COLLOQUE INTERNATIONAL THÉORIE DU POINT FIXE ET APPLICATIONS

5-9 juin 1989
MARSEILLE - LUMINY

International Conference on Fixed Point Theory and Applications

CIRM

Résumés des Conférences

ÉDITÉS PAR
JEAN-BERNARD BAILLON
MICHEL THÉRA

Colloque International : Théorie du Point Fixe et Applications

Organisateurs : J. B. Baillon (Lyon 1)
M. Théra (Limoges)

Comité Scientifique

Thèmes

- Méthodes topologiques
- Point fixes des applications multivoques
- Théorèmes de minimax et inégalités abstraites
- Applications à l'Economie, l'Analyse, la Théorie des jeux, ...

Colloque International : Théorie du Point Fixe et Applications

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Thème Métrique	: K. Goebel (Lublin) W. A. Kirk (Iowa)
Thème Topologique	: L. Gorniewicz (Torun)
Thème Minimax	: S. Simons (Santa Barbara)
Thème Multiapplications	: W. Takahashi (Tokyo)
Thème Applications	: J. M. Borwein (Halifax) H. Brezis (Paris 6) B. Cornet (Paris I)

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Wieczorek (Warsaw)
Wilczynski (Wroclaw)

Préface

L'idée de ce colloque au CIRM nous est venue à la suite de la session sur les points fixes organisée par Robert Brown lors du Congrés International de Mathématiques à Berkeley en 1986.

Cette collection de résumés contient l'essentiel des exposés que vont donner des experts de dix sept pays différents. Elle représente l'état actuel de la recherche sur les points fixes dans les domaines suivants:

- applications contractantes
- applications multivoques
- théorèmes de minimax et inégalités abstraites
- méthodes topologiques et théorie du degré
- structures ordonnées
- applications à l'analyse économique, l'optimisation, la théorie des jeux, etc...

Au nom du Comité d'Organisation de ce symposium, nous exprimons toute notre gratitude à l'égard des auteurs de ces résumés.

Nous tenons aussi à adresser nos remerciements les plus sincères aux secrétaires du Département de Mathématiques de Limoges, Martine Guerletin, pour avoir tapé le programme préliminaire et une abondante correspondance, ainsi que Valérie Piqueras pour toute la préparation de cette publication.

Nous ne saurions oublier Jacqueline Simons dont l'aide, dans les derniers jours qui ont précédé le colloque, nous a été précieuse.

Enfin, nous tenons à remercier tous les organismes qui nous ont apporté leur soutien financier sans lequel ce colloque n'aurait pu avoir lieu.

Preface

The idea for this symposium at CIRM originated after the special session on fixed points at the ICM at Berkeley in 1986.

This collection of abstracts summarizes the presentations given at the symposium by experts from 17 different countries. It represents active current research on problems in at least the following areas:

- Nonexpansive mappings
- Set-valued mappings
- Minimax theorems and other abstract inequalities
- Topological methods and degree theory
- Ordered structures
- Applications to economic analysis, partial differential equations, optimization, game theory, etc.

On behalf of the Symposium committee, we wish to express our gratitude to the authors of these abstracts.

We are also indebted to the secretaries of the Department of Mathematics at Limoges, Ms. Martine Guerletin, who typed the preliminary program and numerous letters, and Ms. Valérie Piqueras who typed the abstracts and prepared this booklet for printing.

We would like to thank Jacqueline Simons for the great help that she gave us during the last few days preceding the conference.

We would also like to express our gratitude to all the organizations who have supported this conference. Without their support this meeting would not have been possible.

Programme des Conférences

Lundi 5 Juin / Monday June 5

9h00 - 10h15	**Accueil / Registration**
10h15 - 11h30	Ouverture officielle / Official opening H. KÖNIG *(Universität Saarbrücken)* Conférencier / Speaker : **W.A. KIRK** *(University of Iowa)* *Krasnosleskii's iteration process and its generalization* Président / Chair : H. KÖNIG
14h15 - 15h15	**K. GOEBEL** (Instytut Matematyki, Lublin) *The minimal displacement and retraction problems* Président / Chair : R. BRUCK

SESSIONS

Session I : Thème Métrique

Président / Chair : B.E. RHOADES

15h40 - 16h00	**J. DESBIENS**
16h10 - 16h30	**N. ASSAD / S. SESSA**
16h40 - 17h00	**R. BRUCK**
17h10 - 17h30	**T. KUCZUMOV / A. STACHURA**

Session II : Applications

Président / Chair : G. ISAC

15h40 - 16h00	**S. SWAMINATHAN**
16h10 - 16h30	**J. BLOT**
16h40 - 17h00	**W. EGERLAND**

Session III : Applications à l'économie

Salle 112-116. Entrée Coul H. 1er Etage (1st floor)

Président / Chair : R. VOHRA

15h40 - 16h00	**M. FLORENZANO**
16h10 - 16h30	**J-M. BONNISSEAU / B. CORNET**
16h40 - 17h00	**H. VAN MAAREN**

Mardi 6 Juin / Tuesday June 6

8h30 - 9h 30 S. SIMONS (University of California at Santa Barbara)
Continuity of the Inf-Sup with applications
Président / Chair : B. L. LIN

SESSIONS

Session I : Thème Minimax
Président / Chair : L. MC LINDEN

10h00 - 10h20	G. GRECO
10h30 - 10h50	J. LEGUT /M. WILCZYNSKI
11h00 - 11h20	J. GUILLERME
11h30 - 11h50	S. ALPERN / S. GAL

Session II : Point fixe et structures discrètes
Président / Chair : J. CONSTANTIN

10h00 - 10h20	F. ROBERT
10h30 - 10h50	M.A. KHAMSI
11h00 - 11h20	K. KEIMEL
11h30 - 11h50	E. COROMINAS

14h30 - 15h30 B. CORNET (Université Paris I Panthéon-Sorbonne)
The existence problem in economics and fixed point theorem
Président / Chair : M. MAGILL

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Président / Chair : M. FLORENZANO

16h00 - 16h20	A. IDZIK
16h30 - 16h50	A. KHAN
17h00 - 17h20	R. VOHRA / L. SHAPLEY

Session II : Thème Topologique
Président / Chair : F.S. DE BLASI

16h00 - 16h20	J. JAWOROWSKI
16h30 - 16h50	M. MAGGIL
17h00 - 17h20	A.WIECZOREK

Mercredi 7 Juin / Wednesday June 7

8h30 - 9h30	H. BREZIS (Université Paris 6) *Nouveaux résultats de minimax motivés par des probèmes de cristaux liquides* Président / Chair : F. E. BROWDER

SESSIONS

Session I : Thème EDP

Président / Chair : N. HIRANO

10h00 - 10h20	A.L. EDELSON
10h30 - 10h50	E. HANEBALY
11h00 - 11h20	A. CHALJUB-SIMON

Session II : Thème Applications multivoques

Président / Chair : J. SAINT-RAYMOND

10h00 - 10h20	S. LEVI / A. LECHICKI
10h30 - 10h50	H. BEN-EL-MECHAIEKH
11h00 - 11h20	A. CELLINA
11h30 - 11h50	S. PARK

Jeudi 8 Juin / Thursday June 8

8h30 - 9h30	W. TAKAHASHI (Tokyo Institute of Technology) *Existence theorems generalizing fixed point theorems for multivalued mapping* Président / Chair : A. CELLINA

SESSIONS

Session I : Thème Inégalités Abstraites
Président / Chair : G.GRECO

10h00 - 10h20	H. KÖNIG
10h30 - 10h50	B-L. LIN / X.-C. QUAN
11h00 - 11h20	J. GWINNER

Session II : Thème Métrique
Président / Chair : S.SWAMINATHAN

10h00 - 10h20	B.E. RHOADES / S. PARK / K.B. MOON
10h30 - 10h50	J. MYJAK
11h00 - 11h20	A. KALINDE
11h30 - 11h50	A. LAU
14h30 - 15h30	P. VOLKMANN (Universität Karlsruhe) *Application d'un théorème de point fixe aux équations différentielles dans les espaces de Banach* Président / Chair : M. Z. NASHED

SESSIONS

Session I : Thème Applications
Président / Chair : J. P. PENOT

16h00 - 16h20	A.S. LEWIS / J.M. BORWEIN
16h30 - 16h50	G. ISAC
17h00 - 17h20	N. HIRANO
17h30 - 17h50	M.Z. NASHED

Session II : Thème Applications multivoques
Président / Chair : S. LEVI

16h00 - 16h20	J. SAINT-RAYMOND
16h30 - 16h50	I. BEG / A. AZAM
17h00 - 17h20	L. Mc-LINDEN
17h30 - 17h50	S.P. SINGH / V.M. SEGHAL

Vendredi 9 Juin / Friday June 9

8h30 - 9h 30	J.M. BORWEIN (Dalhousie University) *Minimal Convex USCOS and their Applications* Président / Chair : W. OETTLI

SESSIONS

Session I: Thème Topologie

Président / Chair : M. LASSONDE

10h00 - 10h20	D. VIOLETTE / G. FOURNIER
10h30 - 10h50	B. MESSANO
11h00 - 11h20	W. KRYSZEWSKI
11h30 - 11h50	M. FURI / M.P. PERA

Session II : Thème Inégalités Abstraites

Président / Chair : J. GWINNER

10h00 - 10h20	J-E. MARTINEZ-LEGAZ / E. MARCHI
10h30 - 10h50	P. DEGUIRE
11h00 - 11h20	K-K. TAN
11h30 - 11h50	P. MAZET

Session III : Point fixe et structures discrètes

Salle 112-116. Entrée Coul H. 1er Etage (1st floor)

Président / Chair : K. KEIMEL

10h00 - 10h20	V. CONSERVA
10h30 - 10h50	R. MANKA
11h00 - 11h20	J. CONSTANTIN

14h00 - 15h00	L. GORNIEWICZ (Universytat Mikolaja Kopemifker, Torun) *The Lefschetz fixed point theorem for multivalued mappings* Président / Chair : M. FURI
15h15	**Session :** Discussion libre : problèmes ouverts Discussion of open problems

FIN DU COLLOQUE / END OF THE SYMPOSIUM

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A MIXED STRATEGY MINIMAX THEOREM

STEVE ALPERN AND SHMUEL GAL

Abstract : In this paper we establish a new mixed strategy minimax theorem for a two person zero-sum game given in the normal form $f : X \times Y \to \mathbb{R}$. This is interpreted in the usual way, so that if the minimizer picks a pure strategy x in X and the maximizer picks a pure strategy y in Y then the payoff (to the maximizer) is f(x, y). We will assume that the pure strategy space X is a compact Hausdorff space, and that the mixed strategies available to the minimizer are the regular Borel probability measures on X, collectively de..oted by B(X). Our approach is assymetric in that we do not assume that the maximizer's pure strategies are necessarily topologized. In this case it is well known that there is a certain arbitrariness about the appropriate definition of the maximizer's mixed strategies. We will initially take a general approach and merely assume that the maximizer may choose any mixed strategy in a given set M of probability measures on Y. We will assume that M is a convex set of measures on a common σ-algebra A, and for interpretive reasons we may assume that M contains all point masses. If f is a real (Borel x A) measurable function on $X \times Y$ which is bounded below, we define the usual mixed extension of f, $F : B(X) \times M \to \mathbb{R}$, by the integral

$$F(\mu, \rho) = \iint f \, d(u \times \rho) = \iint f(x, y) \; d\mu(x) \; d\rho(y) = \iint f(x, y) \; d\rho(y) \; d\mu(x),$$

where the equivalence or the various forms follows from Fubini's theorem. Using a pure strategy minimax theorem of Kneser [7], we prove the following mixed strategy result :

Theorem 1 : Let X be a compact Hausdorff space and let (Y, A) be a measurable space. Let $f : X \times Y \to \mathbb{R}$ be a measurable function which is bounded below and lower semicontinuous in x for all fixed y. Let M be any convex set of probability measures on (Y, A). Then

$$\min_{\mu \in B(X)} \sup_{\rho \in M} F(\mu, \rho) = \sup_{\rho \in M} \min_{\mu \in B(X)} F(\mu, \rho).$$

By specializing the mixed strategies M available to the maximizer, we obtain corrolaries to Theorem 1 which extend known results. For example, if we take Y to be a topological space

with A its Borel algebra and M = B(Y), we obtain the following generalization of Glicksberg's Theorem [6] (without compctness of Y or lower semicontinuity in y).

Corollary 1 : Let X and Y be topological spaces, with X compact Hausdorff. Let $f : X \times Y \to \mathbb{R}$ be a measurable function which is bounded below and lower semi-continuous in x for all fixed y. Then

$$\min_{\mu\in B(X)} \sup_{\rho\in B(Y)} F(\mu, \rho) = \sup_{\rho\in B(Y)} \min_{\mu\in B(X)} F(\mu, \rho).$$

If we take the σ-algebra of Theorem 1 to be the power set of Y and M to be all convex combinations of point masses, then we obtain the following generalization of the theorem of Peck and Dulmage [8] (with their assumption of continuity reduced to semicontinuity). The lower bound for I (required for Fubini's Theorem) is no longer needed.

Corollary 2 : Let X be a compact Hausdorff space and Y be any set. Then for any real funciton f on X × Y which is lower semicontinuous on X for every fixed y in Y, we have

$$\min_{\mu\in B(X)} \sup_{\rho\in M} F(\mu, \rho) = \sup_{\rho\in M} \min_{\mu\in B(X)} F(\mu, \rho).$$

where M is the set of finite mixtures $\rho = \sum_{1\leq k\leq n} \rho_k y_k$ with $\sum_{1\leq k\leq n} \rho_k = 1$, $\rho_k \geq 0$ and y_k in Y, k = 1, ..., n, and

$$F(\mu, \rho) = \sum_{1\leq k\leq n} \rho_k \int_X f(x, y)\, d\mu(x).$$

In [4, Appendix 1], Theorem 1 was established for a class of payoff functions given by the "time required for capture" for certain search games. The existence of a more general result implicit in [4, Appendix 1] was observed in [1], where it was shown how this resut could be used to establish a conjectured minimax for a search game of Baston and Bostock [2].

An application of Theorem 1 is to the problem of getting safely, by a continuous trajectory y, from a given point A to a given point B while avoiding k mines $x_1, \ldots, x_k$ placed by an opponent in a compact set D which separates A form B. The trajectory y is constrained to lie within some set S in $\mathbb{R}^n$ which contains A, B and D. With possible further restrictions, the set of allowable trajectories is called Y. The other pure strategy set X is the compact product of k copies of D. Given an allowable trajectory y in Y and mine placements $x = (x_1, \ldots, x_k)$ in $x = D^k$ the payoff f(x, y) is + 1 if the maximizer y successfully reaches B by always keeping a distance greater than a given danger radius r from any of the points x. Otherwise, f(x, y) is - 1. The payoff function is l.s.c. in x for any fixed trajectory y so Theorem 1 applies regardless of any further restrictions (turning radius, speed) that may be placed on trajectories.

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4

COMMON FIXED POINTS FOR NONSELF COMPATIBLE MAPS ON COMPACTA

A. ASSAD AND S. SESSA

Abstract : Let (M, d) be a complete convex metric space, K be a nonempty compact subset of M, f, T : K → M continuous mappings so that $\partial K \subset TK$, $FK \cap K \subset TK$ and $Tx \in \partial K$ (the boundary of K) ⇒ $fx \in K$. If f, T are compatible and $d(fx, fy) < d(Tx, Ty)$ for all x, y ∈ K such that $Tx \neq Ty$, then f and T have a unique common fixed point in K.

Let (M, d) be a metric space and K be a nonempty subset of M. In a recent paper, Hadzic [3] extended a well known result of Assad and Kirk [2] for single-valued mappings using the following definition [7] :

Definition 1 : Two maps f, T : K → M are weakly commutative iff for every x ∈ K, the implication $fx, Tx \in K \Rightarrow d(fTx, Tfx) \leq d(fx, Tx)$ holds.

In the sequel, we will use the following more general definition [6] :

Definition 2 : Two maps f, T : K ⇒ M are compatible iff $d(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in K for which $\{fx_n\}, \{Tx_n\} \subset K$ and $fx_n, Tx_n \to t$ for some t ∈ K.

Clearly, commuting maps are weakly commuting maps and weakly commuting maps are compatible. On the other hand, examples are given in [4], [5], [6] and in [7] to show neither of the above implications are reversible.

We will use the following (slightly modified) result of Jungck [Theorem 2.2, 6] :

Lemma 1 : Let K be a compact subset of M and f, g : K → M be continuous. Then f, g are compatible iff for any x ∈ K such that fx, gx ∈ K the implication $fx = gx \Rightarrow fgx = gfx$ holds.

Remark : If K = M, Lemma 1 becomes Corollary 2.3 of [6].

We recall that a metric space (**M**, d) is convex if for any distinct points x, y $\in$ **M** there exists z $\in$ **M**, $x \neq y \neq z$, such that $d(x, z) + d(z, y) = d(x, y)$. To prove our theorem, we will use the following fact mentioned explicity in [2].

Lemma 2 : Let **K** be a closed subset of the complete and convex metric space (**M**, d). If $x \in$ **K** and $y \notin$ **K**, then there exists a point $z \in \partial$**K** (the boundary of **K**) such that $d(x, z) + d(z, y) = d(x, y)$.

Now we present our theorem.

Theorem : Let (**M**, d) be a complete convex metric space, **K** be a nonempty compact subset of **M**, f, T : **K** $\to$ **M** be continuous maps such that $\partial K \subset TK$, $fK \cap K \subset TK$ and $Tx \in \partial K \Rightarrow fx \in K$. If f, T are compatible and $d(fx, fy) < d(Tx, Ty)$ for all x, y $\in$ **K** such that $Tx \neq Ty$, then f and T have a unique common fixed point in **K**.

Proof : Following the proof of Theorem 1 of [3], we can show that there exist two sequences $\{P_n\}$, $\{P_{n'}\}$ such that $P_{n+1'} = f(P_n)$ for any n $\in$ N and

(i) $P_{n'} \in K \Rightarrow P_{n'} = TP_n$
(ii) $P_{n'} \notin K \Rightarrow TP_n \in \partial K$ and

$$d(TP_n, TP_{n+1}) + d(TP_{n+1}, fP_n) = d(TP_n, fP_n).$$

For sake of completeness, we give the proof of (i) and (ii). Let $x \in \partial K \subset TK$. Then there exists $P_o \in K$ such that $x = TP_o$. Since $TP_o \in \partial K$, then $fP_o \in K \cap fK \subset TK$. Hence, there exists $P_1 \in K$ so that $TP_1 = fP_o = P_1'$. let $P_{2'} = fP_1$. If $fP_1 \in K$, then $fP_1 \in fK \cap K \subset TK$ and thus there exists $P_2 \in K$ such that $TP_2 = fP_1 = P_{2'}$. If $fP_1 \notin K$, then there exists $q \in \partial K$ such that, by Lemma 2,

$$d(TP_1, q) + d(q, fP_1) = d(TP_1, fP_1).$$

From $\partial K \subset TK$, it follows that there exists $P_2 \in K$ so that $q = TP_2$ and hence $d(TP_1, TP_2) + d(TP_2, fP_1) = d(TP_1, fP_1)$. Continuing in this way, we can prove (i) and (ii).

Suppose that $TP_n = TP_{n+1}$ for some n. If $TP_n \in \partial K$, it follows that $fP_n = TP_{n+1} = P_{n+1'} \in K$. on the other hand, if $TP_n \notin \partial K$, then $TP_{n+1} \notin \partial K$ and again we conclude, by (ii), that $TP_{n+1} = P_{n+1'} = fP_n \in K$. Let $z = TP_n = fP_n$ and since T and f are compatible, we have $Tz = TfP_n = fTP_n = fz$ by Lemma 1. Assume that $z \neq Tz$. Then $d(z, Tz) = d(z, fz) = d(fP_n, fTP_n) < d(TP_n, TTP_n) = d(z, Tz)$, a contradiction. Therefore we conclude that $z = Tz = fz$. Now we assume that $TP_n \neq TP_{n+1}$ for all n = 0, 1, 2...

Let $\mathcal{P} = \{P_i : P_i' = TP_i\}$ and $\mathcal{Q} = \{P_i : P_i' \neq TP_i\}$. We observe that if $P_n \in \mathcal{Q}$, then $P_{n+1} \in \mathcal{P}$. Indeed if $P_n \in \mathcal{Q}$, then $P_n' \neq TP_n$ and, by (i), $TP_n \in \partial\mathbf{K}$. So $fP_n = P_{n+1'} \in \mathbf{K}$, i.e. $P_{n+1'} = TP_{n+1} = fP_n$ (by (i)) and thus $P_{n+1} \in \mathcal{P}$.

We will prove that for every $n \geq 2$,

(1) $$d(TP_n, TP_{n+1}) < \begin{cases} d(TP_{n-1}, TP_n) & \text{or} \\ d(TP_{n-2}, TP_{n-1}). \end{cases}$$

We distinguish three cases.

Case 1. $P_n, P_{n+1} \in \mathcal{P}$. Then $d(TP_n, TP_{n+1}) = d(P_{n'}, P_{n+1'}) = d(fP_{n-1}, fP_n) < d(TP_{n-1}, TP_n)$.

Case 2. $P_n \in \mathcal{P}, P_{n+1} \in \mathcal{Q}$. Then $d(TP_n, TP_{n+1}) \leq d(TP_n, fP_n) = d(fP_{n-1}, fP_n) < d(TP_{n-1}, TP_n)$.

Case 3. $P_n \in \mathcal{Q}, P_{n+1} \in \mathcal{P}$. Then $P_{n-1} \in \mathcal{P}$ and $d(TP_n, TP_{n+1}) \leq d(TP_n, fP_{n-1}) + d(fP_{n-1}, fP_n) < d(TP_n, fP_{n-1}) + d(TP_{n-1}, TP_n) = d(fP_{n-2}, fP_{n-1}) < d(TP_{n-2}, TP_{n-1})$.

Thus (1) is proved in any case.

Observing that $\{TP_n\}$ is a sequence in **K**, by compactness of **K**, we may assume that $\{TP_n\}$ has convergent subsequences $\{TP_{n(k)}\}$ and $\{TP_{n(k)+1}\}$ such that $TP_{n(k)+1} = P_{n(k)+1'}$, i.e. $P_{n(k)+1} \in \mathcal{P}$. We will prove that $\lim_k TP_{n(k)} = \lim_k TP_{n(k)+1}$. Let $TP_{n(k)} \to y_0$, $TP_{n(k)+1} \to y_1$. Consider the function $r : \mathbf{Y} = \mathbf{TK} \times \mathbf{TK} - \Delta \to [0, +\infty)$, where $\Delta = \{(Tx, Ty) : x, y \in \mathbf{K}$ and $Tx = Ty\}$ and $r(Ta, Tb) = d(fa, fb) / d(Ta, Tb)$, $a, b \in \mathbf{K}$. $(y_0, y_1) \in \mathbf{Y}$ follows from $y_0 \neq y_1$ and $y_0, y_1 \in \mathbf{TK}$. Since r is continuous, there exists a neighbourhood **U** of (y_0, y_1) such that $(Ta, Tb) \in \mathbf{U} \cap \mathbf{Y}$ implies $0 \leq r(Ta, Tb) < R < 1$. Let $S_1 = S_1(y_0, \rho)$ and $S_2 = S_2(y_1, \rho)$ be open spheres centered at y_0 and y_1, respectively, and of radius $\rho > 0$ small enough so as to have $3\rho < d(y_0, y_1)$ and $S_1 \times S_2 \subset \mathbf{U}$. Let ν be a positive integer such that $k \geq \nu$ implies $TP_{n(k)} \in S_1$ and $TP_{n(k)+1} \in S_2$. Therefore we have $d(TP_{n(k)}, TP_{n(k)+1}) > \rho$ for all $k \geq \nu$. On the other hand, from the definition of r and the choice of **U**, we may conclude that $d(fP_{n(k)}, fP_{n(k)+1}) < Rd(TP_{n(k)}, TP_{n(k)+1})$ for all $k \geq \nu$. Observe that for $k > 1$, it follows from (1), that one of the following possibilities occurs :

either

(2) $$d(TP_{n(k)}, TP_{n(k)+1}) \leq d(TP_{n(k-1)+1}, TP_{n(k-1)+2})$$

or

(3) $$d(TP_{n(k)}, TP_{n(k)+1}) \leq d(TP_{n(k-1)+2}, TP_{n(k-1)+3})$$

If (2) occurs, we get $d(TP_{n(k)}, TP_{n(k)+1}) \leq d(fP_{n(k)-1}, fP_{n(k-1)+1})$

If (3) holds, we consider two cases :

Case (a). $P_{n(k-1)+2} \in \mathcal{P}$. Then

$d(TP_{n(k-1)+2}, TP_{n(k-1)+3}) \leq d(fP_{n(k-1)+1}, fP_{n(k-1)+2}) < d(TP_{n(k-1)+1}, TP_{n(k-1)+2})$

$= d(fP_{n(k-1)}, fP_{n(k-1)+1})$.

Case (b). $P_{n(k-1)+2} \in \mathcal{Q}$. Then $P_{n(k-1)+3} \in \mathcal{P}$ and we have

$d(TP_{n(k-1)+2}, TP_{n(k-1)+3}) \leq d(fP_{n(k-1)+2}, fP_{n(k-1)+1}) + d(fP_{n(k-1)+1}, fP_{n(k-1)+2})$

$< d(TP_{n(k-1)+2}, fP_{n(k-1)+1}) + d(TP_{n(k-1)+1}, TP_{n(k-1)+2}) = d(fP_{n(k-1)}, fP_{n(k-1)+1})$

Therefore we have proved in any case the following inequality for $k > 1$:

$$d(TP_{n(k)}, TP_{n(k)+1}) \leq d(fP_{n(k-1)}, fP_{n(k-1)+1}).$$

Then we deduce $d(TP_{n(k)}, TP_{n(k)+1}) < Rd(TP_{n(k-1)}, TP_{n(k-1)+1})$ for $k > \nu$ and by repeating this argument we get for a fixed j, $k > j > \nu$:

$$d(TP_{n(k)}, TP_{n(k)+1}) < R^{k-j}\, d(TP_{n(j)}, TP_{n(j)+1}).$$

But $R^{k-j} \to 0$ as $k \to +\infty$ and this implies $d(TP_{n(k)}, TP_{n(k)+1}) \to 0$ as $k \to +\infty$, a contradiction to the fact that $d(TP_{n(k)}, TP_{n(k)+1}) > \rho$ for all $k \geq \nu$. Thus we conclude that $y_0 = y_1 = z$ (say) and since $P_{n(k)+1} \in \mathcal{P}$, we have that $fP_{n(k)} = TP_{n(k)+1}$ and hence $\lim_k fP_{n(k)} = z$. Since f and T are compatible, we obtain that $\lim_k d(TfP_{n(k)}, fTP_{n(k)}) = 0$ and hence $fz = Tz$.

By compactness of **K**, we may assume that $P_{n(k)} \to u$. Since T and f are continuous, $TP_{n(k)} \to Tu$, $fP_{n(k)} \to fu$ and then $fu = Tu = z$. If $z \neq Tz$, i.e. $Tu \neq TTu$, we have that $d(z, Tz) = d(z, fz) = d(fu, fTu) < d(Tu, TTu) = d(z, Tz)$, a contradiction. Hence $z = Tz = fz$. It is easily seen that z is the unique common fixed point of T and f and this concludes the proof.

This theorem generalizes the point-to-point analogue result of [1].

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COINCIDENCE POINTS OF MULTIVALUED MAPPINGS

ISMAT BEG AND AKBAR AZAM

A coincidence theorem in a metric space is proved for a multivalued mapping that commutes with two single valued mappings and satisfies a general multivalued contraction type condition.

Introduction

Jungck [2] generalized the Banach contraction principle by introducing a contraction condition for a pair of commuting self mappings on a metric space. He also pointed out in [3] and [4] the potential of commuting mappings for generalized fixed point theorems. Subsequently a variety of extensions, generalizations and applications of this followed ; e.g. see [5], [7] and [8]. This paper is a continuation of these investigations.

Let (X, d) be a metric space. We shall use the following notation and definition.
$CB(X) = \{A : A \text{ is a nonempty bounded closed subset of } X\}$,
$N(\varepsilon, A) = \{x \in X \mid (\exists\, a \in A)\ (d(x, a) < \varepsilon)\}$, where $\varepsilon > 0$.

$$H(A, B) = \begin{cases} \inf\{\varepsilon > 0 : A \subset N(\varepsilon, B),\ B \subset N(\varepsilon, A)\} & \text{if the infimum exists} \\ \infty & \text{otherwise.} \end{cases}$$

This function H is a metric for $CB(X)$ and is called Hausdorff metric.

Let $T : X \to CB(X)$ be a mapping, then $C_T = \{f : X \to X \mid Tx \subseteq fX \text{ and } (\forall\, x \in X)\ (fTx \subset Tfx)\}$. T and f are said to be commuting mappings if for each $x \in X$, $f(Tx) = fTx \subsetneq Tfx = T(fx)$. A point x is said to be a fixed point of a single valued mapping f (multivalued mapping T) provided $fx = x (x \in Tx)$. The point x is called a coincidence point of f and T if $fx \in Tx$. For details see Nadler [6] and Rhoades, Singh and Chitra [8].

Main results

Lemma 1 : Let X be a metric space and $T : X \to CB(X)$ a continuous mapping. Let $f \in C_T$ be continuous and such that f and T have a coincidence point z in X. If $\lim_{n\to\infty} f^n z = t < \infty$, then t is a common fixed point of f and T.

Theorem 2 : Let X be a complete metric space and $T : X \to CB(X)$ a continuous mapping. Let $f, g \in C_T$ be continuous and such that the following condition is satisfied :

$$\begin{aligned} H(Tx, Ty) \leq & Ad(fx, gx) + B\{d(fx, Tx) + d(gy, Ty)\} \\ & + C\{d(fx, Ty) + d(gy, Tx)\} + D\{d(fx, gy)\}^{-1} \\ & d(fx, Tx)\, d(gy, Ty), \end{aligned}$$

for all $x, y \in X$, $A, B, C, D \geq 0$ and $0 < \frac{A + B + C}{1\text{-}B\text{-}C\text{-}D} < 1$. Then there is a coincidence point of f and T which is also a coincidence of g and T.

Corollary 3 : Let X be a complete metric space and $T : X \to CB(X)$ a continuous mapping. Let $f, g \in C_T$ be continuous and such that (1) is satisfied. Moreover, assume that

$$\{fz, gz\} \subset Tz \text{ implies } \lim_{n\to\infty} f^n z = \lim_{n\to\infty} g^n z = t < \infty.$$

Then t is a common fixed point of f, g and T.

Remark 4 : Several other results may also be seen to follow as immediate corollaries to Theorem 2. included among these are the following, Dube and Singh theorem 1 [1], Jungck [2] Kaneko [5] and Nadler theorem 5 [6].

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12

NOTE ON A CLASS OF SET-VALUED MAPS HAVING CONTINUOUS SELECTIONS

H. BEN-EL-MECHAIEKH

Several authors pointed out the fact that some intersection theorems can be expressed as fixed point theorems for appropriate set-valued maps. In this note, we present some fixed point and coincidence theorems motivated by a generalization of the KKM theorem due to Brézis-Nirenberg-Stampacchia, thus stressing the relation between the KKM principle and the selection problem. Hence, we derive in a simple and standard way a general existence theorem for variational inequalities containing several known results.

LE THEOREME DE MARKOV-KAKUTANI ET LA PRESQUE-PÉRIODICITÉ

JOEL BLOT[1]

Une fonction p.p. (presque-périodique au sens de H. Bohr, c.f. [BE]) $f : \mathbb{R} \to E = \mathbb{R}^n$ ou $\mathbb{C}^n$ possède une moyenne temporelle $m\{f\} := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt \in E$. Si A est un groupe topologique plus général que $\mathbb{R}$, et si $f : A \to E$ est p.p. (au sens défini par Bochner et von Neumann, c.f. [NE]) on ne peut plus définir une moyenne comme ci-dessus. Afin de dégager une bonne notion de moyenne, von Neumann en 1934 a montré que l'enveloppe convexe (uniformément) fermée des translatées de f contient une et une seule fonction constante : c'est celle-ci qui est considéré comme la moyenne de f. Les techniques utilisées par von Neumann relèvent de l'Analyse Elémentaire. Loomis a redémontré ce résultat par des techniques d'Analyse Harmonique dans [LO]. Nous suivons ici la démarche de Smart, dans [SM] ch. 8, qui consiste à faire apparaître une moyenne généralisée comme point fixe commun d'une famille d'opérateurs sur un convexe compact. Smart utilise le théorème de point fixe commun de Kakutani de 1938, c.f. [SM] p. 58, (il pourrait utiliser le théorème de Markov-Kakutani) qui assure l'existence (mais pas l'unicité) de la fonction constante. Nous montrerons comment, dans cette optique de Point Fixe, obtenir l'unicité.

A) Fonctions p.p. d'une variable réelle

Soit $f : \mathbb{R} \to E$ p.p.. Un téorème fondamental de Besicovitch de 1932, c.f. [BE] p. 44, (rarement repris dans les traités récents sur la presque-périodicité) établit que, pour chaque $T \neq 0$, il existe une fonction continue T-périodique f^T t.q. $\| \frac{1}{\gamma} \sum_{k=0}^{\gamma-1} \tau_{kT}(f) - f^T \|_\infty \to 0$ $(\gamma \to \infty)$, où $\tau_r f(t) := f(t + r)$.

[1] Faculté des Sciences de limoges, Département de Mathématiques, 123 av. Albert Thomas, 87060 Limoges Cedex (France).

Signalons que les théorèmes de Smart ([SM] p. 62) et de Besicovitch ont été utilisé pour l'étude des équations d'Euler-Lagrange à lagrangiens convexes dans [BL_1], [BL_2]. On introduit $K := \overline{co}\,\{\tau r\, f / r \in \mathbb{R}\}$, $Fix(T\mathbb{Z}) = \{x \in K / \forall k \in \mathbb{Z}, \tau_{kT}(x) = x\}$ et $Fix(\mathbb{R}) := \{x \in K / \forall r \in \mathbb{R}, \tau_r(x) = x\}$, c'est l'ensemble des fonctions constantes de K.

Theorème 1. i) $Fix(\mathbb{R}) = \{\mathcal{M}\{f\}\}$

ii) $\forall t > 0 \quad Fix(T\mathbb{Z}) = \overline{co}\,\{\tau r\, f^T / r \in \mathbb{R}\}$

On fournit une démonstration de ce théorème basée sur le théorème de Markov-Kakutani, sur le théorème de la projection orthogonale et sur le théorème de Besicovitch.

B) Theoreme abstrait

$(X, |\cdot|)$ est un espace de Banach équipé, c-à-d. sur X est donné un produit scalaire $<\cdot|\cdot>$, $\|x\| := <x|x>^{1/2}$, t.q. existe $\beta > 0 : \|\cdot\| \le \beta|\cdot|$.

Theoreme 2. Soit G un semi-groupe d'opérateurs linéaires continus bijectifs sur X, qui commutent 2 à 2, t.q. tout $g \in G$ est une $\|\cdot\|$-isométrie, et $|g|_{\mathcal{L}} \le 1$. Soit $z \in X$ t.q. $\{g(z) / g \in G\}$ est relativement compact dans $(X, |\cdot|)$; $K := \overline{co}\,\{g(z) / g \in G\}$, $Fix(g) := \{x \in K / g(x) = x\}$ $Fix(G) := \bigcap_{g \in G} Fix(g)$. Alors :

i) card Fix(G) = 1

ii) $\forall g \in G \quad Fix(g) = \overline{co}\,\{h(z^g) / h \in G\}$, où z^g est la limite dans $(X, |\cdot|)$ de la suite $\left(\frac{1}{\gamma}\sum_{k=0}^{\gamma-1} g^k(z)\right)_{\gamma\ge 1}$.

La démonstration de ce théorème s'appuie sur le théorème de Markov-Kakutani et sur le théorème ergodique de von Neumann. ($|g|_{\mathcal{L}} := \sup\{|g(x)| / |x| \le 1\}$).

C) Fonctions p.p. sur un groupe

Soit A un groupe topologique, $f : A \to E$ p.p. On note $\tau_\alpha f(s) := f(\alpha s)$ pour $\alpha, s \in A$. On peut généraliser le théorème de Besicovitch :

Théorème 3. *Pour tout $\alpha \in A$, il existe une fonction continue $f^\alpha : A \to E$, α-périodique* (c-à-d. $\tau_\alpha f^\alpha = f^\alpha$) t.q. $\|f^\alpha - \sum_{k=0}^{\gamma-1} \tau_\alpha^k(f)\|_\infty \to 0 \ (\gamma \to \infty)$

Une conséquence directe du Théorème 2, en utilisant la compactification de Bohr (notion de Anzai, Kakutani et Weil) et l'intégrale de Haar, est

Théorème 4. On suppose A abélien ; $k := \overline{co}\,\{\tau_\alpha f / \alpha \in A\}$, $Fix(\alpha) := \{x \in K / \tau_\alpha x = x\}$, $Fix(A) := \bigcap_{\alpha \in A} Fix(A)$. Alors :

i) card Fix(A) = 1

ii) $\forall A \in G \quad Fix(\alpha) = \overline{co}\,\{\tau_\gamma f^\alpha / \gamma \in A\}$.

L'unique élément de Fix(A) est la moyenne généralisée de f.

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FIXED-POINT THEOREMS AND MORSE'S LEMMA FOR LIPSCHITZIAN FUNCTIONS

JEAN-MARC BONNISSEAU AND BERNARD CORNET

Presentation

In this paper, we prove a fixed-point theorem for set-valued mappings S defined on a compact subset X of a finite dimensional Euclidean space E. We consider the class $\mathcal{C}$ of compact subsets X of E which can be defined by inequality constraints, that is, $X = \{x \in E \mid f(x) \leq 0\}$ where f is a locally Lipschitzian function which verifies the following assumptions:

Assumption 1 (i) $0 \notin \partial f(x)$, *for all* x *such that* $f(x) \geq 0$; *(ii) for all* $t \in R$, *the set* $M_t = \{x \in E \mid f(x) \leq t\}$ *is bounded.*

In the above assumption, ∂f denotes the generalized gradient in the sense of Clarke. We recall that a locally Lipschitzian function is almost everywhere differentiable. If Ω_f is the set on which f is differentiable, $\partial f(x)$ is the closure of the convex hull of the set:

$$\{\lim_\nu \nabla f(x_\nu) \mid \text{for all } (x_\nu) \in \Omega_f,\ (x_\nu) \to x\}.$$

This class of set extands significantly the class of convex, compact subsets of E with a nonempty interior. Indeed let C such a set and let c a point in the interior of C. Let γ be the gauge function of C with respect to c which is defined by:

$$\gamma(x) = \inf\{\lambda > 0 \mid x - c \in \lambda(C - \{c\})\}.$$

One easily checks that the mapping $f = \gamma - 1$ is localy Lipschitzian and verifies Assumption 1 and that

$$C = \{x \in E \mid f(x) \leq 0\}.$$

One can found other examples of nonconvex sets in the class $\mathcal{C}$ in Bonnisseau-Cornet (1988) which studies the existence of marginal cost pricing equilibrium in economies with nonconvex production sets.

We now state our fixed point theorem.

THEOREM 1. *Let X be a set in the class C and let S be an upperhemicontinuous set-valued mapping from X to E, such that, for all $x \in X$, $S(x)$ is a nonempty, convex, closed, subset of E and one of the two following properties is satisfied:*

(I) for all $x \in \partial X,\ S(x) \cap \{x\} + \partial f(x)^o \neq \emptyset$,
(II) for all $x \in \partial X,\ S(x) \cap \{x\} - \partial f(x)^o \neq \emptyset$.

Then, there exists $x^* \in X$ such that $x^* \in S(x^*)$.

In conditions (I) and (II), $\partial f(x)^o$ denotes the negative polar cone ot $\partial f(x)$. This condition are respectively called "inward" and "outward". Indeed, when X is convex, one can write this conditions as follows:

(I) for all $x \in \partial X,\ S(x) \cap cl\{x + \lambda(y - x) \mid \lambda > 0,\ y \in X\} \neq \emptyset$,
(II) for all $x \in \partial X,\ S(x) \cap cl\{x - \lambda(y - x) \mid \lambda > 0,\ y \in X\} \neq \emptyset$.

In this case, the conditions (I) and (II) clealy means that the set-valued mapping S points inward or outward at x. When X is not convex, one has the following inclusion:

$\partial f(x)^o \subset T_X(x)$, for all $x \in \partial X$,

where $T_X(x)$ denotes Clarke's tangent cone to X at x. When f is tangentially regular, the equality holds. Under this latter assumption, the "inward" and "outward" conditions have a natural geometric interpretation.

We now give an equivalent reformulation of this theorem in terms of the existence of critical points of a set valued mapping.

THEOREM 2. *Let X be a set in the class C and let S be an upperhemicontinuous set-valued mapping from X to E, such that, for all $x \in X$, $S(x)$ is a nonempty, convex, closed, subset of E and*

(III) for all $x \in \partial X,\ S(x) \cap \partial f(x)^o \neq \emptyset$.

Then, there exists $x^* \in X$ such that $0 \in S(x^*)$.

One shows the equivalence between this two theorems by considering the set-valued mappings $S - Id$, $Id - S$ and $S + Id$ where Id is defined by $Id(x) = \{x\}$.

We now state a fixed point theorem which is not comparable with Theorem 1 whereas in the convex case, it is a direct consequence.

THEOREM 3. *Let X be a set in the class $\mathcal{C}$ and let S be an upperhemicontinuous set-valued mapping from X to E, such that, for all $x \in X$, $S(x)$ is a nonempty, convex, closed, subset of E and*

(IV) for all $x \in X$, $S(x) \subset X$.

Then, there exists $x^* \in X$ such that $x^* \in S(x^*)$.

The proof of this theorems will be a consequence of Kakutani's Theorem and of a topological property of sets X in the class $\mathcal{C}$. Precisely, every set X in this class is a continuous deformation retracts of a ball. That is, there exists a ball B included in E and a mapping r from B to X which verify: (i) $X \subset B$ and (ii) r is continuous, $r(x) = x$ for all $x \in X$, and $f(r(x)) = 0$ for all $x \notin X$.

This topological property will be proved as a consequence of the following theorem which provides a nonsmooth generalization of the Morse's Lemma.

THEOREM 4. *Let f be a locally Lipschitzian function from E to R, and let a and b two real numbers such that $a < b$. If the set $M_{ab} = \{x \in E \mid a \leq f(x) \leq b\}$ is nonempty and compact, if there exists an upperhemicontinuous set-valued mapping δ from E to E such that, for all $x \in E$, $\delta(x)$ is nonempty, closed, convex and, if for all $x \in M_{ab}$, $\partial f(x) \subset \delta(x)$ and 0 does not belong to $\delta(x)$, then*

a) there exists a neighborhood M of $M_b = \{x \in E \mid f(x) \leq b\}$ and a continuous mapping r from M to $M_a = \{x \in E \mid f(x) \leq a\}$ such that

(i) $r(x) = x$, for all $x \in M_a$;

(ii) $f(r(x)) = a$, for all $x \in M \setminus M_a$;

b)there exists $\varepsilon \in (0, b-a)$ such that for all $(x,y) \in f^{-1}((a, a+\varepsilon]) \times f^{-1}([a, a+\varepsilon])$, with $r(x) = r(y)$, then

$0 < (x - r(x)) \cdot \delta$, for all $\delta \in \delta(y)$.

An important example of a set-valued mapping δ verifying the above assumption of the theorem is clearly given by the generalized gradient $\partial f(.)$. The introduction of the set-valued mapping δ is not only done for a matter of generality, and will be of fundamental use in the proof of Theorem 1. Roughly speaking, the use of δ is due to the fact that, in general, one can not exactly compute the generalized gradient.

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MINIMAL CUSCOS AND SUBGRADIENTS OF LIPSCHITZ FUNCTIONS

J.M. BORWEIN

ABSTRACT. We study the structure of minimal cuscs [upper semi-continuous mappings with nonempty convex compact images]. Particular attention is paid to the case when the cusco is the generalized gradient of a locally Lipschitz function.

KEY WORDS: uscos, convex uscos, upper-semicontinuity, minimality, Lipschitz functions, generalized gradients, generic differentiability, representability, integrability, monotonicity.

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INTRODUCTION: The purpose of this talk is two-fold. One goal is to describe some of the applications of minimal **cuscos** [upper semicontinuous set-valued mappings with nonempty convex compact images]. Particular attention is paid to the case when the cusco is the generalized gradient of a well-behaved locally Lipschitz function on a smooth Banach space. This allows us to exploit David Preisses beautiful recent result on the dense differentiability of Lipschitz functions, which is our second goal.

In absolute generality, the Clarke gradient of a Lipschitz function f can be a somewhat unwieldy beast. By comparison, those of most functions arising in applications (such as smooth functions, convex functions, saddle functions, and reasonable compositions thereof) are considerably better behaved. Much of this behaviour is usefully explained by considering conditions for and consequences of minimality of the underlying multifunction ∂f. This is the primary aim of our lecture.

In a first part some basic properties of minimal cuscos and uscos are set out. In a second part we characterize the minimality of various classes of cuscos. Finally we identify classes of minimal cuscos arising either from operators of monotone-type, or as subgradients of Lipschitz functions. In a last part we observe the consequences for differentiability of a Lipschitz function and for representability and integrability of generalized gradients. In section 5 we obtain some refined results for distance functions (including proximal normal formulae) when the underlying norm is uniformly Gateaux differentiable. Finally, we gather up various limiting examples along with a few special results for Lipschitz functions on the real line.

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PRACTICAL CONDITIONS FOR FENCHEL DUALITY IN INFINITE DIMENSIONS

J. M. BORWEIN AND A. S. LEWIS

Frequently in infinite-dimensional convex optimization problems the easiest condition for duality, the Slater constraint qualification, fails because the underlying constraint set has empty interior. When the additional constraints consist just of a finite number of linear inequalities this condition can be weakened considerably to a "quasi-interior" constraint qualification, giving a duality result which is easy to use and widely applicable.

Let X be a locally convex topological vector space, with dual X*, let $A : X \to \mathbb{R}^n$ be continuous and linear, with adjoint $A^* : \mathbb{R}^n \to X^*$, and suppose $f : X \to (-\infty, +\infty]$ and $g : \mathbb{R}^n \to (-\infty, +\infty]$ are lower semicontinuous and convex. Denote $\mathrm{dom} f := \{x \mid f(x) < +\infty\}$, and assume *domf and domg are nonempty*. As usual, $f^* : X^* \to (-\infty, +\infty]$ denotes the conjugate function :

$$f^*(a) = \sup\{a(x) - f(x) \mid x \in X\}.$$

The relative interior of a convex set C is denoted riC. We consider the Fenchel dual pair (with values V(P) and V(P*), respectively)

(P) $\qquad \inf\{f(x) + g(Ax) \mid x \in X\}$,

(P*) $\qquad \max\{-f^*(A^*\lambda) - g^*(-\lambda) \mid \lambda \in \mathbb{R}^n\}$.

Theorem 1 : [4] Suppose

(2) $\qquad 0 \in \mathrm{ri}(A\,(\mathrm{dom} f) - \mathrm{dom} g)$.

Then V(P) = V(P*).

Notice that since $g = g^{**}$ we could write this in the minimax form,

$$\inf_{x\in X} \sup_{\lambda\in \mathbb{R}^n} K(x\,;\lambda) = \max_{\lambda\in \mathbb{R}^n} \inf_{x\in X} K(x\,;\lambda)$$

where $K(x\,;\lambda) := f(x) - g^*(-\lambda) - \langle Ax, \lambda \rangle$.

Rockafellar [4] observes that (2) is equivalent to $0 \in$ ri(A(domf)) - ri (domg), and if X is finite-dimensional then

(3) $$\text{ri}(A(\text{dom}f)) = A(\text{ri}(\text{dom}g)),$$

so (2) becomes

(sc) there exists $x_0 \in$ ri(domf) with $Ax_0 \in$ ri(domg), a condition which he calls 'strong consistency'. He states : 'strong consistency is unlikely to be of any help in infinite-dimensional spaces, because relative interiors are so badly behaved there ; formulas [such as (3)] fail almost completely, even in Hilbert spaces.'

In fact it is possible to circumvent this difficulty relatively easily by defining the following generalization of the relative interior.

Definition 4 [1]. For convex $C \subset X$, the **quasi relative interior** of C (qri C) is the set of those $x \in C$ for which cl cone $(C - x)$ is a subspace.

Example 5 [1]. Suppose $X = L^p(T, \mu)$ with (T, μ) a σ-finite measure space and $1 \leq p < \infty$. Then

$$\text{qri}\,\{x \mid x(t) \geq 0 \text{ a.e.}\} = \{x \mid x(t) > 0 \text{ a.e.}\}$$

In infinite dimensions the quasi relative interior shares many of the important properties of the relative interior in infinite dimensions. For example, if X is a separable Banach space and C is closed then qri $C \neq \emptyset$. Furthermore, for any X and C, A(qri C) $\subset$ ri(AC) (with equality if qriC $\neq \emptyset$), which leads to the following generalization of strong consistency :

(QSC) there exists $x_0 \in$ qri(domf) with $Ax_0 \in$ ri(domg).

Theorem 6 [1]. If (QSC) holds then $V(P) = V(P^*)$.

This duality result has a wide range of applications (see [1]). To give just one example, consider the following problem which avises in spectral estimation and constrained approximation (see [2] and [3]). let (T, μ) be a σ-finite measure space, $a_i \in L^2(T, \mu)$, $i = 1,\ldots,n$ and let $\langle\cdot, \cdot\rangle$ denote the inner product.

$$\text{(EP)} \quad \begin{cases} \inf & \frac{1}{2} \| x \|_2^2 \\ \text{subject to} & \langle a_i, x \rangle = b_i, \quad i = 1,\ldots,n, \\ & 0 \leq x \in L^2(T, \mu). \end{cases}$$

Applying Theorem 6 and Example 5, the dual problem is

$$\text{(EP*)} \quad \begin{cases} \text{maximize} & b^T\lambda - \frac{1}{2} \| (\sum_{i=1}^{n} \lambda_i a_i)^+ \|_2^2 \\ \text{subject to} & \lambda \in \mathbb{R}^n, \end{cases}$$

where $(\cdot)^+$ denotes the positive part.

Corollary 7 : If there exists $x_0(t) > 0$ a.e., feasible for (EP), then V(EP) = V(EP*).

Thus the duality result holds despite the failure of the standard Slater condition. The dual problem (EP*) is relatively easy to solve numerically (for example, by Newton's method), and if $\bar{\lambda}$ is optimal for (EP*) then the optimal solution of (EP)- is $(\sum_{i=1}^{n} \bar{\lambda}_i a_i)^+$.

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NOUVEAUX RESULTATS DE MINIMAX MOTIVES PAR DES PROBLEMES DE CRISTAUX LIQUIDES

BREZIS

STRUCTURE OF THE APPROXIMATE FIXED-POINT SETS OF NONEXPANSIVE MAPPINGS IN GENERAL BANACH SPACES

RONALD E. BRUCK

Abstract : Let C be a nonempty bounded closed convex subset of a Banach space E, and $T : C \to C$ a nonexpansive mapping. We prove that the contraction semigroup generated by $A = I - T$ satisfies $\|AS(t)x\| = 0(1/\ln t)$ as $t \to \infty$, and use this to prove that for each $\varepsilon > 0$ the ε-approximate fixed-point set $F_\varepsilon(t)$ of points $x \in C$ satisfying $\|x - Tx\| \leq \varepsilon$ is contractible to a point.

Throughout this paper, C denotes a bounded closed convex nonempty subset of a Banach space E, and $T : C \to C$ is a nonexpansive mapping, that is, a mapping with Lipschitz constant 1. We recall some well-known results (cf. [2, Chapter III], for example) : $A = I - T$ generates a contraction *semigroup* $\{S(t) : t \geq 0\}$, *that if, for each* $x \in C$ there is a unique solution of

(1) $$\frac{du}{dt} = - Au(t) \quad \text{for all} \quad t \geq 0$$

satisfying $u(0) = x$; a nonlinear self-mapping S(t) of C is then defined by $S(t)x = u(t)$. S(t) is nonexpansive for each t, and

(2) $$\|AS(t)x\| = \|u'(t)\| \text{ is a non-increasing function of } t.$$

Since we are assuming C is bounded, it follows from [1] that $u'(t) \to 0$ as $t \to \infty$. No rate of convergence was given in [1] because the authors were unable to find one. We begin by supplying an explicit (albcit slow) convergence rate :

Theorem 1. : If C is bounded and u is a solution of (1), then

$$\|u'(t)\| \leq \frac{3 \text{ diam } C}{\ln t} \quad \text{for all } t \geq \varepsilon.$$

Proof : For $\varepsilon > 0$ put

$$u_\varepsilon(t) = x + \int_0^t \frac{S(s+\varepsilon)x - S(s)x}{\varepsilon}\, ds.$$

Thus $u_\varepsilon \in C^2(0, \infty; E)$ and $u_\varepsilon'(t) + u_\varepsilon''(t) = (TS(t+\varepsilon)x - TS(t)x)/\varepsilon$, hence

(3) $$\|u_\varepsilon' + u_\varepsilon''\| \le \|u_\varepsilon'\|.$$

Now let $\varphi : R^+ \to R$ be any C^2 function with $\varphi(0) = 0$. We readily verify the identity

$$\varphi(t)\, u_\varepsilon'(t) = \int_0^t \varphi(s)\, (u_\varepsilon'(s) + u_\varepsilon''(s))\, ds$$
$$+ \int_0^t (\varphi''(s)) - \varphi'(s)\, (u_\varepsilon(s) - u_\varepsilon(t))\, ds + \varphi'(0)\, (u_\varepsilon(0) - u_\varepsilon(t))$$

for any $t > 0$, so with (3) we obtain

(4) $$\varphi(t)\, \|u_\varepsilon'(t)\| \le \int_0^t \varphi(s)\, \|u_\varepsilon'(s)\|\, ds + \int_0^t |\varphi''(s) - \varphi'(s)|\, \|u_\varepsilon(s) - u_\varepsilon(t)\| ds$$
$$+ |\varphi'(0)|\, \|u_\varepsilon(t) - u_\varepsilon(0)\|.$$

As $\varepsilon \to 0_+$ we clearly have $u_\varepsilon(t) \to u(t) = S(t)x$ and $u_\varepsilon'(t) \to u'(t)$ uniformly on bounded sets of t, hence

(5) $$\varphi(t)\, \|u_\varepsilon'(t)\| \le \int_0^t \varphi(s)\, \|u_\varepsilon'(s)\|\, ds + \int_0^t |\varphi''(s) - \varphi'(s)|\, \|u(s) - u(t)\| ds$$
$$+ |\varphi'(0)|\, \|u(t) - u(0)\|.$$

Finally, we estimate $\|u(s) - u(t)\|$ and $\|u(t) - u(0)\|$ by diam C, which we abbreviate to δ, and use (2) to find

(6) $$\varphi(t)\, \|u_\varepsilon'(t)\| \le \int_0^t \varphi(s)\, \|u'(s)\|\, ds + \delta \int_0^t |\varphi'' - \varphi'| + \delta\, |\varphi'(0)|$$
$$\le \left(\int_0^t \varphi\right) \|u'(0)\| + \delta \int_0^t |\varphi'' - \varphi'| + \delta\, |\varphi'(0)|.$$

Note that the technical question of whether u"(t) exists was avoided by our use of (3). In a reflexive space u"(t) exists for a.e. t because u'(t) = - AS(t)x is Lipschitzian, but the theorem is stated for arbitrary Banach spaces.

We now specialize (5) by taking $\varphi(t) = e^t - 1$ to obtain

$$\|u'(t)\| \leq (1 - \lambda(t))\ \|u'(0)\| + \lambda(t)\frac{\delta}{t}$$

where $\lambda(t) = t/(e^t - 1)$ (so $0 < \lambda(t) < 1$). Replacing x by S(s)x therefore results in the estimate

$$\|u'(s + t)\| \leq (1 - \lambda(t)\|\ u'(s)\| + \lambda(t)\frac{\delta}{t}.$$

By induction on this convex combination we find

$$\|u'(mt)\| \leq (1 - \lambda(t))^m\ \|u'(0)\| + (1 - (1 - \lambda(t))^m)\frac{\delta}{t}$$
$$\leq (1 - \lambda(t))^m\ \|u'(0)\| + \frac{\delta}{t}$$

or all positive integers m. Since $\|u'(0)\| = \|x - Tx\| \leq \delta$ and $1 - t < \exp(- t)$ for $t > 0$, we end up with

$$\|u'(mt)\| \leq \delta.\ (\exp(- m\lambda(t)) + \frac{1}{t})$$
$$\leq \delta.\ (\exp(- mte^{-1-t}) + \frac{1}{t})$$

Now for $s \geq e$ put $t = 1/2 \ln s$, $m = [s/t]$ (where $[\cdot]$ denotes the greatest-integer, or floor, function). Evidently $s - t < mt$ and $e^{-t} = 1/\sqrt{s}$, so a very conservative estimate for large s is

$$\exp(- mte^{-t}) \leq \exp(- (s - t) / \sqrt{s}) \leq \frac{1}{\ln s}.$$

In fact, computer graphics make it clear this is valid for $s \geq e$, although we have not actually carried out the details which would be necessary for a formal proof.

By (2), therefore,

$$\|u'(s)\| \leq \|u'(mt)\| \leq \delta(\frac{1}{\ln s} + \frac{1}{t}) = \frac{3 \text{ diam } C}{\ln s}.$$

The usual feature of the proof of Theorem 1 is that rather than regarding t as an infinitesimal in (3) (and estimating $\|u(t) - u(s)\|$ correspondingly), we take t large and estimate $\|u(t) - u(s)\|$ by the diameter of C. Incidentally, we do not know whether the estimate $0(1/\ln s)$ is optimal ; we presented inequality (4) for fully general φ in the hope that some better choice of φ than $e^t - 1$ will be found. In Hilbert space a better estimate is $0(1/\sqrt{s})$, as can be seen by integrating the easy estimate

$$\|u'(s)\|^2 \leq 2\,(-u'(s),\, u(s) - f) = -\frac{d}{ds}\|u(s) - f\|^2$$

for a fixed-point f of T.

Henceforth we shall denote the fixed-point set of T by $F(T) = \{x \in C : Tx = x\}$, and for $\varepsilon > 0$, the ε-approximate fixed point set by

$$F_\varepsilon(T) = \{x \in C : \|x - Tx\| \leq \varepsilon\}.$$

It follows from (2) that $F_\varepsilon(T)$ is invariant under S(t). Our main application of Theorem 1 is :

Corollary 1 : If C is bounded then for any $\varepsilon > 0$ there exists $\tau > 0$ (depending only on ε and diam C) such that $S(t)C \subset F_\varepsilon(T)$ for all $t \geq \tau$.

The main result of this paper can now be stated :

Theorem 2 : $F_\varepsilon(T)$ is contractible ; indeed, there exists a homotopy $H : [0, 1] \times F_\varepsilon(T) \to F_\varepsilon(T)$ connecting the identity on $F_\varepsilon(T)$ to a constant map on $F_\varepsilon(T)$, such that $H(t, \cdot)$ is nonexpansive for all t.

Proof : Fix a point p of $F_\varepsilon(T)$. Using Corollary 1, find $\tau > 0$ so that S(t) maps C into $F_\varepsilon(T)$ for $t \geq \tau$. Define a piecewise linear $\phi : [0, 1] \to [0, 1]$ to be 0 on [0, 1/2], and to increase for 1 at t = 1. We define the homotopy H by

$$H(t, x) = S(2t\tau)\,((1 - \phi(t))x + \phi(t)p).$$

Clearly $H(0, x) = x$, $H(1, x) =$ constant ; for $0 < t \leq 1/2$, $H(t, x) \in F_\varepsilon(T)$ since $x \in F_\varepsilon(T)$ and $F_\varepsilon(T)$ is invariant under $S(2t\tau)$; while for $t > 1/2$, $H(t, x) \in F_\varepsilon(T)$ by the choice of τ from Corollary 1. The nonexpansiveness of $H(t, \cdot)$ is clear. □

We shall give an alternative proof of Theorem 2 which does not rely on Theorem 1. For $0 < \lambda < 1$ we put $T_\lambda = \lambda T + (1 - \lambda)I$ (T_λ is called an averaged mapping). It was proved by Ishikawa [4] and by Edelstein and O'Brien [3] that averaged mappings are asymptotically regular, that is, that $\|T_\lambda^n x - T_\lambda^{n+1} x\| \to 0$ as $n \to \infty$; [3] proved, moreover, that averaged mappings are uniformly asymptotically regular, that is, the limit is approached uniformly for $x \in C$. This is the analogue of Theorem 1 which we shall need.

Since $\|x - T_\lambda x\| = \lambda\|x - Tx\|$, to prove Theorem 2 it evidently suffices to assume that T itself is an averaged mapping, hence is uniformly asymptotically regular. An equivalent way of formulating uniform asymptotic regularity is then to say that for each $\varepsilon > 0$ there exists an integer N such that T^N maps C into $F_\varepsilon(T)$.

We need one other well-known fact about nonexpansive mappings : the function $\varphi(x) = \|x - Tx\|$ satisfies $\varphi(y) \leq \varphi(x)$ for all y in the line segment [x, Tx] connecting x to Tx. This is a simple calculation :

$$\|y - Ty\| \leq \|y - Tx\| + \|Tx - Ty\| \leq \|y - Tx\| + \|x - y\| = \|x - Tx\|,$$

the last equality holding because y is on the segment [x, Tx].

We now homotope the identity map of $F_\varepsilon(T)$ to T by putting

$$H(t, x) = (1 - t)x + tT(x) ;$$

our observation about φ shows that $H(t, x) \in F_\varepsilon(T)$. We can repeat this process on the segments $[Tx, T^2x], \ldots, [T^{N-1}x, T^nx]$ to ultimately connect the identity to T^n. If we follow this with a homotopy of the form $T^N((1 - t)x + tp)$ for a fixed $p \in F_\varepsilon(T)$ (this remains inside $F_\varepsilon(T)$ by the choice of N), we have connected the identity map to a constant.

It is an attractive hypothesis that $F_\varepsilon(T)$ is a retract of C, but we do not known whether this is true even in Hilbert space. We finish with an example which shows that $F_\varepsilon(T)$ may fail to be homeomorphic to C, and that one candidate for a retraction put $R(x) = S(t_0)x$ for the first t_0 such that $S(t_0)x \in F_\varepsilon(T)$ is not even continuous. Let C be the rectangle $[0, 2] \times [-1, 1]$ in the Euclidean space R^2, and define

$$T(x, y) = (x - \min(x, 1), 0).$$

It is easily seen that T is nonexpansive and that the set $F_1(T)$ consists of the closed unit disk intersected with the right half-plane, unioned with a "filament" $\{(x, 0) : 1 \leq x \leq 2\}$. This set is a retract of C (move points in the upper half-plane diagonally leftward and downward until they intersect $F_1(T)$, points in the lower half-plane diagonally upward and leftward ; the retraction is even Lipschitzian). On the other hand, following the semigroup trajectories until they enter $F_1(T)$ doesn't work, because points in the open upper half-plane move to points in the disk, no matter how close they are to the filament.

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THE UPPER SEMICONTINUITY OF THE NEMISKI OPERATOR

A. CELLINA

UN THEOREME DE POINT FIXE DANS LES CONES DES ESPACES DE BANACH, ET APPLICATIONS AUX SOLUTIONS POSITIVES D'EQUATIONS ELLIPTIQUES SEMI-LINEAIRES

ALICE CHALJUB-SIMON

1. Des théorèmes de points fixes dans les cônes des espaces de Banach ont été donnés essentiellement par M. A. Krasnoselskii. On se propose d'établir ici un théorème avec une hypothèse moins forte ; la démonstration est indépendante de la notion de degré topologique ; elle utilise la notion d'application essentielle, due à Granas. Elle est très courte. Ce théorème est dû à P. Volkman. L'application A) figure dans [1]. Soit E un espace de Banach réel, on appellera cône, un fermé, convexe, non vide, tel que : $\lambda C \subseteq C$ pour $\lambda \geq 0$.
La propriété : $C \cap - C = \{0\}$ n'étant pas demandée, le cas : $C = E$ est inclus. On notera :
$B_\rho = [X/ X \in C ; \|x\| < \rho]$, $S_\rho = (X/x \in C ; \|x\| = p)$; $\bar{B}_\rho = B_\rho \cup S_\rho (p > 0)$.

Definition 1 : Soient f, g : $S_\rho \to C$ compactes, sans point fixe ; on pose :

$$f \sim g \text{ s'il existe } h : [0, 1] \times S_\rho \to C \text{ telle que :}$$

$h(0,.) = f$, $h(1,.) = g$ et $h(\lambda,.)$ est sans point fixe.

Definition 2 : Soit f : $S_\rho \to C$ compacte, sans *point fixe*. *On dit* que f est inessentielle, si f admet un prolongement compact F : $B_\rho \to C$ sans point fixe. Sinon f est essentielle.

Exemple : Soit $f(x) = a$ sur S_ρ, alors f est inessentielle, si $a \notin B_\rho$; f est essentielle si $a \in B_\rho$.
On utilise le résultat suivant (Dugundji et Granas) : si $f \sim g$, alors f et g sont simultanément essentielles ou inessentielles.

Théorème d'excision : $0 < r < R$; $U = B_R \setminus \bar{B}_r$

Soit $\phi : \bar{B}_R \to C$ une application compacte, telle que :
(1) ϕ n'a pas de point fixe sur $S_r \cup S_R$
(2) ϕ/S_R *inessentielle* ; ϕ/S_r essentielle

Alors ϕ possède au moins deux points fixes : $x_1 \in B_r$, $x_2 \in U$.
La démonstration utilise le lemme suivant :

Lemme : Soient $\varphi : \bar{U} \to C$ et $\psi : \bar{B}_R \to C$ deux applications compactes, sans point fixe, coïncidant sur S_R. Alors il existe un prolongement compact de φ à $\bar{B}_R$ sans point fixe. Le prolongement se construit à la main [1].

Démonstration du théorème d'excision : $\phi_{/S_r}$ étant essentielle, il existe $x_1 \in B_r$, tel que : $\phi(x_1) = x_1$. Supposons que : $\varphi = \phi_{/\bar{U}}$ $(\bar{U} \to C)$ n'a pas de point fixe, alors φ n'a pas de point fixe dans U. D'autre part, $\phi_{/S_R}$ étant inessentielle, il existe un prolongement compact ψ : $(\bar{B}_R \to C)$ de $\phi_{/S_R}$ sans point fixe. On aurait :

$$\psi(x) = \phi(x) = \varphi(x),\ x \in S_R,\ \text{et}\ \psi(x) \neq x\ \text{dans}\ \bar{B}_R.$$

D'après le lemme, il existerait un prolongement compact de φ en $\tilde{\varphi} : \bar{B}_R \to C$ sans point fixe. En particulier, $\tilde{\varphi}_{/S_R}$ n'a pas de point fixe ; mais :

$\tilde{\varphi}(x) = \varphi(x) = \phi_{/\bar{U}}(x) = \phi(x)$ sur S_r. Donc $\tilde{\varphi}_{/\bar{B}_r}$ est un prolongement compact de $\phi_{/S_r}$ sans point fixe ; ceci contredit : $\phi_{/S_r}$ est essentielle.

Condition suffisante des hypothèses (1) et (2) du théorème d'excision :
Les hypothèses (1) et (2) sont vérifiées si :
(a) $x \neq t\,\phi(x)$, $t \in [0, 1]$, $x \in S_r$
(b) $x \neq \phi(x) + tv$, $t \geq 0$, $x \in S_R$, où $v \in C \setminus \{0\}$
On retrouve les hypothèses de Figuereido- P.L. Lions-Nussbaum [2].

2. Applications

Il se trouve que le théorème d'excision précédent est un outil bien adapté pour montrer l'existence de solutions positives pour certaines équations semi-linéaires elliptiques sur $\mathbb{R}^n$.
A) Soit le problème :

$$(1)\quad \begin{cases} \Delta u - c^2 u + g(x, u) = 0 \quad \text{dans } \mathbb{R}^n \\ u > 0,\ u \to 0 \ \text{à l'}\infty \end{cases}$$

Ici : $g(x, 0) = 0$, et : $t \to g(x, t)$ est surlinéaire.
Le problème correspondant à (1) dans un ouvert borné de $\mathbb{R}^n$ avec une non linéarité forte a donné lieu à un très grand nombre de travaux. Dans le cas non borné, la situation est plus

difficile, à cause du manque de compacité ; les résultats connus sont peu nombreux. Dans [1], on a introduit des espaces de fonctions à poids exponentiel, qui permettent d'obtenir une certaine compacité.

Définition 3 : C^k_δ est l'espace de Banach des fonctions de classe C^k, telles que :

$$\omega_\delta \,|\, D^\alpha u(x) \,|\, \leq C, \; |\alpha| \leq k, \text{ avec : } \omega_\delta(x) = \exp\{\delta\, \sigma(x)\}, \; \sigma(x) = (1 + |x|^2)^{1/2} \quad \delta \geq 0$$

Soit $L = \Delta - c^2$, l'équation s'écrit : $- Lu = g(x, u)$

On montre que L a pour inverse un opérateur intégral $G : C^0_\delta \to C^1_\delta$, continu pour $\delta < c$, et que l'injection $C^1_\delta \to C^0_\rho$ est compacte, si $\delta > \rho$ (l'injection $C^1_\delta \to C^0_\delta$ n'est pas compacte).

L'équation s'écrit alors : $u = \phi(u)$, avec : $\phi(u) = G[\tilde{g}(u)]$, $\tilde{g}(u)\,(x) = g(x, u(x))$.

On suppose que : $\lim_{t \to \infty} g(t, x)t^{-k} = h(x)$, $1 < k < (n + 2) / (n - 2)$, $h(x) > 0$, $\lim_{|x| \to \infty} h(x) = 0$. En outre, on fait des hypothèses de régularité sur g(cf. [1]). Dans ce cas, où $\phi(0) = 0$, il faut écarter la solution triviale : $u = 0$, et pour cela le théorème d'excision est l'outil idéal. Soit C le cône des fonctions positives de C^0_δ, on montre que $\phi : C \to C$ est compact (à cause de la sur linéarité de g). On peut alors montrer que les conditions (a) et (b) données après le théorème sont vérifiées : (a) est vérifiée, si $\tilde{g}(t) = 0\,(t^\alpha)$, avec $\alpha > 1$, au voisinage de 0. (b) est vérifiée s'il existe une majoration à priori pour toutes les solutions du problème (1).

Alors, on déduit l'existence de deux points fixes, l'un dans B_r (on a déjà 0), l'autre dans U qui donne une solution régulière, non triviale du problème.

B) dans A), l'utilité du théorème d'excision est d'obtenir surement une solution non triviale ; on peut utiliser ce théorème dans des cas où : $\phi(0) \neq 0$ pour obtenir deux solutions. Donnons un exemple simple : un problème de pertubation.

(2)
$$- Lu = g(x, u) + \varepsilon\, f(x)$$
$$u > 0, \; u \to 0 \text{ à l'}\infty$$

On se place dans les espaces à poids définis précédemment ; en particulier f a un poids suffisamment grand, on prend aussi : $f > 0$. Les hypothèses de régularité sur g, et les hypothèses de croissance de $t \to g(x, t)$ sont les mêmes que dans A)

L'équation (2) s'écrit :

(3)
$$u = \phi_1\,(u) \quad \text{avec : } \phi_1 = \phi + K, \; K = \varepsilon\, Gf$$

On peut alors montrer que pour ε assez petit, l'équation (2) a deux solutions dans C^0_δ. En effet ϕ_1 est compacte de C^0_δ dans lui-même, avec les hypothèses, et ϕ_1 est compact de C dans lui-même. D'autre part, la majoration à priori est valable pour (2), si elle est vraie pour (1) ; d'où la

condition suffisante (b). Pour vérifier (a), il faut montrer qu'il existe r > 0, tel que : $u \neq t\,\phi_1(u)$, $u \in S_2$, $t \in [0, 1]$. On peut choisir ε assez petit pour que cette condition soit vérifiée.
Ainsi on obtient l'existence de deux solutions du problème.
On peut également étudier l'existence de branches de solutions positives pour des problèmes associés.

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FIXED POINTS AND PARTIAL ORDERS

VINCENZO CONSERVA

Let $(E, \leq)$ be a partially ordered set which admits at least one maximal (respectively minimal) element $\bar{x}$, and let f be a self-mapping of E such that $x \leq fx$ (respectively $fx \leq x$) for all $x \in E$, then $f\bar{x} = \bar{x}$, i.o. f admits at least one fixed point.

If (E, d) is a metric space and $\varphi : E \to [0, \infty[$ is a real valued function on E, we define a partial order $\leq_{\varphi,d}$ on E by putting :

$$x \leq_{\varphi,d} y \quad \text{iff} \quad d(x, y) \leq \varphi(x) - \varphi(y).$$

The purpose of the present note is to show how certain fixed point theorems can be deduced from the above observation.

This kind of procedure is present, for example, in papers [1] - [2].

For $x \in E$, we set :

$S(x) = \{y \in E \mid x \leq_{\varphi,d} y\}$

$d(x) = \operatorname{diam} S(x) = \sup_{x \leq_{\varphi,d} y, z} d(y, z)$

Theorem 1 : Let (E, d) be a metric space. Suppose $f : E \to E$ and let $\varphi : E \to [0, \infty[$ be a function on E which is lower semicontinuous. Assume :

i) there exists an $x \in E$ such that

$$d(y, fy) \leq \varphi(y) - \varphi(fy) \quad \text{for all } y \in S(x)$$

and any Cauchy sequence in S(x) converges to a point in E.

Then f admits at least one fixed point which is a maximal element in $(E, \leq_{\varphi,d})$.

Proof : We define a sequence (x_n) as follows. Let's take $x_1 = x$. By iterative process, when $x_1, x_2, \ldots, x_n$ have been defined, we put :

$$\lambda_n = \inf \varphi(S(x_n))$$

and take $x_{n+1} \in S(x_n)$ such that :

$$\varphi(x_{n+1}) \leq \lambda_n + \frac{1}{n}.$$

We have :

1°) $x_n \leq_{\varphi,d} x_{n+1}$ for all $n \in N$

2°) $\lambda_n \leq \lambda_{n+1}$ for all $n \in N$

3°) $\lambda_n \leq \varphi(y) \leq \varphi(x_n) \leq \lambda_{n-1} + \frac{1}{n-1}$ for all $y \in S(x_n)$

4°) $0 \leq \varphi(x_m) - \varphi(x_n)$ for $m \leq n$.

By means of 4°), we have (x_n) is a Cauchy sequence. Let $\bar{x} \in E$ be such that $\lim_{n\to\infty} x_n = \bar{x}$.

for the lower semicontinuity of φ, we have :

$$d(x_n, \bar{x}) = \lim_{j\to\infty} d(x_n, x_j) \leq \varphi(x_n) - \lim_{j\to\infty} \varphi(x_j)$$
$$\leq \varphi(x_n) - \varphi(\bar{x})$$

i.e. $x_n \leq_{\varphi,d} \bar{x}$.

It follows (see i)) that $d(\bar{x}, f\bar{x}) \leq \varphi(\bar{x}) - \varphi(f\bar{x})$, i.e. $\bar{x} \leq_{\varphi,d} f\bar{x}$.

Now, using 3°), one can easily show that

$$\lim_{n\to\infty} d(x_n) = 0.$$

Then, if (y_n) is a sequence such that $x_n \leq_{\varphi,d} y_n$, we have $\lim_{n\to\infty} y_n = \bar{x}$.

Now, we are able to show that $\bar{x}$ is a fixed point for f and a maximal element in $(E, \leq_{\varphi,d})$.

Let $y \in E$ be such that $\bar{x} \leq_{\varphi,d} y$. We have $x_n \leq_{\varphi,d} y$ for all $n \in N$; it follows that $y = \bar{x}$.

As a consequence of $\bar{x} \leq_{\varphi,d} f\bar{x}$, we have $f\bar{x} = \bar{x}$.

This completes the proof.

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PROPRIÉTÉS DE LA CLIQUE INVARIANTE DANS LES GRAPHES

J. CONSTANTIN

Grâce à un graphe défini sur l'ensemble des applications préservant l'adjacence dans un graphe donné, on peut introduire une notion d'homotopie permettant de parler de graphes démontables infinis et obtenir la propriété de la clique fixe pour ces graphes.

Nous étendons également aux graphes une version discrète du théorème de Hoft-Lefschetz présentée par Baclawski-Björner pour les applications monotones d'un ordonné fini et étendue ensuite par Constantin et Fournier aux applications préservant la comparabilité. Nous introduisons enfin une notion de graphes escamotables qui généralise celle de graphes démontables finis ; ils sont, comme ces derniers, contractiles.

THE FIXED-POINT THEORY FOR INWARD AND OUTWARD CORRESPONDANCE : EXTENSION TO NONCONVE DOMAINS AND APPLICATIONS TO THE THEORY OF ECONOMIC EQUILIBRIUM

BERNARD CORNET

The fixed-point theory for inward and outward correspondences developed by Bergman, Browder, Halpern, Ky Fan... provides important generalizations of Kakutani's theorem which are of particular use in applications, and among others to the theory of economics equilibrium of Kenneth Arrow and Gérard Debreu. This fixed-point theory, however, relies crucially on the convexity of the domain X on which the correspondence is defined, an assumption which is too strong in view of various applications.

The aim of this lecture is to provide an extension of this fixed-point theory by weakening the convexity assumption on the domain of the correspondences. Our presentation will only consider the finite dimensional setting. Applications are given to new developments of the Arrow-Debreu theory in the presence of "nonconvexity" in production.

In the absence of convexity assumptions on the domain X of the correspondence, the inwardness and outwardness conditions are formalized with Clarke's tangent cone. Similarly the class of domain X we consider is defined in terms of Clarke's normal cone ; this class includes both compact convex subsets of R^n and compact submanifolds of R^n with boundaries (corners) and having a nonzero Euler characteristic. We provide also a definition of the Euler characteristic in a nonsmooth framework ; this allows us also to show the relationship between the Poincaré-Hopf theorem and the fixed-point theory for inward and ouward correspondences.

PROPRIETE DE PROJECTION ET PROPRIETE DU POINT FIXE

E. COROMINAS

L'exposé concerne la propriété du point fixe pour les applications croissantes dont le domaine est un produit d'ensembles ordonnés.

On introduit les ensembles projectifs : un ensemble ordonnée H est **projectif** lorsque toute application croissante du produit H × H sur H *qui est l'identité sur la diagonale est forcément* l'une des deux projections. On prouve qu'un ensemble projectif, rétracte d'un produit de deux facteurs, est rétracte d'un des deux facteurs. Un ensemble ordonné est dit **automorphe** s'il possède un automorphisme sans point fixe et il est dit **automorphe minimal** si aucun rétracte propre n'est automorphe. On observe que, dans tous les exemples rencontrés, les automorphes minimaux sont projectifs. Le fait qu'un produit automorphe possède toujours un facteur automorphe permet d'affirmer qu'un automorphe minimal est premier, c'est-à-dire non factorisable. Finalement, on observe que, dans tous les cas connus, les ensembles premiers et de degré d'embranchement multiple sont projectifs. On montre que d'une preuve de ce fait se déduit que l'opération produit respecte la propriété du point fixe.

SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS AND THEIR APPLICATIONS

BUI CONG CUONG

In this lecture we present some fixed point theorems for multivalued mappings in topological spaces. The values of the mappings are not required to be connected. The proofs of these theorems based upon a new intersection theorem.

Some applications of the fixed point theorems to minimax theorems and variational inequalities are considered.

QUELQUES INÉGALITÉS ABSTRAITES

PAUL DEGUIRE

Introduction

Les théorèmes des minimax ou les alternatives non linéaires qui utilisent des applications numériques obéissant à certaines conditions de convexité et de continuité (quasi-convexité ou concavité, semi-continuité supérieure ou inférieure, ...) peuvent toujours être reformulés dans un langage géométrique, sous forme de théorème de point fixe ou d'intersection pour des applications multivoques qui se trouvent ainsi à être associées à nos applications numériques de départ.

Les formulaires géométriques et analytiques sont équivalentes. Les premières s'obtiennent souvent plus facilement grâce à l'élégance et à la simplicité des énoncés géométriques et de leurs preuves, les secondes conservent néanmoins leur intérêt car la plupart des applications utilisent un langage analytique et sont facilement obtenues comme conséquence d'une alternative non linéaire appropriée. Très fréquemment, pour prouver un résultat concret, il suffit de montrer que les fonctions numériques concernées satisfont à des conditions précises de convexité et de continuité.

Nous allons dans cet exposé définir des conditions analytiques abstraites qui, si elles sont vérifiées, nous donnent comme conséquence directe plusieurs des alternatives non linéaires et des théorèmes de minimax habituels.

Dans tout ce qui suit, $\mathbb{P}$ représente un ensemble ordoné abstrait (qui, dans la pratique, sera le plus souvent $\mathbb{R}$) et les espaces considérés sont des sous-ensembles d'espaces linéaires topologiques séparés. Par application numérique, nous entendons application à valeurs dans $\mathbb{P}$ ou dans $\mathbb{P}^n$.

Quelques définitions

Définition 1 : Soient X un espace, Y un espace convexe et $f : X \times Y \to \mathbb{P}$ une application numérique. Nous dirons que f satisfait la condition $(m^*, >)$ pour la constante $\lambda \in \mathbb{P}$ fixée si et seulement si pour chaque compact $K \subseteq X$, il existe un ensemble fini $C = \{y_1, y_2, \ldots, y_n\} \subseteq Y$ et une application univoque $s : K \to \mathrm{Conv}(C)$ telle que pour chaque $x \in K$ on ait : $f(x, s(x)) > \lambda$.

Définition 2 : Soient X un espace convexe, Y un espace et $f : X \times Y \to \mathbb{P}$ une application numérique. Nous dirons que f satisfait la condition $(m, >)$ pour la constante $\lambda \in \mathbb{P}$ fixée si et seulement si pour chaque compact $K \subseteq Y$, il existe un ensemble fini $C = \{x_1, x_2, \ldots, x_n\} \subseteq X$ et une application univoque $s : K \to \mathrm{Conv}(C)$ telle que pour chaque $y \in K$ on ait : $f(s(y), y) > \lambda$.

Ces conditions analytiques sont associées aux applications multivoques de type M* et M que nous avons traitées avec H. Ben El Mechaïekh et A. Granas dans [2]. Nous devons faire remarquer que ces conditions sont au nombre de celles qu'on associe aux classes M* et M et qu'on peut définir d'autres conditions analytiques du même type, par exemple en renversant les *inégalités.*

Quelques applications connues satisfaisant aux conditions abstraites m* et m.

Exemple 1 : Soient X un espace, Y un convexe et f, g de $X \times Y$ dans $\mathbb{P}$ des applications numériques satisfaisant les conditions suivantes :

i) $g(x, y) \leq f(x, y)$ pour chaque $(x, y) \in X \times Y$.
ii) $y \to f(x, y)$ est quasi-concave sur Y pour chaque $x \in X$.
iii) $x \to g(x, y)$ est semi-continue inférieurement sur X pour chaque $y \in Y$.
iv) il existe $\lambda \in \mathbb{P}$ tel que pour chaque $x \in X$ il existe $y_x \in Y$ avec $f(x, y_x) > \lambda$.
Alors l'application f satisfait la condition $(m^*, >)$ pour la constante λ.

Notons que l'on associe à l'application f une application multivoque de type ϕ^*. (voir [1]).

On obtient d'autres exemples liés aux applications multivoques de type ϕ^* en renversant les inégalités et en utilisant semi-continuité supérieure et quasi-convexité. Par ailleurs, étant donné la dualité entre les définitions 1 et 2, il suffit d'intervertir de manière adéquate les rôles de x et y dans les exemples qui concernent la condition m* pour obtenir des exemples satisfaisant la condition m.

Les applications univoques associées aux applications multivoques obtenues en composant un nombre fini d'applications de type ϕ^* ou ϕ nous donnent de nouvelles applications qui satisfont les conditions m* et m. (voir [3]).

Quelques théorèmes

Nous présentons dans cette section deux théorèmes abstraits qui utilisent les conditions m* et m. Nous mentionnons au passage quelques-unes des conséquences immédiates de ces résultats. Par souci de brièveté, nous ne présentons pas nos résultats dans leur plus grande généralité, pour plus de détails on peut consulter [3].

Théorème 1 : Soient X un espace convexe, Y un espace et $\mathbb{P}$ un ensemble ordonné. Si f est une application numérique satisfaisant la condition m pour $\lambda \in \mathbb{P}$ fixée, alors pour chaque application compacte $s \in C(X, Y)$ il existe $x_0 \in X$ tel que :

$$f(x_0, s(x_0)) > \lambda.$$

Corollaire 1.1 : Soient X un espace convexe, Y un espace et $f, g : X \times Y \to \mathbb{R}$ deux fonctions numériques telles que :

i) $g(x, y) \leq f(x, y)$ pour chaque $(x, y) \in X \times Y$

ii) $x \to f(x, y)$ est quasi-concave sur X pour chaque $y \in Y$

iii) $y \to g(x, y)$ est semi-continue inférieurement sur Y pour chaque $x \in X$

A) Alors pour chaque application compacte $s \in C(X, Y)$ et pour chaque $\lambda \in \mathbb{R}$, au moins l'un des énoncés suivants est satisfait :

1) il existe $y_0 \in Y$ tel que $g(x, y_0) \leq \lambda$
pour chaque $x \in X$

2) il existe $x_0 \in X$ tel que $f(x_0, s(x_0)) > \lambda$.

B) $\inf\limits_{y \in Y} \sup\limits_{x \in X} g(x, y) \leq \sup\limits_{x \in X} f(x, s(x))$

Un théorème analogue au corollaire 1.1 a été montré par M. I assonde (1983) qui a utilisé des conditions légèrement plus générales.

Corollaire 1.2 : Prenons dans le résultat précédent (1.1) X et Y convexes compacts, f = g et s la fonction identité sur X. On retrouve alors l'inégalité minimax de Ky Fan (1972).

Nous introduisons maintenant les notations nécessaires à notre second résultat abstrait.

Soient X_i, $i \in I$ une famille d'ensembles. Nous dénotons par X^j le produit de tous les facteurs X_i pour $i \in I$ sauf $i = j$. Nous écrivons $X = X^i \times X_i$. Nous supposons que tous les X_i sont convexes et compacts.

Théorème 2 : Si, pour tout $i \in I$, f_i est une application numérique de $X^i \times X_i \to \mathbb{P}$ satisfaisant la condition m* pour $\lambda_i \in \mathbb{P}$ fixée, alors il existe $x \in X$ tel que $f_i(x) > \lambda_i$ pour chaque $i \in I$.

Corollaire 2.1 : Soient X et Y des convexes compacts et $\mathbb{P}$ un ensemble ordonné. Si f et g sont des applications numériques de $X \times Y \to \mathbb{P}$ satisfaisant les propriétés suivantes pour $\lambda \in \mathbb{P}$ fixée :

i) f satisfait la condition (m, <)

ii) g satisfait la condition (m*, >)

Alors il existe $(x, y) \in X \times Y$ tel que $f(x, y) < \lambda < g(x, y)$.

Notons que dans ces deux résultats on peut supposer que l'un des espaces ne soit pas compact ou encore que d'autres types de conditions de compacité remplacent celles qui ont été énoncés. (voir [3])

Conséquences : Comme conséquence du théorème 2 on trouve le théorème sur les équilibres de Nash ainsi que ses généralisations en dimension infinie de T. W. Ma (1969) et Ky Fan (1984). Comme conséquence du corollaire 2.1 on retrouve, par ordre de généralité décroissante, un théorème minimax de F. C. Liu (1978), l'égalité minimax de Ky Fan (1964) et les égalités minimax de Sion et Von Neumann.

Nous référons ici à certains de nos travaux antérieurs sur le même sujet. Une bibliographie un peu plus complète pourra être trouvée dans [3].

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BIAIS D'UN ESPACE DE BANACH

M. DESBIENS

Notations

$(\mathbf{B}, \|\cdot\|)$	Espace de Banach réel.
Θ	Le vecteur origine de l'espace **B**.
$S(x ; p)$	$\{x \in \mathbf{B} \mid \|x - y\| = \rho\}, \rho > 0$.
$B(x ; p)$	$\{y \in \mathbf{B} \mid \|x - y\| \leq \rho\}, \rho > 0$.
S	$S(\Theta ; 1)$.
$\dim(\mathbf{B})$	La dimension algébrique de **B**.
$[x, y]$	Le segment de doite ayant les vecteurs x et y comme extrémités.
T_γ	$\{x \in \mathbf{B} \mid T(x) = \gamma x\}$ où $T : X \rightarrow \mathbf{B}$ est une fonction donnée.
F(T)	T_1, c'est à dire *l'ensemble des points fixes de l'application* T.

Orthogonalité de Birkhoff-James

Définition : Soit **B** un espace de Banach et $(x, y) \in \mathbf{B}^2$. Nous dirons que x est orthogonal à y au sens de Birkhoff-James, ce que nous noterons par $x \perp_{BJ} y$, si et seulement si : $\|x + \lambda y\| \geq \|x\|, \forall \lambda \in R$.

Biais d'un espace de Banach

Définition : Soit **B** un espace de Banach. On définit le biais de l'espace **B**, que nous dénotons $\beta(\mathbf{B})$, par la formule

$$\beta(\mathbf{B}) = \sup\{\gamma \mid x + \gamma y \perp_{BJ} y \; ; \; 0 < \|x\| \leq \|y\| \; ; \; x, y \in \mathbf{B}\}.$$

Propriétés du biais d'un espace de Banach

Théorème : Soit **B** un espace de Banach, Alors

- $1 \le \beta(\mathbf{B}) \le 2$;
- $\beta(\mathbf{B}) = \sup\{\gamma \mid x + \gamma y \perp_{BJ} y \,;\, 0 < \|x\| = \|y\| \,;\, x, y \in \mathbf{B}\}$;
- $\beta(\mathbf{B}) = \sup\{\gamma \mid x + \gamma y \perp_{BJ} y \,;\, x, y \in S\}$;
- Si $[x, y] \subset S$ alors $\beta(\mathbf{B}) \ge \|x - y\|$;
- Supposons que $\dim(\mathbf{B}) = 2$. Pour que $\beta(\mathbf{B}) = 2$ il faut et il suffit que le cercle unité $C \subset \mathbf{B}$ soit un parallélogramme.

Théorème : Soit **B** un espace de Banach. Pour que la relation d'orthogonalité de Birkhoff-James définie dans cet espace soit symétrique il faut et il suffit que $\beta(\mathbf{B}) = 1$.

Exemple :

- Soit $\mathbf{B}_r = R^2$ et $r, s \ge 1$ deux nombres réels conjugués, c'est-à-dire que $r^{-1} + s^{-1} = 1$. Définissons une norme sur $\mathbf{B}_r$ par la formule :

$$\|(x, y)\|_r = \begin{cases} (|x|^r + |y|^r)^{1/r} & \text{si } \operatorname{sgn}(x) = \operatorname{sgn}(y), \\ (|x|^s + |y|^s)^{1/s} & si\ sgn(x) \neq sgn(y), \end{cases}$$

Il n'est pas difficile de voir que, pour $r \ge 1$, l'orthogonalité de Birkhoff-James est symétrique dans $\mathbf{B}_r$. Donc $\beta(\mathbf{B}) = 1$. Par contre $\mathbf{B}_r$ n'est pas hilbertien ($r \neq 2$).

Définition : Soit **B** un espace de Banach. Nous dirons que **B** a la propriété $\mathcal{P}'$ si, pour tout scalaire $\gamma > 1$ et pour tout couple de vecteurs $(x, y) \in \mathbf{B}^2$, l'implication suivante est vraie :

$$\|x\| \le \|y\| \Rightarrow \|x + y\| \le \|x + \gamma y\|.$$

Théorème : Soit **B** un espace de Banach, Alors

- $\beta(\mathbf{B}) = 1$ entraîne que **B** possède la propriété $\mathcal{P}'$;
- si **B** a la propriété $\mathcal{P}'$ et que $\beta(\mathbf{B}) > 1$ alors **B** n'est pas strictement convexe ;
- le fait que **B** soit strictement convexe et qu'il ait un biais égal à 1 est équivalent au fait que **B** possède la propriété $\mathcal{P}'$.

Une caractérisation métrique des espaces de Hilbert

Définition : Soit **B** un espace de Banach, $0 < \varepsilon \leq 1$, $x \in S$ et $z \in S(\mathcal{O} ; \varepsilon)$. Parcourons le segment $[x, z]$ dans le sens de x à z et notons par y le premier point de contact entre le segment $[x, z]$ et la sphère $S(\mathcal{O} ; \varepsilon)$. Nous dirons alors que **B** a la propriété $\mathcal{P}''$ si, quel que soit le choix du rayon ε et des vecteurs x et z, l'inégalité

$$\|x - y\| \leq 1$$

est toujours vérifiée.

Géométriquement, la propriété $\mathcal{P}''$ signifie que, pour un observateur placé en un point x de la sphère unité S, tous les points visibles de la sphère $S(\mathcal{O} ; \varepsilon)$ $(0 < \varepsilon \leq 1)$ appartiennent à la boule $B(x ; 1)$.

Théorème : Soit **B** un espace de Banach.

- Si, dans cet espace, la relation d'othogonalité de Birkhoff-James est symétrique alors cet espace à la propriété $\mathcal{P}''$;
- si **B** est strictement convexe et s'il possède la propriété de $\mathcal{P}''$, il possède alors la propriété $\mathcal{P}'$.

Théorème : Soit **B** un espace de Banach strictement convexe. Les conditions suivantes sont alors équivalentes :

- La relation d'orthogonalité de Birkhoff-James définie sur **B** est symétrique ;
- $\beta(\mathbf{B}) = 1$;
- **B** possède la propriété $\mathcal{P}'$;
- **B** possède la propriété $\mathcal{P}''$.

Théorème : Soit **B** un espace de Banach strictement convexe avec $\dim(\mathbf{B}) \geq 3$. Les conditions suivantes sont alors équivalentes :

- La norme sur **B** est issue d'un produit scalaire ;
- La relation d'orthogonalité de Birkhoff-James définie sur **B** est symétrique ;
- $\beta(\mathbf{B}) = 1$;
- **B** possède la propriété $\mathcal{P}'$;
- **B** possède la propriété $\mathcal{P}''$.

Géométrie de l'ensemble des vecteurs propres des fonctions pseudo-contractantes

Définition : Soit **B** un espace de Banach, $X \subset \mathbf{B}$ et $T : X \rightarrow \mathbf{B}$ une application. Nous dirons que T est pseudo-contractante si $\forall (x, y) \in X^2$ et $\forall r > 0$ on a que

$$\|x - y\| \leq \|(1 + r)(x - y) - r(T(x) - T(y))\|.$$

Lemme : Une application $T : X \rightarrow \mathbf{B}$ est pseudo-contractante si et seulement si l'opérateur $I - T$ est accrétif, c'est-à-dire si pour tout couple de vecteurs $(x, y) \in X^2$ il existe une transformation linéaire $j \in J(x - y)$ telle que

$$(T(x) - T(y), j) \leq \|x - y\|^2.$$

Théorème : Soit **B** un espace de Banach strictement convexe de biais égal à 1 et $T : X \rightarrow \mathbf{B}$ une application pseudo-contractante telle que $T_\gamma \neq \emptyset$ pour tout $\gamma \in \,]1, \tau[$ avec $\tau > 1$. Il existe alors une application $\phi : \,]1, r[\, \rightarrow X$ définie par

$$\phi(\gamma) = \omega_\gamma \Leftrightarrow \omega_\gamma \in T_\gamma . \forall_\gamma \in \,]1, r]$$

ayant les propriétés suivantes :

1) ϕ est continue sur tout $]1, \tau]$;
2) $\gamma \leq \mu \Rightarrow \|\phi(\mu)\| \,|\, \leq \|\phi(\gamma)\|$;
3) $\sup\{\|\phi(\gamma)\| \mid \gamma \in \,]1, \tau]\} \leq \inf\{\|x\| \mid x \in F(T)\}$;
4) $\oslash \in X$ et $\gamma \rightarrow \infty \Rightarrow \phi(\gamma) \rightarrow \oslash$;
5) la fonction $h : \,]1, r] \rightarrow [0, \infty]$ définie par

$$h(\gamma) = \|\phi(\gamma) - T(\phi(\gamma))\|, \forall \gamma \in \,]1, \tau]$$

est non décroissante.

Lemme : (Morales, 1979) Soit **B** un espace de Banach strictement convexe, X un convexe fermé de **B** et $T : X \rightarrow X$ une application continue et pseudo-contractante. Alors $F(T)$ est un convexe fermé de **B**.

Géométriquement, nous pouvons dire que l'ensemble des vecteurs propres associés aux valeurs propres $\gamma > 1$ d'une application continue et pseudo-contractante T forme une courbe de Jordan dans l'ensemble $X \subset \mathbf{B}$ originant, si X est un convexe fermé de **B**, de l'unique point fixe de norme minimale de l'application T et tendant progressivement vers l'origine de l'espace **B**.

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GENERALIZED DISCRETE DYNAMICAL SYSTEMS : MINIMAL SETS, PERIODIC POINTS, GLOBAL CONVERGENCE

G. DILENA AND B. MESSANO

1. Introduction

According to [4], a **generalized discrete dynamical system** (abbr. **g.d.d.s.**) is a pair (S, f), where S is a topological space and f is a function from S into itself [(1)] ; moreover :

a nonempty subset X of S is said **positively invariant** (resp. **invariant**) **set** of the **g.d.d.s.** (S, f) if $f(x) \subseteq X$ (resp. $f(X) = X$).

a closed invariant set M of the **g.d.d.s.** (S, f) is said **minimal set** of the **g.d.d.s.** (S, f) *if does not exist* a *subset* I of M such that I is a closed invariant set of (S, f).

In this paper we shall use the following proposition (see (3.2) of [4]), where, for each $n \in N$, f^n denotes the nth iterate of f :

(1.1). If S is Hausdorff compact space satisfying the first axiom of countability and f is continuous, then the following statements are true :
1) (S, f) is Birkhoff system (i.e. a subset M of (S, f) is a minimal set of (S, f) if and only if M is a minimal set of the set of all closed positively invariant sets of (S, f)).
2) (S, f) is endowed with a minimal set.
3) A nonempty subset M of S is minimal set of (S, f) if and only if, for each $x \in M$, M is equal to the set of all cluster points of $(f^n(x))_{n\in \mathbb{N}}$.

In section 2 of this paper, amongst other things, we prove that if S is a complete metric space, then each minimal set of (S, f) is finite or uncountable.

In section 3, supposing that S is an arcwise connected tree satistying the first axiom of countability and f is continuous, we obtain further property of minimal sets of (S, f) (see [3.1]) and, moreover, we show the following theorem :

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(1) In the case in which S is a metric space and f is continuous the above definition of g.d.d.s. is substantially that given in [5], pag. 223.

A) If the following condition is satisfied :

(*) for each y ∈ Fixf the set S-{y} is endowed with a finite number of components,

then the following statements are equivalent :

i) For the **g.d.d.s.** (S, f) there is global convergence (i.e. for each x ∈ S the sequence $(f^n(x))_{n\in \mathbb{N}}$ converges to a point of Fixf).

ii) Each minimal set of (S, f) is a singleton

iii) For each x ∈ S, the sequence $(f^n(x))_{n\in \mathbb{N}}$ is endowed with a cluster point belongs to Fixf.

In the end, in section 4, using the results obtained in sections 2 and 3, we prove (see (4.2)) that if f is continous and S is an arcwise connected tree endowed with a countable number of end points satisfying the above condition (*), then the above condition i) is equivalent ot the non existence of periodic points of f.

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METHODE DES POINTS FIXES ET EQUATIONS SEMI LINEAIRES DANS R^n

ALLAN L. EDELSON

Nous considérons le problème de l'existence et les propriétés asymtotiques des solutions entières de l'équation

(1) $$-\Delta u = p(x)u^{\gamma},\ p(x) > 0,\ x \in \mathbb{R}^n$$

L'équation s'appelle singulière si $\gamma < 0$, sublinéaire si $0 < \gamma < 1$, et superlinéaire si $1 < \gamma$. On trouve ces équations dans beaucoup d'applications ; par exemple, les équations nonlinéaires Klein-Gordon et la théorie des fluides visqueux. Pour trouver une solution Berestycki et Lyons ont utilisé une combinaison de la "shooting method" et d'une méthode variationelle. Kusano et Swanson ont utilisé la méthode des sub et super solutions pour démontrer l'existence des solutions positives entières, décroissantes pour les équations sublinéaires et singulières, correspondantes à $0 < \gamma < 1$, et $-1 < \gamma < 0$ respectivement.

Ici nous allons décrire une théorie de l'existence pour l'équation (1), en utilisant les opérateurs intégraux classiques. Nous ne supposons pas que les équations soient symmétiques. Nous écrivons l'équation dans la forme intégrale comme

$$u(x) = Tu(x) = \int \Gamma(x - y)\, p(y)\, u(y)^{\gamma}\, dy,\ y \in \mathbb{R}^n$$

Le solutions de l'équation u = Tu sont des solutions de (1), et nous pouvons calculer la vélocité de décadence directement par l'intégrale. La méthode a plusieurs avantages :

1. Elle s'applique aux problèmes non symmétriques.
2. Elle donne deux estimations plus précises pour la vélocité de décadence lorsque |x| tends vers l'infini.
3. Elle s'applique aux équations d'ordre > 2, où les méthodes du principe du maximum ne s'appliquent pas.

Ces résultats s'appliquent également aux équations de la forme $L[u] = f(x, u)$, où L est un opérateur uniformément elliptique. La méthode a des difficultés à calculer les estimations pour les opérateurs intégraux dans $\mathbb{R}^n$, qu'on peut obtenir par la méthode de la théorie du potentiel. Pour $n = 2$ nous trouvons aussi des théorèmes de l'existence des solutions asymptotiques à des constantes positives, et des solutions asymptotiques à $\log |x|$. Comme la solution fondamentale n'est pas positive partout, les démonstrations deviennent plus difficiles.

Théorème ($n \geq 3$) : Soit (1) sublinéaire ou singulière, et p dans $C^\alpha(\mathbb{R}^n)$. Soit p^* une majoration symétrique pour p, c'est à dire, $0 < p(x) \leq p^*(|x|)$.

Si p^* satisfait la condition asymptotique

$$\int t^{n-1+\gamma(2-n)} p^*(t)\, dt < \infty,$$

alors il existe une solution positive, entière, u dans $C^\alpha{}_{loc}(\mathbb{R}^n)$, telle que $\lim_{|x|\to\infty} |x|^{(2-n)} u(x) = C$.

Si p satisfait la condition,

$$\int t^{1+\gamma(2-n)} p^*(t)\, dt < \infty,$$

alors il existe une solution positive, entière, u dans $C^\alpha{}_{loc}(\mathbb{R}^n)$, telle que $\lim_{|x|\to\infty} u(x) = C$.

Théorème ($n = 2$) : Soit (1) sublinéaire, et p dans $C^\alpha{}_{lcc}(\mathbb{R}^n)$.

Si p satisfait la condition $\int t \log(t)^\gamma p^*(t)\, dt < \infty$, il existe une solution u qui satisfait la condition asymptotique

$$\lim_{|x|\to\infty} (\log(|x|)^{-1})\, u(x) = C.$$

Si p satisfait $\int t\, \log(t)\, p^*(t) dt < \infty$,

il existe une solution u qui satisfait la condition asymptotique $\lim_{|x|\to\infty} u(x) = C$.

Pour le problème superlinéaire l'existence des solutions qui s'approchent à zéro est difficile à démontrer. Elles sont instables comme solutions du problème parabolique correspondant. Dans plusieurs exemples s'il existe une sub solution et une super solution elles sont déjà des solutions du problème. Alors la méthode des sub et super solutions ne s'applique pas. Du point de vue topologique, les opérateurs intégraux associés sont expansifs. C'est le problème de trouver le "middle fixed point" (voir Amann).

En utilisant une méthode variationelle, Noussair et Swanson ont démontré pour l'équation

$$(2) \qquad - \Delta u = p(|x|)u^{\gamma}, \ \ p(|x|) \sim |x|^{a}, - 2 < a < 0, \\ (n + 2 + 2a) / (n - 2) < \gamma < (n + 2) / (n - 2),$$

qu'il existe une solution positive, entière, symmétrique dans l'espace $W^{1,2}(\mathbb{R}^n)$. Ces solutions satisfont les estimations

$$(3) \qquad C_1|x|^{(2 - n)} \leq u(|x|) \leq C_2|x|^{(2-n)/2}.$$

L'estimation inférieure est une conséquence du principe du maximum. L'estimation supérieure du lemme de Berestycki et Lions, qui implique qu'une fonction radiale dans $W^{1,2}$ décroit au moins comme $|x|^{(2-n)/2}$.

En utilisant une combinaison de la méthode variationelle et les points fixes, les résultats préliminaires démontrent que pour le problème superlinéaire aussi nous pouvons démontrer dans certains cas l'existence des solutions minimales, c'est à dire, des solutions qui satisfont la condition précise $\lim_{|x| \to \infty} |x|^{(2 - n)} u(x) = C$.

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FIXED POINTS OF EXPANSIVE ANALYTIC FUNCTIONS

WALTER O. EGERLAND

Let f be an analytic function of the complex variable z in the closed disk D and let f "cover" D, i.e., D is contained in the image f(D). We state results that insure that f has a fixed point in D and show, in particular, that every quadratic function that covers D has a fixed point in D. A brief historical overview is included in this presentation of approximately 15 minutes.

EDGEWORTH EQUILIBRIA, FUZZY CORE AND EQUILIBRIA OF A PRODUCTION ECONOMY WITHOUT ORDERED PREFERENCES

MONIQUE FLORENZANO

I. Definitions and notations

In a vector space L as commodity space, let us consider :

$$\mathcal{E} = ((X^i, P^i, \omega^i)_{i\in M}, \ (Y^j)_{j\in N}, \ (\theta^{ij})_{i\in M, j\in N})$$

a private ownership economy with a finite set M of consumers and a finite set N of producers, standardly defined.

$\forall$ i $\in$ M, $X^i \subset L$ is the consumption set of i,

$P^i : \prod_{k\in M} X^k \to X^i$ is a preference correspondence assigning to each $x = (x^k) \in \prod_{k\in M} X^k$ the set $P^i(x)$ of the elements of X^i which are (strictly) preferred by agent i to x^i when the consumption of each agent $k \neq i$ is equal to x^k,

$\omega^i \in L$ is the initial endowment of i ;

$\forall$ j $\in$ N, $Y^j \subset L$ is the production set of j ;

$\forall$i, $\forall$j, θ^{ij} denotes a contractual claim of consumer i on the profit of producer j and a relative share in the proprietorship of firm j. The θ^{ij} are assumed to verify, for every j $\in$ N, $\sum_{i\in M} \theta^{ij} = 1$.

$X = \prod_{i\in M} X^i$. $Y = \sum_{j\in N} Y^j$ is the total production set. $\hat{X} = \{x \in X \mid \sum_{i\in M} x^i \in \sum_{i\in M} \omega^i + Y\}$ is the set of all attainable allocations of the economy.

Now let $T = [0, 1]^m \setminus \{0\}$. $t = (t^i)_{i\in M} \in T$ is a fuzzy coalition. We define :

$Y^t = \sum_{i\in M} t^i \sum_{j\in M} \theta^{ij} Y^j$, the production set of t, $\hat{X}^t = \{x^t \in \prod_{t^i>0} X^i \mid \sum_{i\in M} t^i x^{it} \in \sum_{i\in M} t^i\omega^i + Y^t\}$, the attainable set of t.

$P^t = \prod_{i\in M} X^i. \to \prod_{t^i>0} X^i$, the preference correspondence of the fuzzy coalition t is defined by unanimity between the consumers who have a strictly positive rate of participation t^i:

$$P^t(x) = \{z^t \in \prod_{t^i>0} X^i \mid z^{it} \in P^i(x) \quad \forall\, i : t^i > 0\}.$$

Let Q denote the set of all rational numbers.
An Edgeworth equilibrium is an element of

$$C^e(\varepsilon) = \{x \in \hat{X} \mid \hat{X}^t \cap P^t(x) = \phi \quad \forall\, t \in T \cap Q^M\}$$

The fuzzy core of ε is the set :

$$C^f(\varepsilon) = \{x \in \hat{X} \mid \hat{X}^t \cap P^t(x) = \phi \quad \forall\, t \in T\}.$$

The idea of fuzzy core is borrowed from Aubin [2]. Under standard convexity assumptions, it can be seen that, as in [4], [12], [13], [1], an Edgeworth equilibrium is an attainable allocation whose r-fold repetition belongs to the core of the r-fold replica of the original economy for any positive integer r.
Obviously, $C(\varepsilon) = \{x \in \hat{X} \mid \hat{X}^t \cap P^t(x) = \phi \quad \forall\, t \in T \cap \{0, 1\}^M\}$ corresponds to the standard definition of the core.

II. Non emptiness theorems

Let σ be a Hausdorff vector space topology on L. We make on ε the following assumptions :

A_1 - $\forall\, i \in M$, X^i is convex and $\omega^i \in X^i$
$\forall\, x \in X$, $P^i(x)$ is convex and $x^i \notin P^i(x)$
$\forall\, x^i \in X^i$, $(P^i)^{-1}(x^i)$ is σ^m-open in X

A_2 - $\forall\, j \in N$, $0 \in Y^j$

A_3 - Y is convex and $\hat{X}$ is σ^m-compact.

Let us consider another vector space topology τ on L. We make the following additional assumption :

A_4 - $\forall\, i \in M$, $\forall\, x \in X$, $P^i(x)$ is τ-open in X^i

Proposition 1 : Under A_1, A_2, A_3, $C^e(\varepsilon) \neq \emptyset$
The proof of proposition 1 depends on a fixed-point theorem borrowed from Gale and Mas-Colell [9].
From proposition 1, one can easily derive

Proposition 2 : Under A_1, A_2, A_3, A_4, $C^f(\mathcal{E}) \neq \emptyset$.

III. Fuzzy core and Walrasian quasi-equilibrium allocations

In this section, we assume that L is endowed with a Hausdorff vector space topology τ. If $p \in (L, \tau)'$ (the conjugate space of (L, τ)), define :

$\gamma^i(p) = \{x^i \in X^i \mid p.x^i \leq p.\omega^i + \sum_j \theta^{ij} \sup p.Y^j\}$, the budget set of i

$\delta^i(p) = \{x^i \in X^i \mid p.x^i < p.\omega^i + \sum_j \theta^{ij} \sup p.Y^j\}$.

$(\bar{x}, \bar{y}, \bar{p}) \in \prod_{i\in M} X^i \times \prod_{j\in M} Y^i \times ((L, \tau)' \setminus \{0\})$ is a Walrasian quasi-equilibrium if

- $\sum_{i\in M} \bar{x}^i = \sum_{i\in M} \omega^i + \sum_{j\in N} \bar{y}^j$ (attainability)
- $\forall i \in M, \bar{x}^i \in \gamma^i(\bar{p})$ and $P^i(\bar{x}) \cap \delta^i(\bar{p}) = \emptyset$
- $\forall j \in N, \bar{p}.\bar{y}^j = \sup \bar{p}.Y^j$.

In this case, $\bar{x}$ is said to be a Walrasian quasi-equilibrium allocation. $QW(\mathcal{E})$ denotes the set of such allocations.

In this section, we make the following assumptions on $\mathcal{E}$:

B_1 - $\forall i \in M$, X^i is convex

$\forall x \in X$, $P^i(x)$ is convex, τ-open in X^i and $x^i \notin P^i(x)$

B_2 - $\forall j \in N$, Y^j is convex and $0 \in Y^j$

B_3 - If $x \in \hat{X}$, $x^i \in \overline{P^i(x)}$ $\forall i \in M$ (where $\overline{P^i(x)}$ denotes the τ-closure of $P^i(x)$).

Proposition 3 : Under B_1, B_2, B_3 and if $L = \mathbb{R}^\ell$, then $C^f(\mathcal{E}) \subset QW(\mathcal{E})$.

Let $\bar{x} \in C^f(\mathcal{E})$; the proof of proposition 3 is based on a separation argument between 0 and G = $co\,(\bigcup_{i\in M} (P^i(\bar{x}) - \sum_j \theta^{ij} Y^j - \omega^i))$.

In the infinite dimensional case, let $P : X \to X$ and $R : X \to X$ be

$$P(x) = \{x' \in X \mid x'^i \in P^i(x) \quad \forall i \in M\} = P^M(x)$$
$$R(x) = \{x' \in X \mid P^i(x') \subset P^i(x) \quad \forall i \in M\}.$$

We set the following additional assumption :

C - There exists a convex cone Z, with a non-empty τ-interior, such that either C_1 : $x \in X$ and $\sum_i x^i \in \sum_i \omega^i + Y + Z \Rightarrow \hat{X} \cap R(x) \neq \emptyset$

or $C_2 : x \in X$ and $\sum_i x^i \in \sum_i \omega^i + Y + Z \Rightarrow \hat{X} \cap (P(x) \cup \{x\}) \neq \emptyset$

and define the auxilliary economy :

$$\varepsilon_Z = ((X^i, P^i, \omega^i)_{i \in M}, (Y^j)_{j \in N}, Z, (\theta^{ij})_{i \in M, j \in N}, (\theta^{iz})_{i \in M})$$

where $(\theta^{iz})_{i \in M}$ satisfies : $\sum_{i \in M} \theta^{iz} = 1$

Proposition 4 : Assume B_1, B_2, B_3 and C.

Under C_1, if $\bar{x} \in C^f(\varepsilon_Z)$, then $\hat{X} \cap R(\bar{x}) \subset (C^f(\varepsilon) \cap QW(\varepsilon))$.

Under C_2, $C^f(\varepsilon_Z) \subset (C^f(\varepsilon) \cap QW(\varepsilon))$.

In other words, if $C^f(\varepsilon_Z) \neq \emptyset$, there exists a member of $C^f(\varepsilon)$ which is a Walrasian quasi-equilibrium allocation of ε.

IV. Applications to the existence of quasi-equilibria of ε

In this section, the two Hausdorff vector space topologies σ and τ are simultaneously considered on the commodity space L.

Proposition 5 : Assume A_1, A_2, A_3, B_1, B_2, B_3, and C. Assume moreover that :

- $\forall$ i $\in$ M, X^i is σ-closed
- $\forall$ j $\in$ N, Y^j is σ-closed and $\hat{Y}^j$ (the attainable set of producer j) is σ-compact
- Z (in assumption C) is σ-closed.

Then $C^f(\varepsilon_Z) \neq \emptyset$ and ε has a Walrasian quasi-equilibrium.

Under a suitable definition of Z in assumption C, proposition 5 extends various existence results in [1], [3], [5], [10], [11], [14].

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ON THE EXISTENCE OF FORCED OSCILLATIONS FOR THE SPHERICAL PENDULUM

MASSIMO FURI AND MARIA PATRIZIA PERA

In spite of the fact that the forced ordinary pendulum does not necessarily possess forced oscillations when acted on by a periodic force, we prove that the forced spherical pendulum (or, more generally, a constrained system, whose configuration space is a compact manifold M with Euler-Poincaré characteristic $\chi(M) \neq 0$), admits forced oscillations (i.e., periodic solutions of the same period as that of the forcing term), provided that the coefficient of friction is non-zero.

This result my be regarded as an extension of the well known fact that any tangent vector field on a compact boundaryless manifold M with $\chi(M) \neq 0$ must vanish at some point. The proof is mainly based on fixed point index theory applied to the Poincaré translation operator acting on the tangent bundle TM of the configuration space M.

We conjecture the existence of forced oscillations even in the case of a frictionless constraint, provided that such a constraint may be regarded as a compact smooth manifold with non-zero Euler-Poincaré characteristic.

THE MINIMAL DISPLACEMENT AND RETRACTION PROBLEMS

KAZIMIERZ GOEBEL

The famous collection of mathematical problems known as "The Scottish Book" contains a question (Problem 36) raised around 1935 by S. Ulam which reads : "Can one transform continuously the solid sphere of a Hilbert space into its boundary such that the transformation should be identity on the boundary of the ball ?"

An addendum states : "There exists a transformation with the required property given by Tychonoff".

Ulam's problem is probably one of the simplest in the Scottish book. In nowadays terminology it reads : Is the unit sphere in a Hilbert space a retract of its unit ball ?

The answer to this question is most commonly attributed to S. Kakutani who, in 1943 presented several examples of continuous self-mappings of the unit ball in Hilbert space without fixed points, any of which may be used to provide the answer to Ulam's question.

In 1955 V. Klee showed that for any bounded closed convex but noncompact subset K of a Banach space there exists a continuous selfmapping of K which is fixed point free. A consequence of this fact is the existence of a retraction of unit ball onto unit sphere in any Banach space. Next in 1979 B. Nowak used a complicated construction to prove that such retraction can be even lipschitzian. This was subsequently somewhat simplified by Y. Benyamini and Y. Sternfeld in 1983. Two years later P. K. Lin and Y. Sternfeld extended Klee's result showing that for any bounded closed and convex subset K of a Banach space X there is a lipchitzian mapping $T : K \rightarrow K$ such that

$$\inf[\|x - Tx\| : x \in K] = d > 0.$$

Not much is known about the dependence of d on the regularity of T. The minimal displacement problems mentioned in the title can be formulated as follows.

Let K be a bounded closed convex subset of a Banch spaceX. Let $T : K \rightarrow K$ be a mapping satisfying Lipschitz condition $\|Tx - Ty\| \leq K\|x - y\|$. How to evaluate $d = \inf\|x - Tx\|$ in terms of

K and some parameters describing the set K ? How this evaluation depends on "the geometry" of the set K ?

If K is taken to be the unit ball the above questions are closely related to the following retraction problem.

In view of the mentioned results of Nowak, Benyamini and Sternfeld, for a given space X there is a constant $k_o(X)$ being infimum of all numbers K such that there exists a retraction R of the unit ball B onto unit sphere S satisfying

$$\|Rx - Ry\| \leq K\|x - y\|.$$

For which space X, if any, $k_o(X)$ is the smallest ? For which, if any, is the largest ? What are the estimates for $k_o(X)$ for classical Banach spaces ?

Not much is known about solutions to this problems. The talk is devoted to the presentation of some recent results and examples.

For instance let us quote some inequalities : $k_o(L^1(0, 1)) < 10$, $k_o(\ell^2) = K_0(H) \geq 4.55\ldots$, $k_o(\ell^2) < 65$, and let us present one sample of a construction in C[0, 1].

For $f \in C[0, 1]$, $\|f\| \leq 1$, define

$$(Af)(t) = |f(t) + 1 - 2(1 - \|f\|)t| - 1 + 2(1 - \|f\|)t,$$
$$Rf = \frac{Af}{\|Af\|}\ .$$

We leave to the reader finding the lipschitz constant for this retraction of B onto S.

THE LEFSCHETZ FIXED POINT THEOREM FOR MULTI-VALUED MAPPINGS

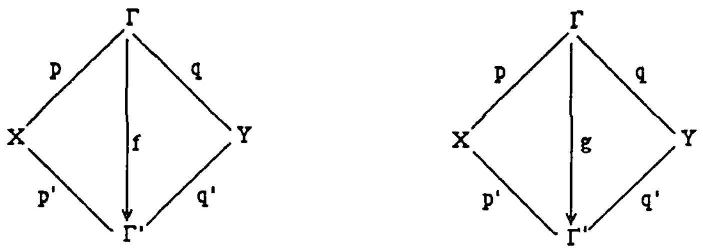
L. GÓRNIEWICZ AND A. GRANAS

The notion of morphism was introduced by A. Granas and L. Gorniewicz in 1981 (cf. [6]). Note that every morphism determines a multi-valued mapping. The class of multi-valued mappings determined by morphisms is quite large and contains, in particular upper semi continuous mappings with convex, contractible or acyclic values and compositions of such maps. The aim of this paper is to present the Lefschetz fixed point theorem for morphisms between approximative neighbourhood extension spaces. Our result contains as a special case the Lefschetz fixed point theorem proved in [1, 2, 3, 4, 5] and hence all known versions of the Schauder fixed point theorem for multi-valued mappings (cf. [7, 8]).

1. Morphisms

By H we shall denote the Cech homology functor with compact carriers and coefficients in the field of rationals Q from the category of Hausdorff topological spaces and continuous maps to be graded vector spaces over Q and linear maps of degree zero. A compact non-empty space X is called acyclic provided (i) $H_0(X) = Q$ and (ii) $H_q(X) = 0$ for $q \geq 1$. Note that any compact contractible space or in particular any compact convex subset of a locally convex space is acyclic. A Hausdorff space X is called of finite type provided (i) $\dim H_q(X) < +\infty$ and (ii) $H_q(X) = 0$ for almost all q. A continuous (single-valued) map $p : \Gamma \to X$ is called a Vietoris map provided p is proper and for every $x \in X$ the set $p^{-1}(x)$ is acyclic. Some important properties of Vietoris mappings are summarized in [5, 6]. Note that if $\xi : X \to Y$ is an upper semi continuous multi-valued mapping with acyclic values, then the natural projection $p : \Gamma_\xi \to X$, $p(x, y) = x$, is a Vietoris map, where $\Gamma_\xi = \{(x, y) \in X \times Y ; y \in \xi(x)\}$ is the graph of ξ.

Given two spaces X and V let D(X, Y) be the set of all diagrams of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ where Γ is a Haudorff space, p is a Vietoris map and q is a continuous map. Every such a diagram we denote briefly by (p, q). Given two digrams (p, q), $(p', q') \in D(X, V)$ are called equivalent (write $(p, q) \approx (p', q')$), if there are two (continuous and single valued) maps $f : \Gamma \to \Gamma$ and $g : \Gamma \to \Gamma$ such that the following two diagrams commute :



The equivalence class of a diagram $(p, q) \in D(X, V)$ with respect to $\approx$ is denoted by

$$\varphi = \{X^p \leftarrow \Gamma \rightarrow {}^q Y\} : X \to Y$$

and is called a morphism from X to Y.
Any morphism $\varphi : X \to Y$ determines a multi-valued map $\xi : X \to Y$ by the following formula :

$$\xi(X) = q(p^{-1}(x)), \text{ for every } x \in X.$$

It is well known (cf. [5, 6]) that the class of multi-valued mappings determined by morphisms is quite large and contains acyclic mapping and also compositions of acyclic mappings. For example, any acyclic map $\xi : X \to Y$ is determined by the morphism $\varphi = \{X^p \leftarrow \Gamma \rightarrow {}^q Y\} : X \to Y$, where $q(x, y) = y$ for every $(x, y) \in \Gamma_\xi$.

The notion of morphism has many good properties, namely it is possible to define the composition law for morphisms (cf. [6]) such that topological spaces and morphisms are a category, moreover the homology functor H extends onto this category (cf. [6]). In what follows we let $H(\varphi) = \varphi_*$.

A morphism $\varphi : X \to X$ is called a Lefschetz morphism provided $\varphi_* : H(X) \to H(X)$ is a Leray endomorphism (see [7] for details) ; for such φ we define the generalized Lefschetz number $\Lambda(\varphi)$ of φ by putting :

$$\Lambda(\varphi) = \Lambda(\varphi_*).$$

A morphism $\varphi : X \to Y$ is called compact (locally compact) if there exists a representative (p, q) of φ such that q is compact (locally compact). A morphism $\varphi : X \to X$ is said a compact absorbing contraction (cf. [2, 7, 8]), if there exists a representative (p, q) of φ and an open subset $U \subset X$ such that : (i) the closure $Cl[q(p^{-1}(U))]$ of $q(p^{-1}(U))$ in U is a compact set and (ii) $X \subset \cup\{(qp^{-1})^{-n}(U) ; n = 0, 1, 2, \dots\}$.

Note that any compact morphism or locally compact with compact iteration φ^n (for some n) or locally compact morphism with compact attractor or asymptotically compact morphism is compact absorbing contraction (cf. [2, 7]).

2. The Lefschetz fixed point theorem

First we define a class of spaces for which we are able to prove the Lefschetz fixed point theorem.

A Hausdorff topological space X is called an approximative neighbourhood extension space for the class of compact spaces ($X \in$ ANES (compact) ; cf. [4, 7]) provided for each open covering α of X, for each pair (Y, A) of compact spaces and for each continuous (single-valued) map $f : A \to X$ there is an open neighbourhood U_α of A in Y and a continuous (single-valued) map $f_\alpha : U_\alpha \to X$ such that for every $x \in X$ there is a member V of α with the following property $f(x) \in V$ and $f_\alpha(x) \in V$. Note that, in particular every convex or open subset in a locally convex space and every metric ANR-space is an ANES (compact) space.
We shall say that a morphism $\varphi : X \to X$ has a fixed point, if there exists a representative (p, q) of φ and a point $x \in X$ such that $x \in q(p^{-1}(x))$.

Now we are able to formulate the main result of this paper.

Lefschetz fixed point theorem

Let $X \in$ ANES (compact) and let $\varphi : X \to X$ be a compact absorbing contraction morphism. Then φ is a Lefschetz morphism and $\Lambda(\varphi) \neq 0$ implies that φ has a fixed point.

Let us remark that if $X \in$ ANES (compact) and X is a contractible or in particular convex, space then for any compact absorbing contraction morphism $\varphi : X \to X$ we have $\Lambda(\varphi) = 1$ and hence our result contains as a special case the Schauder fixed point theorem for multi-valued mappings.

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NEW INSIGHT INTO MINIMAX THEOREMS

GABRIELE H. GRECO

Abstract : This talk concerns some extensions of the Sion minimax theorem (from the u.s.c.//.s.c. quasi concave-convex functions to the wider class of the quasi concave-convex **topologically closed functions**) and of the Kneser-Ky Fan minimax theorem (from the l.s.c. concave-convex functions to the wider class of the quasi concave-convex verifying the **simplex property**). The basic notion is that of **concave-convex multifunctions**. This notion enables us to study geometric counterparts of saddling transformations (that is : transformations mapping games onto games that have saddle values) and subsequently of minimax theorems. The proofs proposed in this talk do not require the usual tools for minimax theorems : the KKM theorem, the Helly theorem, fixed point theorems, separation of convex sets, etc. Instead our method of proof makes use of geometrical properties of concave-convex multifunctions : the **pasting property and the constant selection property, etc.**

1. Concave-convex multifunctions and intersection

Theorems : Let X, Y be non-empty convex subsets of topological vector spaces (not necessarily separated) ; their induced topologies will be denoted by ξ and τ, respectively.

A multifunction $\Omega : X \rightrightarrows Y$ is said to be **concave-convex** if for every $(x_0, y_0) \in X \times Y$, the sets Ωx_0 and $\{x \in X : y_0 \notin \Omega x\}$ are convex. Clearly, a multifunction Ω is concave-convex if and only if the following properties hold :

(1.1) Ωx **is convex for every** $x \in X$,

(1.2) $\Omega x' \subset \Omega x_1 \cup \Omega x_2$ **for every** $x' \in [x_1, x_2] \subset X$,

where $[x_1, x_2]$ denotes the convex hull of the set $\{x_1, x_2\}$.

Let γ be a topology on Y ; the symbol $\mathrm{Li}_{x' \to x}\, \gamma\, \Omega x'$ denotes the usual Kuratowski's lower limit of sets. We recall that $y \in \mathrm{Li}_{x' \to x}\, \gamma\, \Omega x'$ if and only if for every γ-neighbourhood A of y there is a ξ-neighbourhood B of x such that $A \cap \Omega x' \neq \emptyset$ for every $x' \in A$.

Theorem 1.1 (Greco [3]) : Let Y be a convex subset of a locally convex topological vector space with the induced topology τ and let $\Omega : X \rightrightarrows Y$ be a concave-convex *multifunction with* closed values. If Ωx_0 is compact for at least one $x_0 \in X$ and

$$\operatorname{Li}^{\tau}_{x' \to x} \Omega x \neq \emptyset \text{ for every } x \in X, \tag{1.3}$$

then $\bigcap_{x \in X} \Omega x \neq \emptyset$.

Corollary 1.2 (Greco [3]) : Let $\Omega : X \rightrightarrows Y$ be a concave-convex multifunction with closed values. If Ωx is compact for at least one $x_0 \in X$ or $\{x \in X : y_0 \notin \Omega x\}$ is relatively compact for at least one $y_0 \in Y$ and

$$\operatorname{Li}^{\tau}_{x' \to x} \Omega x' \neq \emptyset \text{ for every } x \in X, \tag{1.4}$$

where τ is the discrete topology on Y, then $\bigcap_{x \in X} \Omega x \neq \emptyset$.

Theorem 1.3 (Flam, Greco [5]). Let $\Omega : X \rightrightarrows Y$ be a concave-convex multifunction with closed compact non-empty values. If for every finite dimensional simplex $S \subset X$ with dim $S \geq 1$ and for every vertex $v \in S$ the following implication

$$\bigcap_{x \in S \setminus \{v\}} \Omega x \neq \emptyset \Rightarrow \bigcap_{x \in S} \Omega x \neq \emptyset \tag{1.5}$$

holds, then $\bigcap_{x \in X} \Omega x \neq \emptyset$.

2. Minimax theorems

The previous theorems enable us to give some minimax theorems for quasi-concave-convex functions which are not necessarily u.s.c. //.s.c. For example, by intersection Theorem 1.1 we have the following minimax theorem.

Theorem 2.1 (Greco [3]).: Let X be a convex subset of a topological vector space and let Y be a convex subset of a locally convex topological vector space. Let $f : X \times Y \to \overline{\mathbb{R}}$ be a quasi concave-convex function which is inf-compact at some point. If for every open subset V of Y

$$f \text{ is l.s.c. on } Y \quad \text{and} \quad \inf_V f \text{ is u.s.c. on } X \tag{2.1}$$

then f has a saddle value.

The condition (3.1) is equivalent to the following one :

(2.1')
$$\text{epi } f(x,\cdot) = \underset{x' \to x}{\text{Li}}{}^{\tau\times\nu} \text{ epi } f(x',\cdot) \quad \text{for every } x \in X,$$

where epi $f(x,\cdot) := \{(y, r) \in Y \times \mathbb{R} : f(x, y) \le r\}$ and $\tau\times\nu$ is the product of the topology τ of Y and the usual topology ν of real numbers.

Sion [2] proved that every u.s.c.//.s.c. quasi concave-convex function $f : X \times Y \to \overline{\mathbb{R}}$ has a saddle value, when X or Y is compact. Moreover, he exhibited an example to the effect that semicontinuity cannot be removed in general. Now we give a way to remove it ; namely the larger family of topologically closed functions will take place of that of u.s.c.//e.s.c. ones. A fucntion $f : X \times Y \to \overline{\mathbb{R}}$ is said to be **topologically closed** if for every $(x_o, y_o) \in X \times Y$ by :

(2.2)
$$\liminf_{y \to y_o} \limsup_{x \to x_o} f(x, y) \le f(x_o, y_o) \le \limsup_{x \to x_o} \liminf_{y \to y_o} f(x, y).$$

It is clear that every u.s.c.//.s.c. function is topologically closed. Moreover, for every f, the funcitons f^{+-}, f^{-+} defined for every $(x_o, y_o) \in X \times Y$:

$$f^{+-}(x_o, y_o) = \liminf_{y \to y_o} \limsup_{x \to x_o} f(x, y) \quad \text{and} \quad f^{+-}(x_o, y_o) = \limsup_{x \to x_o} \liminf_{y \to y_o} f(x, y).$$

are also topologically closed. The following equivalent form of Corollary 1.2 entails an improvement of the Sion theorem [2].

Theorem 2.2 (Greco [3]) : Let X, Y be convex subsets of topological vector spaces and let $f : X \times Y \to \overline{\mathbb{R}}$ be a quasi concave-convex function which is either inf-compact or sup-compact at some point. If f is topologically closed, then f has a saddle value.

Theorem 2.3 (Greco [3]) : Let X, Y be convex subsets of locally convex topological vector spaces and let $f : X \times Y \to \overline{\mathbb{R}}$ be a topologically closed quasi concave-convex function. Then

(2.3)
$$\inf_{y\in A} \sup_{x\in B} f(x, y) = \sup_{x\in B} \inf_{y\in A} f(x, y)$$

if A is a open (resp. compact) convex subset of Y and B is a compact (resp. open) convex subset of X.

We say that a function $f : X \times Y \to \overline{\mathbb{R}}$ satisfies the simplex property, if

(2.4)
$$\inf_{y\in Y} \sup_{x\in S\setminus\{v\}} f(x, y) = \inf_{y\in Y} \sup_{x\in S} f(x, y)$$

for every finite dimensional simplex $S \subset X$ with dim $S \geq 1$ and for every vertex $v \in S$. The intersection Theorem 1.3 entails the following minimax theorem.

Theorem 2.4 (Flam, Greco [5]) : Let X, Y be convex subsets of topological vector spaces and let $f : X \times Y \rightarrow \bar{\mathbb{R}}$ be a quasi concave-convex function which is inf-compact at each point. If f is l.s.c. on Y and verifies the simplex property, then f has a saddle value.

If $f(\cdot, y)$ is l.s.c. on segments in X, then for every $y \in Y$ and $v_0, v_1 \in X$ with $v_0 \neq v_1$ we have that

$$\sup_{x \in [v_0, v_1]} f(x, y) = \sup_{x \in [v_0, v_1]} f(x, y) ; \tag{2.5}$$

hence f have the simplex property. On the other hand, every $f : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ concave on X is l.s.c. on segments in X. Thus Theorem 2.3 improve the following Ky Fan theorem.

Theorem 2.4 (Ky Fan [1]) : Let $X \times Y \rightarrow \mathbb{R}$ be a finite concave-convex function. If Y is compact and f is l.s.c. on Y, then f has a saddle value.

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LES INEGALITES "INF-SUP ≤ SUP-INF"

JEAN GUILLERME

Depuis longtemps on s'intéresse naturellement à ce qu'on appelle les théorèmes de min-max. Dans l'énoncé le plus connu peut-être (Sion) la fonction considérée satisfait à l'égalité :

$$\min_{x\in X} \max_{y\in Y} f(x, y) = \max_{y\in Y} \min_{x\in X} f(x, y)$$

pourvu qu'elle possède à la fois des propriétés topologiques (semi-continuité) et des propriétés géométriques (quasi-convexité) et ceci en chaque variable.

Le besoin de séparer toutes ces propriétés apparaît par exemple (G. Greco) lorsque, disposant d'une fonction h quasi-convexe-concave, on souhaite lui appliquer un théorème de min-max ; on peut alors la "'régulariser" topologiquement, par semi-continuité inférieure (s.c.i.) sur X et supérieure (s.c.s.) sur Y mais on dispose alors de trois fonctions $f \leq h \leq g$. De quels théorèmes peut-on faire usage ? Quels sont les outils suffisants pour les obtenir ?

Notre but est de répondre à ces questions.

Précisément, étant donné quatre functions $f \leq s \leq t \leq g$ sur un ensemble produit $X \times Y$, on cherche des conditions topologiques sur f et g et géométriques sur s et t afin d'obtenir l'inégalité :

$$\mathrm{Inf}_{x\in X} \mathrm{Sup}_{y\in Y} f(x, y) \leq \mathrm{Sup}_{y\in Y} \mathrm{Inf}_{x\in X} g(x, y) \quad .$$

Les fonctions non constantes les plus simples sont celles qui ne prennent que deux valeurs, autrement dit ce sont les fonctions caractéristiques d'ensembles ; le problème précédent devient donc dans ce cas : étant donné quatre relations $F \subset S \subset T \subset G$ dans $X \times Y$, donner des conditions topologiques sur F et G et géométriques sur S et T afin d'obtenir l'implication :

$$[\forall x \in X \quad Fx \neq \emptyset] \Rightarrow \bigcap_{x\in X} Gx \neq \emptyset.$$

Evidemment, pour arriver à ce résultat, il suffit de supposer les coupes Gx fermées ($x \in X$) dont au moins une compacte, de façon à se limiter à montrer ce que l'on appelle la propriété de l'intersection finie :

"pour toute partie finie A de X : $\bigcap_{x \in A} Gx \neq \emptyset$" .

Deux approches sont utilisées pour obtenir cette propriété.

La première consiste à montrer ce résultat pour un ensemble à 2 éléments, puis 3, etc... C'est-à-dire à raisonner par récurrence. Mais on utilise alors à l'étape (n + 1) les résultats de l'étape n ce qui conduit nécessairement à imposer aux relations F à / et G d'être identiques. On obtient ainsi de nouveaux résultats d'égalités inf-sup = sup-inf dont des raffinements du Théorème de Sion. De là découlent de plus les premières inégalités souhaitées avec, disons brièvement, « s "convexe-like", t "concave-like", g scs »

La deuxième approche, directe, pour obtenir la propriété de l'intersection finie se scinde en deux suivant les moyens utilisés : le théorème de l'intersection de Berge ou le lemme de KKM ; ces deux énoncés donnent des conditions suffisantes pour que des ensembles en nombre fini se rencontrent. Grâce au Théorème de Berge (facile à montrer) on obtiendra l'inégalité souhaitée avec « f s.c.i., s = t quasi-convexe, quasi-concave, g s.c.s. »

Enfin le lemme (difficile) de KKM permet de dissocier toutes les propriétés géométriques et topologiques (« f s.c.i., s quasi-convexe, t quasi-concave, g s.c.s. ») en ayant affaibli les contraintes géométriques (on n'a plus besoin de supposer s = t comme à partir de l'application du théorème de Berge).

On voit ainsi, suivant H. Tuy, le rôle primordial de la connexité, puis vient celui de la convexité (Berge) et enfin, les propriétés les plus délicates sont donc obtenues par la méthode de K. Fan d'utilisation du lemme de KKM.

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AN INF-SUP INEQUALITY ON NONCOMPACT SETS

J. GWINNER

Extended Abstract : It is well known that minimax and fixed point theorems are very useful in different problems of nonlinear analysis. However for a more direct use in some applications, variants that dispense with compactness of the underlying sets can be more useful.

Therefore we present an inf-sup-inequality for two payoff functions $L_1, L_2 : C \times D \to \mathbb{R}$, where C is an arbitrary nonvoid subset of U, D is a nonvoid, closed convex subset of V, U and V are topological vector spaces in duality $<\cdot, \cdot>$. Here U is endowed with the weak topology $\sigma(U, V)$. Moreover we introduce

$$F_i(x, u) = \sup_{v \in D} \{<u, v> - L_i(x, v)\} \quad x \in C, u \in U ; \qquad i = 1, 2 ;$$

$$S_i = \{(u, t) \in U \times \mathbb{R} : F_i(x, u) \leq t \text{ for some } x \in C\} \qquad i = 1, 2.$$

Then we can establish the following result.

Theorem A : Let $L_1 \leq L_2$ on $C \times D$. Suppose that, for each $x \in C$, $L_1(x, \cdot)$ is convex and lower semicontinuous, and that S_2 is nonvoid, closed, and convex. Suppose, there Exists $u \in U$ such that

$$\sup_{x \in C} \inf_{y \in D} \{L_2(x, y) + <u, y>\}$$

is finite. Then this Sup Inf is a Max Inf and

$$\inf_{y \in D} \sup_{x \in C} \{L_1(x, y) + <u, y>\} \leq \max_{x \in C} \inf_{y \in D} \{L_2(x, y) + <u, y>\}.$$

Let us compare our Theorem A to the related inf-sup result of Simons in [7, Theorem 5]. There the convexity assumptions are relaxed to convex-likeness conditions. However, the inf-sup inequality on $C \times D$ is established only for compact D. But we do not conceal that our situation of paired topological vector spaces is more special than that of Simons.

Choosing $u = 0$; $D = C$ we arrive at the following theorem of alternative.

Theorem B : Let the assumptions of Theorem A with $u = 0$, $D = C$ be satisfied. Then for any $\mu \in \mathbb{R}$ the following alternative holds :

1°. There exists $x_0 \in C$ such that the inequation $L_2(x_0, y) < \mu$ does not admit a solution y in C.

or

2°. There exists $w \in C$ such that $L_1(w, w) < \mu$.

This latter result compares to the principal result of Deguire and Granas in [2], which is based on Ky Fan's extension of the classical Knaster-Kuratowski-Mazurkie vics theorem and which requires compactness of the set C.

To give an application, we treat variational inequalities of monotone kind in the sense of Minty [5] : for some given monotone-convex function $\varphi : C \times C \to \mathbb{R}$ one looks for some $\hat{x} \in C$ that satifies $\varphi(\hat{x}, y) \geq 0$ for all $y \in C$. In virtue of hemicontinuity, it is enough to find $\hat{x} \in C$ such that $\varphi(y, \hat{x}) \leq 0$ for all $y \in C$. By monotony $L_1(x, y) := \varphi(x, y) \leq -\varphi(y, s) =: L_2(x, y)$, and Theorem B applies. We note that already [2] gives an application to monotone variational inequalities of the type $\varphi(x, y) = (f(x), y - x)$ deriving the Hartman-Stampacchia theorem on weakly compact sets in reflexive Banach spaces. More related to our approach, Oettli [6] and Simons [7] established variational inequality results as a consequence of the Hahn-Banach theorem.

The cost of dropping the compactness hypothesis is the seemingly complicated closedness assumption on S_2. On the other hand, this latter closedness can be tackled in some more special situations. In particular if C is locally compact, an abstract closedness theorem in [1, 3], or if, as in Banach spaces, the Krein-Smulian theorem is available, a theorem on weak*closedness in [4] can be applied leading to more concrete assumptions.

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UN THEOREME DU POINT FIXE ET SOLUTIONS PERIODIQUES D'EQUATIONS DIFFERENTIELLES V-DISSIPATIVES

E. HANEBALY

§.1. Espaces de Banach à V-structure-Normale. Un théorème du point fixe.

1.1. La notion de Structure-Normale, introduite par Brodski-Milman caractérise certaines classes d'espaces de Banach sur lesquels une application contractante pourrait avoir un point fixe. Nous proposons dans la suite, une généralisation de cette notion, introduite par l'auteur dans [1], en utilisant une fonctionnelle V(x, y), au lieu de la fonction norme $|x - y|$; ceci dans l'esprit de Liapounov et de certains articles de F. Browder et P. Hartmann.

Définitions

• Soit E un espace de Banch réel, $V \in C[E \times E, \mathbb{R}^+]$ (= l'espace des fonctions continues de $E \times E$ dans $\mathbb{R}^+$) convexe par rapport à chacun de ses arguments et telle que :

$$V(x, x) = 0 \qquad \text{et} \qquad V(x, y) > 0 \ \text{ si } \ x \neq y$$

• Soit K un sous-ensemble de E fermé borné convexe et non-réduit à un point.

• On dira que K possède un point non diamétral par rapport à V s'il existe un $u \in K$ tel que :

$$\sup_{x \in K} V(x, u) < \sup_{(x, y) \in K \times K} V(x, y)$$

• On dira que K possède une V-Structure-Normale (V-S-N) si tout sous-ensemble de K, fermé convexe et non-réduit à un point, possède un point non-diamétral par rapport à V.

• On dira, enfin, que E possède une V-S-N si tout sous-ensemble de E, fermé borné convexe et non-réduit à un point, possède une V-S-N.

Exemples

1) Tout ensemble compact convexe et non-réduit à un point, d'un espace de Banach quelconque possède une V-S-N. En particulier l'espace $\mathbb{K}^n (= \mathbb{R}^n$ ou $\mathbb{C}^n)$ muni d'une norme quelconque possède une V-S-N.

2) Un espace de Banach uniformément convexe possède une V-S-N, où $V(x, y) = |x - y|$. Belluce-Kirk-Steiner ont donné l'exemple d'un espace de Banach reflexif non-uniformément convexe et qui possède une V-S-N, avec $V(x, y) = |x - y|$.

Notation : Soit $V \in C\,[E \times E, \mathbb{R}^+]$. On désignera par P1, P2, P3, P4 et P5 les propriétés suivantes :

P1 : $V(x, x) = 0$ et $V(x, y) > 0$ si $x \neq y$

P2 : V est convexe par rapport à chacun de ses arguments.

P3 : Il existe une constante $K > 0$ telle que :

$$|V(x, y) - V(\bar{x}, \bar{y})| \leq K\{|x - \bar{x}| + |y - \bar{y}|\}$$

P4 : $\lim\limits_{|x| \to +\infty} V(x, o) = +\infty$ ou $\lim\limits_{|y| \to +\infty} V(o, y) = +\infty$

P5 : $V(x_n, y_n) \to 0 \Rightarrow |x_n - y_n| \to 0$

1.2. Solution par itération des équations fonctionnelles.

Soit E un espace de Banach reflexif réel. Soit T une application de E dans E. Pour résoudre l'équation suivante :

$$x = Tx \tag{1.1}$$

on peut utiliser des conditions pour la suite d'itérations

$$x_{n+1} = Tx_n \; ; \; x_0 \text{ donné} \in E \tag{1.2}$$

Le théorème suivant généralise un théorème de Browder-Petryshyn :

Théorème : Soit $V \in C\,[E \times E, \mathbb{R}^+]$ vérifiant les propriétés P1-P4. On suppose que E possède une V-S-N et que T vérifie :

$$V(Tx, Ty) \leq V(x, y) \; ; \; \forall\, x, y \in E$$

Supposons enfin, qu'il existe un $x_0 \in E$ tel que la suite d'itérations $\{x_n\}_{n=0}^{\infty}$ dans (1.2) est bornée. Alors, l'équation (1.1) possède, au moins un point fixe.

La démonstration de ce théorème est basée essentiellement sur le théorème du point fixe suivant, généralisant le théorème connu de Browder-Cohde-Kirk (1965) :

Théorème : Soit $V \in C[E \times E, \mathbb{R}^+]$ vérifiant seulement les propriétés P1 et P2. On suppose que E possède une V-S-N. Soit B un sous-ensemble de E, fermé borné et convexe. Soit T une application V-contractante de B dans B, c'est-à-dire que :

$$V(Tx, Ty) \leq V(x, y) \;\; ; \;\; \forall\, x, y \in B$$

Alors T possède, au moins, un point fixe.

§.2. Equations différentielles V-Dissipatives.

2.1. Soit E un espace de Banach réel muni d'une norme notée |.|. Considérons le problème de Cauchy suivant :

(2.1) $$x' = f(t, x)$$

(2.2) $$x(a) = z \; ; \; (a, z) \in \mathbb{R} \times E \; ;$$

Où $f : \mathbb{R} \times E \to E$, $(t, x) \to f(t, x)$ est continue et bornante.

Définition

On dit que f est V-Dissipative, s'il existe une fonction $V \in C[I \times E \times E, \mathbb{R}^+]$, où $I = [a, +\infty)$, telle que, pour tout $(t, x, y) \in I \times E \times E$, on a :

$$D^-V(t, x, y) = \operatorname*{Lim\,Sup}_{h \to 0^-} h^{-1}\{V(t + h, x\; hf(t, x), y + h\, f(t, y)) - V(t, x, y)\} \leq 0$$

Exemples

1) Supposons qu'il existe une constante $L \geq 0$ telle que pour tout $(t, x, y) \in I \times E \times E$, on ait :

$$(f(t, x) - f(t, y), x - y)_- \leq L\, |x - y|^2$$

où $(.,.)_-$ denote le semi-produit-scalaire inférieur. Rappelons que :

$$(x, y)_\pm = \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \{y^*(x), y^* \in J_y\} \; ;$$

J désigne l'application de dualité, définie par :

$$J_y = \{y^* \in E^* : \langle y^*, y \rangle = |y|^2 = |y^*|^2\}$$

Alors, si on considère $V(t, x, y) = e^{-2Lt}|x - y|^2$, f est V-Dissipative.
2) Soit H un espace de Hilbert réel muni d'un produit scalaire noté (.,.) et de la norme associée à ce produit scalaire noté |.|. Considérons l'équation différentielle :

(*) $$x'' + x' \varphi(|x'|) + Qx = f(t)$$

où
- $Q \in \mathcal{L}(H)$, autoadjoint et défini positif.
- $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$., est continue monotone croissante
- $f : \mathbb{R}^+ \to H$, est continue.

Considérons les deux systèmes suivants associés à l'équation (*)

$$\begin{cases} x' = y \\ y' = - y\, \varphi(|y|) - Qx + f(t) \end{cases} \qquad \begin{cases} u' = v \\ v' = - v\, \varphi(|v|) - Qu + f(t) \end{cases}$$

Soit $V \in C[H^2 \times H^2, \mathbb{R}^+]$, où $V(x, y, u, v) = (|y - v|^2 + |V\bar{Q}(x - u)|^2)^{1/2}$
Alors $D^-V(x, y, u, v) \leq 0$

2.2. <u>Bornage et périodicité</u>

Théorème : Soit $V \in C\,[E \times E, \mathbb{R}^+]$ vérifiant P1-P5. On suppose que l'espace E est réflexif et posséde une V-S-N. On suppose aussi que f(t, x) est V-Dissipative et que f(t, x) est ω-périodique en t(ω > 0). Alors l'équation (2.1) possède une solution ω-périodique si, et seulement si, l'équation (2.1) possède une solution bornée sur [a, + ∞)

Ce théorème est comparé à des résultats de Baillon-Haraux, Browder et Mawhin-Willem.

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FIXED POINT THEOREMS FOR PSEUDO-MONOTONE MAPPINGS AND ITS APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

NORIMICHI HIRANO

In this talk, we show the existence of solutions for nonlinear elliptic equations by using fixed point theorems and the degree of pseudo-monotone operators. Let X be a Banach space and X^* be the dual of X. A multivalued mapping $T : X \to 2^{X^*}$ is said to be **pseudo-monotone** if it satisfies the following condition : For any sequence $\{x_n\}$ in X satisfying $x_n \to x$ weakly in X and any sequence $\{z_n\}$ in X^* with $z_n \in Tx_n$ for $n \geq 1$, for which

$$\limsup_{n\to\infty} < z_n, x_n - x > \leq 0,$$

there exists $z \in X^*$ such that $z \in Tx$ and

$$< z, x - y > \leq \liminf_{n\to\infty} < z_n, x_n - v >$$

for all $v \in X$(cf. [1]). it is known that a broad class of mappings (compact mappings, monotone mappings, ...etc. (cf. [2] and [3]) satisfy the definition of pseudo-monotone mapping.

We make use of degree theory for pseudo-monotone mappings to show the existence of nontrivial solutions of nonlinear elliptic equations. Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We denote by $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ the Sobolev space and its dual space. Let L be a linear (or quasilinear) elliptic operator on Ω and $g : \mathbb{R} \to \mathbb{R}$ be a continuous mapping. We consider the existence of the solutions of the problem

$$(*) \qquad \begin{cases} Lu - g(u) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is a mapping in $H^{-1}(\Omega)$.

The existence and multiplicity of the solutions of the problem (*) has been studied by many authors (cf. [1], [2]). Here we put

$$T(u) = Lu - g(u) - f \qquad \text{for each } u \in H_0^1(\Omega).$$

then T is a pseudo-monotone mapping under a certain condition for L and g. Then we can make use of fixed point theorems and degree theory of pseudo-monotone mappings for solving this problem. We will show some existence results for the problem (*).

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NONCOOPERATIVE GAMES DEFINED BY RELATIONS

ADAM IDZIK

A new fixed point theorem of the Schauder-Tychonoff type, proved for general topological vector spaces by Idzik [5], is applied to show the existence of the Nash equilibrium of a noncooperative game defined by relations.

In this paper we apply a new fixed point theorem for general topological vector spaces to show the existence of an equilibrium of a noncooperative game defined by preference relations for an arbitrary number of players. Such games are a natural generalization of games with constraints. Some theorems of this type for noncooperative games with constraints were presented by Idzik [4].

Let X, Y be topological spaces and let 2^Y denote, as usual, the space of closed subsets of Y with the exponential topology. A function $g : X \to 2^Y$ is **u.s.c.** (**l.s.c.**) iff for every closed (open) subset $B \subset Y$ the set $\{x \in X : g(x) \cap B \neq \emptyset\}$ is closed (open). The function g is **continuous** iff it is u.s.c. and l.s.c.

Throughout this paper E denotes a real Hausdorff topological vector space.

A set $B \subset E$ is **convexly totally bounded** (c.t.b. for short), if for every neighbourhood V of $0 \in E$ there exist a finite subset $\{x_i : i \in I\} \subset B$ and a finite family of convex sets $\{C_i : i \in I\}$ such that $C_i \subset V$ for $i \in I$ and $B \subset \cup\{x_i + C_i : i \in I\}$.

A set $A \subset E$ is **almost convex**, if for every neighbourhood V of $0 \in E$ and every finite set $\{w_1,\dots,w_n\} \subset A$ there exists a finite set $\{z_1,\dots,z_n\} \subset A$ such that for every $i \in \{1, \dots, n\}$ $z_i - w_i \in V$ and $co\{z_1,\dots,z_n\} \subset A$.

By a **noncooperative game** defined by relations we understand a family

$$\Gamma = \{(A_i, g_i <^i) : i \in I\}.$$

The set I is a nonempty subset of players and for each $i \in I : A_i$ is a decision set for the player i, the function $g_i : A \to 2^{A_i}$ constraints decisions of the player i ($A = \Pi\{A_i : i \in I\}$, 2^{A_i}

denotes here the family of all subsets of A_i) and the relation function $<^i : A \to 2^{A_i \times A_i}$ is such that for every $x \in A$, $\overset{i}{<_x}$ on the set $g_i(x)$. Let $\overline{g}_i(x)$ denote the set of maximal elements of the relation $\overset{i}{<_x}$ on the set $g_i(x)$.

An element x satisfying the condition $x \in \Pi\{\overline{g}_i(x) : i \in I\}$ is called an **equilibrium** of the game Γ.

The theorem below generalizes Theorem 2 of Flam [1], Theorem 7 of Idzik [3] and Theorem 17.1 of Makarov and Rubinov [6].

Theorem : let $\{E_i : i \in I\}$ be a family of real Hausdorff topological vector spaces. If for every $i \in I$: A_i is a nonempty almost convex subset of E_i ; $g_i : A \to 2^{A_i}$ is a continuous function, $A = \Pi\{A_i : i \in I\}$; C is a dense subset of A ; $\overline{g_i(A)}$ is the compact, c.t.b. subset of A_i ; for every $x \in A$ $\overset{i}{<_x}$ is an irreflexive and transitive relation on $g_i(x)$ such that the set $\{(x, y, y') \in A \times A_i \times A_i : y, y' \in g_i(x)$ and non $y \overset{i}{<_x} y'\}$ is closed in $A \times A_i \times A_i$; $\overline{g}_i(x)$ (the set of maximal elements of the relation $\overset{i}{<_x}$ on the set $g_i(x)$) is convex for every $x \in C$, then there exists an equilibrium of the game Γ.

For a proof of our theorem we apply Theorem 3 on p.29 in Hildenbrand [2] and Theorem 4.3 in Idzik [5].

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FIXED POINT THEOREMS ON CONVEX CONES, GENERALIZED PSEUDO-CONTRACTIVE MAPPINGS AND THE COMPLEMENTARITY PROBLEM

G. ISAC

In this paper we use the **Multivalued Complementarity Problem** and, respectively, the **Implicit Complementarity Problem** to obtain some fixed point and, respectively, some coincidence theorems on convex cones in Hilbert spaces.

In particular we obtain some fixed point theorems for generalized pseudo-contractive multivalued mappings.

The Complementarity Problem is much studied this time, since it has important applications in : Optimization, Game Theory, Mechanics, Elasticity Theory, Engineering etc and its connections with the Fixed Point Theory *is an interesting mathematical fact.*

Let $(H, <,>)$ be a Hilbert space and let $K \subset H$ be a closed convex cone.

We denote by K^* the dual of K.

Given a *point-to-set mapping* $f : K \rightarrow H$ the **Generalized Multivalued Complementarity Problem** associated to f and K is :

$$\text{G.M.C.P.} \quad (f, x) : \quad \left\| \begin{array}{l} \text{find } x_o \in K \text{ and } y_o \in H \text{ such that} \\ y_o \in f(x_o) \cap K^* \text{ and} \\ <x_o, y_o> = 0 \end{array} \right.$$

Now, supposing given two mappings $S : K \rightarrow K$; $T : K \rightarrow H$ the **Implicit Complementarity Problem** associated to T, S and K is :

$$\text{I.C.P.} \quad (T, S, K) : \quad \left\| \begin{array}{l} \text{find } x_o \in K \text{ such that} \\ T(x_o) \in K^* \text{ and } < S(x_o), T(x_o) > = 0 \end{array} \right.$$

Proposition 1.

Suppose that f has the form $f(x) = x - S(x)$, where $S : K \rightarrow$ is a point-to-set mapping.

If $(x_o, y_o) \in K \times H$ is such that $y_o \in f(x_o)$ and (x_o, y_o) is a solution of the problem G.M.C.P. (f, K) then x_o is a fixed point of S in K. □

Consider also the following special variational inequality :

$$\textbf{S.V.I.} \quad \textbf{(T, S, K)} : \quad \left\| \begin{array}{l} \text{find } x_o \in K \text{ such that} \\ \langle x - S(x_o), T(x_o) \rangle \geq 0 \, ; \, \forall \, x \in K. \end{array} \right.$$

Proposition 2.

The problem S.V.I. (T, S, K) is equivalent to the problem I.C.P. (T, S, K). □

Proposition 3.

The problem S.V.I. (S-T, S, K) is equivalent to the coincidence equation :

$$\text{C.I.} \quad \textbf{(S, T, K)} : \quad \left\| \begin{array}{l} \text{find } x_o \in K \text{ such that} \\ S(x_o) = T(x_o) \, . \end{array} \right.$$

Using **Propositions 1, 2, 3,** The **Eilenberg-Montgomery Fixed Point Theorem** the **Ky Fan's Section Theorem** and other techniques based on the **Complementarity *Theory*** *we obtained* several fixed point and coincidence theorems on convex cones in Hilbert spaces.

Between the obtained results we have the following theorems. (The notions are defined in our papers [1], [2]).

Theorem 1.

Let $K \subset H$ be a locally compact cone and $S : K \to K$ an upper semicontinuous point-to-set mapping with S(x) nonempty and contractible for each $x \in K$.

If S is generalized pseudo-contractive and the problem G.M.C.P. (f, K) (where f(x) = x - S(x)) is strictly feasible then S has a fixed point in K. □

Theorem 2.

let $K(K_n)_{n \in N}$ be a pointed closed Galerkin cone in a Hilbert space (H, < , >) and let $S : K \to K$ be a point-to-set mapping.

Denote $f(x) = x - S(x) \, ; \, \forall \, x \in K$.

If the following assumptions are satisfied :

1) S is completely upper seconitnuous,

2) for every $x \in K$, S(x) is nonempty and contractible,

3) one of the following is satisfied :

a) f satisfies the condition (G.K.C.) with an equibounded family $\{D_n\}_{n\in N}$,

b) f satisfies the condition (G.K.C.) and S is φ-asymptotically bounded with $\lim_{r\to\infty} \varphi(r) \neq +\infty$, then S has a fixed point in K. □

Theorem 3.

Let (H, < , >) be a Hilbert space and $K \subset H$ a closed convex cone.

Suppose given $S, T : K \to K$ continuous mappings and $f : K \to K$ a positive homogeneous mapping of order $p_2 > 0$.

If the following assumptions are satisfied :

1) the mapping $x \to < f(x), S(x) >$ is upper semicontinuous in K and $< f(x), S(x) > \; > 0$ for every $x \in K$, with $\|x\| = 1$,

2) there exist $c_1 > 0$, $r_1 > 0$ and $p_1 > 0$ such that, $S(\lambda x) <_{K^*} c_1.\lambda.^{p_1} S(x)$, for every $x \in K$ with $\|x\| \geq r_1$ and $0 < \lambda \leq 1$,

3) there exist $c_2 < 0$ and $r_2 > 0$ such that $<f(x), S(x)> \leq c_2.<S(x), S(x)>$, for every $x \in K$ with $\|x\| \geq r_2$,

4) $\lim_{\|x\| \to\infty} \sup \frac{<S(x), T(x)>}{\|x\|^{p_1+p_2}} \leq 0$,

5) there exists $c_3 > 0$ such that $c_3.<S(x), S(x) - T(x)> \leq <x, S(x) - T(x)>$; $\forall x \in K$ then there exists $x_* \in K$ such that $S(x_*) = T(x_*)$. □

The results presented here are proved in our papers [1], [2].

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GRASSMANN MANIFOLDS AND A BORSUK-ULAM THEOREM FOR THE ORTHOGONAL GROUP

JAN JAWOROWSKI

We start with the following version of the classical Borsuk-Ulam Theorem : Let $f : S^n \to \mathbb{R}^k$ be a map and let $A_f = \{x \in S^n \mid f(x) = f(-x)\}$. Then, if $k \leq n$, $A_f \neq \emptyset$. In fact, results due to Yang give an estimation of the size of A_f of the cohomology index, which is an integer. This classical theorem is about the antipodal action of the group $G = \mathbb{Z}_2$ on S^n. A tool used in estimating the size of A_f in terms of the cohomology index is the first Stiefel-Whitney class of a space with a free involution. Analogous results for free actions of the groups $G = S^1$ and $G = S^3$ use the first Chern class, and the first Pontriagin class, respectively (compare [5] and [6]). The question of Borsuk-Ulam type theorems obtained by using other characteristic classes was studied by the author in [7] and [8]. In order to do this we need to have the concept of index defined in a much more general setting, for a space X with an action of a compact Lie group G. This general idea of index, defined as an ideal in a ring, was defined by Fadell [3, and [4] ; it was also arrived at independently by this author [7]. In the general setting, the index is no longer an integer, but an ideal in a ring.

Suppose that G is a compact Lie group acting on a (paracompact) space X. Let $H^*_G X$ be the Borel G-cohomology of X (the coefficients in $\mathbb{Z}_2$ will be used throughout). If (.) is a one-point space, then $H^*_G(.)$ can be identified with $H^*(BG)$, the ordinary cohomology of the classifying space of G. If G acts freely on X, then $H^*_G X$ is just the ordinary cohomology of the orbit space X/G of the action. The G-index, $\mathrm{Ind}^G X$, of X is defined to be the kernel of the G-cohomology map $H^*_G(.) = H^*(BG) \to H^*_G X$ induced by the constant map $c : X \to (.)$ of X into a one-point space. In the classical case, when $G = \mathbb{Z}_2$ is acting freely on X, the generator of $\mathbb{Z}_2$ represents a free involution on X. In this case, $EG = \mathbb{R}P^\infty$, $H^*_G(.) = H^*(\mathbb{R}P^\infty)$ is a polynomial algebra on one generator of dimension one, the first Stiefel-Whitney class of the involution, $w_1 \in H^1_{\mathbb{Z}_2}(.) = H_1(\mathbb{R}P^\infty)$. Its image under c^* in $H^1_{\mathbb{Z}_2} X = H^1(X/\mathbb{Z}_2)$ is the characteristic class of the involution. The kernel of c^* is the ideal generated by $w_1{}^{n+1}$, for some integer n, and thus $\mathrm{Ind}^{\mathbb{Z}_2} X$ can be identified with this integer. This corresponds to the classical definition of index.

In Borsuk-Ulam type theorems for a general compact Lie group G we usually consider a map $f : X \to W$ of a G-space X into another G-space W, for instance, into a representation space for G ; and we try to estimate the size of the set A_f where the G-symmetry becomes degenerated under f. For instance, if : $X \to W$ is an equivariant map of X into a representation space, we may want A_f to be the set of zeros of f. More generally, for any invariant subset W' of W, we can set $A_f(W') := f^{-1}W'$. If we don't necessarily want to start with an equivariant tmap $f : X \to W$, we can apply the averaging construction, replace f by is average Av $f : X \to W$ and define $A_f(W') := (\text{Av } f)^{-1}W'$. The classical Borsuk-Ulam theorem asserts that for any map $f : S^n \to \mathbb{R}^n$ there exists a point where the average of f (with respect to the antipodal actions on the source space and the target space) is zero.

The following is a direct generalization of the Borsuk-Ulam-Yang situation of $f : S^n \to \mathbb{R}^k$ from the group $G = \mathbb{Z}_2$ to $G = 0(m)$:

Let X be the Stiefel manifold $V_m(\mathbb{R}^{m+n})$ of orthonormal m-frames in $\mathbb{R}^{m+n}$ and let $f : X = V_m(\mathbb{R}^{m+n}) \to (\mathbb{R}^{m+k-1}) = W$ be a map. In order words, f assigns to every m-frame in $\mathbb{R}^{m+n}$ an m-tuple of vectors in $\mathbb{R}^{m+k-1}$. Let W' be the subset consisting of all m-tuples which are not linearly independent. We are asking about the size of $A_f = (\text{Av } f)^{-1}W'$; i.e., this is the degeneracy set in our example. Here the group $G = 0(m)$ acts on $V_m(\mathbb{R}^{m+n})$ and on $\mathbb{R}^{m+k-1}$ in the standar way (on the right). Under this action, W-W' is exactly the part of W where 0(m) acts freely. of course, if $m = 1$ then $X = S^n$, $W = \mathbb{R}^k$, and we are in tne Borsuk-Ulam situation.

In [7] and [8] we studied the size of the degeneracy set A_f in this case and obtained estimates of its size. In order to do this we needed a workable description of the 0(m)-index of $V_m(\mathbb{R}^{m+n})$. Our results, however, were incomplete because we did not have sufficient information about appropriate bases for the 0(m)-cohomology of this pace. This question, and its solution, will be discussed here.

Let $V_m = V_m(\mathbb{R}^\infty)$ be the infinite Stiefel manifold of orthonormal m-frames in $\mathbb{R}^\infty$ and $G_m = V_m / 0(m)$ be the infinite Grassmanniann of m-planes in $\mathbb{R}^\infty$. The 0(m)-cohomology of V_m and $V_m(\mathbb{R}^{m+n})$ or, which is the same, the ordinary cohomology ring of the Grassmann manifolds G_m and $G_m(\mathbb{R}^{m+n}) = V_m(\mathbb{R}^{m+n}) / 0(m)$, has at least two quite different descriptions. On the one hand, Chern [2] gave a description of the cohomology rings of the Grassmannians by means of a specific cellular decomposition constructed by Ehresmann. On the other hand, Borel in this thesis [1] showed that the cohomology ring of G_m is a polynomial algebra $\mathbb{Z}_2[w_1,\ldots,w_m]$ on the Stiefel-Whitney classes $w_1,\ldots,w_m$ and $H^*G_m(\mathbb{R}^{m+n})$ is a quotient of this algebra by an ideal ; this ideal turns out to be exactly the 0(m)-index of $V_m(\mathbb{R}^{m+n})$.

It can be shown that Chern's description of the cohomology of Grassmannians implies that the dimension of the vector space H^rG_m is equal to the number of partitions of r into m non-negative integers ; and the dimension of $H^rG_m(\mathbb{R}^{m+n})$ is equal to the number of such partitions in which each part is at most n. The question mentioned above is what monomials in the Stiefel-Witney classes form a basis for this vector space. We answer this question by proving the following theore

Theorem : The set of all monomials $w_1^{r_1},\dots, w_m^{r_m}$ in $w_1,\dots,w_m$ satisfying $r_1+\dots+r_m \le n$ forms a basis for $H^rG_m(\mathbb{R}^{m+n})$.

This theorem can, in particular, be used to obtain a more precise information about the size of the degeneracy set A_f for maps $f : V_m(\mathbb{R}^{m+n}) \to (\mathbb{R}^{m+k-1})$ of the Stiefel manifolds into a Euclidean space.

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ON THE CONVERGENCE OF THE MANN AND GENERALIZED ISHIKAWA ITERATIVE PROCESSES IN BANACH SPACE

ALBERT K. KALINDE

I. Introduction

Let T be a self-mapping of a nonempty, closed, convex and bounded subset K of a Banach space E. For an arbitrary element v_0 of K, the **Mann iteration process of** T is defined by $v_{n+1} = (1 - d_n)v_n + d_n Tv_n$ for all the natural numbers n, where the sequence $(d_n)_{n\in N}$ of real numbers satisfies the following conditions : (m_1) $d_0 = 1$, (m_2) $0 \leq d_n \leq 1$ for all the integers $n > 0$ and (m_3) $\sum_{n=0}^{\infty} d_n = \infty$ (cf. [5]). The **generalized Ishikawa iteration scheme of** T is defined by $u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tv_n$ for all $n \in N$ where $v_n = (1 - \beta_n)u_n + \beta_n Tu_n$ and the two sequences $(\alpha_n)_{n\in N}$, $(\beta_n)_{n\in N}$ of real numbers are subject to the conditions that (i_1) $0 \leq \alpha_n$, $\beta_n \leq 1$ for all $n \in N$, (i_2) $\lim_{n\to\infty} \beta_n = 0$ and (i_3) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$. If $\beta_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $(u_n)_{n\in N}$ gives rise to a Mann iterative process of T, whereas if $\beta_n = 0$ and $\alpha_n = 1$ for all $n \in N$, $(u_n)_{n\in N}$ is simply the usual sequence of iterates of u_0 (see [9] p. 253).

The Mann iteration process introduced above can also be represented by $v_{n+1} = \sum_{k=0}^{n} a_{nk} Tv_k$ (cf. [8]), where a_{nk} is given by

$$a_{nk} = \begin{cases} \prod_{j=k+1}^{n} (1 - d_j) & \text{if } k < n \\ d_n & \text{if } k = n \\ 0 & \text{if } k > n \end{cases}$$

and defines a summability triangular matrix A that is regular.

Mann iteration processes have been extensively studied by many authors and for various types of mappings in some Banach spaces. For pseudocontractive operatorsT in Banach space E however, that is for operators T from $\mathcal{D}(T)$ in E into E such that

$$\|u - v\| \leq \|(1 + r)(u - v) - r(Tu - Tv)\| \tag{1.1}$$

holds for all u, v ∈ $\mathcal{D}(T)$ and all real numbers $r > 0$, it is not yet known (even in the Hilbert space setting) whether or not the associated Mann iteration process converges to a fixed point of T, even when T satisifes a Lipschitzian condition (cfr. [3]).

The purpose of this paper is to investigate the convergence of the Mann and generalized Ishikawa iteration processes to a fixed point of operators T satisfying the condition

$$\|Tu - Tv\| \leq \beta\|(u - Tu) - (v - Tv)\| \tag{1.2}$$

for all u, v ∈ $\mathcal{D}(T)$ in E, with $\beta > 0$. Such mappings are Lipschitzian strictly pseudocontractive for $0 < \beta < 1$ and both pseudocontractive and nondemicontractive for $\beta = 1$. We show that under a nonLipschitzian condition, the Mann and generalized Ishikawa iterative processes of T with $\beta = 1$ converge in E to a fixed point of T in uniformly convex and general Banach spaces respectively.

The details of the proofs and some other results of this work will be published elsewhere.

II. Preliminaries

The operator T defined by (1.2) has at most one fixed point, as I-T is injective. When T is closed, $(I-T)^{-1}$ *is continuous* on the range $\mathcal{R}(I-T)$ of I-T. In addition to that, for $0 < \beta < 1$, not only T is strictly pseudocontractive, but also T is Lipschitzian of constant $\frac{\beta}{1-\beta}$.

Definition 2.1. : A mapping T from $\mathcal{D}(T)$ in Hilbert space H into itself is said to be **demicontractive** (cf. [5]) if there exists a nonnegative real number $k < 1$ such that

$$\|Tu - p\|^2 \leq \|u - p\|^2 + k\|u - Tu\|^2 \tag{2.3}$$

for each fixed point p of T and each u ∈ $\mathcal{D}(T)$. If $k = 0$, T is called a **quasi-nonexpansive mapping.**

Definition 2.2. : A *self-mapping* T of a Banach space E is said to be **quasi-demicontractive** if there exists $\alpha > 0$ such that

$$\|u - Tu\| \leq \alpha\|u - p\|$$

holds for each $u \in \mathcal{D}(T)$ and each fixed point p of T.

In Hilbert space, any demicontractive operator is quasi-demicontractive, since it can be shown that

$$\|u - Tu\| \leq \frac{2}{1 - \sqrt{k}} \|u - p\|$$

for each $u \in \mathcal{D}(T)$ and each fixed point p of T, where k is a constant defined in (2.3). Any quasi-nonexpansive and any Lipschitzian operators having fixed point in Banach space are quasi-demicontractive.

III. Main results

Our first result is a necessary and sufficient condition for the generalized Ishikawa iterative process of an operator T defined by (1.2) with $\beta = 1$ to converge to a fixed point of T when condition (2.4) with $\alpha = 1$ is satisfied.

Theorem 3.1 : Let K be a nonempty, closed, convex subset of a Banach space E and $T : K \to K$ be an operator such that

$$\|Tu - Tv\| \leq \|u - Tu) - (v - Tw)\|. \tag{3.5}$$

If T satisfies the quasi-demicontractive condition for $\alpha = 1$, then the sequence $(u_n)_{n\in N}$ of generalized Ishikawa iterates converges to a fixed point of T if and only if $\inf_{n\in N} \|u_n - p\| = 0$.

The necessary condition is obvious. For the sake of brevity, the sketch of the proof of the sufficient condition consists of showing that the sequence $(\|u_n - p\|)_{n\in N}$ is monotone decreasing in R and of showing that $(u_n)_{n\in N}$ is a Cauchy sequence.

Remark : The condition $\inf_{n\in N} \|u_n - p\| = 0$ is obviously weaker than $\lim_{n\to\infty} d(u_n, \{p\}) = 0$ obtained from the given condition in [9].

We remember that a Banach space E is said to be **uniformly convex** if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $\|u\| \leq 1$ and $\|v\| \leq 1$, $\|u - v\| \geq \varepsilon$ implies that $\|u + v\| \leq 2(1 - \delta(\varepsilon))$.

Theorem 3.2. : Let K be a closed and convex subset of a uniformly convex Banach space E and $T : K \to K$ be an operator defined by (3.5) and satisfying the quasi-demicontractive condition (2.4) with $\alpha = 1$. Then the sequence $(u_n)_{n \in N}$ defined by $u_{n+1} = (1 - d_n)u_n + d_n Tu_n$ for an arbitrary $u_0 \in K$, where the sequence $(d_n)_{n \in N}$ satisfies the conditions : (i) $d_0 = 1$, (ii) $0 \leq d_n \leq 1$ for all $n > 0$ and (iii) $\sum_{n=0}^{\infty} d_n(1 - d_n) = \infty$ converges to the unique fixed point of T.

This theorem is an extension of theorem 1 in [5] and theorem 4 in [8] to the class of pseudocontractive operators that satisfy (3.5). Its proof is an adaptation of the proof of theorem 4 in [8] and makes use of the lemma established in [4] (see p. 370).

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CONTINUOUS LATTICES, SEMICONTINUOUS FUNCTIONS, AND A KAKUTANI TYPE FIXED POINT THEOREM

KLAUS KEIMEL

The theory of continuous lattices (cf. G. Gierz, K. H. Fofmann, K. Keimel, J. D. Lawson, M. Mislove, D. Scott : A compendium of continuous lattices, Springer Verlag 1980) and their intrinsic topological properties catches essential features of hyperspaces like spaces of closed sets, spaces of closed convex sets, etc., as well as of spaces of semicontinuous (real valued and set-valued) functions. One only needs sufficient compactness assumptions around. Also spaces of superharmonic functions can be dealt with in this framework.

It will be the purpose of this talk to present this order theoretical approach with some applications. In particular, we will present a transfer principle for the fixed point property which can be viewed as an order theoretical formulation of Ky Fan's extension of Kakutani's fixed point theorem from the finite to the infinite dimensional case. This principle also works for fixed point theorems on trees. The results on fixed points are joint work with A. Wieczorek, Warsaw.

UN THEOREM DE POINT FIXE SUR LES RETRACTS L-LOCAUX

M. A. KHAMSI ET M. POUZET

Abstract : In this work we give an abstract definition of a metric space. We prove can analoguous theorem to Kirk's result. We therefore, look to the extension to any commutative family of mappings.
The fact of uniting under the same theorem ordered and linear sets. Thus presents a clear advantage of this formulation.

Résultats et définitions

On commence par définir un espace métrique abstrait.

Définition 1 : Soit M un ensemble. On dira que $d : M \times M \to \vartheta$ définit une structure métrique abstraite sur M ssi

1) ϑ est un ensemble ordonné avec un élément minimum, que l'on notera 0.
2) Toute partie de ϑ a un inf.
3) Toute partie majorée de ϑ a un sup.
4) d satisfait :
 (i) $d(x, y) = 0$ ssi $x = y$
 (ii) $d(x, y) = \tau(d(y, x))$, où τ définit une involution sur ϑ qui préserve l'ordre.

Exemples : (1) Tout espace métrique a une structure métrique abstraite.
(2) Tout espace modulaire [2] a une structure métrique abstraite.
(3) Tout espace métrique généralisé (voir [1] [4]) a une structure métrique abstraite.

Définition 2 : Soient (M, d, ϑ) un espace métrique abstrait et N une partie de M.
(i) On écrira

$$\delta(N) = \mathrm{Sup}\{d(x, y) \,;\, x, y \in N\}$$
$$r(x, N) = \mathrm{Sup}\{d(x, y) \,;\, y \in N\}$$
$$R(N) = \inf\{r(x, N) \,;\, x \in N\}$$
$$\mathcal{C}(N) = \{x \in N \,;\, r(x, N) = R(N)\}$$

(ii) On dira que N est une boule ssi $\exists$ a $\in$ M et r $\in$ ϑ tel que

$$N = \{b \in M ; d(a, b) \leq r\}$$

(iii) On dira que N est une partie admissible ssi N est une intersection de boules.
On notera par $\mathcal{A}(M)$ la famille des parties admissibles.

Comme ce travail traite des résultats de point fixe, on aura besoin de la définition suivante.

Définition 3 : Soit (M, d, ϑ) un espace métrique abstrait. On dira que T : M - M est une contraction si

$$d(Tx), T(y)) \leq d(x, y) \qquad \text{pour } x, y \in M.$$

On notera par Fix(T) l'ensemble des points fixes de T.

Avant de donner l'énoncé du théorème de Kirk [3] dans les espaces métriques abstraits, on aura besoin de la définition suivante.

Définition 4 : Soit (M, d, ϑ) un espace métrique abstrait. On dira que
(i) $\mathcal{A}(M)$ est normal ssi pour toute partie A $\in$ $\mathcal{A}(M)$, qui a plus de deux éléments, vérifie $\mathcal{C}(A) \neq A$.
(ii) $\mathcal{A}(M)$ est compacte ssi $\mathcal{A}(M)$ a la propriété d'intersection finie-infinie.

Notre premier résultat peut s'énoncer de la manière suivante

Théorème 1 : Soit (M, d, ϑ) un espace métrique abstrait. On suppose que $\mathcal{A}(M)$ est normal et compact, alors toute contraction définie sur M a un point fixe.

Pour étendre ce résultat au cas d'une famille quelconque, on démontrera que l'ensemble des points fixes d'une contraction a une propriété de retraction.

Théorème 2 : Sous les hypothèses du théorème 1, toute famille commutative de contractions a un point fixe commun.

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COURNOT-NASH EQUILIBRIUM DISTRIBUTIONS FOR GAMES WITH DIFFERENTIAL INFORMATION

ALI KHAN AND ALDO RUSTICHINI

In recent work, Khan-Rustichini (1989) have extended Mas-Colell's (1984) formulation of Cournot-Nash equilibria of "large" anonymous games to a setting with uncertainty and imperfect information. However, they assume that the imperfect information is commonly (publicly) held. In this paper, we show that their results extend to one-shot games in which each agent is allowed information which is particular to him or her.

There are two essential ideas underlying this extension. The first is to make information part of the characteristics of each agent. This is, of course, the motivation for the work of Allen (1983) and Cotter (1986) on the study of topologies on information. The second idea responds to the difficulty that in the presence of differential information each player's optimal strategy is measurable only in terms of his/her information. We simply imbed all these strategies in a larger space derived from a suitable union of the variety of information i.e., the smallest space of information containing all individual information.

In terms of technicalities, we need three results in order to extend the proofs in Khan-Rustichini to our generalized set-up. The first is to show that this larger space in which all the individual strategies are imbedded nevertheless remains compact. Second, we have to ensure that a pointwise limit of a sequence of strategies each measurable with respect to a particular information (sub-σ-algebra) retains measurability with respect to the limit of the sequence of informations (sub-σ-algebras). The third is to show that Cotter's pointwise topology retains the property of complete regularity in our generalized set-up.

We now develop some preliminary notation. Let $(\Omega, \mathcal{F}, \mathrm{Pr})$ be an abstract probability space with Ω a countable set. Let A be a compact metric space. Let **F** be the set of sub-σ-algebras of $\mathcal{F}$ with any particular sub-σ-algebra $\mathcal{G} \in \mathbf{F}$ denoting incomplete information. We endow the set F with a topology proposed by Cotter (1986) under which

$$\mathcal{F}^{\nu} \to \mathcal{F} \Leftrightarrow E_{\mathcal{F}^{\nu}}f \to E_{\mathcal{F}}f \text{ for all } f \in L_1(\mathrm{Pr}, R),$$

where $\{\mathcal{F}^\nu\}$ is a net chosen from F. Let Meas (Ω, Pr ; A) be the space of measurable functions from (Ω, $\mathcal{F}$, Pr) to A and endowed with the topology of convergence in measure. When we consider a subset of $\mathcal{G}$-measurable functions, $\mathcal{G} \in$ **F**, we shall abbreviate this set to $A^{\mathcal{G}}$. Let $\mathcal{M}^1_+ (A^{\mathcal{F}})$ be the space of Radon probability measures on $A^{\mathcal{F}}$ endowed with the weak* topology.

Next, we turn to the space of payoffs. let $C(A^{\mathcal{F}})$ be the space of continuous functions from $A \times \mathcal{M}^1_+ (A^{\mathcal{F}})$ to the real line R and endowed with the sup-norm topology. Let Meas(Ω, PR ; $C(A \times \mathcal{M}^1_+ (A^{\mathcal{F}}))$) be the space of measurable functions from (Ω, $\mathcal{F}$, Pr) to $C(A^{\mathcal{F}})$ and endowed with the topology of convergence in measure. We shall abbreviate this space to $\mathcal{U}_A$. All that remains to be discussed is conditioning due to imperfect information $\mathcal{G} \in$ **F** which is available to an individual player. We simply take the conditional expectation with respect to $\mathcal{G}$ of random variables taking values in $C(A^{\mathcal{F}})$. Let $L_1(Pr, C(A^{\mathcal{F}}))$ be the space of Bochner integrable functions from (Ω, $\mathcal{F}$, Pr) to $C(A \times \mathcal{M}^1_+ (A^{\mathcal{F}\cdot}))$.

For any $f \in L_1(Pr, C(A^{\mathcal{F}}))$, let $E_{\mathcal{G}}f$ be the conditional expectation of f with respect to $\mathcal{G}$. Recall from Diestel-Uhl (1977, Chapter V.1) that this is well-defined.

A player then is an element (u, $\mathcal{G}$) of the space ($\mathcal{U}_A \times$ F). We shall denote this space by $\mathcal{P}_m$, m for *measurable functions*. From what has been said so far, certaintly the space of players is a topological space.

Definition 1 : A game μ with imperfect and differential information is a Radon probability measure on $\mathcal{P}_m$.

For any game μ, let $\text{supp}_2\, \mu$ be the projection of the support of μ on F. Let $\mathcal{H}$ be the smallest σ-algebra that contains $\cup_{\mathcal{F} \in \text{supp}_2 \mu}\, \mathcal{F}$. We can now present

Definition 2 : A Borel probability measure τ on $A^{\mathcal{H}} \times \mathcal{P}_m$ is a Cournot-Nash equilibrium distribution of a game μ if

(i) $\tau p_m = \mu$, and (ii) $\tau(Br) = 1$,

where subscripts on τ denote marginals and where

$B_r \equiv \{(a, (u, \mathcal{G})) \in A^{\mathcal{G}} \times \mathcal{P}_m : ((E\mathcal{G}\ u)\ (\omega))\ (a(\omega)\ ;\ \tau_A x) \geq ((E\mathcal{G}\ u)\ (\omega))\ (x\ ;\ \tau_A x)\ \forall x \in A,$ almost every $\omega \in \Omega\}$.

The payoffs are defined, in part, on $\mathcal{M}^1_+ (A^{\mathcal{F}})$ but a player can be more specific and focus on elements of $\mathcal{M}^1_+ (A^{\mathcal{H}})$. Since $\mathcal{M}^1_+ (A^{\mathcal{H}}) \subseteq \mathcal{M}^1_+ (A^{\mathcal{F}})$, everything is well-defined. However, note that we do not allow a player to use an element of $\mathcal{M}^1_+ (A^{\mathcal{H}})$ 'o modify his/her imperfect information $\mathcal{G}$.

Assumption 1 : A game μ is said to have uniformly bounded payoffs if there exists a real valued Lebesgue integrale function g on $(\Omega, \mathcal{F}, Pr)$ such that for any $u \in$ supp μ,

$$\| u(\omega) \| \leq g(\omega) \text{ almost every } \omega \in \Omega.$$

Theorem 1 : For any game satisfying Assumption 1, there exists a Cournot-Nash equilibrium.

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KRASNOSELSKII'S ITERATION PROCESS AND ITS GENERALIZATIONS

W. A. KIRK

1. Introduction : Suppose X is a Banach space, K a closed convex subset of X, and $T : K \to K$ nonexpansive. In 1955, M. A. Krasnoselskii [11] proved that if X is uniformly convex and if T has pre-compact range, then the mapping $f : K \to K$ defined by $f = (1/2)(I+T)$ has the property that for each $x \in K$ the sequence $\{f^n(x)\}$ converges to a fixed point of T. H. Schaefer [14] noted almost immediately that Krasnoselskii's result extends to mappings of the form $f_\alpha = (1 - \alpha)I + \alpha T$, $\alpha \in (0, 1)$, and in 1986 M. Edelstein [3] proved that the assumption of uniform convexity (in Schaefer's modification) could be replaced with the assumption that X has strictly convex norm. Our purpose here is to discuss some generalizations and applications of the above ideas.

2. generalizations of the Krasnoselskii Process : Major generalizations of the Krasnoselskii process occurred in 1976 and 1978 when, respectively, Ishikawa [7] and Edelstein and O'Brien [4] prove, in different ways, that the uniform convexity hypothesis can be removed completely. Each proved even more.

Suppose K is a bounded convex subset of a Banach space and $T : K \to K$ nonexpansive. As before, for $\alpha \in (0, 1)$ set $f_\alpha = (1 - \alpha)I + \alpha T$. In [4] it is shown that $\{\|f_\alpha^n(x) - f_\alpha^{n+1}(x)\|\}$ always converges to 0 uniformly for $x \in K$, and in [7] the following more general iteration process is considered : For $\{\alpha_n\} \subset (0, 1)$ and $x_0 \in K$, let

$$\text{(I)} \qquad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n),\ n = 0, 1, \ldots.$$

Ishikawa proved in [7] that if $\alpha_n \leq b < 1$ and if $\sum_{n=0}^{\infty} \alpha_n = +\infty$, then the sequence defined by (I) has the property :

$$\lim_{n \to \infty} \|T(x_n) - x_n\| = 0.$$

Thus if the range of T is pre-compact, both the iterations $\{f_\alpha^n(x)\}$ and $\{T(x_n)\}$ converge to a fixed point of T, and the former does so uniformly on K.

In [6] Goebel and Kirk unifed the above two approaches and extended them even further by proving :

Theorem 2.1 ([6]) : Let X be a Banach space, let K be a bounded closed and convex subset of X, and let $\mathcal{F}$ denote the collection of all nonexpansive self-mappings of K.

Suppose $b \in (0, 1)$ and suppose $\{\alpha_n\} \in [0, b]$ satisfies $\sum_{n=0}^{\infty} \alpha_n = +\infty$. Then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $x_0 \in K$, if $T \in \mathcal{F}$, and if $\{x_n\}$ is the Ishikawa process defined by (I), then $\|x_n - T(x_n)\| \leq \varepsilon$ for all $n \geq N$.

The above result is actually formulated in [6] in a more general setting -for a space X possessing a metric of 'hyperbolic type'. Earlier, in extending some of the previous results to the same setting, the author established the following inequality.

Lemma 2.1 ([8]) : Let K be convex subset of X and let $\alpha \in (0, 1)$. let $x_0 \in K$ and suppose $\{x_n\}$ and $\{y_n\}$ are sequences in K satisfying for all n :

(i) $x_{n+1} = (1 - \alpha)x_n + \alpha y_n$;

(ii) $\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|$.

Then for all i, $k \in \mathbb{N}$,

$$(1 + n\alpha) \leq \|y_{i+n} - x_i\| + (1 - \alpha)^{-n} [\|y_i - x_i\| - \|y_{i+n} - x_{i+n}\|].$$

3. An application of a Generalized Ishikawa Process. Our central result is the following.

Theorem 3.1 : Let K be a bounded closed and convex subset of a Banach space and suppose $T : K \to K$ is continuous and weakly directionally nonexpansive. Then $\inf\{\|x - T(x)\| : x \in K\} = 0$.

We base the proof of Theorem 3.1 on the following transfinite extension of the Ishikawa process. This extension takes account of the fact that Ishikawa's assumption $\sum_{n=0}^{\infty} \alpha_n = +\infty$ may not be easy to check.

Lemma 3.1 : Let X be a Banach space $D \subset X$, Ω_1 the set of countable ordinals, and $\gamma \in \Omega_1$. For each $\alpha < \gamma$, let $t_\alpha \in (0, 1)$, and suppose $\{x_\alpha\}$ and $(y_\alpha\}$ in D satisfy :

(i) $x_{\alpha+1} = (1 - t_\alpha)x_\alpha + t_\alpha y_\alpha$;

(ii) $\|y_\alpha - y_{\alpha+1}\| \leq \|x_\alpha - x_{a+1}\|$;

(iii) if $\mu < \gamma$ is a limit ordinal, then

$$\lim_{\alpha \uparrow \mu} x_\alpha = x_\mu \quad \text{and} \quad \lim_{\alpha \uparrow \mu} y_\alpha = y_\mu.$$

Suppose further that for each $\mu < \gamma$, $\sum_{\alpha<\mu} t_\alpha < \infty$. Then

$$\text{(I)} \qquad (1 + \sum_{\alpha \leq s \leq \alpha+\beta} t_\alpha) \|y_\alpha - x_\alpha\| \leq \|y_{\alpha+\beta+1} - x_\alpha\| + \prod_{\alpha \leq s \leq \alpha+\beta} (1 - t_s)^{-1} [\|y_\alpha - x_\alpha\| - \|y_{\alpha+\beta+1} - x_{\alpha+\beta+1}\|].$$

Moreover, if D is bounded and $t_\alpha \leq b < 1$ for each $\alpha < \gamma$, then the assumption $\sum_{\alpha \leq \gamma} t_s = + \infty$ implies

$$\text{(II)} \qquad \lim_{\alpha \uparrow \gamma} \|y_\alpha - x_\alpha\| = 0$$

The 'sequential' version of Lemma 3.1 is found in Goebel and Kirk [6] where, as noted in the previous section, Ishikawa's result is extended to spaces with possess a metric of hyperbolic type. (In fact, the results presented below also extend routinely to such spaces.)

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RECENT RESULTS ON CONVEX AND SUPERCONVEX ANALYSIS

HEINZ KÖNIG

More than ten years ago Rodé [6] lifted the Hahn-Banach theorem to a new level of abstraction. His abstract Hahn-Banach theorem is so powerful and flexible that it appears to meet all requirements one could think of, and at the same time is of pure and simple character. However, it did not find universal acceptance, perhaps because its proof in [6] was by far not as plain and transparent as it has to be excepted in such a situation. It was not until almost ten years later that the present author [4] found a short and simple proof of the theorem. The main point was to disclose an appropriate fortified statement. For this reason the present talk wants to draw renewed attention to the Rodé theorem, and to describe an improved version of [4].

The scene is a nonvoid set E. We consider operations $\sigma : E^m \to E$ for all numbers of places $m \in \mathbb{N}$. We write

$$\sigma(x) = \sigma(x_1, \ldots, x_m) = \overset{m}{\underset{k=1}{[\,\sigma\,]}} x_k \quad \text{for } x = (x_1, \ldots, x_m) \in E^m.$$

One defines $\sigma : E^m \to E$ and $\tau : E^n \to E$ to commute iff

$$\overset{m}{\underset{k=1}{[\,\sigma\,]}}\ \overset{n}{\underset{l=1}{[\,\tau\,]}}\, x_{kl} = \overset{n}{\underset{l=1}{[\,\tau\,]}}\ \overset{m}{\underset{k=1}{[\,\sigma\,]}}\, x_{kl} \quad \text{for all matrices } (x_{kl})_{\substack{k=1,\ldots,m\\ l=1,\ldots,n}} \text{ from } E\,;$$

for $m = n = 1$ this means that $\sigma \circ \tau = \tau \circ \sigma$ in the usual sense. For $m \in \mathbb{N}$ let $\Omega^m(E)$ consist of all pairs (σ, a) of an operation $\sigma : E^m \to E$ and an $a = (a_1, \ldots, a_m)$ with real $a_1, \ldots, a_m \geq 0$ and let $\Omega(E) := \bigcup_{m=1}^{\infty} \Omega^m(E)$. One defines $\Pi \subset \Omega(E)$ to be **commutative** iff for all $(\sigma, a), (\tau, b) \in \Pi$ the operations σ and τ commute.

Let $\Pi = \bigcup_{m=1}^{\infty} \Pi^m \subset \Omega(E)$. We define a function $f : E \to [-\infty, \infty[$ to be **convex** Π iff

$$f\left(\overset{m}{\underset{k=1}{[\,\sigma\,]}}x_k\right) \le \sum_{k=1}^{m} a_k\, f(x_k) \quad \forall(\sigma, a) \in \Pi^m \text{ and } x_1, \ldots, x_m \in E\ \forall\, m \in \mathbb{N},$$

with the usual convention $0(-\infty) := 0$; and likewise to be **concave** Π iff $\ge$ each time, and to be **affine** Π iff = each time. We can now formulate the Rodé theorem.

1. Theorem (Rodé) : Let $\Pi \subset \Omega(E)$ be commutative and $\vartheta : E \to \mathbb{R}$ be convex Π. Define $M(\Pi, \vartheta)$ to consist of all functions $f : E \to [-\infty, \infty[$ which are concave Π and $f \le \vartheta$. Then each $\varphi \in M(\Pi, \vartheta)$ which is maximal with respect to the pointwise order relation $\le$ is affine Π.

It is vital for the applications that the numbers of $M(\Pi, \vartheta)$ are allowed to attain the value $-\infty$. Since $M(\Pi, \vartheta)$ is upwand inductive in $\le$, the above theorem combined with the Zorn theorem furnishes the sandwich theorem below.

2. Corollary : Let $\Pi \subset \Omega(E)$ be commutative. Assume that $\vartheta : E \to \mathbb{R}$ is convex Π and $f : E \to [-\infty, \infty[$ is concave Π with $f \le \vartheta$. Then there exists a function $\varphi : E \to [-\infty, \infty[$ affine Π such that $f \le \varphi \le \vartheta$.

We pass to the fortified result. An operation $\sigma : E^m \to E$ on E produces for each $r \in \mathbb{N}$ the componentwise operation $\sigma : (E^r)^m \to E^n$, defined by

$$\overset{m}{\underset{k=1}{[\,\sigma\,]}}x^k = \left(\overset{m}{\underset{k=1}{[\,\sigma\,]}}x^k_1, \ldots, \overset{m}{\underset{k=1}{[\,\sigma\,]}}x^k_r\right) \quad \text{for } x^1, \ldots, x^m \in E^r.$$

Thus for $\Pi \subset \Omega(E)$ and $f : E^r \to [-\infty, \infty[$ the notions of being convex Π, concave Π and affine Π are defined.

We need one more notion. For $F : E^r \to [-\infty, \infty[$ and $G : E^s \to [-\infty, \infty[$ we define $F \oplus G : E^{r+s} \to [-\infty, \infty[$ by.

$$F \oplus G\,(x_1, \ldots, x_{r+s}) = F(x_1, \ldots, x_r) + G(x_{r+1}, \ldots, x_{r+s}) \quad \text{for } x_1, \ldots, x_{r+s} \in E.$$

If for $\Pi \subset \Omega(E)$ the functions F and G are convex Π then $F \oplus G$ is convex Π as well ; the same applies to the concave and affine cases. We can new formulate the new result.

3. Theorem : Let $\Pi \subset \Omega(E)$ be commutative. Assume that $P : E^r \to \mathbb{R}$ and $Q : E^s \to \mathbb{R}$ are convex Π, and that $F \in M(\Pi, P)$ and $G \in M(\Pi, Q)$ are as defined above ; thus $F \oplus G \in M(\Pi, P\oplus Q)$. If $F \in M(\Pi, P)$ and $G \in M(\Pi, Q)$ are maximal, then $F \oplus G \in M(\Pi, P\oplus Q)$ is maximal as well.

The Rodé theorem 1 is an immediate consequence of theorem 3. In fact, let $\vartheta : E \to \mathbb{R}$ be convex Π and $\varphi : E \to [-\infty, \infty[$ be a maximal member of $M(\Pi, \vartheta)$. Fix$(\sigma, a) \in \Pi^m$ for some $m \in \mathbb{N}$. In view of theorem 3 then $(a_1 \varphi)\oplus\ldots\oplus(a_m\varphi)$ is a maximal member of $M(\Pi, (a_1 \varphi)\oplus\ldots\oplus(a_m\varphi))$, of course with the obvious notion. Now

$$\sum_{k=1}^{m} a_k \varphi(x_k) \leq \varphi\left(\sum_{k=1}^{m}[\sigma]\, x_k\right) \leq \vartheta\left(\sum_{k=1}^{m}[\sigma] x_k\right) \leq \sum_{k=1}^{m} a_k \vartheta(x_k) \quad \text{for } x_1, \ldots, x_m \in E \text{ ;}$$

and one verifies that th. function $x = (x_1, \ldots, x_m) \to \left(\sum_{k=1}^{m}[\sigma]\, x_k\right)$ is concave Π on E^r. It follows that

$$\sum_{k=1}^{m} a_k \varphi(x_k) = \varphi\left(\sum_{k=1}^{m}[\sigma] x_k\right) \qquad \text{for } x_1, \ldots, x_m \in E \text{ ;}$$

thus φ is affine Π as claimed.

The proof of theorem 3 follows the patten of that of [4] theorem 1.3. It is thus based on the method of auxiliary functionals, as demonstrated in conventional situations for example in [1].

After Rodé [6] corollary 2 furnishes the classical and fortified conventional Hahn-Banach versions collected in [1]. The talk then proceeds to describe the beautiful application of corollary 2 due to Kuhn [5] which in a sense requires the full power of the Rodé version.

If time permits, the talk furthermore reports on recent progress in the theory of superconvex spaces and its applications [2] [3].

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HOMOTOPY AND APPROXIMATION APPROACH TO THE FIXED-POINT INDEX THEORY OF SET-VALUED MAPS

WOJCIECH KRYSZEWSKI

Methods of algebraic topology (started by Eilenberg and Montgomery) provide probably the most powerful tool in the fixed-point theory of set-valued maps. However, in order to develop the Lefschetz or the fixed-point index theory for these maps one needs quite a complex homological machinery (see [11], [9] and [12])

On the other hand, there is another technique available in the theory of multivalued maps ; namely that of single-valued approximation and, consequently, the homotopy approach (see e.g. [5], [7]). Unfortunately, generally speaking uniform single-valued approximations rarely exist and from that point of view are rather useless. Therefore, by approximation we mean an approximation on the graph. Precisely :

Let X, Y be metric spaces and $\varphi : X \multimap Y$ be a set-valued map, $\varepsilon > 0$.

Definition 1 : A (single-valued) map $f : X \to Y$ is an ε-approximation (on the graph) of φ (written $f \in a(\varphi, \varepsilon)$) if

$$Gr(f) \subset N_\varepsilon(Gr(\varphi)) \ ^{*})$$

where $Gr(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$, $Gr(\varphi) = \{x, y) \in X \times Y \mid y \in \varphi(x)\}$.

Having sufficiently close approximations of φ one can study the fixed points, zeros of φ by means of them. This approach goes back to works of J. von Neumann and was successfully applied for convex-valued maps. In 1969, Cellina and Lasota [6] defined the topological degree for convex-valued maps using approximations.

However, the approximation method was never systematically used in the fixed point theory and, in particular, in the fixed-point index theory of maps with not necessarily convex values.

It is my main purpose to present some recent result of mine together with L. Gorniewicz and A. Granas already announced in C.R.Acad. Sci. de Paris [10].

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* If Z is a metric space, $A \subset Z$, $\varepsilon > 0$, then $N_\varepsilon(A) = \{z \in Z \mid dist(z, A) < \varepsilon\}$.

In what follows by a **space** we understand a metric space, by a **map**-a single-valued continuous transformation and by a **set-valued map**- an upper semicontinuous map with compact values

Let X, Y be spaces. By $A_0(X, Y)$ we denote the totality of set-valued maps from X to Y having continuous single-valued ε-approximations for each $\varepsilon > 0$; i.e.

$$A_0(X, Y) : = \{\varphi : X \to Y \mid \forall\, \varepsilon > 0\ \exists\, f : X \text{ A } Y,\ f \in a(\varphi, \varepsilon)\}.$$

We say that $\varphi : X \to Y$ is an A_0-map if $\varphi \in A_0(X, Y)$.

Example 1 : Assume that $\varphi : X \to Y$ and

(i) Y is a normed space, $\varphi(x)$ is a convex set (Cellina [5]) ;

(ii) X is a separable space, $Y = L^1([0, 1])$, $\varphi(x)$ is a decomposable set *) (Bressan, Colombo [4]) ;

(iii) X is a compact polyhedron, $Y = \mathbb{R}^n$, $\varphi(x)$ is a contractible set (Anichini, Conti, Zecca [1])

for any $x \in X$, then $\varphi \in A_0(X, Y)$.

Observe that in the above cases, sets $\varphi(x)$, $x \in X$, are contractible. In a while we shall show that the class A_0 is quite large.

Already for A_0-maps we may say something about fixed points.

Let Y be a compact ANR-space and let $\varphi \in A_0(Y, Y)$. For each $\varepsilon > 0$, define a set of integer numbers

$$\lambda_\varepsilon(\varphi) = \{\lambda(f) \mid f \in a(\varphi, \varepsilon)\}$$

where $\lambda(f)$ denotes the Lefschetz number of a map f. Next we define a set

$$\Lambda(\varphi) = \cap\{\lambda_\varepsilon(\varphi) \mid \varepsilon > 0\}.$$

Theorem 1 : If $\Lambda(\varphi) \neq \{0\}$, then the set of fixed points of φ $Fix(\varphi) \neq \emptyset$. If Y is a compact AR-space, then $\Lambda(\varphi) = \{1\}$.

The above assertions generalize the Lefschetz and the Kakutani fixed point theorems.

Similarly, we may define a kind of a substitute of the fixed-point index. Let Y be a compact ANR-space, U be an open subset of V and let $\varphi \in A_0(clU, Y)$ be such that $Fix(\varphi) \cap bdU = \emptyset$. It can be proved that there is $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon \le \varepsilon_0$ and $f \in a(\varphi, \varepsilon)$, $Fix(f) \cap bdU = \emptyset$. For $0 < \varepsilon \le \varepsilon_0$, define set

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*) Recall that a set $A \subset L^1$ is called **decomposable** if, for each measurable $J \subset [0, 1]$ and any $u, v \in A$, the function $uX_j + vX_{[0,1]\setminus J} \in A$.

$$\mathrm{ind}_\varepsilon(\varphi, U) = \{\mathrm{ind}(f, U) \mid f \in a(\varphi, \varepsilon)\} \subset Z$$

and

$$\mathrm{Ind}(\varphi, U) = \cap\{\mathrm{ind}_\varepsilon(\varphi, U) \mid 0 < \varepsilon \leq \varepsilon_0\}.$$

Theorem 2 : If $\mathrm{Ind}(\varphi, U) \neq \{0\}$, then $\mathrm{Fix}(\varphi) \neq \emptyset$.

However, the above definition of the index is not fully satisfying not only in that $\mathrm{Ind}(\varphi, U)$ is not an integer.

In order to define the integer-valued Lefschetz number or the fixed-point index which satisifes all the standard properties we need certain additional assumptions.

Let X, Y be spaces. We say that set-valued map $\varphi : X \to Y$ is an **A-map** (written $\varphi \in A(X, Y)$) if

(i) $\varphi \in A_0(X, Y)$;

(ii) for any $\delta > 0$, there is $\varepsilon > 0$ such that if $f, g \in a(\varphi, \varepsilon)$, then there exists a map $h : X \times [0, 1] \to Y$ such that $h(.,0) = f$, $h(.,1) = g$ and $h(.,t) \in a(\varphi, \delta)$ for each $t \in [0, 1]$.

Suppose that sets $X, D \subset X$ are closed in Y. We write $\varphi \in A_D(X, Y)$ if $\varphi \in A(X, Y)$ and $\mathrm{Fix}(\varphi) \cap D = \emptyset$.

Moreover, we say that set-valued maps $\varphi, \gamma \in A_D(X, Y)$ are **homotopic** (written $\varphi \simeq_D \gamma$) if there exists $X \in A_0(X \times [0, 1], Y)$ such that $X(.,0) = \varphi$, $X(.,1) = \gamma$ and $x \in X(x, t)$ for $x \in X$, $t \in [0, 1]$.

Theorem 3 : If the space X is compact, then the relation "$\simeq_D$" is an equivalence in $A_D(X, Y)$.

For any $\varphi \in A_D(X, Y)$, by $|\varphi|_D$ we denote the homotopy class of φ. By $[A_D(X, Y)]$ we denote the totality of all homotopy classes. For the later convenience, let $S_D(X, Y)$ be the set of all maps from X to Y without fixed points on D and $[S_D(X, Y)]$ is the totality of all (ordinary) homotopy classes of maps from $S_D(X, Y)$.

Theorem 4 : (Bijection theorem) : There is a bijection

$$F : [A_D(X, Y)] \to [S_D(X, Y)]$$

provided Y is a compact ANR-space.

F is already defined and injective if Y is a space and X is compact. To have surjectivity what we actually need is the ULC (uniform local contractibility) property of ANR Y.

Let Y be a compact ANR-space.

Theorem 5 : (i) If $\varphi \in A(Y, Y)$, then $\Lambda(\varphi)$ is a singleton ; viz. $\Lambda(\varphi) = \{\lambda(f)\}$ where $f \in F(|\varphi|)$ $(D = \emptyset)$.

(ii) If $\varphi \in A_{bdU}(clU, Y)$ where U is open in Y, then $Ind(\varphi, U)$ is a singleton ; viz. $Ind(\varphi, U) = \{ind(f, U)\}$ where $f \in F([\varphi]_{bdD})$.

The defined index has all of the standard properties of the index ; i.e. homotopy, additivity, existence, normalization and contraction properties.

Now, we shall define a class of set-valued maps which satisfy the conditions of the class A.

Let X, Y be spaces.

Definition 2 : (i) A compact set $K \subset Y$ is **∞-proximally connected** (Dugundji [8]) if, for each $\varepsilon > 0$, there are $\delta > 0$ $(\delta < \varepsilon)$ and a point $a \in K$ such that the inclusion

$$N_\delta(K) \to N_\varepsilon(k)$$

induces a trivial homomorphism

$$\pi_n(N_\delta(K), a) \to \pi_n(N_\varepsilon(K), a)$$

for each $n \geq 0$.

(ii) We say that a set-valued map $\varphi : X \to Y$ is a **J-map** (written $\varphi \in J(X, Y)$) if $\varphi(x)$ is ∞-proximally connected for each $x \in X$.

Example 2 : Let Y be an ANR-space and $K \subset Y$ be compact. If

(i) $K \in FAR$ (fundamental absolute retract) (see [3]) ;

(ii) the shape sh(K) is trivial (see [3]) ;

(iii) $K \in R_\delta$ (i.e. $K = \bigcap\{K_i \mid K_{i+1} \subset K_i$, K_i is a compact AR-space$\}$) ;

(iv) K is an AR-space ;

(v) K is contractible ;

(vi) $K = \bigcap\{K_i \mid K_{i+1} \subset K_i$, K_i is ∞-proximally connected$\}$;

then K is ∞-proximally connected.

Thus we see that the class J(X, Y) is quite large.

Theorem 6 : If X is a compact ANR-space, Y is a space, then the following inclusion holds :

$$J(X, Y) \subset A(X, Y).$$

Therefore we are able to construct the fixed-point index theory for maps of compact ANR-spaces with values being, for example, R_δ-sets.

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BRUCK'S RETRACTION METHOD

T. KUCZUMOW AND A. STACHURA

In 1973 R. E. Bruck proved that if C is a closed convex subset of the Banach space X, $T : C \to C$ is nonexpansive and satisfies a conditional fixed point property, then the fixed point set of T is a nonexpansive retract of C. He applied his result to obtain the existence of a common fixed point of a finite family of commuting nonexpansive mappings. The method of the proof given in Bruck's paper is so universal that by using it we can obtain new results.

If B is an open unit ball in the Hilbert space H and if $T : B^n \to B^n$ is holomorphic with $Fix(T) \neq \emptyset$, then Fix(T) is a holomorphic retract of B^n. It gives that for commuting holomorphic $T_1, T_2 : B^n \to B^n$ with $Fix(T_1) \neq \emptyset$ and $Fix(T_2) \neq \emptyset$ we have $Fix(T_1) \cap Fix(T_2) \neq \emptyset$.

Let C be a convex weakly compact subset of the Banach space X. C is said to satisfy the generic fixed point property if every nonexpansive mapping $T : C \to C$ has a fixed point in each nonempty closed convex subset it leaves invariant. Suppose C_1, C_2 are convex weakly compact subsets of Banach spaces X_1 and X_2 (respectively), C_1 has the generic FPP and $T : C_1 \times C_2 \to C_1$ is nonexpansive (in $X_1 \times X_2$ we have the maximum norm). Then there exists a nonexpansive mapping $R : C_1 \times C_2 \to C_1$ such that $T(R(x, u), u) = R(x, u)$ for every $(x, u) \in C_1 \times C_2$. This result allows us to prove that if C_1 and C_2 have the above properties and additionally C_2 has FPP for nonexpansive mappings, then $C_1 \times C_2$ has FPP for nonexpansive mappings.

INEQUALITY SYSTEMS AND OPTIMIZATION FOR SET FUNCTIONS

HANG-CHIN LAI

Extended abstract

The classical inequality systems of linear functionals are extended to the case of convex set functions with values in ordered vector spaces. The Farkas theorem is in turn generalized to his case. This result is applied to establish necessary conditions of minimal/weakly minimal points for convex programming problems with set functions.

Farkas' theorem [8, pp.5-7] is reformulated by Fan [7, Theorem 4] as follows

Theorem A : Let f(x) be a linear functional on e, a real vector space. The inequality $f(x) \geq \alpha$ has a solution satisfying the system of linear inequalities

$$g_i(x) \geq \beta_i, \qquad i = 1, 2, \ldots, p,$$

where α and β_i are real numbers, if and only if there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, p$ such that

$$f = \sum_{i=1}^{p} \lambda_i g_i \qquad \text{and} \quad \alpha \leq \sum_{i=1}^{p} \lambda_i \beta_i \ .$$

Equivalently, Theorem A is rewritten as follows

Theorem B : The system of linear inequalities

$$\begin{cases} f(x) - \alpha < 0 & & (1) \\ \beta_i - g_i(x) \leq 0 & i = 1, 2, \ldots, p & (2) \end{cases}$$

has no solution if and only if there exist $\lambda_i \geq 0$, $1 \leq i \leq p$ such that

$$(f(x) - \alpha) + \sum_{i=1}^{p} \lambda_i(\beta_i - g_i(x)) \geq 0. \qquad (3)$$

We will extend this result to the case of convex set functions with values in ordered vector spaces.

We consider an atomless finite measure space (X, Γ, μ) with $L^1(X, \Gamma, \mu)$ separable, $\mathcal{S} \subset \Gamma$ is a convex subfamily of measurable subset of X (see [3], [4], [10-11], [13-16], [17]). Let Y and Z be locally convex Hausdorff real vector spaces, $C \subset Y$ and $D \subset Z$ be closed convex pointed cones which determine the partial orders of Y and Z respectively. We assume further that Y and Z are ordered complete vector lattices (see [18]). Then the Farkas' type theorem (Theorem B) can be generalized to the following cases of convex set functions, we will call these results the generalized Farkas theorems. We establish some new results as follows.

Theorem 1 : Let $F : \mathcal{S} \to \bar{Y} = Y \cup \{\infty\}$ and $G : \mathcal{S} \to \bar{Z} = Z \cup \{\infty\}$ be, respectively, C-convex and D-convex set functions, where $\mathcal{S}$ is a convex subfamily of Γ. Then the inequalities system

(4) $\qquad F(\Omega) <_C \theta, \quad G(\Omega) <_D \theta \qquad \theta$ = zero vector

has no solution in $\mathcal{S}$ if and only if there exists nonzero $(y^*, z^*) \in C^* \times D^*$ such that

(5) $\qquad \langle y^*, F(\Omega) \rangle + \langle z^*, G(\Omega) \rangle \geq 0$ *for all* $\Omega \in \mathcal{S}$

where C^* and D^* are dual cone of C and D respectively.

Theorem 2 : F, G and $\mathcal{S}$ are the same as in Theorem 1. We assume further that $(C^*)^0 = \text{int } C^* \neq \emptyset$ and there exists a $\tilde{\Omega} \in$ & such that $G(\tilde{\Omega}) <<_D \theta$, that is, $G\tilde{\Omega} \in (-D)^0$. Then the system

(6) $\qquad F(\Omega) <_C \theta, \quad G(\Omega) \leq_D \theta$

has no solution in $\mathcal{S}$ if and only if there exists W_0 Q $\in B^+(Z, Y)$, the set of positive linear operators from Z into Y, such that

(7) $\qquad F(\Omega) + W_0 \circ G(\Omega) <_C \theta$ is not true for any $\Omega \in \mathcal{S}$

Theorem 3 : Under the assumptions of Theorem 2 except the condition $(C^*)^0 \neq \emptyset$, the system

(8) $\qquad F(\Omega) <<_C \theta$, (i.e. $F(\Omega) \in (-C)^0$), $G(\Omega) \leq_D \theta$,

has no solution in $\mathcal{S}$ if and only if there exists $W_0 \in B^+(Z, Y)$ such that

(9) $\qquad F(\Omega) + W_0 \circ G(\Omega) <<_C \theta$ does not hold for any $\Omega \in \mathcal{S}$

Corollary 4 : If $Y = \mathbb{R}$ in Theorem 2, then the system

(10) $$F(\Omega) < 0, \qquad G(\Omega) \leq_D \theta$$

has no solution in $\mathcal{S}$ if and only if there exists $z^* \in D^*$ such that

(11) $$F(\Omega) + < Z^*, G(\Omega) > \geq 0 \text{ for all } \Omega \in \mathcal{S}.$$

If $Z = \mathbb{R}$ in Corollary 4, then it reduces to Theorem 3.1 of [1].

Applications : Applying the generalized Farkas theorem, we can establish the necessary conditions for the existence theorem of programming problem for set functions. Consider the problem

(P) minimize $F(\Omega)$
subject to : $\Omega \in \mathcal{S} \subset \Gamma$ and $G(\Omega) \leq_D \theta$

where $F : \mathcal{S} \rightarrow Y$ and $G : \mathcal{S} \rightarrow Z$ are cone convex set functions. Then we have

Theorem 5 : Let $\mathcal{S}$ be a convex subfamily in Γ, $(C^*)^0 \neq \emptyset$, F and G cone convex functions. Further assume that there exists an $\tilde{\Omega} \in$ & such that $G(\tilde{\Omega}) <<_D \theta$. If Ω_0 is a minimal point of *problem (P), then there exists* $W_0 \in B^+(Z, Y)$ such that $W_0 \circ G(\Omega_0) = \theta$ and (Ω_0, W_0) is a saddle point of the Lagrangian $L(\Omega, W) = f(\Omega) + W \circ G(W)$.

Theorem 6 : As the assumptions of Theorems 5 except the condition $(C^*)^0 \neq \emptyset$, if Ω_0 is a weakly minimal point of (P), then there exists $W_0 \in B^+(Z, Y)$ such that (Ω_0, W_0) is a weakly saddle point of the Lagrangian $L(\Omega, W)$.

Here a **saddle** (or weakly saddle) **point** (Ω_0, W_0) of $L(\Omega, W)$ means that if $W_0 \in B^+(Z, Y)$, $\Omega_0 \in \mathcal{S}$ there does not exist $W \in B^+(Z, Y)$ such that

$$L(\Omega_0, W_0) <_C L(L_0, W) \quad (\text{or } L(\Omega_0, W_0) <<_C L(\Omega_0, W))$$

and there is no $\Omega \in \mathcal{S}$ such that

$$L(\Omega, W_0) <_C L(\Omega_0, W_0) \quad (\text{or } L(\Omega, W_0) <<_C L(\Omega_0, W_0)).$$

Corollary 7 : In Theorem 6, if $Y = \mathbb{R}$, then Ω_0 is a minimal point of (P) if and only if there exists $z_0^* \in D^*$ such that $< z_0^*, G(\Omega_0) > = 0$ and (Ω_0, z_0^*) is a saddle point of $L(\Omega, z^*)$.

In Corollary 7, if $Z = \mathbb{R}$, then it reduces to Theorem 3.2 of [1].

If we drop the condition $(C^*)^0 \neq \emptyset$ and replace $G(\tilde{\Omega}) <<_D \theta$ by $G(\tilde{\Omega}) <_D \theta$ in Theorem 5, then we have

Theorem 8 : Under the assumptions of Theorem 5 with some modification which have show as above, then if Ω_0 is a minimal point of problem (P), then there exists $(y_0^*, z_0^*) \in C^* \times D^*$ such that

$$<z_0^*, G(\Omega_0)> = 0 \text{ and } (\Omega_0, z_0^*) \text{ is a saddle point}$$

of the Lagrangian $\tilde{L}(\Omega, z^*) = < y_0^*, F(\Omega) > + < z^*, G(\Omega) >$ for all $\Omega \in \mathcal{S}$ and $z^* \in D^*$.

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SUR LE PASSAGE DU CADRE UNIVOQUE AU CADRE MULTIVOQUE DANS LES PROBLEMES DE COINCIDENCE

MARC LASSONDE

1. Introduction

Les applications univoques sont notés par des lettres minuscules les applications multivoques (multiapplications) par des lettres majuscules. Applications et multiapplications sont identifiées à leurs graphes.

Soient X et Y deux espaces topologiques, $\gamma : X \to Y$ une application continue, $\Delta : X \to Y$ une multiapplication fermée (c'est-à-dire Δ est fermé comme sous-ensemble de $X \times Y$). On dit que γ et Δ ont une coïncidence si $\exists\, x \in X$ tel que $\gamma(x) \in \Delta x$, c'est-à-dire :

(1) $$\gamma \cap \Delta \neq \emptyset$$

Exemples

(i) $X = Y$, $\Delta = \{(x, x) : x \in X\}$. Alors (1) signifie que γ est un point fixe.

(ii) Y un espace de Banach, X compact $\subset$ Y,

$$\Delta x = \{y \in Y : \| y - x\| \leq \|y - z\| \quad \forall z \in X\}$$

(ensemble des points qui se projettent en x sur X). Alors (1) est un problème de meilleure approximation.

$$\exists\, x \in X \text{ tel que } \|\gamma(x) - x\| = \min_{z \in X} \|\gamma(x) - z\|$$

(iii) X = convexe dans un e.l.c. E, Y = E' dual de E,

$\Delta x = \{x' \in E' : \sup\limits_{z \in X} < x', z - x > \leq 0\}$ (cône normal à X en x). Alors (1) est une inéquation variationnelle

$$\exists x \in X \text{ tel que } < \gamma(x), z - x > \leq 0 \quad \forall z \in X.$$

Soit maintenant $\Gamma : X \to Y$ une multiapplication s.c.s. (semi continue supérieurement). La version multivoque du problème (1) s'écrit :

(2) $$\Gamma \cap \Delta \neq \emptyset.$$

Le problème (2) recouvre en particulier les problèmes de point fixe, de meilleure approximation et d'inéquation variationnelle pour applications multivoques. Naturellement, (2) contient (1). Notre objectif est de montrer que, réciproquement, la validité de (1) pour toute application γ continue implique la validité de (2) pour une multiapplication Γ appartenant à une classe assez vaste de multiapplications.

2. Passage du cadre univoque au cadre multivoque sur un simplexe

Soit Y un convexe dans un e.v.t. (espace vectoriel topologique)

$\mathcal{C}(X, Y) = \{\gamma : X \to Y : \gamma$ est une application (univoque) continue$\}$

$\mathcal{K}(X, Y) = \{\Gamma : X \to Y : \Gamma$ est une multiapplication s.c.s. à valeurs convexes compactes non vides$\}$

$\mathcal{K}_c(X, Y) = \{\Gamma : X \to Y : \exists X_1, \ldots, X_n$ convexes dans un e.v.t., $X_o = X$, $X_{n+1} = Y$, et $\Gamma_i \in \mathcal{K}(X_i, X_{i+1})$ pour $i = 0, \ldots, n$ tels que $\Gamma = \Gamma_n \Gamma_{n+1}, \ldots, \Gamma_o\}$

$\mathcal{K}(X, Y)$ est l'ensemble des multiapplications de Kakutani ; $\mathcal{K}_c(X, Y)$ est l'ensemble des multiapplications Kakutani-factorisables.

Naturellement, $\mathcal{C}(X, Y) \subset \mathcal{K}(X, Y) \subset \mathcal{K}_c(X, Y)$.

Lemme 1 : Soient X un simplexe, Y un convexe dans un e.v.t., $\Delta : X \to Y$ une multiapplication fermée. Alors les propriétés suivantes sont équivalentes :

(i) $\forall \gamma \in \mathcal{C}(X, Y), \quad \gamma \cap \Delta \neq \emptyset$;

(ii) $\forall \gamma \in \mathcal{K}(X, Y), \quad \Gamma \cap \Delta \neq \emptyset$;

(iii) $\forall \gamma \in \mathcal{K}_c(X, Y), \quad \Gamma \cap \Delta \neq \emptyset$.

Remarque : Les multiapplications $\Gamma \in \mathcal{K}(X, Y)$ ne possèdent pas en général de sélection continue "approchée" (car Y n'est pas supposé localement convexe). La démonstration de (i) $\Rightarrow$ (ii) reprend la construction originale de Kakutani (voir aussi Ha) ; la démonstration de (ii) $\Rightarrow$ (i) se fait par induction.

Comme l'application immédiate, on obtient une généralisation du théorème de point fixe de Kakutani :

Théorème 1 : Soient X un simplexe et $\Gamma \in \mathcal{K}_c(X, X)$. Alors Γ possède un point fixe.

3. Passage de la dimension finie à la dimension infinie

On utilise la technique des partitions de l unité pour démontrer le lemme suivant :

Lemme 2 : Soient X un simplexe, Y un espace normal. Notons
$\ddot{X}$ les sommets de X. Soient $F : \ddot{X} \to Y$ et $\Gamma : X \to Y$ telle que
(i) pour tout $A \subset \ddot{X}$, $\Gamma(\text{co } A) \subset F(A)$;
(ii) pour tout $\alpha \in \mathcal{C}(Y, X)$, $\alpha\Gamma$ a un point fixe.

Alors, $\cap\{\overline{F}x : x \in \ddot{X}\} \neq \emptyset$.
En combinant le lemme 2 avec le Théorème 1, on obtient :

Théorème 2 : Soient X un convexe, Y un convexe normal dans un e.v.t. et F, $\Gamma : X \to Y$ telles que :
(i) pour toute partie finie $A \subset X$, $\Gamma(\text{co}A) \subset F(A)$;
(ii) $\Gamma \in \mathcal{K}_c(X_f, Y)$.

Alors, la famille $\{\overline{F}x : x \in X\}$ a la propriété d'intersection finie.

(X_f dénote X muni de la topologie finie)

Remarque : Le Théorème 2 généralise la généralisation de Ky Fan du lemme de Knaster-Kuratowski-Mazurkiewiez.

En prenant le complémentaire de F dans le Théorème 2, on obtient un théorème de coïncidence :

Théorème 3 : Soient X un convexe, Y un convexe normal dans un e.v.t., S, $\Gamma : X \to Y$ telles que :
(i) $\forall x \in X$, Sx est ouvert dans Y ; $\forall y \in Y$, Sy est convexe
(ii) $\Gamma \in \mathcal{K}_c(X, Y)$ et $\Gamma(X)$ est inclus dans un compact de Y.
Alors, il existe $x \in X$ tel que $Sx \cap \Gamma x \neq \emptyset$.

Remarque : Le Théorème 3 est un cas particulier d'un théorème de Ben-El-Mechaiekh-Deguire-Granas, qui dépend du théorème de point fixe de Lefschetz.

Corollaire : Soit X un convexe normal non vide dans un e.v.t. E ; soit $\Gamma \in \mathcal{K}_c(X_f, Y)$ telle que $\Gamma(X)$ est inclus dans un compact de X. Alors, pour tout voisinage convexe ouvert V de l'origine dans E, il existe $x_V \in V$ tel que $\Gamma x_V \cap (x_V + V) \neq \emptyset$.

Du corollaire précédent on obtient immédiatement la généralisation du Théorème 1 en dimension infinie :

Théorème 4 : Soit X un convexe normal non vide dans un e.l.c. E ; soit $\Gamma \in \mathcal{K}_c(X, Y)$ telle que $\Gamma(X)$ est inclus dans un compact de X. Alors Γ possède un point fixe.

Remarque : Le Théorème 4 généralise le théorème de point fixe de Fan-Glicksberg et le théorème d'intersection de von Neumann ; il est à comparer avec un théorème de point fixe de Gorniewicz qui est de nature homologique.

FIXED POINT PROPERTY FOR WEAK*-COMPACT CONVEX SETS IN DUAL BANACH SPACES ASSOCIATED TO A LOCALLY COMPACT GROUP

ANTHONY TO-MING LAU

Abstract of talk

Let K be a bounded closed convex subset of a Banach space. A point x in K is called a diametral point of K if

$$\sup\{\|x - y\| ; y \in K\} = \text{diam}(K)$$

where diam(K) dentoes the diameter of K. The set K is said to have normal structure if every non-trivial convex subset H of K contains a non-diametral point of H. A dual Banach space E is said to have weak*-normal structure if every non-trivial weak*-compact convex subset K of E has normal structure ; E is said to have fixed point property (FFP*) if every weak*-compact convex subset K of E and every non-expansive mapping $T : K \to K$, T has a fixed point in K. Lim [3] showed that ℓ_1 has weak* normal structure. He also observed that a simple modification of Kirk's proof in [1] yiels : If a dual Banach space E has weak*-normal structure, then E has fixed point property (FFP*).

Let G be a locally compact group. In this talk, we shall give conditions on G when certain dual Banach spaces associated to G (i.e. the measure algebra, the Fourier Stieltjes algebra, dual space of the space of (weakly) almost periodic functions, or the space of bounded left uniformly continuous functions) have weak*-normal structure (and hence the fixed point property (FFP*)) and other related properties.

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EXTENSIONS OF SEMICONTINUOUS MULTIFUNCTIONS

A. LECHICKI AND S. LEVI

Introduction :

We study the following problem : let X and Y be topological spaces, A a dense subset of X and F : A $\rightarrow$ Y a semicontinuous multifunction ; under which conditions is it possible to extend F to a semicontinuous multifunction on X, or, at least, on a G_δ of X containing A ?

We treat upper (usc) and lower (lsc) semicontinuous multifunctions separately but both instances reduce, in the single-valued case, to the extension problem for continuous functions.

There are two well known results concerning the extension of continuous functions defined on dense subspaces of topological spaces :

Theorem A : [1] Let A be a dense subset of a topological space X, Y a regular space and f : A $\rightarrow$ Y a continuous function. Let

$$B = \{x \in X \mid f \text{ has a continuous extension } f_x : A \cup \{x\} Q\!\!\rightarrow Y\}.$$

Then f has a continuous extension to B.
It must be noted that theorem A fails if Y is only a Hausdorff space.

Theorem B : [3] Let A be a dense subspace of a topological space X, Y a completely metrizable space and f : A $\rightarrow$ Y a continuous function. Then f has a continuous extension to a G_δ containing A.

As a complement to theorem B we also quote the following :

Theorem C : [2] A completely regular space Y is Cech-complete and has a G_δ diagonal if and only if it has the property that given a dense subspace A of any topological space X and a continuous f : A $\rightarrow$ Y, there is a G_δ subset of X containing A over which f has a continuous extension.

We will give some applications to topology and funcitonal analysis.

The extension theorems.

Let X and Y be topological spaces, A a dense subspace of X and $F : A \to Y$ a multifunction. Define the upper limit of F at x as

$\theta(x) = \cap\{\overline{F(A \cap U)} : U \in N(x)\}$ where N(x) is a local base at x.
θ is a multifunction defined on X, possibly empty-valued, which extends F if Y is regular and F is closed-valued and usc.
θ need not be usc even in the case F is such, but if usc, it is the smallest closed-valued usc extension of F.

Theorem 1 : Let Y be normal, $F : A \to Y$ a usc closed-valued multifunction.Define $B = \{x \in X \mid \theta : A \cup \{x\} \to Y$ is a usc extension of F$\}$. Then $A \subseteq B$ and $\theta : B \to Y$ is a usc extension of F to B.

A multifunction $F : A \to Y$ is subcontinuous at $x \in X$ if for every net $(a_i) \subset A$, $a_i \to x$, every net $(y_i) \subset Y$, with $y_i \in F(x_i)$ $\forall i$ has a convergent subnet.

Theorem 2 : Let Y be regular and let $F : A \to Y$ be a closed-valued usc multifunction. Then for $x \in X - A$ the following facts are equivalent :
(1) $\theta(x)$ is compact and $\theta : X \to Y$ is usc at x
(2) $\theta(x)$ is compact and $\theta : A \cup \{x\} \to Y$ is usc at x
(3) F has a usc extension to $A \cup \{x\}$ which is compact-valued at x
(4) F is subcontinuous at x.

We can obtain Theorem A of the introduction as a simple corollary to the previous result.

Lemma 3 : For every multifunction $F : A \to Y$ with values in a Cech-complete space, the set of points at which F is subcontinuous is a G_δ of X.

Proposition 4 :
(1) Let Y be Cech-complete and $F : X \to Y$ a graph-closed multifunction with compact values. Then the set of points at which F is usc is a G_δ of X ;
(2) A single-valued function with a closed graph and values in a Cech-complete space is continuous at the points of a G_δ.

Theorem 5 : Let $F : A \to Y$ be a usc compact-valued multifunction with values in a Cech-complete space. Then F has a usc compact-valued extension to a G_δ of X containing A.

We remark that there are examples of usc compact-valued multifunctions with values in metric spaces that admit no usc extension to any G_δ.

The property described in the above theorem characterizes Cech-completeness.

A multifunction $F : X \to Y$ is quasi-lsc at $x \in X$ if $\forall y \in F(x)$, $\forall W \in N(y)$ and $\forall U \in N(x)$, there exists a non-empty open set $V \subset U$ such that $\forall t \in V$, $F(t) \cap W \neq \emptyset$.

Theorem 6 : Suppose $F : A \to Y$ is lsc and either
(1) Y is regular and F subcontinuous at each $x \in X - A$ or
(2) Y is locally compact.
Then the extension F* of F to X defined as F on A and θ on X - A is lsc on A and quasi-lsc on X - A.

We note that if the lower limit of F at every $x \in X - A$ agrees with the upper limit, the extension F* of the previous theorem is lsc on X. If, in addition to the hypotheses of theorem 6, Y is separable metrizable, F admits a lsc extension to a residual subset containing A.

Theorem 7 : Let X be a completely regular space, Y a locally compact space and $F : X \to Y$ a lsc graph-closed multifunction. Then the upper limit $\theta : \beta X \to Y$ is an extension of F to the Stone-Cech compactification of X which is lsc (and graph-closed).

Applications

A topological space X is a Namioka space if for every compact space Y and every continuous function $f : X \to C_p(Y)$ (the space of complex continuous functions on Y with the topology of pointwise convergence), there exists a dense G_δ subset A of X such that $f|_A : A \to C(y)$ is norm-continuous. Labuda [4] discusses the class of Namioka spaces for minimal usc compact-valued (usco) multifunctions (minimality refers to the partial order induced by the relation of inclusion of graphs) and shows that most Namioka spaces are also Namioka spaces for minimal usco maps.

Theorem 8 : Let X be a Namioka space for minimal usco maps, Y a compact space and $F : X \to C(y)$ a multifunction. Then F is minimal usco with respect to the norm topology of C(Y) if and only if F is minimal usco with respect to pointwise convergence in C(Y) and $F|_A$ is norm-subcontinuous at each $x \in X - A$, where A is the dense G_δ given by the definition of a Namioka space.

The following is an application to topology :

Theorem 9 : If X and Y are completely regular spaces and there exists a non-empty valued surjective multifunction $F : X \to Y$ which is subcontinuous together with its inverse, then X is Cech-complete (locally compact) if and only if Y is Cech-complete (locally compact).

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AN APPLICATION OF MINIMAX THEOREMS TO OPTIMAL PARTITIONING OF A MEASURABLE SPACE

JERZY LEGUT AND MACIEJ WILCZINSKI

Let $(\mathcal{X}, \mathcal{B})$ be a measurable space and let $\mu_1, \mu_2, \ldots, \mu_n$ denote nonatomic probability measures defined on the same σ-algebra $\mathcal{B}$. By an ordered partition $P = \{A_i\}_{i=1}^n$ of the measurable space $(\mathcal{X}, \mathcal{B})$ is meant a collection of disjoint subsets $A_1, A_2, \ldots, A_n$ of $\mathcal{X}$ satisfying $A_i \in \mathcal{B}$ for all $i \in I := \{1, 2, \ldots, n\}$ and $\bigcup_{i=1}^{n} A_i = \mathcal{X}$. Let $\mathcal{P}$ denote the set of all measurable partitions $P = \{A_i\}_{i=1}^n$ of $\mathcal{X}$. Suppose that $\alpha \in S := \{s = (s_1, s_2, \ldots, s_n) \in R^n : s_i > 0$ for all $i \in I$ and $\sum_{i=1}^{n} s_i = 1\}$.

Definition : A partition $P^\alpha = \{A_i^\alpha\}_{i=1}^n \in \mathcal{P}$ is considered to be an α-optimal if

$$\min_{i\in I} [\alpha_i^{-1} \mu_i (A_i^\alpha)] = \sup \{\min_{i\in I} [\alpha_i^{-1} \mu_i (A_i)] : P = \{A_i\}_{i=1}^n \in \mathcal{P}\}.$$

where α_i is the i-th coordinate of $\alpha \in S$.

The problem of α-optimal partitioning of a measurable space $(\mathcal{X}, \mathcal{B})$ can be interpreted as the well known problem of fair division of an object $\mathcal{X}$ (e.g. a cake) among n participants (cf. [2]). Here, each μ_i, $i \in I$, represents the individual evaluation of sets from $\mathcal{B}$.

The existence of the α-optimal partition follows from the following theorem due to Dvoretzky et al. [3].

Theorem 1 : Let $\bar{\mu} : \mathcal{P} \to R^n$ denote the division vector valued function defined by

$$\bar{\mu}(P) = (\mu_1 (A_1), \mu_2(A_2), \ldots, \mu_n(A_n)) \in R^n, \quad P = \{A_i\}_{i=1}^n \in \mathcal{P}.$$

Then the range $\bar{\mu}(\mathcal{P})$ of $\bar{\mu}$ is convex and compact in R^n.

For all $i \in I$, let $f_i = d\mu_i / dv$, where $v = \sum_{i=1}^{n} \mu_i$. With every $p \in \bar{S}$, where $\bar{S}$ stands for the closure of S in R^n, we will associate $\mathcal{B}$-measurable subsets $B_1(p), B_2(p), \ldots, B_n(p)$ and $C_1(p)$, $C_2(p), \ldots, C_n(P)$ of $\mathcal{X}$ defined by

$$B_i(p) = \bigcap_{i=1}^{n} \{x \in \mathcal{X} : p_i \alpha_i^{-1} f_i(x) > p_j \alpha_j^{-1} f_j(x)\},$$

$$C_i(p) = \bigcap_{\substack{i=1 \\ i \neq j}}^{n} \{x \in \mathcal{X} : p_i \alpha_i^{-1} f_i(x) \geq p_j \alpha_j^{-1} f_j(x)\}, \quad \text{for } i \in I.$$

Now we may state (see [5])

Theorem 2 : For all $\alpha \in S$ there exist a point $p^\alpha = (p_1^\alpha, \ldots p_n^\alpha) \in \bar{S}$ and a corresponding partition $P = \{A_i\}_{i=1}^{n} \in \mathcal{P}$ which satisfies

(i) $$B_i(p^\alpha) \subset A_i^\alpha \subset C_i(p^\alpha),$$

(ii) $$\mu_1(A_1^\alpha)/\alpha_1 = \mu_2(A_2^\alpha)/\alpha_2 = \ldots = \mu_n(A_n^\alpha)/\alpha_n$$

and is α-optimal. Moreover, any partition satisfying (i) and (ii) is α-optimal.

Proof : It is obvious that

$$\sup_{a \in \bar{\mu}(\mathcal{P})} \min_{i \in I} \alpha_i^{-1} a_i = \sup_{a \in \bar{\mu}(\mathcal{P})} \min_{p \in \bar{S}} \sum_{i=1}^{n} p_i \alpha_i^{-1} a_i ,$$

where a_i is the i-th coordinate of $a \in \bar{\mu}(\mathcal{P})$. Moreover, it follows from the minmax theorem of Sion (cf. [1]) that there exists a point $(p^\alpha, a^\alpha) \in \bar{S} \times \bar{\mu}(\mathcal{P})$ for which

$$\sup_{a \in \bar{\mu}(\mathcal{P})} \min_{p \in \bar{S}} \sum_{i=1}^{n} p_i \alpha_i^{-1} a_i = \min_{p \in \bar{S}} \sum_{i=1}^{n} p_i \alpha_i^{-1} a_i^\alpha = \sum_{i=1}^{n} p_i^\alpha \alpha_i^{-1} a_i^\alpha =$$

(1)

$$\sup_{a \in \bar{\mu}(\mathcal{P})} \sum_{i=1}^{n} p_i^\alpha \alpha_i^{-1} a_i = \min_{p \in \bar{S}} \sup_{a \in \bar{\mu}(\mathcal{P})} \sum_{i=1}^{n} p_i \alpha_i^{-1} a_i .$$

It is clear that any partition $P^\alpha = \{A_i^\alpha\}_{i=1}^n \in \mathcal{P}$ with $\mu_i(A_i^\alpha) = a_i$ for $i \in I$ is α-optimal. Now, since by (1)

$$\sum_{i=1}^n P_i^\alpha \alpha_i^{-1} \mu_i(A_i^\alpha) = \sup_{P\in\mathcal{P}} \sum_{i=1}^n p_i^\alpha \alpha_i^{-1} \mu_i(A_i)$$

and since

$$\min_{p\in\bar{S}} \sup_{p\in\mathcal{P}} \sum_{i=1}^n p_i^\alpha \alpha_i^{-1} \mu_i(A_i) = \min_{p\in\bar{S}} \sum_{i=1}^n p_i^\alpha \alpha_i^{-1} \mu_i(A_i^\alpha)$$

it follows that (i) and (ii) must be fulfilled, respectively. This completes the proof of the first part of the Theorem. The proof of the second part is straightforward.

Suppose now we are given countably many nonatomic probability measures $(\mu_i)_{i=1}^\infty$ on the same measurable space $(\mathcal{X}, \mathcal{B})$. Let $\mathcal{P}$ denote the set of all measurable partitions $P = \{A_i\}_{i=1}^\infty$ of the space $\mathcal{X}$ (i.e. sequences of countably many disjoint subsets from $\mathcal{B}$). Let $(\alpha_i)_{i=1}^\infty$ be a sequence of positive numbers with $\Sigma_{i=1}^\infty \alpha_i = 1$.

Similarly to the finite case, a partition $P^* = \{A_i^*\}_{i=1}^\infty \in \mathcal{P}$ is said to be an α-optimal if it maximizes the number

$$\inf_{i\in N} [\alpha_i^{-1} \mu_i(A_i)] \text{ with } P = \{A_i\}_{i=1}^\infty \in \mathcal{P} \text{ ranging over } \mathcal{P}, \text{ i.e.}$$

$$\inf_{i\in N} [\alpha_i^{-1} \mu_i(A_i^*)] = \sup\{\inf_{i\in N} [\alpha_i^{-1} \mu_i(A_i)] : P = \{A_i\}_{i=1}^\infty \in \mathcal{P}\},$$

where $N = \{1, 2, \ldots\}$.

A proof of existence of such partition can be obtained from the following result due to Eisele [4].

Theorem 3 : If $\{\mu_i\}_{i=1}^\infty$ is a sequence of nonatomic measures then the range of the division sequence-valued function $\phi : \mathcal{P} \to \ell^\infty$ defined by

$$\phi(P) = (\mu_i(A_i))_{i=1}^\infty \qquad P = \{A_i\}_{i=1}^\infty \in \mathcal{P}$$

is convex and weakly compact.

As in the finite case, we put for each $i \in N$, $f_i = d\mu_i/dv$ where $v = \sum_{i=1}^\infty \mu_i$. Moreover, let $\{B_i^*\}_{i=1}^\infty$ and $\{C_i^*\}_{i=1}^\infty$ be two sequences of measurable sets from $(\mathcal{X}, \mathcal{B})$ defined by

$$B_i^* = \bigcap_{\substack{j=1\\ j\neq i}}^{\infty} \{x \in \mathcal{X} : \overset{\circ}{p}_i \alpha_i^{-1} f_i(x) > \overset{\circ}{p}_j \alpha_j^{-1} f_j(x)\}, \quad i \in N,$$

$$C_i^* = \bigcap_{\substack{j=1\\ j\neq i}}^{\infty} \{x \in \mathcal{X} : \overset{\circ}{p}_i \alpha_i^{-1} f_i(x) \geq \overset{\circ}{p}_j \alpha_j^{-1} f_j(x)\}, \quad i \in N.$$

As a consequence of the following new minmax theorem (see [6])

Theorem 4 : Let $P^* = \{A_i^*\}_{i=1}^{\infty} \in \mathcal{P}$ be the α-optimal partition. Then

$$\inf_{p \in S^*} \sum_{i=1}^{\infty} p_i \alpha_i^{-1} \mu_i(A_i^*) = \sum_{i=1}^{\infty} \overset{\circ}{p}_i \alpha_i^{-1} \mu_i(A_i^*) = \sup_{P \in \mathcal{P}} \sum_{i=1}^{\infty} \overset{\circ}{p}_i \alpha_i^{-1} \mu_i(A_i) = t^*$$

where $S^* = \{s = (s_1, s_2, \dots) \in \ell^1 : s_i > 0$ for all $i \in N$ and $\sum_{i=1}^{\infty} s_i = 1\}$ we obtain an extension of theorem 2 to infinite case (see [6]).

Theorem 5 : For all $\alpha \in S^*$ there exist a point

$$p^\circ \in S_0 = \{s \in S^* : s_i \leq 2t^* \alpha_i \text{ for all } i \in N\}$$

and a corresponding α-optimal partition $P^* = \{A_i^*\}_{i=1}^{\infty} \in \mathcal{P}$ satisfying

(i) $B_i^* \subset A_i^* \subset C_i^*$,

(ii) $\mu_1(A_1^*) / \alpha_1 = \mu_2(A_2^*) / \alpha_2 = \dots = t^*$.

Moreover, any partition $P^* = \{A_i^*\}_{i=1}^{\infty} \in \mathcal{P}$ which satisfies (i) and (ii) is the α-optimal.

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A SYMMETRIC MINIMAX THEOREM WITHOUT LINEAR STRUCTURE

BOR-LUH LIN AND XIU-CHI QUAN

For a topological space X, let $\mathcal{U}(X)$ be the set of all real-valued upper semi-continuous functions on X. Given $0 < t < 1$, a set $\mathcal{F}$ in $\mathcal{U}(X)$ is said to be t-convex on X if for any f, g in $\mathcal{F}$, there exists an element h in $\mathcal{F}$ such that $h(x) \leq t \max \{f(x), g(x)\} + (1 - t) \min \{f(x), g(x)\}$ for all x in X. Similarly, X is said to be t-concave on $\mathcal{F}$ if for any x, y in X, there exists an element z in X such that $f(x) \geq t \max \{f(x), f(y)\} + (1 - t) \min \{f(x), f(y)\}$ for all f in $\mathcal{F}$.

Theorem : Let X be a compact space and let $\mathcal{F}$ be a set in $\mathcal{U}(X)$. If there exist $0 < s, t < 1$ such that X is s-concave on $\mathcal{F}$ and $\mathcal{F}$ is t-convex on X, then $\inf_{\mathcal{F}} \sup_{X} f(x) = \sup_{X} \inf_{\mathcal{F}} f(x)$. The result is a generalization of the minimax theorem of H. König (*Arch. Math.* **19**, (1968), 482-487) and Ky Fan (Proc. nat. Acad. Sci. **39**, (1953), 42-47).

ON A GRASSMANIAN FIXED POINT THEOREM AND A GENERALISATION OF THE BORSUK-ULAM THEOREM

MICHAEL MAGILL

Let me report briefly on a class of theorems obtained in the three references cited below. These results have their origin in a problem of mathematical economics : proving the existence of equilibrium with incomplete markets.

Let $G^{n,r}$ denote the Grassmanian manifold of K-dimensional subspaces of $\mathbb{R}^n$.

Theorem A : If $\psi^i : G^{n,r} \to \mathbb{R}^n$, i = 1, ..., K are continuous fonctions then there exists $L \in G^{n,r}$ such that

$$\psi^i(\bar{L}) \in \bar{L} \quad , i = 1, \ldots, K$$

Let $O^{n,r}$ denote the Stiefel manifold of K-frames in $\mathbb{R}^n$ and let O_k denote the orthogonal group in K letters.

Theorem B : If $\phi : O^{n,r} \to (R^r)^{n-r}$ is a continuous map satisfying $\psi(gx) = g\psi(x)\ \forall\ g \in O_k$, $\forall\ x \in O^{n,k}$ then there exists $\bar{x} \in O^{n,k}$ such that

$$\psi(\bar{x}) = 0$$

Theorem A and B are equivalent to the geometric property that a naturally induced vector bundle over the Grassmanian has a non-zero Euler class (non-zero mod z Euler number). The proof can thus be obtained either via techniques of algebraic topology (reference [1]) or technique of differential topology, namely mod z intersection theory (references [2], [3]).

Theorem A can be generalized to the case where $G^{n,k}$ is replaced by the flag-manifold $F^{n,k}$ of K mutually orthogonal one dimensional subspaces of $\mathbb{R}^n$. Theorem B can be generalized to the case where the orthogonal group O_k is replaced by the group T_k of diagonal matrices with $\pm$ 1 on the main diagonal. Theorem A has as a corollary Brower's Theorem. Theorem B is a generalization of the Borsuk-Ulam theorem. The vector space $\mathbb{R}^n$ can be replaced by $\mathbb{C}^n$ or $\mathbb{H}^n$, the n-dimensional vector spaces over the complex numbers or the quaternions.

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ORDERING IN THE FIXED POINT THEORY

ROMAN MANKA

Let X be an arbitrary partially ordered set and $\mathcal{A}$ denote a family of non-empty chains, i.e. well ordered subsets, in X which satisfies :

i) if $S \in \mathcal{A}$ and y is an upper bound of S in X, then $S \cup \{y\} \in \mathcal{A}$,

ii) if $S \in \mathcal{A}$ and $T < S$, then $T \in \mathcal{A}$;

where $T < S$ means : $T \subsetneq S$ and every $y \in S - T$ is an upper bound of T. A function $\sigma : \mathcal{A} \to X$ is called a sup-function iff for each $s \in \mathcal{A}$ the value $\sigma(S)$ is an upper bound of S in X but $\sigma(S)$ is the greatest element of S whenever it exists in S. It is true that σ is "least upper bound" iff σ is monotone, i.e. $T \subset S$ and $S, T \in \mathcal{A}$ imply $\sigma(T) \leq \sigma(S)$.

Given a sup-function $\sigma : \mathcal{A} \to X$, a function $f : X \to X$ and a $a \in X$, such that $a \leq f(a)$, define $\mathcal{C}_a(f, \sigma)$ to be the family of all chaine $C \in \mathcal{A}$ such that a is the smallest element of C and :

iii) if $x \in C - \{\sigma(C)\}$, then f(x) is an immediate successor of x in C,

iv) if $D < C$, then $\sigma(D)$ is a least upper bound of D in C.

Then the union $D_a(f, \sigma)$ of all members of $\mathcal{C}_a(f, \sigma)$ is a chain in X and supposing that for every two chains $C, D \in \mathcal{A}$

v) $\sigma(D) = \sigma(C)$ whenever D is a cofinal subset of C,

and that the function $f : X \to X$ satisfies

vi) $x \leq f(x)$ for all $x \in X$,

we have

Theorem : The orbit $0_a(f, \sigma)$ is a common part of all (f, σ) - invariant sets containing the point a / here S is (f, σ) - invariant iff $f(S) \subset S$ and $\sigma(C) \in S$ for every $C \in \mathcal{A}$ satisfying $C \subset S$/.

Proof is provided within Zermelo's minimal theory of sets, without any form of the axiom of choice, so that we obtain a constructive extention of the iteration procedure having among.

Consequences :

1. Zermelo's and Knaster-Tarski fixed point theorems in ordered sets ;
2. Caristi's theorem and Turinici's theorem for Hausdorff spaces ;
3. Borsuk's theorem on the fixed point property of acyclic curves ;

4. Cantor-Bernstein theorem and Kuratowski-Zorn lemma ;
5. Equivalence of the axiom of choice with some extentions of Dedekind continuity principle / concerning the set of real numbers ;
6. Equivalence of dependent choices with Ekeland's extention of Weierstrass theorem / concerning extremes of real functions ;
7. The function $a \rightarrow \sigma(0_a(f, \sigma))$ is a closure operator in X, on condition that $0_a(f, \sigma) \in \mathcal{A}$.

APPLICATIONS OF SOME RECENT FIXED POINT THEOREMS TO THE STUDY OF A CLASS OF PARTIAL DIFFERENTIAL INCLUSIONS

SALVATORE ANGELO MARANO

The aim of the talk I would like to give is to provide an application of the following three fixed point theorems to the study of a class of partial differential inclusions.

Theorem 1 : Let X be a non-empty closed subset of a Banach space E and F a multi-valued ! Then, the set of fixed points of F is an non-empty absolute extensor for paracompact spaces. (B. Ricceri, une propriété topologique de l'ensemble des points fixes d'une contraction multivoque à valeurs convexes, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **81**, (1987), 283-286).

Theorem 2 : Let (X, d) a complete metric space and let F, F_1, F_2, ... be a sequence of closed-valued contractions with the same constant k. Suppose that there exists a dense subset D of X, such that $F(x) \subseteq \operatorname{Li}_{n\to\infty} F_n(x)$ for every $x \in D$. Then, $\operatorname{Fix}(F) \subseteq \operatorname{Li}_{n\to\infty} \operatorname{Fix}(F_n)$. (O. Naselli Ricceri, A fixed points of multi-valued contractions, J. Math. Anal. Appl. **135**, (1988), 406-418.)

Theorem 3 : Let (X, d) be a complete metric space and F_1, F_2 two closed-valued multifunctions from X into itself which are multivalued contractions with the same constant k. Then, $d_H(\operatorname{Fix}(F_1), \operatorname{Fix}(F_2)) \leq \frac{1}{1-k} \sup_{x\in X} d_H(F_1(x), F_2(x))$. (T.-C. Lim, On fixed point stability for set-valued contractive mappings with applications to generalized differential equations, J. Math. Anal. Appl. **110**, (1985), 436-441).

Precisely, if a, b are two positive real numbers Q, the rectangle $[0, a] \times [0, b]$, $(X, \|\cdot\|)$ a real Banach space, F a multifunction from $Q \times X^4$ into X, $f \in C^1([0, a], X)$, $g \in C^1([0, b], X)$, $f(0) = g(0)$, consider the problem

$$(P)\quad \begin{cases} \dfrac{\partial^2 z}{\partial x\, \partial y} \in F(x, y, z, \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial x\, \partial y}) \\ z(x, 0) = f(x) \\ z(0, y) = g(y). \end{cases}$$

Besides Problem (P), given $z_0 \in F(0, 0, f(0), f'(0), g'(0), z_0)$ consider also the following

$$(P_0)\quad \begin{cases} \dfrac{\partial^2 z}{\partial x\, \partial y} \in F(x, y, z, \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial x\, \partial y}) \\ z(x, 0) = f(x) \\ z(0, y) = g(y) \\ \dfrac{\partial^2 z(0, 0)}{\partial x\, \partial y} = z_0. \end{cases}$$

We say that a function $z : Q \to X$ is a classical solution of (P) if $z, \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial x \partial y} \in C^0(Q, X)$ and, for every (x, y) Q one has $\dfrac{\partial^2 z(x,y)}{\partial x \partial y} \in F(x, y, z(x, y), \dfrac{\partial z(x, y)}{\partial x}, \dfrac{\partial z(x, y)}{\partial y}, \dfrac{\partial^2 z(x,y)}{\partial x \partial y})$, $z(x, 0) = f(x)$, $z(0, y) = g(y)$. A classical solution of (P_0° is, of course, any classical solution of (P) whose second mixed derivative assumes at (0, 0) the prescribed value z_0. Denote by $\Gamma(f, g, F)$ (resp. $\Gamma(f, g, z_0, F)$) the set of all classical solutions of (P) (resp. of (P_0)). Applying Theorems 1-3 above we prove that, under suitable hypothesis, each of the sets $\Gamma(f, g, F)$, $\Gamma(f, g, z_0, F)$ is a retract of an appropriate function space. Under the same assumptions, we prove also that these sets depend in a Lipschitzian way on f, g, z_0, F.

Furthermore, given four sequences (f_n), (g_n), (z_n^0), (F_n), we give some sufficient conditions under which each $z \in \Gamma(f, g, F)$ (resp. $z \in \Gamma(f, g\ z_0, F)$) is the limit, with respect to an appropriate metric, of a sequence (z_n) such that $z_n \in \Gamma(f_n, g_n, F_n)$ (resp. $z_n \in \Gamma(f_n, g_n, z_n^0, F_n)$) for all $n \in N$.

SOME RESULTS ON APPROXIMATE CONTINUOUS SELECTIONS, FIXED POINTS AND MINIMAX INEQUALITIES

EZIO MARCHI AND J. E. MARTINEZ-LEGAZ

Inspired in the proof of Browder fixed point theorem we first present a lemma stating the existence of a continuous selection for a multivalued mapping satisfying appropriate assumptions. Secondly, from this result we get a theorem which states the existence of continuous approximations, in same sense, to arbitrary functions defined on a dense subset of a non-empty compact subset of a locally convex Hausdorff topological vector space, with values into a vecto. ... When the vector space is the real line $\mathbb{R}$, it is illustrating that the previous theorem says th. given an arbitrary function $f : D \rightarrow \mathbb{R}$ where D is dense subset of a non-empty compact subset $K \subset E$, for any neighborhood V of 0 in E there exists a continuous function $p_V : K \rightarrow \mathbb{R}$ such that for any $x \in K$ one can find $x', x'' \in (x + V) \cap D$ for which $f(x') \leq p_V(x) \leq f(x'')$.

Next we introduce the class of D-mappings from a non-empty compact subset K of a locally convex Hausdorff topological vector space into another vector space. The class of D-mappings includes that of convex multi-valued mappings (on convex domains). In particular, when K is a closed real interval and the D-mapping is a single real valued function T, a sufficient condition for T being a D-mapping is that it be Darboux continuous. This means that for any p, q with $a \leq p \leq q \leq b$ and any $c \in R$ between T(q) and T(q) there is an $s \in [p, q]$ such that $T(s) = c$.

We prove that the D-mappings satisfy an approximate continuous selection theorem. As a consequence of this result, for the case of single valued functions we obtain that any Darboux continuous function can be made continuous by an arbitrary small perturbation of the independent variable.

Our next result presented in this section, which is based on the first lemma, can be interpreted in terms of continuous selections of ε-solutions of parametric constrained optimization problems. As a corollary of this theorem it is deduced a result which is related to a well known theorem in game theory concerning optimal continuous decision rules. A further theorem with the same conclusion of the previous one but with weaker assumptions except in the requirement that the inverse images of certain multivalued function to be open is also presented. As a direct consequence of the last theorem a further minimax theorem is given.

In order to derive a counterpart to the above mentioned theorem on continuous decision rules in terms of those of the other player we present a lemma which is concerned with the equality between the minsup, in which the allowed strategies of one player are constrained by those of the other, by a multivalued function G, and an amount related to the minimax obtained by introducing the family of the finite sets such that the intersection with G(x) is non-empty for all points in the space of strategies of the first player. Another theorem determines an equality between that minsup and the infsup of a certain expression obtained from the original payoff function considering fixed points of G and the set of functions whose restrictions to any polytope are continuous. The key result in proving this theorem has been a generalization of Ky Fan's inequality. Our version takes into account a multivalued function which restricts the points for the second variable.

The Brouwer fixed point theorem in connection with the existence of continuous selections is useful to obtain fixed point theorems for multivalued mappings as e.g. in the case of the Browder fixed point theorems for mappings from a non-empty compact convex subset of a topological vector space into itself which have non-empty convex images and open inverse images. We employ this technique to derive approximate fixed point theorems from the approximate continuous selections results given previously. By approximate fixed points of a multi-valued mapping we mean two points which are close one to the other and such that one of them belongs to the image of the other one. Our first approximate fixed point theorem is valid for an arbitrary function defined on a dense set D in K, a non-empty compact convex subset of a locally convex Hausdorff topological vector space. As a particular case when the set K is a closed real interval [a, b], it says that for any $\varepsilon > 0$ one can find $x_\varepsilon \in [a, b]$ and $x_\varepsilon', x_\varepsilon'' \in D$ such that $|x_\varepsilon' - x_\varepsilon| < \varepsilon$, $|x_\varepsilon'' - x_\varepsilon| < \varepsilon$ and $f(x_\varepsilon') \leq x_\varepsilon \leq f(x_\varepsilon'')$. This theorem could be regarded as too weak in view of its great generality, due to the absence of hypothesis on the function ; however in order to demonstrate that this is not so, we derive from it, as a corollary, the finite-dimensional version of Kakutani fixed point theorem.

Using one of our previous theorems we obtain an approximate fixed point theorem for D-mappings. In the particular case of a single valued function which is Darboux continuous from an interval [a, b] into itself this result assures that for each $\varepsilon > 0$ there exist $x_\varepsilon, x_\varepsilon' \in [a, b]$ such that $|x_\varepsilon' - x_\varepsilon| < \varepsilon$ and $x_\varepsilon = f(x_\varepsilon')$.

As a further consequence of the approximate fixed point theorem for D-mappings T we obtain that the smallest upper semicontinuous mapping with closed values which contains T pointwise has a fixed point.

Next, we give some minimax inequalities and similar results related to Ky Fan's inequality involving several functions. The first considers the existence of a kind of "equilibrium" for an arbitrary set of continuous functions defined an non-empty compact convex subsets of locally convex Hausdorff topological vector spaces with a quasiconcavity property. As a corollary of this result we obtain the infinite version, in locally convex spaces of the Nash theorem on the existence of equilibrium points in non-cooperative games. Next we present a theorem concerning an equality involving infmax of several functions in a similar context as in the preceding theorem. We prove it using the Browder fixed point theorem. As a particular case we present a result concerning an infmax expression which involves all fixed points of a multivalued function.

Finally, from a previous theorem we derive a vector version of Ky Fan's inequality, involving n functions, which is stated in terms of weakly maximal elements of the closure of the image set. For the particular case corresponding to n = 1, this result coincides with our generalization of Ky Fan's inequality.

POINTS FIXES DE FONCTIONS HOLOMORPHES

PIERRE MAZET

Soient Ω un ouvert borné d'un espace de Banach E et f holomorphe de Ω dans lui-même. Alors, si E est réflexif (et même dans de nombreux autres cas) l'ensemble Fix f des points fixes de f est vide ou est une sous-variété directe de Ω.

Si en outre Ω est convexe Fix f est un rétracte holomorphe de Ω.

Nous montrons comment prouver ces résultats en utilisant diverses techniques (outre la théorie élémentaire des fonctions holomorphes) :

1. Théorème des fonctions implicites.
2. Moyenne ergodique.
3. Transformation du problème en un problème linéaire par isomorphismes locaux.

ACYCLIC MULTIFUNCTIONS WITHOUT METRIZABILITY

LYNN MCLINDEN

A multifunction is called acyclic provided it is upper semicontinuous with acyclic images. Since acyclic sets greatly generalize compact convex sets, results involving acyclic multifunctions are of basic interest and offer a wide range of applications. Nearly all previous work on acyclic multifunctions has been restricted to the metrizable setting. An exception to this is the following notable result of Nikaidô, which is patterned after the celebrated coincidence theorems of Eilenberg-Montgomery (1946) and Begle (1950).

Nikaidô Coincidence Theorem (1959). Let M be a compact Hausdorff topological space, let N be a finite-dimensional compact convex set, and let σ and τ be continuous functions from M into N. If the inverse image $\tau^{-1}(g)$ is acyclic for each $g \in N$, then there exists some $p \in M$ such that $\sigma(p) = \tau(p)$.

The present work builds on Nikaidô's result in order to develop a variety of existence results for acyclic multifunctions without requiring metrizability. Thus, for example, Browder's coincidence theorem (1968) is extended beyond the locally convex setting and to allow acyclic images :

Theorem : Let S be a multifunction from a nonempty compact convex subset C of a Hausdorff topological vector space into a nonempty convex subset D of a Hausdorff topological vector space. Assume that Sz is nonempty convex for each $z \in C$ and that $S^{-1}w$ is open for each $w \in D$. If T is an acyclic multifunction from D into C, then there exist some $z \in C$ and $w \in D$ such that $z \in Tw$ and $w \in Sz$.

As another example, the Fan-Glicksberg fixed point theorem (1952) is extended to permit acyclic images :

Theorem : If T is an acyclic multifunction from a nonempty compact convex subset C of a locally convex Hausdorff topological vector space into C, then there exists some $z \in C$ such that $z \in Tz$.

Our proof of the latter result relies on a certain somewhat involved "tool theorem", from which additional results are deduced.

The various results obtained toush on several topics in optimization, including variational inequalities, complementarity problems, Walrasian equilibrium and Nash equilibrium, as well as fixed point and coincidence theory.

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GLOBAL CONVERGENCE AND NON EXISTENCE OF PERIODIC POINTS OF PERIOD 4

B. MESSANO

1. Introduction

Let S be a compact metric space and $C^\circ(S)$ the set of all continous functions from S into itself.

If $F \in C^\circ(S)$ it is said that for the pair (S, F) the successive approximations method (abbr. s.a.m.) converges globally if for each point $x \in S$ the sequences $(F^n(x))_{n \in \mathbb{N}}$ converges (to a fixed point of F).

A theorem of global convergence relative to the case S = [0, 1] is the following (see [1, 2, 7, 8]) :

(1.1) Whatever $F \in C^\circ([0, 1])$ be, for the pair ([0, 1], F) the s.a.m. converges globally if and only if F has no periodic point of period 2.

A theorem similar to (1.1) is the following (see [3]) :

(1.2) Whatever $F \in C^\circ(S^1)$ (S^1 denotes the unit circle) be, for the pair (S^1, F) the s.a.m. converges globally if and only if F has no periodic point of even period.

The theorems (1.1) and (1.2) can be expressed by saying that [0, 1] and S^1 are examples of compact metric spaces S for which the following proposition is true.

A) There exists a subset $\mathcal{M}$ of $\mathbb{N}$ such that, whatever $F \in C^\circ(S)$ be, for the pair (S, F) the s.a.m. converges globally if and only if F has no periodic point whose period belongs to M.

Another example of compact metric space for which the proposition A) is true is given by the following theorem (see [10]) which generalizes the theorem (1.1) :

(1.3) If X is an arcwise connected tree endowed with a finite number, m, of end points and $F \in C^\circ(X)$, then for the pair (X, F) the s.a.m. converges globally if and only if F has no periodic point whose period belongs to {2, ..., m}.

It is not known if the proposition A) is true when $S = [0, 1]^2$. Then, it is interesting the problem to give meaningful examples of nonempty subsets $\mathcal{J}$ of $C^\circ([0, 1]^2)$ and subsets M of $\mathbb{N}$ for which the following proposition is true :

B) Whatever $F \in \mathcal{J}$ be for the pair $([0, 1]^2, F)$ the s.a.m. converges globally if and only if F has no periodic point whose period belongs to M.

The theorem (3.2), which will be proved in section 3, gives a solution of above problem. 1)

In fact from (3.2) trivially follows that the proposition B) is true if M is equal to {4} and $\mathcal{J}$ is the set of all functions F from $[0, 1]^2$ into itself of kind :

$$(*) \qquad F(x, y) = (f(x, y), x), \qquad f \in C^{\circ}([0, 1]^2, [0, 1]),$$

such that :

1) f is decreasing with respect to both variables ;

2) Set, for each $x \in [0, 1]$, $\varphi(x) = f(x, x)$ it results :

$$(\varphi(x) - x)\ (\varphi^2(x) - x) \geq 0 \quad \forall x \in [0, 1] ;$$

3) There do not exist a periodic point ξ of φ of period 2 and a point $P \in [0, 1]^2$ such that the point $(\xi, \varphi(\xi))$ is a cluster point of the sequence $(F^n(P))_{n \in \mathbb{N}}$.

In the end, let us observe that theorem (3.2) has been proved using some results, relate to the existence of periodic points of period 4 for a function F of kind (*) satisfying 1), showed in section 2.

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ON THE POROSITY OF THE SET OF CONTRACTIONS WITHOUT FIXED POINTS

J. MYJAK

Let C be a noempty closed bounded subset of a Banach space. For a contraction (α-contraction, ω-contraction) $f : C \to C$ consider the fixed point problem

$$(*) \qquad f(x) = x$$

consisting in finding all $x \in C$ such that (*) is satisfied. (Here α means the Kuratowski's *measure of noncompactness*, and ω the De Blasi's *measure of weak noncompactness*).

The problem (*) is said to be well posed if it has a unique solution, say x_o, and every sequence $\{x_n\} \subset C$ satisfying $\lim_{n \to +\infty} \|fx_n - x_n\| = 0$, converges to x_o.

The problem (*) is said to be weakly (resp. weakly*) well posed if the set of all solutions of (*) is compact (resp. weakly compact) and every sequence $\{x_n\} \subset C$ satisfying $\lim_{n \to +\infty} \|fx_n - x_n\| = 0$ is compact (resp. weakly compact).

A subset X of E is said to be porous on E if there is $0 < \alpha \leq 1$ and $r_o > 0$ such that for every $x \in E$ and $0 < r \leq r_o$, there is $y \in E$ such that $B(y, \alpha r) \subset B(x, r) \cap (E \setminus X)$. (Here $B(u, \varphi)$ stands for the open ball in E with center at u and radius $\varphi > 0$).

A set X is called G-porous on E if it is a countable union of porous subset of E.

Note that every G-porous set on E is meager, and that, there are meager subsets of E which are not G-porous.

The following results can be proved :

(i) Let $\mathcal{M}$ be the space of all contractions $f : C \to C$ endowed with the metric of uniform convergence. Let $\mathcal{M}^o$ be the set of all $f \in \mathcal{M}$ such that f has a unique fixed point, say x_o, and the sequence $\{f^n x\}$ converges to x_o for every $x \in C$. Then $\mathcal{M} \setminus \mathcal{M}^o$ is a G-porous subset of $\mathcal{M}$. In particular $\mathcal{M}^o$ is a residual subset of $\mathcal{M}$.

(ii) Let $\mathcal{M}^*$ be the set of all $f \in \mathcal{M}$ such that the problem (*) is well posed. Then $\mathcal{M} \setminus \mathcal{M}^*$ is a G-porous subset of $\mathcal{M}$. In particular $\mathcal{M}^*$ is a residual subset of $\mathcal{M}$.

(iii) Let $\mathcal{K}$ be the set of all α-contractions (resp. ω-contractions) $f : C \to C$, endowed with the metric of uniform convergence. Let $\mathcal{K}^*$ be the set of all $f \in \mathcal{K}$ for which the problem (*) is weakly (resp. weakly*) well posed. the the set $\mathcal{K} \setminus \mathcal{K}^*$ is a G-porous subset of $\mathcal{K}$. In particular $\mathcal{K}^*$ is a residual subset of $\mathcal{K}$.

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BY VARIATIONAL AND SINGULAR VARIATIONAL DERIVATIVES IN NONLINEAR FUNCTIONAL ANALYSIS

M. Z. NASHED

FIXED POINTS AND COINCIDENCE POINTS OF MULTIFUNCTIONS

DEVIDAS PAI

Let p be a continuous seminorm on a Hausdorff locally convex space X and let C be a nonempty convex subset of X, which is not necessarily compact. Given a function $g : C \to X$ and a multifunction $F : C \to CC(X)$ with nonempty closed convex values, we investigate here conditions ensuring existence of points x_o in C satisfying

$$d_p(gx_o, Fx_o) = d_p(C, Fx_o).$$

Hence

$$d_p(gx_o, Fx_o) := \inf\{p(gx_o - v) : v \in Fx_o\} \text{ and}$$
$$d_p(C, Fx_o) := \inf\{p(u - v) : u \in C, v \in Fx_o\}.$$

Using these approximation results, we explore suitable boundary conditions yielding existence of points x_o in C satisfying $gx_o \in Fx_o$.

GENERALIZED BROUWER-KAKUTANI TYPE FIXED POINT THEOREMS

SEHIE PARK

The Brouwer or Kakutani fixed point theorems have numerous generalizations. Many of them deal with weakly inward (outward) maps. Recently, J. Jiang [11] generalized the notion of such maps and obtained generalizations of known fixed point theorems for new class of multimaps defined on paracompact convex subsets of a Hausdorff topological vector space (t.v.s). In this paper, we obtain extended versions of Jiang's theorems.

A convex space X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a c-compact set if for each finite set $S \subset X$ there is a compact convex set $L_s \subset X$ such that $L \cup S \subset L_s$ [14].

Let E be a real t.v.s and E* its topological dual. A multimap $F : X \to 2^E$ is said to be upper hemi-continuous (u.h.s) if for each $\phi \in E^*$ and for any real a, the set $\{x \in X : \sup \phi(Fx) < a\}$ is open in X. Note that an upper semi-continuous (u.s.c) map is upper demi-continuous (u.d.c.), and that an u.d.c. map is u.h.c. [21].

Let cc(E) denote the set of nonempty closed convex subsets of E and kc(E) the set of nonempty compact convex subsets of E. Bd and -will denote the boundary and closure, resp.

Let $X \subset E$ and $x \in E$. The inward and outward sets of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows :

$$I_X(x) = x + \cup_{r>0} r(X - x), \quad O_X(x) = x + \cup_{r<0} r(X - x).$$

A map $F : X \to 2^E$ is said to be weakly inward (outward, resp.) if $Fx \cap \bar{I}_X(x) \neq \emptyset$ [$Fx \cap \bar{O}_X(x) \neq \emptyset$, resp.] for each $x \in \mathrm{Bd}\, X \setminus Fx$.

For $A \subset \mathbf{R}$, let $|A|$ denote the set $\{|a| : a \in A\} \subset \mathbf{R}$.

Theorem 1 : Let X be a nonempty paracompact convex set in a real t.v.s. E, L a c-compact subset of X, and K a nonempty compact subset of X. Let $F : X \to 2^E$ be a map such that, for any $\phi \in E^*$, $\{x \in X : \emptyset x \geq \inf\phi(Fx)\}$ is closed, and either

(A) E^* separates points of E and $F : X \to kc(E)$, or

(B) E is locally convex and $F : X \to cc(E)$.

(i) If $\inf|\phi(Fx - \bar{I}_X(x))| = 0$ for every $x \in K \cap BdX$ and $\phi \in E^*$, and $\inf|\phi(Fx - \bar{I}_L(x))| = 0$ for every $x \in X\backslash K$ and $\phi \in E^*$, then F has a fixed point.

(ii) If $\inf|\phi(Fx - \bar{O}_X(x))| = 0$ for every $x \in K \cap BdX$ and $\phi \in E^*$, and $\inf|\phi(Fx - \bar{O}_X(x))| = 0$ for every $x \in X\backslash K$ and $\phi \in E^*$, then F has a fixed point and $F(X) \supset X$.

For a metrizable t.v.s. E with metric d, the paracompact assumption on X is redundant in Theorem 1. Moreover, in a normed space E, for any $A, B \in cc(E)$, we have

$$d(A, B) = 0 \text{ iff } \inf|\phi(A - B)| = 0 \text{ for all } \phi \in E^* \text{ [11].}$$

Therefore, from Theorem 1, we have

Theorem 2 : Let X be a nonempty convex set in a real normed space E, L a c-compact subset of X, and K a nonempty compact subset of X. Let $F : X \to cc(E)$ be a map such that, for any $\phi \in E^*$, $\{x \in X : \emptyset x \geq \inf \emptyset(Fx)\}$ is closed.

(i) If $d(Fx, \bar{I}_X(x)) = 0$ for every $x \in K \cap BdX$ and $d(Fx, \bar{I}_L(x)) = 0$ for every $x \in X\backslash K$, then F has a fixed point.

(ii) If $d(Fx, \bar{O}_X(x)) = 0$ for every $x \in K \cap BdX$ and $d(Fx, \bar{O}_L(x)) = 0$ for every $x \in X\backslash K$, then F has a fixed point and $F(X) \supset X$.

If F is u.h.c., then $\{x \in X : \phi x \geq \inf \phi(Fx)\}$ is closed for any $\phi \in E^*$, but not conversely. Theorems 1(B) and 2 for u.h.c. maps are due to Jiang [11, Corollaries 2.3 and 2.4] with different proofs.

Let W(x) denote any one of weakly inward (outward) sets in Theorem 1. Then

$$Fx \cap W(x) \neq \phi \text{ implies } \inf|\phi(Fx - W(x))| = 0,$$

but not conversely. For an example, see [11]. Therefore, Theorem 1 contains known results on weakly inward (outward) maps.

In fact, in this case, Theorem 1(B) is due to Shih and Tan [21, Theorems 4 and 5] for u.h.c. maps, and, for K = L, to Ky Fan [7, Corollary 2] for u.d.c. maps.

For a compact X = L = K, there have been appeared a number of results on weakly inward (outward) maps. In this case, Theorem 1(A) is due to Park [15, Theorem 6] for u.s.c. maps and (B) to Rogalski [1, Theorems 6.4.14 and 6.4.15] for u.h.c. maps. Note that these generalizes many of well-known Browder or Kakutani type fixed point theorems due to Schauder [20], Tychonoff [22], Kakutani [13], Bohnenblust and Karlin [2], Ky Fan [4], [5], [6], Glicksberg [8], Browder [3], Halpern and Bergman [9], Halpern [10], Reich [17], [18], Kaczynski [12], and Park [15]. for details, see [15].

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ON GENERALIZATIONS OF THE MEIR-KEELER TYPE CONTRACTION MAPS

B. E. RHOADES, S. PARK AND K. B. MOON

In 1969, Meir and Keeler [2] obtained a remarkable generalization of the Banach contraction principle. Since then, a number of generalizations of their result has appeared. In 1981, the second author and Bae [3] extended the Meir-Keeler theorem to two commuting maps by adopting Jungck's method. This influenced many authors, and consequently, a number of new results in this line followed. Recent works of Sessa and others ([5] and [6]) contain common fixed point theorems of four maps satisfying certain contractive type conditions.

In the present paper, we give a new result which encompasses most of such generalizations of the Meir-Keeler theorem. Further our result also includes many other generalizations of the Banach contraction principle.

Some authors have obtained fixed point theorem on 2-metric spaces. However, 2-metric versions are easily obtained from metric ones by an obvious modification. Therefore, for simplicity, we have confined this work to metric spaces.

Previous to this paper, Meir-Keeler type conditions have required continuity of the maps involved. In our Theorem we remove this restriction. We also replace the condition of commutativity, or weakly commutative, by a weaker condition called compatible. As a consequence, our theorem is the most general fixed point result of its type and includes over fifty theorems in the literature as special cases.

Let (X, d) be a metric space, A and S selfmaps of X. A and S are said to be weakly commuting at a point x if $d(ASx, SAx) \leq d(Sx, Ax)$. This property was first defined by Sessa [4], and is strictly weaker than the condition that A and S commute at x. A property weaker than that of weakly commuting is compatability [1] or preorbitally commuting [7]. Two maps A and S are compatible if, whenever there is a sequence $\{x_n\} \subset X$ satisfying $\lim Ax_n = \lim Sx_n = u$, then $\lim d(SAx_n, ASx_n) = 0$. Every weakly commuting map is compatible, but there are examples to show that the converse is false.

Our main result is the following

Theorem 1 : Let (X, D) be a complete metric space, S, T selmaps of X with S or T continuous. Suppose there exists a sequence $\{A_i\}$ of selmaps of X satisfying

(i) either $A_i : X \to SX \cap TX$ for each i, or

(i') $T : X \to \cap_i A_i X$,

(ii) each A_i is compatible with S and T,

(iii) each A_i weakly commutes with S at each point ξ for which $A_i\xi = S\xi$ and each A_i weakly commutes with T at each point η for which $A_i\eta = T\eta$.

and

(iv) for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each x, y $\in$ X,

$$\varepsilon \leq M_{ij}(x, y) < \varepsilon + \delta \text{ implies } d(A_ix, A_jy) < \varepsilon,$$

where

$$M_{ij}(x, y) = \max\{d(Sx, Ty), d(Sx, A_ix), d(Ty, A_jy), [d(Sx, A_jy) + d(Ty, A_ix)]/2\}.$$

Then all the A_i, S and T have a unique common fixed point.

The oral presentation of this paper will include a listing of some of the many results that this theorem generalizes.

The complete text of this paper, including a proof of Theorem 1 will appear in the Journal of Mathematical Analysis and Applications.

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ITERATIONS DISCRETES

FRANÇOIS ROBERT

Nous nous proposons d'entreprendre l'analyse du comportement dynamique d'**itérations discrètes**, c'est à dire conduites sur un ensemble fini (en général très grand : des millions d'éléments). C'est le cadre mathématique naturellement sous-jacent aux modèles actuels du type réseaux de processeurs, résaux d'automates, réseaux neuronaux (Ref. [6] à [8], [10], [11], [14] à [18]).

Le contexte est le suivant : n automates binaires (prenant pour seuls états 0 et 1) sont reliés entre eux par un graphe de connexion donné. On note f_i la fonction de transition de l'automate numéro i (i = 1, 2, ..., n) : C'est une fonction booléenne des n variables booléennes x_j représentant l'état de l'automate j (j = 1, 2, ..., n).

$F = (f_i)_{i=1,2,...,n}$ est alors la **fonction de transition globale du système**. C'est une application du n-cube $\{0, 1\}^n$ dans lui-même, et l'on va s'intéresser au système **dynamique discret :**

$$x^0 \in \{0, 1\}^n \, ; \, x^{r+1} = F(x^r) \; (r = 0, 1, 2\ldots) \quad \text{(itération parallèle sur F)}$$

ou plus généralement :

$$x^0 \in \{0, 1\}^n \, ; \, x^{r+1} = H(x^r) \; (r = 0, 1, 2\ldots) \quad \text{où H est un opérateur du n-cube}$$

dans lui-même correspondant au **mode opératoire** choisi pour activer le réseau (parallèle (H = F), série parallèle, séquentiel, chaotique, asynchrone...)

Exemple : On place un automate binaire en chaque noeud d'un maillage plan (p, p) : il y a $n = p^2$ automates, donc $N = 2^{p^2}$ états possibles du système. pour p = 5, N = 33 554 432 ; pour p = 10, $N = 10^{30}$ environ. Les itérations discrètes considérées évoluent dans un ensemble de N éléments.

Pour des modes opératoires réguliers (parallèle, série-parallèle, séquentiel) les attracteurs sont des points fixes ou des cycles, toujours atteints en un nombre fini d'itérations. Pour les modes opératoires chaotiques ou asynchrones, les choses sont bien plus compliquées.

Nous avons développé des outils métriques pour l'analyse et la comparaison de ces différents modes opératoires : distance vectorielle booléenne, contraction discrète, dérivée (Jacobienne) discrète. On obtient des résultats de convergence globale ou locale de telles itérations vers un point fixe ou un cycle. Les résultats de convergence globale proviennent de la notion de contraction discrète ; les résultats locaux, eux proviennent de l'utilisation de la Jacobienne discrète : points fixes ou cycles attractifs dans un voisinage premier, second, ou plus généralement massif. Une méthode de Newton discrète peut aussi être définie et étudiée du point du vue de sa convergence (discrète). Actuellement, l'accent est mis sur l'analyse du comportement d'itérations **chaotiques** ou plus généralement **asynchrones**.

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POINTS FIXES DES CONTRACTIONS MULTIVOQUES

J. SAINT RAYMOND

Soient E un espace de Banach, C un convexe fermé non vide de E et ϕ une multiapplication définie sur C à valeurs fermées contenues dans C. On dit que ϕ est une **contraction** s'il existe une constante $q < 1$ telle que pour tout couple (x, y) de points de C, la distance de Hausdorff entre $\phi(x)$ et $\phi(x)$ soit majorée par $q.\|x - y\|$. On dit aussi qu'un point x de C est un **point fixe** de ϕ si $x \in \phi(x)$.

Le résultat fondamental est que toute contraction multivoque a au moins un point fixe, et plus précisément :

Théorème 1 : Si ϕ est une multiapplication q-lipschitzienne (avec $q < 1$), et si a est un point de C tel que $d(a, \phi(a)) < \delta$, il existe un point fixe x_0 de C avec

$$\|x_0 - a\| < \frac{1}{1-q} \cdot \delta$$

Le but de cet exposé est de donner quelques résultats qualitatifs sur l'ensemble des points fixes d'une contraction multivoque à **valeurs convexes**.

C'est B. Ricceri qui m'a posé un certain nombre de questions sur ce sujet lors d'un séjour que j'ai fait, en juin 1987, à l'Université de Catane, et qui est à l'origine de ce travail.

Il avait remarqué, notamment, qu'en utilisant le théorème de sélection de Michaël et la méthode indiquée ci-dessus pour prouver l'existence de points fixes, on obtenait une rétraction continue de C (donc aussi de E) sur l'ensemble des points fixes de ϕ, c'est-à-dire que cet ensemble de points fixes est un rétracte absolu.

Ce résultat contient le seul résultat connu antérieurement, à savoir que l'ensemble des points fixes est connexe par arc.

I. Caractérisation des ensembles de points fixes

Nous appellerons, dans toute la suite, **ensemble de points fixes** tout ensemble fermé X de l'espace E pour lequel existe une contraction multivoque ϕ à valeurs convexes dont X est l'ensemble des points fixes.

Un premier résultat est que les ensembles de points fixes ne sont pas caractérisés par le fait qu'ils sont des rétractes absolus.

En particulier, le graphe de la fonction $t \to t.\sin\frac{\pi}{t}$ sur [0, 1] est un arc simple, donc un rétracte absolu, mais ne peut être un ensemble de points fixes pour aucune norme sur $\mathbb{R}^2$.

Une des seules conditions suffisantes connues pour qu'un ensemble X soit un ensemble de points fixes est la suivante :

Théorème 2 : Si E est un espace de Banach, C un convexe fermé de E, f une fonction lipschitzienne : $C \to \mathbb{R}$ et $E_1 = E \times \mathbb{R}$ muni de la norme $|||(x, t)||| = (||x||^p + |t|^p)^{1/p}$ $(1 \leq p \leq +\infty)$, le graphe X de f est un ensemble de points fixes.

Dans le cas où X est un arc simple dans un espace euclidien E de dimension finie, on peut trouver une condition géométrique sur X qui est nécessaire et suffisante pour que X soit, dans E, un ensemble de points fixes.

Cette condition est assez compliquée pour décourager toute tentative de caractériser géométriquement, dans le cas général, les ensembles de points fixes.

Théorème 3 : Soit X un arc simple (compact) dans un espace euclidien E de dimension finie p. Pour que X soit un ensemble de points fixes, il faut et il suffit que, pour tout x de X, le compact de la sphère unité défini par

$$P(x) = \lim \sup_{z \to x} \left\{\frac{y - z}{||y - z||} : y, z \in X, y \neq z\right\}$$

soit disjoint d'un hyperplan de E (pouvant dépendre de x).

Il en résulte que tout arc de cercle dans le plan, distinct du cercle entier (qui n'est pas un rétracte du plan) est un ensemble de points fixes.

On peut remarquer aussi que la sphère unité S de l'espace de Hilbert E de dimension infinie est un rétracte de l'espace, mais n'est pas un ensemble de points fixes.

II. La dimension de l'ensemble des points fixes

Quand le convexe C est compact, et ϕ seulement supposée continue, au lieu de contractante, nous avons prouvé que si ϕ prend en chaque point de C une valeur de dimension $\geq n$, la dimension topologique de l'ensemble des points fixes est au moins égale à n. (Cf. [1]).

On peut espérer, dans le cas des contractions multivoques, des résultats analogues, reliant la dimension de l'ensemble des points fixes à celle des valeurs de ϕ. Malheureusement, même si $\phi(x)$ est **non-borné** pour tout x, on ne sait pas prouver l'existence de **plusieurs** points fixes, dans le cas général.

On peut, cependant, dans quelques cas particuliers, démontrer un résultat de ce genre

Théorème 4 : Soient C un convexe fermé de l'espace de Banach E et ϕ une contraction multivoque de C dans C, de rapport $q < \frac{1}{2}$. Si a est un point fixe de ϕ et si $\phi(a) \neq \{a\}$, il existe des points fixes distincts de a.

Théorème 5 : Si a est un point fixe de ϕ, et si $\phi(a)$ contient une demi-droite sans contenir la droite entière, l'ensemble des points fixes n'est pas borné.

Dans le cas où E est un espace hilbertien, on peut donner des résultats plus précis.

Théorème 6 : Si ϕ est une contraction multivoque définie sur un convexe fermé de l'espace de Hilbert H, si a est un point fixe de ϕ et si $\phi(a)$ contient a + u, il existe pour tout $\varepsilon > 0$ un point fixe b avec $\|b - a\| < \varepsilon$ et $<b - a, u> > 0$.

Plus précisément, si ϕ est q-lipschitzienne et $0 \leq t \leq 1$, il existe un point fixe b tel que

$$\begin{cases} \|b - a\| \leq \dfrac{1 - q}{2(1+q)} \; t \, \| u \| \\ <b - a, u> \geq t . \left[\dfrac{(1-q)^2 . \|u\|}{4q\,(1-q)} \right]^2 \end{cases}$$

Corollaire 7 : Si a est un point fixe et $\phi(a) \neq \{a\}$, l'ensemble des points fixes n'est pas réduit à $\{a\}$.

Il existe en effet un $u \neq 0$ tel que $a + u \in \phi(a)$.

Corollaire 8 : Si a est un point fixe et $\phi(a)$ non borné, l'ensemble des points fixes n'est pas borné.

Théorème 9 : Soient H un espace hilbertien, C un convexe fermé de H et ϕ une contraction multivoque de C. Si a est un point fixe de ϕ et si $\phi(a)$ est de dimension $\geq p$, il existe une variété affine V de dimension p et un compact T de l'ensemble X des points fixes de ϕ dont la projection orthogonale sur V est d'intérieur non vide.
En particulier, la dimension de Hausdorff de X est au moins p.

Théorème 10 : Soient H un espace hilbertien, C un convexe fermé de H, ϕ une contraction multivoque de C dont les valeurs sont de dimension finie et a un point fixe de ϕ. Si $\phi(a)$ est de dimension $\geq p$, l'ensemble X des points fixes de ϕ a, au voisinage de a, une dimension topologique $\geq p$.

On voit que les problèmes les plus intéressants, hors du cas hilbertien, restent ouverts. A titre d'exemple, le problème suivant, un des plus simples parmi ceux qui ne sont résolus ici, est toujours sans solution.

Si E est un espace de Banach (non hilbertien et de dimension infinie) et V un sous-espace vectoriel fermé de E non réduit à 0, si π désigne la projection canonique de E sur E/V et φ une application linéaire de E dans E/V de norme < 1, l'équation $\pi(x) = \varphi(x)$ a-t-elle dans E d'autres solutions que 0 ?

Le problème est, en effet, de chercher les points fixes de la contraction multivoque.

$$x \rightarrow \phi(x) = \pi^{-1}(\varphi(x))$$

pour laquelle $0 \in \phi(0) = V$.

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ON KAKUTANI'S FIXED POINT THEOREM, THE K-K-M-S THEOREM AND THE CORE OF A BALANCED GAME

LLOYD SHAPLEY AND RAJIV VOHRA

Abstract : Shapley (1973) provided an extension of the K-K-M theorem, henceforth referred to as the K-K-M-S theorem. This result allowed for an alternative, and simpler, proof of Scarf's (1967) result on the non emptiness of the Core of a non transferable utility game. While alternative proofs of these results are available, so far, we did not have any proof based on Kakutani's fixed point theorem (a theorem which is typically used in econonmics to prove existence results). Here we provide elementary proofs of Scarf's theorem and the K-K-M-S theorem based on Kakutani's Fixed Point theorem.

A cooperative game in characteristic function form is defined as follows. Let $N = \{1,\ldots,n\}$ denote the set of players and let $\mathcal{N} = 2^N \setminus \{\phi\}$ denote the set of all non-empty subsets of N. An element of $\mathcal{N}$ is reffered to as a coalition. For any coalition $S \in \mathcal{N}$, let R^s denote the $|S|$ dimensional Euclidean space with coordinates indexed by the elements of S. For $u \in R^N$, u_s will denote its projection on R^s. We shall use the convention >>, >, ≥ to order vectors in R^N, R^N_+ refers to the positive orthant of R^N and for any [illegible] R^N, Co(Y) and $\partial(Y)$ will denote its convex hull and boundary respectively. Each coalition S has a feasible set of payoffs or utilities denoted $\bar{V}(S) \subseteq R^s$. It is convenient to describe the feasible utilities of a coalition as a set in R^N. For $S \in \mathcal{N}$ let $V(S) = \{u \in R^n \mid u_s \in \bar{V}(S)\}$; i.e. V(S) is a cylinder in R^N . With this interpretation in mind, we can now define :

Definition 1 : A non-transferable utility (NTU) game is a pair (N,V) where $V : \mathcal{N} \rightarrow R^N$ satisfies the following

(A1) V(S) is non-empty, closed and comprehensive (in the sense that $V(S) = V(S) - R^N_+$ for all $S \in \mathcal{N}$,

(A2) if $x \in V(S)$ and $y \in R^N$ such that $y_s = x_s$, then $y \in V(S)$ for all $S \in \mathcal{N}$,

(A3) there exists $p \in R^n$ such that for every $j \in N$, $V(\{j\}) = \{x \in R^N \mid x_j \leq p_j\}$, (A4) there exists a real number $q > 0$, such that if $x \in V(S)$ and $x_S \geq p_S$, then $x_i < q$ for all $i \in S$.

Definition 2 : The core of a game (N,V) is defined as

$$C(N,V) = \{u \in V(N) \mid \nexists S \in \mathcal{N} \text{ and } \bar{u} \in V(S) \text{ such that } \bar{u}_S >> u_S\}.$$

For any $S \in \mathcal{N}$ let e^S denote the vector in R^N whose i th coordinate is 1 if $i \in S$ and 0 otherwise. We shall also use the notation e for e^N.
Let A be the unit simplex in R^N. Define

$$A^S = Co\{e^i \mid i \in S\}.$$

Finally, for each $S \in \mathcal{N}$, define

$$m^S = \frac{e^S}{|S|}.$$

Definition 3 : A set $B \subseteq \mathcal{N}$ is said to be balanced if

$$m^N \in Co\{m^S \mid S \in B\}.$$

Definition 4 : A game (N,V) is said to be balanced if $\cap_{S\in B} V(S) \subseteq V(N)$ for any balanced collection B.

Theorem 1 (Scarf) : A balanced game has a non-empty core.

Notice that if $u \in \partial(\cup_{S\in \mathcal{N}} V(S))$, then by (A1) there does not exist a coalition $S \in \mathcal{N}$ and $\bar{u} \in V(S)$ such that $\bar{u} >> u$. It will, therefore, suffice to show that there exists such a u which also belongs to V(N). Let $\Omega = \partial(\cup_{S\in \mathcal{N}} V(S))$. let $G : \Omega \rightarrow A$ be defined as

$$G(u) = \{m^S \mid S \in \mathcal{N} \text{ and } u \in V(S)\}.$$

The idea of our proof is to use Kakutani's fixed point theorem to look for $u \in \Omega$ such that $m^N \in Co(G(u))$.

This would imply that $\{S \in \mathcal{N} \mid u \in V(S)\}$ is a balanced collection. By the hypothesis that the game is balanced, this will imply that $u \in N(V)$. Since $u \in \Omega$ as well, $u \in C(N,V)$.

Shapley (1973) proved the following generalization of the K-K-M theorem.

Theorem 2 (K-K-M-S) : Let $\{C^S \mid S \in \mathcal{N}\}$ be a family of closed subsets of A such that $\cup_{S \subseteq T} C^S \supseteq A^T$ for each $T \in \mathcal{N}$. Then there is a balanced set **B** such that $\cap_{S \in \mathbf{B}} C^S \neq \{\phi\}$.

The idea of the proof is similar to that used in proving theorem 1. We look for $x \in A$ such that $m^N \in G(x) = Co\{m^S \mid x \in C^S)\}$.

Finally, we observe that both theorems 1 and 2 relate to obtaining a 'coincidence' ; $\{m^N\} \cap Co(G(u)) \neq \emptyset$ for some u. We show that simple alternative proofs of the two theorems can be obtained by appealing to a coincidence theorem of Fan (Theorem 6, (1969)).

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THE CONTINUITY OF INFSUP, WITH APPLICATIONS

STEPHEN SIMONS

Let X and Y be nonempty sets and a, b: X x Y $\to \mathbb{R}$. We give conditions under which the map defined by

$$(\lambda,\mu) \to \inf_X \sup_Y (\lambda a + \mu b)$$

is continuous on the line-segment

$$\mathbb{L} := \{(\lambda,\mu) : \lambda \geq 0, \mu \geq 0, \lambda + \mu = 1\} \subset \mathbb{R}^2.$$

Our results imply infinite dimensional generalizations both of a result of Fan on the equilibrium value of a system of convex and concave functions and also of a result of Aubin on eigenvalues of a multifunction. Our results also have applications to the von Neumann-Kemeney Theorem and the Perron-Frobenius Theorem in matrix theory. We use the Hahn-Banach theorem and do not use any fixed-point related concepts.

A GENERALIZATION TO MULTIFUNCTIONS OF FAN'S BEST APPROXIMATION THEOREM

V. M. SEHGAL AND S. P. SINGH

Let E be a locally convex Hausdorff topological vector space, S a nonempty subset of E and p a continuous seminorm on E. It is a well-known result that if S is compact and convex and $f : S \to E$ is a continuous map, then there exists an $x \in S$ satisfying

(1) $$p(fx - x) = d_p(fx, S) = \min\{p(fx, - y) \mid y \in S\}.$$

Since then a number of authors have provided either an extension of the above theorem to set valued mappings or have weakened the compactness condition therein. Some of these results are

(a) Reich (1978). If S is approximately compact and $f : S \to E$ is continuous with f(S) relatively compact, then (1) holds.

(b) Lin (1979). If S is a closed unit ball of a Banach space X and $f : S \to X$ is a continuous condensing map, then (1) holds when p is the norm on X [4].

(c) Waters (1984). If S is a closed and convex subset of a uniformly convex Banach space E and $f : S \to 2^E$ is a continuous multifunction with convex and compact values and f(S) is relatively compact, then (1) holds.

(d) Sehgal and Singh (1985). Let $S \subseteq E$ with $int(S) \neq \emptyset$ and cl(S) convex and let $f : S \to 2^E$ be a continuous condensing multifunction with convex, compact values and with a bounded range. Then for each $w \in int(S)$, there exists a continuous seminorm $p = p(w)$ satisfying (1).

Our aim is this presentation is to prove (a) for multifunctions and derive some corollaries.

Definition : A subset S of E is approximatively p-compact iff for each $y \in E$ and a net $\{x_\alpha\}$ in S satisfying $p(x_\alpha - y) \to d_p(y, S)$ there is a subset $\{x_\beta\}$ and an $x \in S$ such that $x_\beta \to x$. Clearly a compact set in E is approximatively compact. The converse, however, may fail. For example, the closed unit ball of an infinite dimensional uniformly convex Banach space is approximatively norm compact but not compact.

Some consequence of the definition follow.

1. An approximatively p-compact set S in E is closed. Let y be a cluster point of S and let a net $\{x_\alpha\} \subseteq S$ satisfy $p(x_\alpha - y) \to d_p(y, S) = 0$. Since S is a approximatively p-compact, $\{x_\alpha\}$ contains a subset $x_\beta \to x \in S$. Since $x_\beta \to y$ also and E is Hausdorff, $x = y \in S$.

2. If S is a closed and convex subset of a uniformly convex Banach space then S is approximatively norm compact.

Definition : let E and F be topological vector spaces and let 2^F denote the family of nonempty subsets of F. The mapping $T : E \to 2^F$ is upper semicontinuous (u.s.c.) iff $T^{-1}(B) = \{x \in E \mid Tx \cap B \neq \emptyset\}$ is closed for each closed subset B of F.

3. If S is an approximatively p-compact subset of E then for each $y \in E$, $Q(y) = \{x \in S \mid p(y - x) = d_p(y, S)\}$ is nonempty and the mapping defined by $y \to Q(y)$ is an upper semicontinuous (u.s.c.) multifunction on E.

The main result is the following :

Theorem 1 : Let S be an approximatively p-compact, convex subset of E and let $F : S \to 2^E$ be a continuous multifunction with closed and convex values. If $FS = \bigcup_{x \in S} Fx$ is relatively compact then there exists an $x \in S$ with

$$d_p(x, Fx) = d_p(Fx, S).$$

Further, if $d_p(x, Fx) > 0$, then $x \in \partial S$.

Note that $d_p(A, B) = \inf\{p(x - y) \mid x \in A, y \in B\}$.

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FIXED POINT THEORETIC PROOFS OF THE CENTRAL LIMIT THEOREM

S. SWAMINATHAN

We consider the following form of the Central Limit Theorem :

Let x_1, x_2..., be a sequence of independent random variables with the same distribution as a random variable x, which has mean 0 and variance 1. Define

$$x_n = \frac{x_1 + x_2, + \ldots + x_n}{\sqrt{n}}.$$

Then the sequence of random variables (x_n) converge in distribution to a standard normal variate.

The usual proof of this theorem uses characteristic functions. H. F. Trotter [2] and G. G. Hamedani-G. G. Walter [1] have interpreted and proved it as a fixed point theorem. The former considers operators on a space of continuous functions, associates an operator with each random variable which turns out to be a contraction and the one associated the normal distribution has it a fixed point. The approach of the latter authors is to introduce a metric on the space of distribution functions and define a self-mapping on the space which is then shown to be a contraction ; the fixed point resulting from the contraction mapping principle yields the central limit theorem. In this paper we point out how Caristi's fixed point theorem can be used in this connection.

Caristi's theorem : Let M be a complete metric space and $F : M \to M$ satisfy $d(x, F(x)) \leq \phi(x) - \phi(F(x))$ for every x in M, where $\phi : M \to R$ is a lower semi-continuous function. Then F has a fixed point.

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ON THE EXISTENCE OF FIXED POINTS AND ERGODIC RETRACTIONS FOR NONEXPANSIVE MAPPINGS

WATARU TAKAHASHI

Let S be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $t \to t \cdot s$ and $t \to s \cdot t$ of S *into itself are continuous*. Let B(S) be the Banach space of all bounded real valued functions on S with supremum norm and let X be a subspace of B(S) containing constants. Then, an element u of X^* (the dual space of X) is called a mean on X if $\|u\| = u(1) = 1$. Let u be a mean on X and $f \in X$. Then, according to time and circumstances, we use $u_t(f(t))$ instead of u(f). For each $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in B(S) given by

$$(l_s f)(t) = f(st) \text{ and } (r_s f)(t) = f(ts)$$

for all $t \in S$. Let X be a subspace of B(S) containing constants which is l_s-invariant (r_s-invariant), i.e., $l_s(X) \subset X$ ($r_s(X) \subset X$) for each $s \in S$. Then a mean u on X is said to be left invariant (right invariant) if

$$u(f) = u(l_s f) \quad (u(f) = u(r_s f))$$

for all $f \in X$ and $s \in S$. An invariant mean is a left and right invariant mean. Let C be a nonempty subset of a Banach space E. Then a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself is called a Lipschitzian semigroup on C if it satisfies the following :

(1) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;

(2) for each $x \in C$, the mapping $s \to T_s x$ is continuous on S ;

(3) for each $s \in S$, T_s is a Lipschitzian mapping on C Into itself, i.e., there is $k_s \geq 0$ such that

$$\|T_s x - T_s y\| \leq k_s \|x - y\|$$

for all x, y ∈ C. A Lipschitzian semigroup $\mathcal{S} = \{T_t : t \in S\}$ on C is said to be nonexpansive if $k_s = 1$ for every s ∈ S. For a Lipschitzian semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C. we denote by F(S) the set of common fixed points of T_s, s ∈ S.

Let C(S) be the Banach space of all bounded continuous functions on S and let RUC(S) be the space of all bounded right uniformly continuous functions on S, i.e., all f ∈ C(S) such that the mapping $s \to r_s f$ is continuous. Then RUC(S) is a closed subalgebra of C(S) containing constants and invariant under l_s and r_s. On the other hand, a semitopological semigroup S is left reversible if any two closed right ideals of S have nonvoid intersection. In this case, (S, ≤) is a directed system when the binary relation "≤" on S is defined by a ≤ b if and only if $\{b\} \cup \overline{bS} \subset \{a\} \cup \overline{aS}$. A right reversible semigroup is similarly defined.

In this talk, we first introduce the notion of submean which generalizes "mean" on RUC(S) and "lim sup" in the case when S is directed and prove fixed point theorems and nonlinear ergodic theorems for Lipschitzian semigroups in Hilbert spaces. Secondly, in Banach spaces, we prove a fixed point theorem for nonexpansive semigroups in the case when RUC(S) has a left invariant mean and prove the existence of nonlinear ergodic retractions for nonexpansive semigroups in the case when S is commutative. Last, we consider some problems concerning the asymptotic behavior of almost-orbits of nonexpansive semigroups.

A MINIMAX INEQUALITY WITH APPLICATIONS TO EXISTENCE OF EQUILIBRIUM POINT AND FIXED POINT THEOREMS

XIE PING DING AND KOK-KEONG TAN

The following very general minimax inequality is first obtained which generalizes minimax inequalities of Allen [1, Theorem 2], Bae-Kim-Tan [3, Theorem 1], Fan [7, Theorem 1 and 8, Theorem 4], Tan[13, Theorem 1] and Yen[15, Theorem 1].

Theorem 1 : Let X be a non-empty convex subset of a topological vector space and $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that

(i) for each fixed $x \in X$, $f(x, y)$ is a lower semi-continuous function of y on C for each non-empty compact subset C of X :

(ii) for each $A \in F(X)$ and for each $y \in co(A)$, $\min_{x \in A} f(x, y) \leq 0$;

(iii) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $f(x, y) > 0$. Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

The following is one of three equivalent formulations of Theorem 1 :

Theorem 2 : (Maximal element version) Let X be a non-empty convex subset of a topological vector space and $G : X \to 2^X \cup \{\emptyset\}$ be a set-valued map such that

(i) for each $x \in X$ and for each non-empty compact subset C of X, $G^{-1}(x) \cap C$ is open in C (where $G^{-1}(x) = \{y \in X : x \in G(y)\}$) ;

(ii) for each $A \in F(x)$ and for each $y \in co(A)$, there exists $x \in X$ such that $x \notin G(y)$;

(iii) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x \in G(y)$. Then there exists $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$.

The following three equivalent fixed point theorems are easy consequences of Theorem 2 :

Theorem 3 : Let X be a non-empty convex subset of a topological vector space and $G : X \to 2^X$ be a set-valued map such that

(i) for each $y \in X$ and for each non-empty compact subset C or X, $G^{-1}(y) \cap C$ is open in C ;

(ii) there exist a non-empty compact convex subset x_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(x_0 \cup \{y\})$ with $x \in G(y)$.

Then there exists an $\hat{y} \in X$ such that $\hat{y} \in co(G(\hat{y}))$.

Theorem 3' : Let X be a non-empty convex subset of a topological vector space and $G : X \to 2^X$ be a set-valued map such that

(i) for each $x \in X$ and for each non-empty compact subset C or X, $G^{-1}(x) \cap C$ is open in C ;

(ii) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x \in G(y)$;

(iii) for each $y \in X$, G(y) is convex.

Then there exists an $\hat{y} \in X$ such that $\hat{y} \in co(G(\hat{y}))$.

Theorem 3" : Let X be a non-empty convex subset of a topological vector space and $G : X \to 2^X$ be a set-valued map such that

(i) for each $x \in X$ and for each non-empty compact subset C or X, $G^{-1}(x) \cap C$ is open in C ;

(ii) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in co(X_0 \cup \{y\})$ with $x \in G(y)$.

Then there exists an $\hat{y} \in X$ such that $\hat{y} \in co(G(\hat{y}))$.

Theorem 3' generalizes Theorem 1 of Browder in [4] in several aspects.

As an application of Theorem 3", we have the following existence theorem of an equilibrium point for a one-person game :

Theorem 4 : Let X be a non-empty convex subset of a topological vector space. Let $P : X \to 2^X$ be of class L_C and $A, B : X \to 2^X$ be such that

(i) for each $x \in X$, A(x) is non-empty and $co(A(x)) \subset B(x)$;

(ii) for each $y \in X$ and for each non-empty compact subset C of X, $A^{-1}(y) \cap C$ is open in C ;

(iii) the map $cl\,B : X \to 2^X$ defined by $(cl\,B)(x) = cl_X B(x)$ is upper semi-continuous ;

(iv) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that each $y \in X \setminus K$, $co(X_0 \cup \{y\}) \cap co(A(y) \cap P(y)) \neq \emptyset$.

Then there exists an equilibrium choice $\hat{x} \subset X$, i.e. $\hat{x} \in cl\,B(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

As another application of Theorem 1, we have the following two fixed point theorems :

Theorem 5 : let X be a non-empty convex subset of a topological vector space E and $G : X \to 2^E$ be continous on C for each non-empty compact subset C of X such that for each $x \in X$, G(x) is compact and convex. Let $p : X \times E \to \mathbb{R}$ be such that (a) p is continuous on $C \times E$ for each non-empty compact subset C of X ; (b) for each $x \in X$, $p(x, \cdot)$ is a convex function on E. Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that

(i) for each $y \in K$ with $y \notin G(y)$, there exist $x \in y^+ \overline{\bigcup_{\lambda>0} \lambda(X - y)}$ (resp. $x \in y^+ \overline{\bigcup_{\lambda<0} \lambda(X - y)}$) and $v \in G(y)$ such that

$$p(y, x - v) < \inf_{U \in G(y)} p(y, y - u) ;$$

(ii) for each $y \in X \setminus K$ with $y \notin G(y)$, there exist $x \in y^+ \overline{\bigcup_{\lambda>0} \lambda(X_0 - y)}$ (resp. $x \in y^+ \overline{\bigcup_{\lambda<0} \lambda(X_0 - y)}$) and $v \in G(y)$ such that

$$p(y, x - v) < \inf_{U \in G(y)} p(y, y - u) .$$

Then G has a fixed point in X.

Theorem 5 generalizes Theorem 3.3 of Jiang in [9], Theorem 7 of Fan in [8], Theorem 10 and Corollary 1 of Shih-Tan in [12] and Theorem 1 and Corollary 2 of Browder in [5] (resp. Theorem 2 and Corollary 2' of Browder in [5] and Corollary 3.4 of Jiang in [9]).

Theorem 6 : let X be a non-empty convex subset of a topological vector space E and $G : X \to 2^E$ be upper semi-continuous on C for each non-empty compact subset C of X such that for each $x \in X$, G(x) is compact. Let $p : X \times E \to \mathbb{R}$ be continuous on $C \times D$ for each non-empty compact subsets C and D of X and E respectively such that for each $x \in X$, $p(x, \cdot)$ is a convex function on E. Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact K of X such that

(i) for each $y \in K$ with $y \notin G(y)$, there exists $x \in y + \bigcup_{\lambda>0} \lambda(X_0 - y)$ (resp. $x \in y + \bigcup_{\lambda<0} \lambda(X_0 - y)$) such that $p(y, x - u) < p(y, y - u)$ for all $u \in G(y)$.

Then G has a fixed point in X.

Theorem 6 generalizes Theorem 1 of Browder in [5] and Theorem 10 of Shih-Tan in [12].

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COMPUTING FIXED POINTS BY GLOBAL OPTIMIZATION METHODS

HOANG TUY

It has long been noticed that finding a fixed point of a mapping $F : C \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ can be viewed as a nonlinear optimization problem :

(P) minimize $\|F(x) - x\|$ subject to $x \in C$.

But, to the best of our knowledge, so far no serious attempt has been made to solve fixed point problems by this approach (though some work has been done in the converse direction). in fact, the global optimization problem (P) to which the fixed point problem has been reduced is itself inherently difficult and cannot be handled by local optimization methods which have been the main concern of optimization theory over the past decades.

In this lecture we shall show that, under rather general conditions, a fixed point problem can be converted into a parametric d.c. minimization problem. More precisely, there exists, for any given mapping F satisfying certain conditions, a funciton $h(\alpha, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ convex in x, such that the problem (P) is equivalent to finding a value α satisfying

$$0 = \inf\{h(\alpha, x) - \|x\|^2 : x \in \mathbb{R}^n\}.$$

By this way the problem (P) is reduced to a connected sequence of quadratic concave minimization problems, solvable by currently available methods.

FIXED POINT THEORY AND COALITIONS

HANS VAN MAAREN

As is generally known, Brouwer's fixed point theorem can be shown using Sperner's lemma. Doing such, the proof consists of two parts. First, a combinatorial argument locates a number of points which are mapped into various directions and which are "close" together. Secondly, a topological argument shows the existence of one single point, which is mapped into all indicated directions simultaneously, that is, a fixed point.

From the viewpoint of applicability only the combinatorial part is important. Approximating a fixed point of a continuous mapping on a simplex consists of the determination of a simplicial subdivision of sufficiently small gridsize on which Sperner's lemma is applied. Algorithms locating the desired subsimplex have been developed by various authors, and computational efficiency has increased considerably.

The same applies to the approximation of fixed points of multifunctions (Kakutani's theorem), where the algorithms are steered by a vector-labeling device.

Sofar, triangulation methods are used on simplices, on products of simplices and on polytopes, covering a wide range of applications in economic theory and game theory. However, applications along this line seem to be restricted to Euclidean space.

In [1] an algorithm is developed which establishes the combinatorial parts of the quoted theorems in a much wider setting, that of multiply ordered spaces.

A multiply ordered space is a set X supplied with a finite set of preference orderings (transitive, reflexive and complete relations) indexed by a set P. The elements of X may be interpreted as alternatives for the persons of P, each person having his particular preferences concerning these alternatives. Thus $x \leq_p y$ means that person p prefers alternative y above alternative x. A coalition Q is a subset of P. If all members of Q prefer y above x we shall write $x \leq_Q y$.

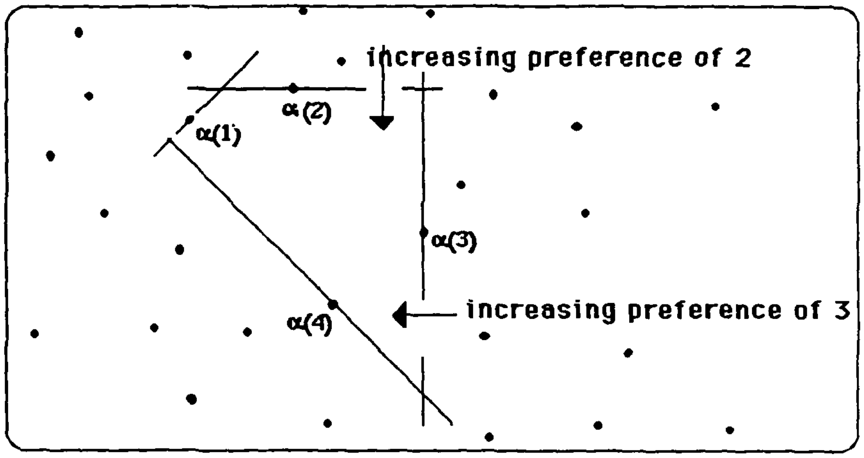
A labeling of X is a mapping $L : X \to P$.

By a random grid A we mean a finite subset of X on which the preferences become linear orderings. Further, an agreement of coalition Q is a mapping $a: Q \to A$ with the following properties:

(1) $\alpha(q)$ is the minimum of $\alpha(Q)$ with respect to the preference relation of q.

(2) No $a \in A$ exists such that for all $q \in Q : \alpha(q) <_q a$.

Agreements may be visualized in a following manner:



In the above picture the points of the grid are dotted and an agreement of persons 1, 2, 3 and 4 is illustrated. The idea of an agreement is that the points of α(Q) are close, in some sense, because of condition (2). This condition states that, restricted to the points of the grid considered, the points of the agreement are Pareto-optimal for the coalition involved.

Now the algorithm of [1] establishes in any multiply ordered space, for any random grid and any labeling a coalition and an agreement, the points of which are differently labeled by elements of the coalition involved, that is

(*) There exists Q, agreement a : $Q \to A$ with
$L(\alpha(Q)) = Q$

The above statement can be regarded as a generalized version of Sperner's lemma as shall be explained more extensively in the lecture.
Also, we shall shortly discuss a vector-labeling variant of this result, which is needed for a Kakutani-type fixed point theorem.
The main goal of the lecture however is concerned with the topological arguments which one wants to apply to obtain an "infinite" version of (*), that is a true fixed point result in a multiply ordered space.
We investigate the well-known standard arguments in the new context and discuss sufficient conditions on

(1) topologies associated with the preference relations

(2) the labelings device, the corresponding mapping and its behaviour on the boundary of X

(3) the structure of X with respect to the preferences given.

We introduce a related concept of (topological) dimension and associate to each coalition a new notion of boundary in order to state the restrictions on the labelings properly.

It turns out that the imposed conditions make the possibility of an embedding of the multiply ordered space considered into Euclidean Space very near.

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APPLICATIONS MULTIVOQUES DIFFÉRENTIABLES ET LA FORMULE DE LERAY-SCHAUDER

DONALD VIOLETTE ET GILLES FOURNIER

1. L'indice de point fixe pour les composés de fonctions multivoques acycliques dans des espaces de Banach

Dans cette section, nous rappelons brièvement l'indice de point fixe pour des composées d'applications multivoques acycliques dans des espaces de Banach. Pour plus de détails, nous conseillons au lecteur de consulter [2].

Soit $f = (F_n, ..., F_0)$ une suite de fonctions acycliques $F_i : X_i \to X_{i+1}$ où chaque X_i est Hausdorff $\forall$ $i = 0, 1, ..., n$ et $X_0 = U$ est une partie ouverte de $X_{n+1} = X$. On dit que f est admissible si Fixe $(F) = \{x \in U \mid x \in F(x)\}$ où $F = F_n \circ ... \circ F_0 : U \to X$, est une partie compacte de U. La suite f est appelée une décomposition acyclique de F.

Soit $\mathcal{F}$ la classe des réunions localement finies $X = \bigcup_{j \in J} C_j$ de fermés convexes C_j d'un espace de Banach E et soit $\mathcal{F}_0$ la sous-classe de $\mathcal{F}$ pour laquelle J est fini.

Définition 1.1 : Une suite admissible de fonction acycliques $f = (F_n, ..., F_0)$ est compactifiante, s'il existe un ensemble ouvert W et une suite $\{K_n\}_{n \in \mathbb{N}^+} \subset \mathcal{F}_0$ tels que

(1.1.1) Fixe $(F) \subset W \subset \bar{W} \subset U$ où $F = F_n \circ ... \circ F_0 : U \to X$ et $X \in \mathcal{F}$;

(1.1.2) $W \subset K_1 \subset X$;

(1.1.3) $F(W \cap K_n) \subset K_{n+1} \subset K_n \quad \forall\, n \in \mathbb{N}^+$;

(1.1.4) $\lim_{n \to \infty} \gamma(K_n) = 0$ où γ est la mesure de non-compacité de K_n

Nous avons le théorème suivant :

Théorème 1.2 : Si $f = (F_n, ..., F_0)$ est une suite compactifiante, alors l'indice de point fixe de f, ind(U, f, X) est défini.

Cet indice de point fixe possède les propriétés habituelles de l'indice (voir [2]), y incluant une propriété d'homotopie qui s'avère être un résultat nouveau même dans le cas univoque. Nous avons également un théorème de Lefschetz.

Théorème 1.3 : (Théorème de Lefschetz). Soit $(F = F_n)$ une suite compactifiante avec W et $\{K_n\}_{n \in \mathbb{N}^+}$ satisfaisant aux conditions de (1.1). Supposons que $F(U) \subset U$ et $F = F_n \circ \ldots \circ F_o$: $U \to U$ possède un attracteur compact M tel que $M \subset W$. Alors le nombre de Lefschetz $\wedge(f_U)$ est défini et $\wedge(f_U) \neq 0$ implique que f possède un point fixe dans U.

2. Application de l'indice aux applications multivoques différentiables

Nous allons rappeler brièvement nos notions d'applications multivoques différentiables et continûment différentiables (voir [3]). Il s'agit d'un nouveau concept de différentiabilité d'une application multivoque. Nous rappellerons également que l'indice tel que défini dans la section 1 est défini pour une application multivoque continûment différentiable à itérée condensante et possédant une décomposition acyclique, résultat qui est vrai aussi pour toute sélection d'une telle fonction.

Soient E. E' des espaces de Banach réels et U une partie ouverte de E.

Définition 2.1 : Soit $T : U \to E'$ une application multivoque telle que T(x) soit compact $\forall\, x \in U$. T est différentiable au point $x \in U$ s'il existe une application multivoque semi-continue supérieurement (s.c.s) $S_x : \begin{matrix} T(x) & E \to E' \\ (z,h) & \to \; S_x(z,h) \end{matrix}$ telle que la fonction $S_{x,z} : E \to E'$ soit s.c.s., homogène et demi-linéaire positive et telle que $\forall\, \epsilon > 0, \exists\, \delta(\varepsilon, x) > 0$ tel que si $\|h\| < \delta$, *alors*

$$d_H\Big[T(x+h), \bigcup_{z \in T(x)} (z + S_x(z,h))\Big] \leq \varepsilon \|h\|$$

où d_H désigne la métrique de Hausdorff.

La *fonction* S_x *est appelée une différentielle de* T *au point* x. *Si* T *est différentiable en* tout point de U, alors T est différentiable dans U. Notons qu'on n'a pas nécessairement l'unicité de la différentielle en un point. En revanche, notre définition de différentiabilité, est à notre connaissance, la seule ne requérant pas que la différentielle soit à valeur convexe.

Soient T, F : $E \to E'$ deux applications multivoques s.c.s. et homogènes. Posons $\bar{d}(T, F) = \sup_{\|h\| = 1} d_H(T(h), F(h))$, alors $\bar{d}$ est une métrique sur l'espace des applications multivoques s.c.s. et homogènes de E dans E'.

Définition 2.2 : Soit $T : U \to E'$ une application multivoque différentiable dans U. On dit que T est continûment différentiable dans U s'il existe une fonction $(x, z) \to S_{x,z}$ telle que S_x soit une différentielle de T en x et la fonction est continue dans l'ensemble $\bigcup_{x \in U} \{x \times T(x)\}$ i.e. quels que soient $x \in U$ et $z \in T(x)$, alors $\forall\, \varepsilon > 0, \exists\, \delta > 0$ tel que $\|x - y\| < \delta$ et $\|z - z'\| < \delta$ entraînent $\tilde{d}\,(S_{x,z}, S_{y,z'}) < \varepsilon\ \forall\, y \in U$ et $\forall z' \in T(y)$.

Dans ce qui suit, nous allons voir que notre classe d'applications multivoques continûment différentiables et à itérée condensante ainsi que toutes leurs sélections ont un indice de point fixe tel que défini à la section 1. Pour les théorèmes suivants, consultez [3].

Théorème 3 : Soit $f = (F_n, \ldots, F_o)$ une suite admissible de fonctions acycliques $F_i = X_i \to X_{i+1}$ où chaque X_i est Hausdorff et $X_o = U$ est une partie ouverte de $X_{n+1} = X \in \mathcal{F}$. Si $F = F_n \circ \ldots \circ F_o : U \to X$ est continûment différentiable dans U et est à itérée condensante, alors f est compactifiante et ind (U, f, X) est défini. De plus, pour tout compact K stable par F tel que Fixe $(F) \subset K \subset U$, on peut choisir W et $\{K_n\}_{n \geq 1}$ satisfaisant aux conditions de (1.1) avec $K \subset W$.

Théorème 2.4 : (Théorème de Lefschetz). Soit $f = (F_n, \ldots, F_o)$ une suite admissible de fonctions acycliques $F_i : X_i \to X_{i+1}$ où chaque X_i est Hausdorff et $X = X_o = X_{n+1}$ est une partie ouverte d'un élément de $\mathcal{F}$. Si $F = F_n \circ \ldots \circ F_o : X \to X$ est continûment différentiable dans X et est à itérée condensante, alors $\Lambda(f)$ est défini et $\Lambda(f) \neq 0$ implique que F possède un point fixe.

3. Le résultat principal

Dans cette section, nous allons obtenir pour les applications continûment différentiables à itérée condensante et possédant une décomposition acyclique, la formule de Leray-Schauder [4] pour le calcul de l'indice en un point fixe isolé d'une application continûment différentiable. Dans le cas univoque, ce résultat est dû à Eells-Fournier [1]. Nous verrons qu'en plus d'une légère modification de la condition habituelle (la différentielle au point fixe ne possède pas la valeur propre 1), certaines conditions techniques doivent être ajoutées pour le cas multivoque, notamment la convexité de l'application au point fixe isolé x et aussi des images de la différentielle en x et l'espace de Banach doit être strictement convexe.

La proposition suivante est nécessaire pour démontrer le résultat principal.

Proposition 3.1 : Soit $F : U \to X$ une application multivoque continûment différentiable dans U et à itérée condensante avec $S_{x,z}$ satisfaisant à la condition qui rend F continûment différentiable dans U. Alors la fonction $S_{x,x}$ est à itérée condensante.

Nous allons maintenant énoncer le théorème principal et donner un aperçu de la démonstration.

Théorème 3.2 : Soit $f = (F_n, \ldots, F_o)$ une suite admissible de fonctions acycliques $F_i : X_i \to X_{i+1}$ où chaque X_i est Hausdorff $\forall$ i = 0, 1, ..., n et $X_o = U$ une partie ouverte de $X_{n+1} = X \in \mathcal{F}$, un sous-espace d'un espace de Banach strictement convexe E. Soit $F = F_n \circ \ldots \circ F_o$: $U \to X$ continûment différentiable dans U et à itérée condensante avec $S_{x,z}$ satisfaisant à la condition qui rend F continûment différentiable dans U. Si

(3.2.1) $\{x\} = N \cap \text{Fixe}(F)$ où N est un voisinage de x dans U ;

(3.2.2) F(x) est convexe ;

(3.2.3) $S_{x,x}(h)$ est convexe $\forall$ h ;

(3.2.4) $\exists d > 0$ tel que

$$d(t(F(x) + x), S_{x,x}(h) - h) \geq d \quad \forall \|h\| = 1, \forall t > 1$$
$$[\text{i.e. } x + h \notin N_{dr}(F(x) + S_{x,x}(h)) \; \forall \|h\| = r] ;$$

(3.2.5) $1 \notin \text{co}\, \lambda(h)\ \forall\, h \in E \setminus \{0\}$ où $\lambda(h) = \{\lambda \mid \lambda h \in S_{x,x}(h)\}$;

(3.2.6) $\lambda v \in S_{x,x}(v) \Rightarrow \lambda > 1$ pour toute combinaison linéaire de vecteurs propres généralisés correspondant à des valeurs propres supérieures à 1.

Alors $\text{ind}(N, f, X) = (-1)^{\alpha(x)}$

où $\alpha(x)$ est la somme des multiplicités des valeurs propres de $S_{x,x}$ contenues dans $(1, \infty)$.

Remarque 3.3 : La condition habituelle sur la différentielle dans le cas univoque devient dans le cas multivoque les conditions (3.2.5) et (3.2.6).

Esquisse de la démonstration : Il y a trois étapes importantes dans la démonstration de ce résultat.

a) réduction du calcul de l'indice à celui de la différentielle de F en x:

En vertu des conditions (3.2.1) à (3.2.4) et par la stricte convexité de E, il existe $\varepsilon > 0$ tel que

$$\text{ind}(f|_{N_\varepsilon(x)}) = \text{ind}(S_{x,x})$$

où $f|_{N_\varepsilon(x)} = (F_n, \ldots, F_o|_{N_\varepsilon(x)})$ et $N_\varepsilon(x) = \{y \in E \mid \|x - y\| < \varepsilon\}$.

b) réduction à un espace de dimension finie :

En imitant la démonstration du lemme 2, section 7 de [1], on montre à l'aide de (3.2.5) que

$$\mathrm{ind}(S_{x,x}) = \mathrm{ind}(E^1, S_{x,x}, E^1)$$

où $E^1 = \bigcup_{n\geq 1} E^1_n$, E^1_n est le sous-espace vectoriel de E engendré par l'ensemble $\{v \in E \mid \exists\, \lambda \geq 1$ tel que $S_{x,x}(x) \cap (\lambda\, v + E^1_{n-1}) \neq \emptyset\}$ et $E^1_0 = \{0\}$. Remarquez que $\dim E^1 < \infty$ car $S_{x,x}$ est à itérée condensante par 3.1.

c) réduction au cas univoque :

En se servant de l'homotopie, on montre que

$$\mathrm{ind}(E^1, S_{x,x}, E^1) = \mathrm{ind}(E^1, h', E^1)$$

où h' est une application linéaire univoque et donc par Leray-Schauder [4], $\mathrm{ind}(E^1, h', E^1) = (-1)^{\alpha(x)}$ car E^1 est de dimension finie et $\dim E^1 = \alpha(x)$.

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UN THEOREME D'EXISTENCE POUR LES ÉQUATIONS DIFFÉRENTIELLES DANS LES ESPACES DE BANACH ORDONNÉS

PETER VOLKMANN

Soient E un espace de Banach réel, $a \in E$, $\rho > 0$ et

$$S(a\,;\rho) = \{x \mid x \in E, \|x - a\| \le \rho\}.$$

Nous supposons que $T > 0$ et

(1) $$f(t, x) : [0, T] \times S(a\,;\rho) \to E$$

soit une fonction continue. Cette note a comme but d'établir l'existence d'une solution locale du problème de Cauchy.

(2) $$u(0) = a, \quad u' = f(t, u),$$

c.-à-d. on fera des hypothèses par rapport à la fonction (1), telles qu'il existe $\tau \in (0, T]$ et une solution $u : [0, \tau] \to E$ du problème (2).

Soit $K \subseteq E$ un cône (au sens de Kreìn et Rutman [2]). On pose

$$x \le y \Leftrightarrow y - x \in K \quad (x, y \in E)$$

et

$$[x, y] = \{z \mid z \in E, x \le z \le y\}.$$

Le cône K est dit normal, si tous les $[x, y]$ $(x, y \in E)$ sont des ensembles bornés. Soient E^* le dual topologique de E et

$$K^* = \{\varphi \mid \varphi \in E^*\,; x \in K \Rightarrow \varphi(x) \ge 0\}.$$

Une fonction

$$g(t, x) : [0, T] \times S(a ; \rho) \to E \tag{3}$$

est dite monotone croissante par rapport à la variable x, si

$$x, y \in S(a ; \rho), x \le y, \quad 0 \le t \le T \Rightarrow g(t, x) \le g(t, y) ;$$

elle est dite quasi-monotone croissante par rapport à x (voir [8]), si

$$x, y \in S(a ; \rho), x \le y, \varphi \in K^*, \varphi(x) = \varphi(y), \quad 0 \le t \le T \Rightarrow \varphi(g(t, x)) \le \varphi(g(t, y)).$$

La fonction (3) est dite α-lipschitzienne, s'il existe une constante $L \ge 0$, telle que

$$\alpha(g([0, T]) \times A)) \le L\,\alpha(A) \quad (A \subseteq S(a ; \rho)),$$

où α désigne la mesure de non-compacité de Kuratowski [3]. Finalement, g est dite dissipative, si elle satisfait à la condition

$$[x - y, g(t, x) - g(t, y)]_- \le L\|x - y\| \quad (x, y \in S(a ; \rho),\ 0 \le t \le T),$$

où par $[\cdot, \cdot]_-$ on entend l'expression

$$[x, y]_- = \lim_{h \uparrow 0} \frac{1}{h}(\|x + hy\| - \|x\|) \qquad (x, y \in E).$$

Théorème 1 : Soit E un espace de Banach, ordonné par un cône K normal d'intérieur non-vide et satisfaisant à la condition suivante :
(C) Si $\emptyset \ne D \subseteq E$ et D est totalement ordonné et borné [1)] par rapport à $\le$, alors supD existe (dans E).
Soit $f = g + h$, où

$$g(t, x), h(t, x) : [0, T] \times S(a ; \rho) \to E$$

sont des fonctions continues, g quasi-monotone croissante et h monotone croissante. Supposons en plus, que g soit α-lipschitzienne ou dissipative. Alors le problème de Cauchy (2) admet une solution locale.

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1) K étant normal et d'intérieur non-vide, un ensemble $D \subseteq E$ est borné par rapport à l'ordre, si et seulement si D est borné par rapport à la norme.

Démonstration : Soient

$$\Gamma = C([0, T], E) = \{v \mid v : [0, T] \to E \text{ continue}\}$$

et

$$\Lambda = \{v \mid v : [0, T] \to E,\ v(0) = a,\ \|v(s) - v(t)\| \leq |s - t| \quad (s, t \in [0, T])\}$$

(donc $\Lambda \subseteq \Gamma$). Si $v, w \in \Gamma$, on pose

(4) $$v \leq w \iff v(t) \leq w(t) \quad (0 \leq t \leq T).$$

et ainsi on obtient une relation d'ordre dans Γ. En utilisant un théorème de point fixe de Bourbaki [1], p. 37, Lemmert [4] a démontré le théorème 1 avec (C) remplacée par l'hypothèse suivante :

(H) Si $\emptyset \neq \Delta \subseteq \Lambda$ et Δ est totalement ordonné par rapport à l'ordre $\leq$ donné par (4), alors $\sup\Delta$ existe dans Γ.

Pour la démonstration de notre théorème 1, il suffit maintenant d'introduire dans E une norme, équivalente à $\|\cdot\|$, telle que par rapport à cette nouvelle norme

(5) $$(C) \Rightarrow (H)$$

soit vraie. Prenons un élément p de l'intérieur de K et posons

$$B = [-p, p].$$

K étant normal, B est un ensemble borné dans E, et à cause de cela B peut être pris comme boule unité d'une norme $\|\cdot\|_1$ équivalente à $\|\cdot\|$. Montrons (5) par rapport à cette norme. Maintenant on a

$$\Lambda = \{v \mid v : [0, T] \to E,\ v(0) = a,\ -p \leq \frac{v(s) - v(t)}{|s - t|} \leq p \ (s, t \in [0, T], s \neq t\}.$$

Soit $\Delta \neq \emptyset$ un sous-ensemble totalement ordonné de Λ. Alors pour $t \in [0, T]$, $D_t\{v(t) \mid v \in \Delta\}$ devient un ensemble totalement ordonné et borné dans E. Grâce à (C) la borne supérieure

$$\omega(t) = \sup D_t = \sup_{v \in \Delta} v(t)$$

existe. On a évidement $\omega(0) = a$, et de la double-inégalité

$$-p \leq \frac{v(s) - v(t)}{|s - t|} \leq p \quad (s, t \in [0, T], s \neq t)$$

pour chaque v ∈ Δ il découle aisément

$$- p \leq \frac{\omega(s) - \omega(t)}{|s - t|} \leq p \quad (s, t \in [0, T], s \neq t).$$

On a donc $\omega \in \Lambda$ et $\omega = \sup\Delta$. La condition (H) est vérifiée.

Remarques

1. Les hypothèses du théorème 1 par rapport à K sont satisfaites dans deux cas particuliers : Premièrement si K est un cône régulier d'intérieur non vide et deuxièment si

$$E = B(A) = \{x \mid x : A \to \mathbb{R}, \|x\| = \sup_{\lambda \in A} |x(\lambda)| < \infty\}$$

(A étant un ensemble arbitraire et $\mathbb{R}$ désignant les réels), ordonné par

$$K = B_+(A) = \{x \mid x \in B(A), x(\lambda) \geq 0 \ (\lambda \in A)\}.$$

Dans ces cas le théorème 1 avec g = 0 est bien connu de Stecenko [7] et de [9], respectivement. Dans [5] on trouve une première unification de ces deux théorèmes. Remarquons que sans la condition (C) le théorème 1 devient faux ; voir [9] pour un contre-exemple avec E = C[0, 1], ordonné par son cône naturel.

2. Il n'est pas connu, si le théorème 1 reste valable sans l'hypothèse que K soit normal. D'après Lemmert [4], cela est vrai si l'on remplace (C) par (H).

3. D'après un contre-exemple de Schmidt [6], le théorème 1 devient faux si l'on omet l'hypothèse que g soit quasi-monotone croissante. Dans cet exemple g est simultanément lipschitzienne et compacte.

4. Si la fonction continue f n'est que α-lipschitzienne ou dissipative, alors (2) admet une solution locale ; voir p. ex. [10], où l'auteur propose (implicitement) d'utiliser en plus des conditions de monotonie pour obtenir des résultats plus généraux. D'après l'exemple de Schmidt mentionné ci-dessus, de telles généralisations ne sont pas immédiates.

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KAKUTANI PROPERTY OF SPACES WITH ABSTRACT CONVEXITY

ANDRZEJ WIECZOREK

The paper deals with the usual fixed point property and the following Kakutani property of a space X : for every upper semi-continuous function ϕ from X to nonempty closed convex subsets of X there exists x_0 such that $x_0 \in \phi(x_0)$. We derive this property of X from various separation properties of convex subsets of X and from a kind of local convexity of X.

The convexity in our setup is given in an abstract axiomatic way. Special emphasis is given to the case where X has the form of a product. The obtained results cover several known theorems : Fy Fan-Glicksberg, Wallace and certain special cases of Eilenberg-Montgomery as well as many new instances.

We shall now formulate the results more precisely :

A **convexity** on a topological space X is a family $\mathcal{K}$ of closed subsets of X with $X \in \mathcal{K}$ and $\cap\mathcal{K}' \in \mathcal{K}$ for any $\mathcal{K}' \subseteq \mathcal{K}$; elements of $\mathcal{K}$ are **convex** sets.

We say that a convexity $\mathcal{K}$ is **normal** if for every $K \in \mathcal{K}$ and every closed set F disjoint from K there exists $K' \in \mathcal{K}$ also disjoint from F, such that $K \subseteq \text{Int } K'$; the convexity is **local** if $\mathcal{K}$ is a topological base for X.

We say that a family of sets $\mathcal{K}$ **screens** a family of sets $\mathfrak{G}$ whenever, for every $A \in \mathcal{F}$ and $G, G' \in \mathfrak{G}$ such that $A \cap G \cap G' = \emptyset$, there exist $F, F' \in \mathcal{F}$ such that $A \cap G \subseteq F \setminus F'$, $A \cap G' \subseteq F' \setminus F$ and $F \cup F' = A$; a family of sets $\mathfrak{G}$ **penetrates** a family of sets $\mathcal{F}$ whenever, for every $F, F' \in \mathcal{F}$ such that $F \cup F' \in \mathcal{F}$ and every $G \in \mathfrak{G}$, the conditions $G \cap F \neq \emptyset$ and $G \cap F' \neq \emptyset$ imply $G \cap F \cap F' \neq \emptyset$. The main result in the paper is the following theorem :

Theorem : Let $\mathcal{K}$ be a convexity on a nonempty compact spaces X. Assume that there exists a familly $\mathfrak{G}$ of subsets of X such that :

1. $\mathfrak{G}$ is finitely multiplicative ; 2. $\mathcal{K}$ screens $\mathfrak{G}$; 3. $\mathfrak{G}$ penetrates $\mathcal{K}$; and 4. $\mathfrak{G}$ is a topological base for X.

a. If $\mathcal{K}$ is local then X has the usual fixed point property.

b. If $\mathcal{K}$ is normal then X has the Kakutani property.

The proof of the Theorem is long and rather complicated. Its main idea is in approximating a given correspondence ϕ by some correspondences with finitely many values. A fixed point theorem for finite posets yields a fixed point for each of those correspondences. Their cluster point occurs to be a fixed point for ϕ.

Very often in applications the family $\mathfrak{G}$ appearing in the formulation of the Theorem coincides with $\mathcal{K}$.

If X is taken to be a compact convex set in a locally convex TVS while $\mathcal{K}$ consists of all closed convex subsets of X, we obtain, as a special case, the Theorem of Ky Fan and Glicksberg. If X is taken to be a tree while $\mathcal{K}$ consists of its all closed connected subsets, we obtain, as a special case, the Theorem of Wallace.

Another interesting special case obtains if a topological space X admits a **connecting** function, i.e. a continuous function $c : [0 ; 1] \times X^2 \to X$ with $c(0, x, y) = c(1, y, x) = c(\alpha, y, y) = y$ for all $\alpha \in [0 ; 1]$, $x, y \in X$; in this case a closed set K is defined to be **convex** whenever, for all $x, y \in K$ and $\alpha \in [0 ; 1]$, $c(\alpha, x, y)$ is also in K.

Finally, let us mention still another special case, very important for applications in game theory : Suppose that, for i in some set of indexes I, X_i, $\mathcal{K}_i$ and $\mathfrak{G}_i$ are, respectively, topological spaces and families of their subsets satisfying the assumptions of the Theorem. Then the product space ΠX_i equipped with the "box" convexity consisting of all boxes ΠC_i where $C_i \in \mathcal{K}_i$ for all i has the fixed point property (whenever all $\mathcal{K}_i$ are local) or the Kakutani property (whenever all $\mathcal{K}_i$ are normal).

Liste des participants / List of participants

Afrique du Sud / Southern Africa

Prof. Kalinde Albert K.
University of Fort Hare
Department of Mathematics
Private Bag X1314
Alice/CISKEI

Allemagne Fédérale / F.R.G

Prof. Gwinner Jochem
Fachbereich Matematik AG 6
Technische Hochschule Darmstadt
Schlossgartenstr.,7
D-6100 Darmstadt

Prof. Keimel Klans
Fachbereich Mathematik
Technische Hochschule
D-6100 Darmstadt

Prof. König Heinz
Universität des Saarlandes
Fachbereich Mathematik
D-6600 Saarbrücken

Prof. Oettli W
Lehrstuhl für Mathematik VII
Universität Mannheim
Postfach 103462
D-6800 Mannheinn 1

Prof. Volkmann Peter
Math. Inst I.
Univiversität Karlsruhe
Postfach 6980
7500 Karlsruhe

Angleterre / England

Prof. Alpern Steve
London School of Economics
Houghton St WC2A 2AE
London

Australie / Australia

Prof. Sims Brailey
Department of Mathematics
University of New England
Armidale N.S.W 2351

Canada

Prof. Ben-El-Mechaiekh H.
Mathematics
Brock University
St Catharines
Ontario L2S 3A1

Prof. Borwein Jonathan
Department of Mathematics
Dalhousie University
Halifax NS B3H3J5

Prof. Constantin Julien
Faculté des Sciences
Université de Sherbrooke
Québec J1K2R1

Prof. Deguire Paul
Université de Moncton
Département de Mathématiques
Moncton EIA3E9

Prof. Desbiens Jocelyn
Collège Militaire Royal de Saint-Jean
St-Jean/Richelieu
Québec J0J1RO

Prof. Donald Violette
Département de Mathématique
Physique et Informatique
Université de Moncton
Moncton EIA 3E9

Prof. Isac Georges
Département de Mathématiques
Collège Militaire Royal
St Jean Québec JOJ1RO

Prof. Lau Anthony T
Department of Mathematics
University of Alberta
Edmonton, Alberta T6G - 2G1

Prof. Lewis A.S.
Department of Mathematics Statistics
and Computing Science
Dalhousie University
Halifax, N.S., B3H 3J5

Prof. Singh S.P
Department of Mathematics
Memorial University of Newfoundland
St John's NF A1C 5S7

Prof. Tan Kok-Keong
Department of Mathematics
Dalhousie University
Halifax NS B3H3J5

Prof. Tingley Daryl
Department of Mathematics
University of New Branswick
Fredericton NB E3B5A3

Corée / Korea

Prof. Park Sehie
Math. Sciences Research Institut
Seoul National University
151 Seoul

Espagne / Spain

Prof. Arandiga F.
Universidad de Valencia
Departamento de Analisis Matematico
Doctor Moliner, 50
46100 Burjasot (Valencia)

Prof. Falset Jesus Garcia
Universidad de Valencia
Departamento de Analisis Matematico
Dr. Moliner 50
46100 Burjassot

Prof. Llaudes Francesco Arandiga
Universidad de Valencia
Departamento de Analisis Matematico
Dr. Moliner 50
46100 Burjassot

Prof. Llorens E.
Departamento de Analisis Matematico
Universitat de Valence
Dr Moliner 50
46100 Burjassot

Prof. Marchi Ezio
Dept. of Mat. Apl. y Anal.
Universitat de Barcelona
Gran Via, 585
0807 Barcelona

Prof. Martinez-Legaz J.E
Dept. of Mat. Apl. y Anal.
Universitat de Barcelona
Gran Via, 585
0807 Barcelona

Etats Unis / United States

Prof. Browder Felix
Department of Mathematics
Rutgers University
New Brunswick
New Jersey 08903

Prof. Bruck Ronald
Mathematics
University of Southern California
Los Angeles California 90089

Prof. Edelson Allan
1015 Department of Mathematics
U.C Davis Ca 95616

Prof. Egerland Walter
US Army Ballistic Research
Laboratory
Aberdeen Proving
Ground Aberdee, MD

Prof. Jaworowski Jan
Department of Mathematics
Indiana University
Bloomington IN 47405

Prof. Khamsi Amine
Department of Mathematics
University of Rhode Island
Kingston RI 02881

Prof. Khan M. Ali
Department of Economics
Johns Hopkins University
Baltinore Md 21218

Prof. Kirk W. A.
Department of Mathematics
University of Iowa
Iowa City 52242

Prof. Lin Bohr-Luh
Department of Mathematics
University of Iowa
Iowa City 52242

Prof. Magill Michael
Department of Economics
University of Southern California
Los Angeles CA, 900089

Prof. Mc Linden Lynn
Mathematics
University of Illinois
1409 W Green St.
Urbana Illinois 61801

Prof. Nashed M.Z.
Department of Mathematics
University of Delaware
Delaware Newark 1917

Prof. Rallis N.E
Department of Mathematics
Boston College
Chestnut Hill MA 02167

Prof. Rhoades B.E
Department of Mathematics
Indiana University
Bloomington IN 47405

Prof. Simons Stephen
Department of Mathematics
UC Santa Barbara
CA 93106

Prof. Vohra R.
Department of Economics
Brown University
Providence RI 62912

France

Prof. Baillon Jean-Bernard
72 Quai de la Loire
75019 Paris

Prof. Benoist Joël
Département de Mathématiques
Faculté des Sciences
126, rue Albert Thomas
87060 Limoges Cedex

Prof. Blot Joël
6 Avenue René Coty
87100 Limoges

Prof. Bonnissean Jean-Marc
CERMSEM
Université Paris I
90, Rue de Tolbrac
75634 Paris Cédex 13

Prof. Brezis Haim
Institut de Math.Pures et Appl.
Université P et M Curie
4,Place Jussieu
75230 Paris Cédex 05

Prof. Chaljub-Simon Alice
Département de Mathématiques
et d'informatique
Faculté des Sciences
B.P 6759
45067 Orléans Cédex

Prof. Charlot François
Université de Rouen
Faculté des Sciences et des Techniques
Boîte Postale n° 67
76130 Mont-Saint-Aignan

Prof. Cornet Bernard
39, Rue Saint Placide
75006 Paris

Prof. Corominas Ernest
ISM
Université Claude-Bernard
43 Bd du 11 Novembre 1918
69622 Villeurbanne Cédex

Prof. Florenzano Monique
3, Rue Lion St Paul
75004 Paris

Prof. Jouini Elyes
18, Rue de la Belle Feuille
92100 Boulogne

Prof. Grangier Gérard
I.M.I
Université C.Bernard
43,bd du 11 Novembre 1918
69622 Villeurbanne Cédex

Prof. Guillerme Jean
Département de Mathématiques
Faculté des Sciences
123, rue Albert Thomas
87060 Limoges Cédex

Prof. Lassonde Marc
Département de Mathématiques
Université Blaise Pascal
BP 45 63177 Aubière Cédex

Prof. Mazet Pierre
Université Paris VI
Analyse Complexe et Géométrie
4 Place Jussieu
75252 Paris

Prof. Nivat M.
ISM
Université Claude-Bernard
43 Bd du 11 Novembre 1918
69622 Villeurbanne Cédex

Prof. Penot Jean-Paul
Département de Mathématiques
Faculté des Sciences de Pau
Avenue de l'Université
64000 Pau Cédex

Prof. Pouzet Maurice
14, rue du Boulevard
69100 Villeurbanne

Prof. Robert François
IMAG / TIM 3 BP 53x
38041 Grenoble Cédex

Prof. Saint Raymond Jean
Equipe d'Analyse
Université P et M Curie
4,Place Jussieu
75252 Paris Cédex 05

Prof. Sonntag Yves
Département de Mathématiques
Université de Provence
13000 Marseille

Prof. Théra Michel
Dept de Mathématiques
Université de Limoges
123 rue Albert Thomas
87060 Limoges Cédex

Israel

Prof. Assad Nadim, A
Department of Mathematics
Birzeit University
Birzeit, P.O. Box 14
West Bank

Italie / Italy

Prof. Cellina Arigo
Mathematics
Sissa
Strada Castiera 11
34100 Trieste

Prof. Chersi Franco
Dipartimento di Scienze Matematiche
Università di Trieste
34100 Trieste

Prof. Conserva Vincenzo
Dipartimento di Matematica
via Arnesano
73100 Lecce

Prof. De Blasi F.S.
Università di Roma
Dipartimento di Matematica
via O. Raimondo
00173 Roma

Prof. Di Lenna G.
Dipartimento di Matematica e
Applicationi
Università degli Studi di Napoli
via Claudio 21
Cap. 80125 Napoli

Prof. Faina Loris
Mathematics
Sissa
Strada Castiera 11
34100 Trieste

Prof. Greco Gabriele
Dipartimento di Matematica
Universita di Trento
38050 Povo (Trento)

Prof. Furi Massimo
Istituto Matematico "U. Dini"
Università
via Morgani 67 A
50134 Firenze

Prof. Messano Basilio
Dipartimento di Matematica e
Applicationi
Università degli Studi di Napoli
via Claudio 21 Cap. 80125 Napoli

Prof. Myjak Jozef
Dipartimento di Energetica
Università degli Studi de L'Aquila
Via Roma 33
67100 L' Aquilla

Prof. Roux Delfina
Università di Milano
Dipartimento di Matematica
via C. Saldini 50
20133 Milano

Prof. Sandro Levi
Università di Milano
Dipartimento di Matematica
via C. Saldini 50
20133 Milano

Prof. Sessa S.
Facoltà di Architettura
istituto Matematico
via monteoliveto 3
80134 Napoli

Prof. Silvio Massa
Università di Milano
Dipartimento di Matematica
via C. Saldini 50
20133 Milano

Prof. Vitolo P.
via Pelicelli 1
43100 Parma

Japon / Japan

Prof. Hirano Norimichi
Department of Mathematics
Faculty of Engineering
Yokohama National University
Hodogayaku Yokohama

Prof. Takahashi Wataru
Dept of Information Sciences
Tokio Institut of Technology
Meguro-ku Tokyo 15
Tokyo

Maroc / Morocco

Prof. Bouslous Hammadi
Département de Mathématiques
Faculté des Sciences
B.P. S15 Marrakech

Prof. Hanebaly
Département de Mathématiques
Faculté des Sciences
B.P 1014 Rabat

Prof. Hbid Hassan
Département de Mathématiques
Faculté des Sciences
B.P. S15 Marrakech

Prof. Khaladi Moh^m^d
Département de Mathématiques
Faculté des Sciences
B.P. S15 Marrakech

Pakistan

Prof. Beg Ismat
Department of Mathematics
Quaid-I-Azam University
Islamabad

Pays-Bas / The Netherlands

Prof. Van Maaren Hans
Erasmus Universiteit Rotterdam
Econometrisch Instituut, H2-25
Postbus 1738
3000 DR Rotterdam

Pologne / Poland

Prof. Goebel K.
Instytut Matemtyki
UMCS 20-031
Lublin

Prof. Gorniewicz L
Universytat Mikolaja Kopernifker
Matematyki
Choguna 12
87100 Torun

Prof. Idzik Adam
Institut of Computer Sciences
Polish Academy of Sciences
00-901 Varsovie

Prof. Kryszewski W
Institut Matematyki
Universytat Lodzki
Banacha 22
90-238 Lodz

Prof. Kuczumow T.
Instytut Matematyki UMCS
Plac Mavii Curie-Stodoskiej 1
20-031 Lublin

Prof. Legut Jerzy
Instytut Matematyki
Politechniki Wroclavskiej
Wybrzeze Wyspianskiego 27
JO-370 Wroclaw

Prof. Manka Roman
Instiytut Matematyczny
Polish Academy of Sciences
P.O. Box 22
00-901 Warsaw

Prof. Stachura A.
Instytut Matematyki UMCS
Plac Mavii Curie-Stodoskiej 1
20-031 Lublin

Prof. Wieczorek Andrzej
Institute of Computer Science
Polish Academy of Sciences
P.O. Box 22
00-901 Warsaw

Taiwan

Prof. Lai Hang-Chin
Department of Mathematics
National Tsing Hua University
Hsinchu

Vietnam

Prof. Cuong Bui Cong
Institute of Mathematics
P.O. Box 631BO HO
Hanoi