}{\partial x_i} b_i(t);$$

here  $a = \{a_{ij}\} = \sigma \sigma^T$ .

Define  $Y_t = \sigma\{y_s: 0 \leq s \leq t\}$  for the  $\sigma$ -field generated by the observations up to time  $t$ . There exists an unnormalized density  $q_t$  of  $x_t$  conditioned on  $Y_t$ ;  $q_t$  satisfies

$$(2.4) \quad q_0 = p_0$$

and the following stochastic partial differential equation

$$(2.5) \quad q_t(x) = p_0(x) + \int_0^t L_s^* q_s(x) ds + \sum_{j=1}^m \int_0^t H_j(s) q_s(x) dy_s^j,$$

for  $t \in [0, T]$ . Equation (2.5) is precisely the Zakai equation (see, e.g., [5], [10] for more details).

### 3. Operator Splitting Methods for Time Dependent Problems.

Let's consider a (deterministic) dynamical system modelled by

$$(3.1) \quad \frac{du}{dt} + A(u) = 0, \text{ for } t > 0,$$

$$(3.2) \quad u(0) = u_0.$$

In (3.1), (3.2),  $A$  is a (possibly nonlinear) operator from a Hilbert space  $H$  into itself, and  $u_0 \in H$ . We suppose that  $A$  has a nontrivial decomposition of the following type

$$(3.3) \quad A = A_1 + A_2;$$

by nontrivial we mean that  $A_1, A_2$  are individually simpler than  $A$  (indeed  $A_1$  and/or  $A_2$  can be multivalued operators).

There exist many schemes taking advantage of decomposition (3.3) (see, e.g., [1], [2], [3] for such schemes). We shall present here some of them:

### 3.1 Fractional Step Schemes

#### 1st. Scheme:

$$(3.4) \quad u^0 = u_0;$$

then for  $n \geq 0$ ,  $u^n$  being known we compute  $u^{n+1/2}$  and then  $u^{n+1}$  as follows

$$(3.5) \quad \frac{u^{n+1/2} - u^n}{\Delta t} + A_1(u^{n+1/2}) = 0,$$

$$(3.6) \quad \frac{u^{n+1} - u^{n+1/2}}{\Delta t} + A_2(u^n) = 0.$$

This scheme is at most first order accurate in  $\Delta t$ . To obtain a better accuracy, we can "symmetrize" scheme (3.4) - (3.6) by using the following scheme

#### 2nd Scheme:

$$(3.7) \quad \frac{u^{n+1/3} - u^n}{\Delta t/2} + A_1(u^{n+1/3}) = 0,$$

$$(3.8) \quad \frac{u^{n+2/3} - u^{n+1/3}}{\Delta t} + A_2(u^{n+2/3}) = 0,$$

$$(3.9) \quad \frac{u^{n+1} - u^{n+2/3}}{\Delta t/2} + A_1(u^{n+1}) = 0.$$

It follows from, e.g., [3] (see also [4]), that in many situations, the above scheme is second order accurate with respect to  $\Delta t$ .

### 3.2 Alternating Direction Schemes.

The above schemes are not too well suited to capture steady state solutions of (3.1), i.e., solutions of

$$(3.10) \quad A(u) = 0$$

(if such solutions exist). This drawback is corrected if one uses the schemes below:

Peaceman-Rachford Scheme:

$$(3.10) \quad u^0 = u_0;$$

then for  $n \geq 0$ ,

$$(3.11) \quad \frac{u^{n+1/2} - u^n}{\Delta t/2} + A_1(u^{n+1/2}) + A_2(u^n) = 0,$$

$$(3.12) \quad \frac{u^{n+1} - u^{n+1/2}}{\Delta t/2} + A_1(u^{n+1/2}) + A_2(u^{n+1}) = 0.$$

Douglas-Rachford Scheme:

$$(3.13) \quad u^0 = u_0;$$

then for  $n \geq 0$ ,

$$(3.14) \quad \frac{u^{n+1/2} - u^n}{\Delta t} + A_1(u^{n+1/2}) + A_2(u^n) = 0,$$

$$(3.15) \quad \frac{u^{n+1} - u^n}{\Delta t} + A_1(u^{n+1/2}) + A_2(u^{n+1}) = 0.$$

$\theta$ -Scheme: With  $\theta \in (0, .5)$ , this scheme is defined by (3.13) and

$$(3.16) \quad \frac{u^{n+\theta} - u^n}{\theta \Delta t} + A_1(u^{n+\theta}) + A_2(u^n) = 0,$$

$$(3.17) \quad \frac{u^{n+1-\theta} - u^{n+\theta}}{(1-2\theta)} + A_1(u^{n+\theta}) + A_2(u^{n+1-\theta}) = 0,$$

$$(3.14) \quad \frac{u^{n+1} - u^{n+1-\theta}}{\theta \Delta t} + A_1(u^{n+1}) + A_2(u^{n+1-\theta}) = 0.$$



All these schemes are first order accurate with respect to  $\Delta t$  in general, the last one being (for  $\theta$  well chosen) better suited than the first two to capture steady states.

#### 4. Application to the solution of the Zakai equation.

Using the notation of Section 2, we are considering the solution of

$$(4.1) \quad dy + A^*(t)y dt = B(t)y \cdot dw.$$

The idea here, is to consider

$$(4.2) \quad A^*(t)y dt - B(t)y \cdot dw$$

as the sum of two operators and using the approach associated to scheme (3.4) - (3.6), consider a sequence of problems of the form

$$(4.3) \quad d\varphi + A(t)\varphi dt = 0,$$

$$(4.4) \quad d\Psi = B(t)\Psi \cdot dw,$$

which are considerably simpler than (4.1); indeed, the  $\varphi$  equation is deterministic and the  $\Psi$  equation has a closed form solution.

The algorithm investigated has been analyzed in [5] and generalized to more complicated stochastic partial differential equations in [6]; it is described as follows:

Consider first (4.1) on the time interval  $[0, T]$  and define  $\Delta t = k = T/(N+1)$ . We shall define two processes  $y_{1k}, y_{2k}$  depending on  $k$ . We split therefore  $[0, T]$  in  $[0, k], [k, 2k], \dots, [Nk, (N+1)k]$ . Consider now  $[rk, (r+1)k]$  with  $r=0, 1, \dots, N$ ; then  $y_{1k}, y_{2k}$  are defined on this interval by the relations

$$(4.5) \quad dy_{1k} + A^*(t)y_{1k}dt = 0,$$

$$(4.6) \quad dy_{2k} = B(i) y_{2k} \cdot dw,$$

$$(4.7) \quad y_{1k}(rk) = y_k^r;$$

$$(4.8) \quad y_{2k}(rk) = y_k^{r+1/2}$$

the sequences  $y_k^r, y_k^{r+1/2}$  are defined as follows

$$(4.9) \quad y_k^{r+1/2} = y_{1k}((r+1)k-0),$$

$$(4.10) \quad y_k^{r+1} = y_{2k}((r+1)k-0).$$

Relations (4.5) - (4.10) define completely  $y_{1k}, y_{2k}$  on  $[0, T[$ . The processes  $y_{1k}, y_{2k}$  are right continuous and their discontinuity points are  $k, \dots, Nk$  (on  $[0, T[$ ); we observe that (4.5) is *deterministic*.

The convergence of  $y_{1k}, y_{2k}$  to the solution  $y$  of (4.1) is proved in [5], justifying therefore the splitting approach.

In practice, problem (4.5) is an advection-diffusion problem whose solution will be discussed in Section 7.

##### 5. Application to the solution of obstacle type parabolic inequalities.

It follows from [7], that applications in *impulse control*, lead to the solution of parabolic inequalities of the following type

$$(5.1) \quad \left( \frac{\partial u}{\partial t} + Au - f \right) (\Psi - u) = 0, \text{ a. e. on } \Omega \times (0, T),$$

$$(5.2) \quad \frac{\partial u}{\partial t} + Au - f \leq 0, \text{ a. e.,}$$

$$(5.3) \quad u - \Psi \leq 0, \text{ a. e.,}$$

$$(5.4) \quad u = u_0 \text{ at } t = 0;$$

in (5.1) - (5.4),  $A$  is an advection-diffusion elliptic operator, and  $\Psi$  defines the obstacle. From a numerical point of view, it may be more convenient to formulate (5.1) - (5.4) as a *parabolic variational inequality* such as

$$(5.5) \quad \int_{\Omega} \left( \frac{\partial u}{\partial t} + Au - f \right) (v - u) dx \geq 0, \quad \forall v \in K(t),$$

$$(5.6) \quad u(t) \in K(t), \quad \forall t \geq 0,$$

$$(5.7) \quad u(0) = u_0 \left( \in K(0) \right),$$

with

$$(5.8) \quad K(t) = \left\{ v | v \in V, v(x,t) \leq \Psi(x,t) \text{ a.e. on } \Omega \times (0,T) \right\};$$

in (5.8),  $V$  is a Hilbert space of Sobolev type.

In (5.5) - (5.8) we have implicitly assumed that  $\Psi$ , and therefore the (convex) set  $K$ , vary continuously with  $t$ . Concerning the numerical solution of (5.5) - (5.8) by operator splitting methods, let's introduce

$$(5.9) \quad C(t) = \left\{ v | v \in L^2(\Omega), v(x) \leq \Psi(x,t) \text{ a.e. on } \Omega \right\},$$

and the *indicator functional*  $I_c$  of  $C(t)$ , defined by

$$(5.10) \quad I_c(v, t) = 0 \text{ if } v \in C(t), I_c(v, t) = +\infty \text{ if } v \notin C(t).$$

Problem (5.5) - (5.8) can then be formulated as a *nonlinear parabolic "equation"* (it is in fact a *multivalued equation*) by

$$(5.11) \quad 0 \in \frac{\partial u}{\partial t} + Au + \partial I_c(u) - f,$$

$$(5.12) \quad u(0) = u_0;$$

in (5.11),  $\partial I_C(u)$  is the *subdifferential* of the convex functional  $I_C$  at  $u$ ; it is defined,  $\forall v \in L^2(\Omega)$ , by

$$(5.13) \quad w \in \partial I_C(v, t) \Leftrightarrow I_C(z, t) - I_C(v, t) \geq \int_{\Omega} (z-v) w \, dx, \quad \forall z \in L^2(\Omega).$$

Problem (5.11), (5.12) can be approximated by the following nonlinear parabolic equation (with  $\varepsilon > 0$ ):

$$(5.14) \quad \frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \Psi)^+ = f,$$

$$(5.15) \quad u_\varepsilon(0) = u_0,$$

which corresponds to a *penalty* approximation of condition (5.6). Operator

$$(5.16) \quad v \longrightarrow (v - \Psi)^+$$

is (for  $t$  given) a ("nice") *monotone* operator from  $L^2(\Omega) \rightarrow L^2(\Omega)$ .

Both problems (5.11), (5.12) and (5.14), (5.15) can be solved by operator splitting methods. For simplicity, we shall just describe the application of scheme (3.4)-(3.6), but the other schemes described in Section 3 also apply. We obtain then for problem (5.11), (5.12)

$$(5.17) \quad u^0 = u_0;$$

then for  $n \geq 0$ , we compute  $u^{n+1/2}$  and  $u^{n+1}$  via

$$(5.18) \quad \frac{u^{n+1/2} - u^n}{\Delta t} + \partial I_C(u^{n+1/2}, (n+1)\Delta t) = 0,$$

$$(5.19) \quad \frac{u^{n+1} - u^{n+1/2}}{\Delta t} + Au^{n+1} = f^{n+1}.$$

Problem (5.19) is an advection-diffusion elliptic problem whose solution is discussed in Section 7; on the other hand problem (5.18) has a *closed form* solution which is given by

$$(5.20) \quad u^{n+1/2}(x) = \min(u^n(x), \Psi(x, (n+1)\Delta t)).$$

A similar conclusion would hold for problem (5.14), (5.15). In fact from the simplicity of scheme (5.17), (5.19) it seems that penalty is useless in this context; we have mentioned it however since it may be useful in some other situations. We can switch of course the splitting order and solve first the advection diffusion part, and then project as in (5.20).

#### 6. Application to the Navier-Stokes equations.

Another advection-diffusion problem of interest is associated to the *Navier-Stokes equations* for viscous fluids. Concentrating on the *incompressible* case we obtain

$$(6.1) \quad \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega.$$

$$(6.2) \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \text{ (incompressibility condition),}$$

$$(6.3) \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega,$$

with appropriate boundary conditions; for simplicity we shall assume that

$$(6.4) \quad \mathbf{u} = \mathbf{g} \text{ on } \Gamma (= \partial \Omega).$$

In (6.1) - (6.4),  $\mathbf{u} = \{u_i\}_{i=1}^N$  is the *velocity*,  $p$  is the *pressure*,  $\Omega$  is the *flow region*,  $\nu (> 0)$  is a *viscosity coefficient* and  $\mathbf{f}$  a density of external forces and

$$(6.5) \quad (\mathbf{v} \cdot \nabla) \mathbf{w} = \left\{ \sum_{j=1}^N v_j \frac{\partial w_i}{\partial x_j} \right\}_{i=1}^N.$$

Following the approach in [8] (see also [9]) we shall discretize (6.1) - (6.4) by the following  $\theta$ -scheme, which decouples *nonlinearity* and *incompressibility*:

$$(6.6) \quad \mathbf{u}^o = \mathbf{u}_o,$$

and then for  $n \geq 0$ ,

$$(6.7)_1 \quad \frac{\mathbf{u}^{n+\theta} - \mathbf{u}^n}{\theta \Delta t} - \alpha \nu \Delta \mathbf{u}^{n+\theta} + \nabla p^{n+\theta} = \mathbf{f}^{n+\theta} + \beta \nu \Delta \mathbf{u}^n - (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n \text{ in } \Omega,$$

$$(6.7)_2 \quad \nabla \cdot u^{n+\theta} = 0 \quad \text{in } \Omega,$$

$$(6.7)_3 \quad u^{n+\theta} = g^{n+\theta} \quad \text{on } \Gamma,$$

$$(6.8)_1 \quad \frac{u^{n+1-\theta} - u^{n+\theta}}{(1-2\theta)\Delta t} - \beta \nu \Delta u^{n+1-\theta} + (u^{n+1-\theta} \cdot \nabla) u^{n+1-\theta} = f^{n+1-\theta} + \alpha \nu \Delta u^{n+\theta} - \nabla p^{n+\theta} \quad \text{in } \Omega,$$

$$(6.8)_2 \quad u^{n+1-\theta} = g^{n+1-\theta} \quad \text{on } \Gamma,$$

$$(6.9)_1 \quad \frac{u^{n+1} - u^{n+1-\theta}}{\theta \Delta t} - \alpha \nu \Delta u^{n+1} + \nabla p^{n+1} = f^{n+1} + \beta \nu \Delta u^{n+1-\theta} - (u^{n+1-\theta} \cdot \nabla) u^{n+1-\theta} \quad \text{in } \Omega,$$

$$(6.9)_2 \quad \nabla \cdot u^{n+1} = 0 \quad \text{in } \Omega,$$

$$(6.9)_3 \quad u^{n+1} = g^{n+1} \quad \text{on } \Gamma.$$

A good choice for  $\theta$  is  $\theta = 1-1/\sqrt{2}$  and for  $\alpha$  and  $\beta$

$$\alpha = \frac{\theta}{1-\theta}, \quad \beta = \frac{1-2\theta}{1-\theta}.$$

Numerical results obtained using the above scheme, combined to finite element methods are given in [8]; the above methods has been extended to two-phase flow problems in [9].

## 7. Numerical Solution of Advection - Diffusion Problems

### 7.1. Synopsis. Motivation

We follow here the presentation in [8], completed by some recent findings concerning upwinding methods of order three. In order to concentrate on Zakai equation related problems we shall illustrate the subsequent developments by considering an advection-diffusion problem associated to the following *stochastic ordinary differential equation*

$$(7.1) \quad dx = V(x) dt + dw,$$

where  $x$  is an  $N$ -dimensional vector and  $w$  a noise. Assuming convenient hypotheses on the noise  $w$ , the probability of finding at time  $t$  the state vector  $x$ , in neighborhood of  $\xi \in \mathbb{R}^N$  of measure  $d\xi = d\xi_1 \dots d\xi_N$ , is  $p(\xi, t) d\xi$  where

$$(7.2) \quad p(\xi, t) = \frac{q(\xi, t)}{\int_{\mathbb{R}^N} q(\eta, t) d\eta},$$

where the unnormalized probability density  $q$  satisfies a parabolic equation (see Section 2 for details). In the particular case where  $V$  is *divergence free* and for simple noise models, this parabolic equation reduces to

$$(7.3) \quad \frac{\partial q}{\partial t} - \varepsilon \nabla^2 q + V \cdot \nabla q = 0.$$

An interesting and difficult case is the one where the level of noise is "weak", implying that  $\varepsilon$  is "small". We have then an *advection dominated advection-diffusion equation*.

Using the splitting methods described in Section 4, the solution of such equations plays a fundamental role in the implementation of some solution methods for the Zakai equation.

In the sequel, we consider the following *initial boundary value problem*.

$$(7.4)_1 \quad \frac{\partial p}{\partial t} - \varepsilon \nabla^2 p + V \cdot \nabla p = f \text{ in } \Omega \times (0, T),$$

$$(7.4)_2 \quad p = g \text{ on } \partial\Omega \times (0, T),$$

$$(7.4)_3 \quad p(x, 0) = p_0(x) \text{ in } \Omega,$$

with  $\Omega \subset \mathbb{R}^N$ .

For solving such problems, particularly the ones associated to filtering, one has to face two outstanding difficulties, namely

- (i) When  $\varepsilon$  is small, the problem is *advection-dominated*,  
(ii) For practical problems, we have  $N > 3$  (in fact we may have  $N \sim 10$  and more).  
The methods described here have been applied in [8] to  $N=2$  to 6.

## 7.2 Solution of problem (7.4) by finite differences and upwinding methods.

The methods we need have to be *accurate* and *robust*; in our opinion, *we have to avoid those methods requiring parameter tuning.*

For simplicity, we consider the case where  $\Omega = (0,1)^N$ , with  $N=2$ ; the extension to  $N > 2$  is straight forward. With  $I$  a *positive integer*, we define  $h$  by  $h=1/I+1$  and consider over  $\bar{\Omega} = \Omega \cup \partial\Omega$  the mesh points

$$(7.5) \quad M_{i,j} = \{ih, jh\}; \quad 0 \leq i,j \leq I+1.$$

At the points  $M_{i,j}$  interior to  $\Omega$  (i.e.,  $1 \leq i,j \leq I$ ) we approximate (7.4) by the following *finite difference scheme* (with  $V = \{V_1, V_2\}$ ):

$$(7.6)_1 \quad \left\{ \begin{array}{l} \frac{p_{ij}^{n+1} - p_{ij}^n}{\Delta t} - \varepsilon \frac{p_{i+1j}^{n+1} + p_{i-1j}^{n+1} + p_{ij+1}^{n+1} + p_{ij-1}^{n+1} - 4p_{ij}^{n+1}}{h^2} \\ + V_1^+(M_{i,j}) \frac{p_{ij}^{n+1} - p_{i-1j}^{n+1}}{h} - V_1^-(M_{i,j}) \frac{p_{i+1j}^{n+1} - p_{ij}^{n+1}}{h} \\ + V_2^+(M_{i,j}) \frac{p_{ij}^{n+1} - p_{ij-1}^{n+1}}{h} - V_2^-(M_{i,j}) \frac{p_{ij+1}^{n+1} - p_{ij}^{n+1}}{h} \\ = f(M_{i,j}, (n+1)\Delta t), \end{array} \right.$$

with , in (7.6),  $\Delta t (> 0)$  a *time discretization step*,  $p_{ij}^n \sim p(M_{i,j}, n\Delta t)$ ,  $a^- = \max(0, a)$ ,  $a^+ = \max(0, -a)$ ,  $\forall a \in \mathbb{R}$ , and

$$(7.6)_2 \quad p_{kl}^{n+1} = g(M_{kl}, (n+1)\Delta t) \text{ if } M_{kl} \in \partial\Omega,$$



$$(7.6)_3 \quad p_{kl}^0 = p_0(M_{kl}).$$

Scheme (7.6) is of the *backward Euler* type for the *time discretization* and of the *first order upwind type* for the *space discretization*. Probabilists favor the above finite difference scheme because it satisfies a *discrete maximum principle* and therefore possesses a *probabilistic interpretation*. Unfortunately the above scheme is only *first order accurate*, *very dissipative* and not well-suited for those situations where  $\epsilon$  is small and  $V$  has a fast variation over  $\Omega$ .

An interesting alternative to (7.6) is obtained through a space/time discretization which is second order accurate and also of the upwind type (however, it does not satisfy the discrete maximum principle). Such a scheme is obtained as follows:

$$(7.7)_1 \quad p_{kl}^0 = p_0(M_{kl}), p_{kl}^1 \text{ is obtained (for example) via (7.6);}$$

then for  $n \geq 1$  and  $2 \leq i, j \leq I-1$ , discretize (7.4)<sub>1</sub> by

$$(7.7)_2 \quad \left\{ \begin{array}{l} \frac{\frac{3}{2} p_{ij}^{n+1} - 2 p_{ij}^n + \frac{1}{2} p_{ij}^{n-1}}{\Delta t} - \epsilon \frac{p_{i+1j}^{n+1} + p_{i-1j}^{n+1} + p_{ij+1}^{n+1} + p_{ij-1}^{n+1} - 4 p_{ij}^{n+1}}{h^2} \\ + V_1^+(M_{ij}) \frac{\frac{3}{2} p_{ij}^{n+1} - 2 p_{i-1j}^{n+1} + \frac{1}{2} p_{i-2j}^{n+1}}{h} + V_1^-(M_{ij}) \frac{\frac{3}{2} p_{ij}^{n+1} - 2 p_{i+1j}^{n+1} + \frac{1}{2} p_{i+2j}^{n+1}}{h} \\ + V_2^+(M_{ij}) \frac{\frac{3}{2} p_{ij}^{n+1} - 2 p_{ij-1}^{n+1} + \frac{1}{2} p_{ij-2}^{n+1}}{h} + V_2^-(M_{ij}) \frac{\frac{3}{2} p_{ij}^{n+1} - 2 p_{ij+1}^{n+1} + \frac{1}{2} p_{ij+2}^{n+1}}{h} \\ = f(M_{ij}, (n+1)\Delta t). \end{array} \right.$$

If  $M_{ij} \in \Omega$  with either  $M_{i,j \pm 1}$  or  $M_{i \pm 1, j}$  belonging to  $\Gamma$ , it is possible that  $M_{i,j \pm 2}$  or  $M_{i \pm 2, j}$  are outside  $\bar{\Omega}$ ; in such a case we can use to discretize  $V \cdot \nabla p$  at  $M_{ij}$ , a first order scheme like in (7.6)<sub>1</sub>, or alternatively a *centered second order approximation* like

$$(7.8) \quad (V \cdot \nabla p)(M_{ij}) \sim V_1(M_{ij}) \frac{p_{i+j} - p_{i-1j}}{2h} + V_2(M_{ij}) \frac{p_{ij+1} - p_{ij-1}}{2h}$$

The boundary conditions are treated as in (7.6)<sub>2</sub>. The fact that the problems under consideration may have a fast dynamics, requires the use of small  $h$  and  $\Delta t$ ; indeed, we can take advantage of the fact that  $\Delta t$  is small, to solve the above discrete elliptic problems by successive over-relaxation, since the method has good vectorization and parallelization properties (in practice few iterations will insure convergence at each time step).

Numerical experiments (see, e.g., [8]) show the superiority of the second order upwinding method, over the first order method (it is more accurate, less dissipative and almost as easy to implement).

### 7.3. A third order in space upwinded scheme.

Influenced by third order in space, upwinded finite difference schemes, recently developed in Japan, for Computational Fluid Dynamics purposes, we shall describe here a variant of scheme (7.7) which uses a third order accurate space discretization of the advection and which is second order accurate in time. The basic principle of this scheme is to combine two second order accurate schemes, one centered (in fact, it is scheme (7.8)) and the one used in (7.7). The resulting scheme is *more accurate* than scheme (7.7) but *less robust*; we have then (7.7)<sub>1</sub> and

$$(7.9) \left\{ \begin{array}{l} \frac{\frac{3}{2} p_{ij}^{n+1} - 2 p_{ij}^n + \frac{1}{2} p_{ij}^{n-1}}{\Delta t} - \epsilon \frac{p_{i+1j}^{n+1} + p_{i-1j}^{n+1} + p_{ij+1}^{n+1} + p_{ij-1}^{n+1} - 4p_{ij}^{n+1}}{h^2} + \\ V_1^+(M_{ij}) \frac{\frac{1}{3} p_{i+1j}^{n+1} + \frac{1}{2} p_{ij}^{n+1} - p_{i-1j}^{n+1} + \frac{1}{6} p_{i-2j}^{n+1}}{h} + V_1^-(M_{ij}) \frac{\frac{1}{3} p_{i-1j}^{n+1} + \frac{1}{2} p_{ij}^{n+1} - p_{i+1j}^{n+1} + \frac{1}{6} p_{i+2j}^{n+1}}{h} \\ + V_2^+(M_{ij}) \frac{\frac{1}{3} p_{ij+1}^{n+1} + \frac{1}{2} p_{ij}^{n+1} - p_{ij-1}^{n+1} + \frac{1}{6} p_{ij-2}^{n+1}}{h} + V_2^-(M_{ij}) \frac{\frac{1}{3} p_{ij-1}^{n+1} + \frac{1}{2} p_{ij}^{n+1} - p_{ij+1}^{n+1} + \frac{1}{6} p_{ij+2}^{n+1}}{h} \\ = f(M_{ij}, (n+1)\Delta t); \end{array} \right.$$

we shall use (7.8) for those grid points at distance  $h$  of the boundary.

## 8. Further Comments. Conclusion

Operator splitting methods provide a systematic way to obtain efficient solution methods for parabolic equations and inequalities in high dimension. This point of view has been illustrated by the design of efficient methods for fundamental problems such as the Zakai equation, the Navier-Stokes equations and time dependent obstacle problems (other applications, like *liquid crystal* problems are discussed in [8]). Concerning now more specifically the Zakai equation we would like to point out the following facts:

- (a) Due to the very high space dimension associated to practical problems, there is no hope that fast solution can be achieved for several years to come. Indeed the methods described here are quite fast, since the experiments in [8] show that the solution cost per time step is of the order of the number of grid points; however this number is so large that the task is still formidable, requiring among other things highly massive parallelism and huge computer memory.
- (b) It is perplexing to realize that several partial differential equation specialists advocate to solve deterministic systems of advection-diffusion type (such as the Navier-Stokes equations) by *particle methods* where the advection is treated via the integration of a (large) system of ordinary differential equations, and the diffusion by a random walk method. Indeed the stochastic dynamical systems leading to the Zakai equation are already in the above form since they couple the system of ordinary differential equations modelling the dynamic of the problem to a noise "generator", and this suggests therefore to deal more directly with the original form of the problem.
- (c) Since the Zakai equation is linear even if the associated dynamical system is not, we can see it as a simplification. This is a quite superficial conclusion since in the case of a treatment by particle or characteristic methods we shall have nevertheless to solve a very large number of nonlinear ordinary differential systems very close to the original one in order to accurately construct the characteristic curves associated to the advection term.

Our very conclusion concerning the Zakai equation will be therefore that due to lack of computing power it is hopeless that genuine real life systems can be treated by this approach; (at least for several years to come) in fact the situation is very close to the one encountered in the solution of the time dependent Schrodinger equation where only the simplest molecules can be investigated, precisely due to the dimension difficulty.

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