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Clyde A. Morrison

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The solution is given to Laplace's equation for a conducting sphere above a ground plane. The solution includes the presence of a uniform electric field perpendicular to the ground plane. The arbitrary constants in the solution are determined by applying the boundary conditions on the plane and on the sphere. Formulas are given for the evaluation of the potential and electric fields at an arbitrary field point. All the resulting formulas are expressed in a style suitable for computation. The computational results will form the basis of a future comparison report.

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Foreword

Thirty-some years ago Nick Karayianis and I tackled a problem in electrostatics which was similar to the one discussed here. At that time, I was fresh from a course in special functions taught by E. D. Rainville at the University of Michigan and I imparted my newly acquired wisdom to Nick. A few years later, Nick finished his doctorate at Indiana University and returned the favor by teaching me Racah algebra and angular momentum theory. In that exchange, I received more than I gave. During the course of "breaking the neck" of this problem, I frequently thought of our earlier problem and missed Nick's able guidance. This report would have been much better with his assistance.

Contents

Appendices

1. Introduction

In this report we derive expressions for the electric potential and electric fields produced by a charged conducting sphere above a conducting plane. We were unable to find the problem done (potential and electric fields at an arbitrary field point) in a number of standard textbooks on electrostatics [1-7].* Further, textbooks on spherical harmonics did not give the solution [8-10]. Also, a preliminary search of the literature revealed a number of related problems, of which two are cited here [11, 12]. These latter references approach the problem using bipolar coordinates and refer to a number of internal memoranda which are difficult to obtain.

In most of those references where the geometry of the problem was similar to the one considered here, the discussion centered around finding the capacitance of the system. In a few references, the problem of the electric field at a very particular point (for example, the surface of the sphere) was considered. Here we are interested in finding the potential and, consequently, the electric fields at an arbitrary field point. Thus the possibility of finding the problem done in a textbook or monograph is rather slim, but an extensive literature search would quite possibly find the problem. Since an extensive search would take considerable time, we decided to do the problem while the search was underway, and as of this date we have not located a usable solution. In our analysis, we use spherical coordinates throughout.

2. Solution to Laplace's Equation

We consider a conducting sphere of radius a, at potential V, located at a height h above a conducting plane. The potential of the plane is chosen to be zero. To simplify the problem, we replace the conducting plane by an identical sphere at potential $-V$, located at $-h$ below the plane. The coordinate system is shown in figure 1.

References are listed at the end of this report.

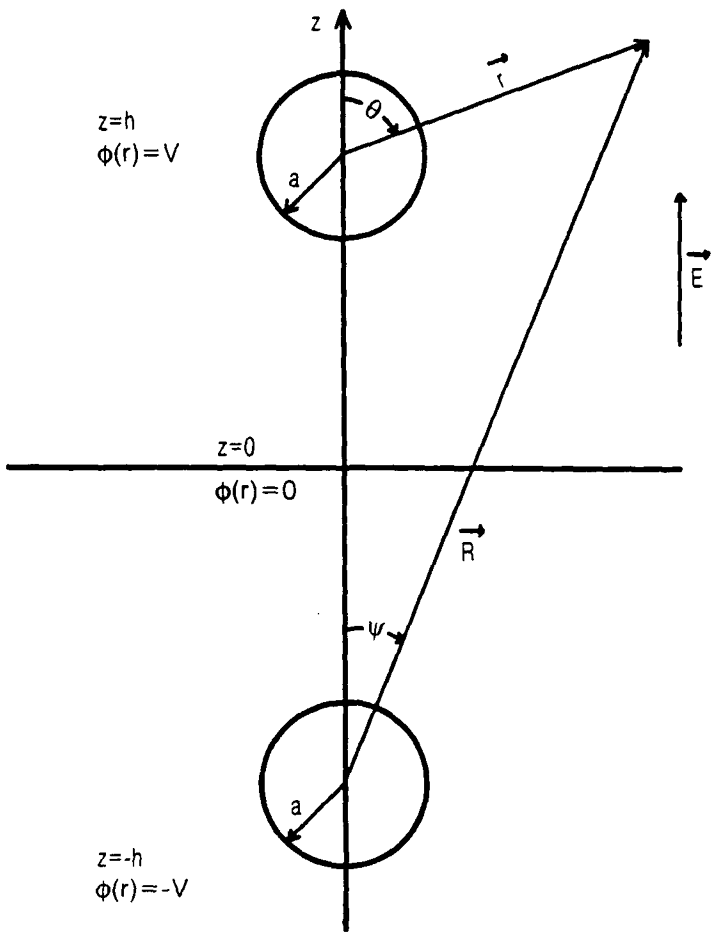

Figure 1. Coordinate system of the two spheres.

We consider the upper sphere first. The two linearly independent solutions to Laplace's equation ($\nabla^2\phi = 0$) in a coordinate system centered on the upper sphere are

$$r^n P_n(\cos\theta) \tag{1}$$

and

$$\frac{1}{r^{n+1}} P_n(\cos\theta) \quad , \tag{2}$$

where $n = 0, 1, 2, \ldots$, and the $P_n(\cos\theta)$ are the Legendre polynomials [13]. For convenience, we list the first few Legendre polynomials:

$$P_0(\cos\theta) = 1 \quad ,$$
$$P_1(\cos\theta) = \cos\theta \quad ,$$
$$P_2(\cos\theta) = (3\cos^2\theta - 1)/2 \quad ,$$
$$P_3(\cos\theta) = [(5\cos^2\theta - 3)\cos\theta]/2 \quad ,$$

and the general relations,

$$P_n(-\cos\theta) = (-1)^n P_n(\cos\theta) \ ,$$

$$nP_n(\cos\theta) = (2n-1)\cos\theta P_{n-1}(\cos\theta) - (n-1)P_{n-2}(\cos\theta) \ .$$

This last relation is used recursively to generate $P_n(\cos\theta)$ for $n > 2$ using the $n = 0$ and $n = 1$ polynomials as initial values.

The solutions given in equation (1) diverge for large r and are discarded. The solutions in equation (2) have no singularities in the region $a \leq r < \infty$ and are retained. Similarily, the retained solutions for the lower sphere are

$$\phi \sim \frac{1}{R^{n+1}} P_n(\cos\psi) \ . \tag{3}$$

The potential associated with the uniform electric field is given by

$$\phi_E = -Ez \ . \tag{4}$$

The general solution for the potential is given by multiplying the solutions given by equations (2) and (3) by arbitrary constants and summing over n to obtain

$$\phi = \sum_{n=0}^{\infty} \frac{A_n}{r^{n+1}} P_n(\cos\theta) + \sum_{n=0}^{\infty} \frac{B_n}{R^{n+1}} P_n(\cos\psi) - Ez \ . \tag{5}$$

The problem then is to evaluate A_n and B_n by using the boundary conditions

$$\begin{aligned} \phi &= V \ , \quad \text{for} \quad r = a \ , \\ \phi &= -V \ , \quad \text{for} \quad R = a \ , \end{aligned} \tag{6}$$

and

$$\phi = 0 \ , \quad \text{for} \quad z = 0 \ .$$

In equations (2) through (6) we have avoided introducing the arguments in the potential ϕ because the particular variables will depend on which boundary condition is considered.

3. Evaluation of the Constants A_n and B_n

The simplest boundary condition is the last of equation (6). On the plane $z = 0$, we have $r = R$ and $\theta = \pi - \psi$ so that

$$\cos\theta = -\cos\psi$$

and

$$P_n(-\cos\psi = (-1)^n P_n(\cos\psi) \ .$$

Then from equation (5) we have

$$0 = \sum_n [(-1)^n A_n + B_n] \frac{P_n(\cos\psi)}{r^{n+1}} \ ; \tag{7}$$

and since each $P_n(\cos\psi)/r^{n+1}$ is linearly independent for each n, we have

$$B_n = (-1)^{n+1} A_n \ . \tag{8}$$

To obtain explict values for A_n we shall use the first boundary condition in equation (6). In order to do this we need the expansion of $P_n(\cos\psi)/R^{n+1}$ in terms of $P_n(\cos\theta)$. This expansion is given in a number of places [14, 15] in very general form, and the result is derived heuristically in appendix A. Since we are to use the first boundary condition in equation (6), we use the expansion given in equation (A-12) of appendix A which, with obvious changes, is

$$\frac{P_k(\cos\psi)}{R^{k+1}} = \sum_n (-1)^n \binom{n+k}{n} \frac{r^n}{(2h)^{n+k+1}} P_n(\cos\theta) \ , \tag{9}$$

where $\binom{n+k}{n}$ is a binomial coefficient.

Substituting equation (9) into equation (5) gives

$$\begin{aligned} \phi(r,\theta) = \sum_n \left[\frac{A_n}{r^{n+1}} + \sum_{k=0}^{\infty} B_k (-1)^n \binom{n+k}{n} \frac{r^n}{(2h)^{n+k+1}} \right] P_n(\cos\theta) \\ - E[h + rP_1(\cos\theta)] \ , \end{aligned} \tag{10}$$

where we have written

$$\begin{aligned} z &= h + r\cos\theta \\ &= h + rP_1(\cos\theta) \ . \end{aligned}$$

From the first of equation (6), the potential at the surface of the upper sphere, $\phi(a, \theta) = V$, we obtain

$$\begin{aligned} V = \sum_n \left[\frac{A_n}{a^{n+1}} + \sum_k (-1)^{n+k+1} \frac{a^n}{(2h)^{n+k+1}} \binom{n+k}{n} A_k \right] P_n(\cos\theta) \\ - E[h + aP_1(\cos\theta)] \ , \end{aligned} \tag{11}$$

where we have used equation (8) to eliminate B_k in favor of A_k. Since the $P_n(\cos\theta)$ are orthogonal, for each n we can write

$$V = U_0 - \lambda \sum_k (-\lambda)^k U_k - Eh \ ,$$

$$0 = U_1 - \lambda \sum_k (-\lambda)^{k+1} \binom{k+1}{k} U_k - Ea \ ,$$

and (12)

$$0 = U_n - \sum_k (-\lambda)^{k+n} \binom{n+k}{k} U_k \quad for \quad n > 1 \ .$$

In equation (12), U_n and λ are defined by

$$U_n = \frac{A_n}{a^{n+1}}$$

and

$$\lambda = \frac{a}{2h} \ .$$

The system of equations given in equation (12) can be written compactly in matrix form as

$$\vec{F} = (\underset{\sim}{1} - \lambda\, \underset{\sim}{G}) \cdot \vec{U} \ , \tag{13}$$

where $\vec{F}$ is the column vector with components

$$F_0 = V + Eh \ , \tag{14}$$

$$\begin{aligned} F_1 &= Ea \ , \\ F_n &= 0 \ , n > 1 \ , \end{aligned} \tag{15}$$

$\underset{\sim}{1}$ is the unit matrix, with elements δ_{nm}, $\underset{\sim}{G}$ is the symmetric matrix with components

$$G_{nm} = (-\lambda)^{n+m} \binom{n+m}{n} \ , \tag{16}$$

and $\vec{U}$ is the column vector with components U_n given in equation (12).

The formal solution to equation (13) is

$$\vec{U} = \underset{\sim}{B} \cdot \vec{F} \ , \tag{17}$$

where $\underset{\sim}{B}$ is

$$\begin{aligned} \underset{\sim}{B} &= (\underset{\sim}{1} - \lambda \underset{\sim}{G})^{-1} \\ &= \underset{\sim}{1} + \lambda \underset{\sim}{G} + \lambda^2 \underset{\sim}{G}^2 + \lambda^3 \underset{\sim}{G}^3 + \ldots \ . \end{aligned} \tag{18}$$

The result given in equation (18) can be quite deceptive in that not all the λ dependence is explicit; the matrix $\underset{\sim}{G}$ also contains λ as shown in equation (16). All the constants (V, a, E, and h) are contained in F_0 and F_1, and equation (17) determines each U_n in terms of these constants. Also, since

$\underset{\sim}{B}$ is symmetric, equation (17) can be written $\vec{U} = \vec{F}^T \bullet \underset{\sim}{B}$, if convenient ($\vec{F}^T$ in the transpose vector — a row vector).

For a given potential, V, on the sphere, the charge Q on the sphere can be determined* from equation (12). By Gauss′ law, the charge is determined by A_0 or $A_0 = Q$ (and therefore ($U_0 = Q/a$)). Hence,

$$\frac{Q}{a} = B_{00} F_0 + B_{01} F_1 \quad ,$$

and from (14),

$$\frac{Q}{a} = B_{00}(V + Eh) + B_{01} Ea \quad . \tag{19}$$

Thus, a knowledge of the two components of $\underset{\sim}{B}$ determine the charge on the sphere. We shall discuss equation (19) further when we consider the case when the charge on the sphere is fixed.

4. The Potential and the Electric Fields

Having determined the constants U_n by equation (12), it is necessary to determine the potential ϕ and the field components $E_r (= -\partial\phi/\partial r)$, and $E_\theta (= -\partial\phi/r\partial\theta)$ using the U_n. Substituting U_n for A_n and $B_n = (-1)^{n+1} A_n$ into equation (10), we have

$$\phi = \sum_n \left[\left(\frac{a}{r}\right)^{n+1} U_n - \left(\frac{r}{a}\right)^n \lambda \sum_m G_{nm} U_m \right] P_n - E\left[h + r(P_1)\right] \quad , \tag{20}$$

$$E_r = \sum_n \left[(n+1)\left(\frac{a}{r}\right)^{n+2} U_n + n\left(\frac{r}{a}\right)^{n-1} \lambda \sum_m G_{nm} U_m \right] \frac{P_n}{a} + E P_1 \quad , \tag{21}$$

$$E_\theta = \sum_n \left[\left(\frac{a}{r}\right)^{n+2} U_n - \left(\frac{r}{a}\right)^{n-1} \lambda \sum_m G_{nm} U_m \right] \frac{T_n}{a} - E T_1 \quad , \tag{22}$$

where

$$T_n = -\frac{dP_n}{d\theta} \quad .$$

**In MKS units replace Q by $Q/(4\pi\varepsilon_0)$, where $4\pi\varepsilon_0 = 1.112650 \times 10^{-12}$ F/m.*

From equation (13) we can write $\lambda \sum_m G_{nm} U_m = U_n - F_n$, and using this result in equations (20), (21), and (22) we can express these equations in terms of U_n and F_n.

We now let

$$X_0 = \left(\frac{a}{r} - 1\right) U_0 + V \ ,$$

$$X_n = \left[\left(\frac{a}{r}\right)^{n+1} - \left(\frac{r}{a}\right)^{n}\right] U_n \ , n > 0 \ , \tag{23}$$

and

$$Y_0 = \left(\frac{a}{r}\right)^2 \frac{U_0}{a} \ ,$$

$$Y_n = \left[(n+1)\left(\frac{a}{r}\right)^{n+2} + n\left(\frac{r}{a}\right)^{n-1}\right] \frac{U_n}{a} \ , n > 0 \ . \tag{24}$$

Z_0 is set to zero since $T_0 = 0$ (see eq (27)),

$$Z_0 = 0 \ \text{ and}$$

$$Z_n = \left[\left(\frac{a}{r}\right)^{n+2} - \left(\frac{r}{a}\right)^{n-1}\right] \frac{U_n}{a} \ , n > 0 \ . \tag{25}$$

Using equations (23), (24), and (25) in (20) through (22), we have

$$\begin{aligned} \phi(r, \theta) &= \vec{X}(r) \bullet \vec{P}(\theta) \\ E_r(r, \theta) &= \vec{Y}(r) \bullet \vec{P}(\theta) \\ E_\theta(r, \theta) &= \vec{Z}(r) \bullet \vec{T}(\theta) \end{aligned} \tag{26}$$

for $r < 2h$.

In equation (26), the components of $\vec{P}$ are the Legendre polynomials and the components of $\vec{T}$ are $-dP_n/d\theta$. The first few values of $\vec{T}$ are

$$\begin{aligned}
T_0 &= 0, \\
T_1 &= \sin\theta, \\
T_2 &= 3\sin\theta\cos\theta, \\
&\text{and the recursion relation is} \\
T_n &= n(P_{n-1} - \cos\theta\, P_n)/\sin\theta \ .
\end{aligned} \tag{27}$$

The above results are for $r < 2h$; for $r > 2h$ we have to use the expansion for $P_k(\cos\psi)/R^{k+1}$ in terms of $P_n(\cos\theta)$ given in appendix A in equation (A-13). The solution for U_n given in equation (17) still holds, so we can write

$$\phi = \sum_n [U_n - W_n]\left(\frac{a}{r}\right)^{n+1} P_n - Ea\left(\frac{h}{a} + \frac{r}{a}P_1\right) , \tag{28}$$

where

$$W_n = \frac{1}{(-\lambda)^n}\sum_{m=0}^{n} \lambda^m \binom{n}{m} U_m \ ,$$

and the field components are given by

$$\mathrm{E_r} = \sum_n (n+1)[U_n - W_n]\left(\frac{a}{r}\right)^{n+2} \frac{P_n}{a} + EP_1 \ , \tag{29}$$

and

$$E_\theta = \sum_n [U_n - W_n]\left(\frac{a}{r}\right)^{n+2} \frac{T_n}{a} - ET_1 \ . \tag{30}$$

In equations (28), (29), and (30), we let

$$X_0' = -Eh \ ,$$

$$X_1' = \left(\frac{a}{r}\right)^2 [U_1 - W_1] - \mathrm{Er} \ , \tag{31}$$

$$X_n' = \left(\frac{a}{r}\right)^{n+1} [U_n - W_n], n > 1 \ ,$$

$$Y_0' = 0 \ ,$$

$$Y_1' = 2\left(\frac{a}{r}\right)^3 (U_1 - W_1)\frac{1}{a} + E \ , \tag{32}$$

$$Y_n' = (n+1)\left(\frac{a}{r}\right)^{n+2} (U_n - W_n)\frac{1}{a}, n > 1 \ ,$$

and

$$Z_0' = 0 \ ,$$

$$Z_1' = \left(\frac{a}{r}\right)^3 (U_1 - W_1)\frac{1}{a} - E \ , \tag{33}$$

$$Z_n' = \left(\frac{a}{r}\right)^{n+2} (U_n - W_n)\frac{1}{a}, n > 1 \ .$$

Then, as in equation (26), we have

$$\phi = \vec{X}'(r) \bullet \vec{P}(\theta) \ ,$$

$$\mathrm{E_r} = \vec{Y}'(r) \bullet \vec{P}(\theta) \ , \tag{34}$$

$$E_\theta = \vec{Z}'(r) \bullet \vec{T}(\theta) \ ,$$

for $r > 2h$.

The $\vec{P}(\theta)$ and $\vec{T}(\theta)$ in equation (34) are the same as in equation (26).

The results given in equations (26) and (34) constitute a solution for the entire region $a < r < \infty$ and $0 < \theta < \pi$. The procedure consists in finding the coefficients U_n for given V, a, h, and E by using equation (17). After this, we find the components of $\vec{X}$ ($\vec{X}'$), $\vec{Y}$ ($\vec{Y}'$), and $\vec{Z}$ ($\vec{Z}'$) for $r < 2h$ ($r > 2h$) using equation (26) (eq (34)).

As discussed following equation (19), when the charge Q on the sphere is fixed, it is necessary to determine V from equation (19) as

$$V = \frac{Q}{aB_{00}} - Eh - \frac{B_{01}}{B_{00}} Ea \ . \tag{35}$$

In order to obtain V, we need to evaluate B_{00} and B_{01}, and the result from appendix B for general B_{0n} is given as

$$B_{0n} = \delta_{0n} + (-\lambda)^n \sum \lambda^p \frac{(A_{p-1})^n}{(A_p)^n} \ , \qquad \text{[B-24]}$$

where

$$A_p = A_{p-1} - \lambda^2 A_{p-2} \qquad \text{[B-20]}$$

with

$$A_0 = A_1 = 1 \ . \qquad \text{[B-21]}$$

The values of B_{00} and B_{01} can be calculated using these results.

The solution to the problem where the charge Q is fixed is then obtained by determining V by using equation (35). We then insert this V into equations (14) and (15) to determine $\vec{F}$ and continue as in the case in which V is given (eq (26) for $r < 2h$; eq (34) for $r > 2h$).

5. Conclusion

The results derived here constitute a formal solution to the potential and the fields for a conducting sphere at a given potential V (or charge Q) at height h above a charged conducting plane. Preliminary calculations show that the potential and fields are sufficiently represented by equations (26) and (34) even for a modest number of terms (~10). However, the region near $r = 2h$ is not well represented by these equations. The reason for this can be traced back to equations (A-12) and (A-13). The series represented in these equations become conditionally convergent near $r = 2h$ and diverge at $r = 2h$. If it is important to evaluate equations (26) and (34) in this region, either analytical methods or computational methods will have to be devised. One way of handling this problem is to compute, at values of θ, a number of values for ϕ, E_r, and E_θ to be evaluated for $r < 2h$ and $r > 2h$. An interpolation formula is used in the immediate vicinity of $r = 2h$. The results given in equations (26) and (34) will be used in a following report where we calculate $\phi(r, \theta)$, $E_r(r, \theta)$, and $E_\theta(r, \theta)$ for a number of interesting cases.

Acknowledgements

The author wishes to thank Don Wortman, John Bruno, and Greg Turner for various suggestions during the process of development of this work. Also, for the contributing discussion and encouragement I thank Dave Hull, Mike Wren, and Barry Stann. Particularly, since the problem originated from the last three coworkers, I wish to thank them for the hours of enjoyment the problem has provided.

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Appendix A. — Expansion of $P_k(\cos\psi) / R^{k+1}$ in terms of $P_n(\cos\theta)$

In the main body of the report, we used the formal expressions from Judd[1] and Morrison[2] for the expansion of $P_k(\cos\psi)/R^{k+1}$ in terms of $P_n(\cos\theta)$. Because of the very general nature of the expansion it is useful to have an independent check. To do this, we consider the expansion (from eq (13))

$$\frac{1}{R} = \sum_{l=0}^{\infty} (-1)^l \frac{r^l}{x^{l+1}} P_\ell(\cos\theta) \qquad r < x \qquad \text{(A-1)}$$

$$= \sum_{l=0}^{\infty} (-1)^l \frac{x^l}{r^{l+1}} P_\ell(\cos\theta) \qquad r > x \ , \qquad \text{(A-2)}$$

where

$$R = \sqrt{x^2 + r^2 + 2xr\cos\theta} \qquad \text{(A-3)}$$

and

$$x + r\cos\theta = R\cos\psi \ . \qquad \text{(A-4)}$$

(In the final result we let $x = 2h$.)

Our method consists of taking successive derivatives of $1/R$ with respect to x and using (A-4) to cast the result into the desired form. First consider

$$\frac{d}{dx}\left(\frac{1}{R}\right) = -\frac{R_x}{R^2} \ , \qquad \text{(A-5)}$$

where

$$R_x = \frac{dR}{dx} \ .$$

[1] *B. R. Judd, Angular Momentum Theory for Diatomic Molecules, Academic Press, New York (1975).*

[2] *C. A. Morrison, Angular Momentum Theory Applied to Interactions in Solids, Springer-Verlag (1988).*

From (A-3) we have

$$R_x = \frac{x + r\cos\theta}{R} \quad ,$$

and using (A-4) we obtain $R_x = \cos\psi$. Since $P_1(\cos\psi) = \cos\psi$, we have

$$\frac{d}{dx}\left(\frac{1}{R}\right) = \frac{-P_1(\cos\psi)}{R^2} \quad . \tag{A-6}$$

Repeating this process we obtain

$$\frac{d^2}{dx^2}\left(\frac{1}{R}\right) = \frac{2\,P_2(\cos\psi)}{R^3} \quad , \tag{A-7}$$

$$\frac{d^3}{dx^3}\left(\frac{1}{R}\right) = \frac{-\,6\,P_3(\cos\psi)}{R^4} \quad . \tag{A-8}$$

In obtaining these results, it is convenient to express all higher derivatives in terms of R_x:

$$R_{xx} = (1 - R_x^2)/R \quad ,$$
$$R_{xxx} = (3\,R_x^2 - 3\,R_x)/R^2 \quad .$$

This procedure keeps the size of the algebraic expressions under control.

On the right side in expression (A-1) we consider only the x-dependent part as

$$
\begin{aligned}
&\frac{1}{R} \sim \frac{1}{x^{l+1}} \\
&\frac{d}{dx}\left(\frac{1}{R}\right) \sim -(l+1)\,\frac{1}{x^{l+2}} \\
&\frac{d^2}{dx^2}\left(\frac{1}{R}\right) \sim (l+1)(l+2)\,\frac{1}{x^{l+3}} \\
&\frac{d^3}{dx^3}\left(\frac{1}{R}\right) \sim -\,(l+1)(l+2)(l+3)\,\frac{1}{x^{l+4}}
\end{aligned}
\tag{A-9}
$$

and consider the binomial coefficient $\binom{l+k}{l}$, which is l, $l+1$, $(l+1)(l+2)/2$, $(l+1)(l+2)(l+3)/6$ for $k = 0, 1, 2,$ and 3, respectively. So we write

$$\frac{d^k}{dx^k}\left(\frac{1}{R}\right) = (-1)^k\, k!\, \frac{P_k\,(\cos\psi)}{R^{k+1}} \;, \tag{A-10}$$

and

$$\frac{d^k}{dx^k}\left(\frac{1}{x^{l+1}}\right) = (-1)^k\, k! \binom{l+k}{l} \frac{l}{x^{l+k+1}} \;; \tag{A-11}$$

thus, with x set to $2h$, we obtain

$$\frac{P_k\,(\cos\psi)}{R^{k+1}} = \sum_{l=0}^{\infty} (-1)^l \binom{l+k}{l} \frac{r^l}{x^{l+k+1}}\, P_l\,(\cos\theta) \;, \qquad r < x. \tag{A-12}$$

For $x < r$ in (A-5) the result corresponding to equation (A-11) becomes

$$\frac{d^k}{dx^k}\, x^l = k! \binom{l}{l-k} x^{l-k} \;,$$

with $k < l$. Then we obtain, after the substitution $x = 2h$,

$$\frac{P_k\,(\cos\psi)}{R^{k+1}} = \sum_{l=k}^{\infty} \frac{(2\mathrm{h})^{l-k}}{r^{l+1}}\, (-1)^{k+l} \binom{l}{k} P_l\,(\cos\theta) \;, \qquad r > 2h. \tag{A-13}$$

Appendix B. — Fixed Charge on the Sphere

When the charge on the sphere is specified rather than the voltage, it is useful to write the boundary conditions given in equations (14) and (15) as

$$\begin{aligned} U &= B_{00}F_0 + B_{01}F_1 \\ U &= B_{00}(V - Eh) + B_{01}Ea \ , \end{aligned} \tag{B-1}$$

and by Gauss′ law the charge on the sphere, Q, determines A_0; that is,

$$A_0 = Q \ . \tag{B-2}$$

Then from equation (12), $U_0 = Q/a$ and from (B-1) we obtain*

$$V = [Q/a - (B_{00}h + B_{01}a)\ E]/B_{00} \ , \tag{B-3}$$

and we notice that when $E = 0$ we obtain

$$Q = cV \ , \tag{B-4}$$

where $c = aB_{00}$.

As $\lambda \to 0$ ($h \to \infty$) we know that $B_{00} = 1$ and the capacitance becomes $c = a$, which is the usual expression for the capacitance for a sphere of radius a. From equation (B-3) we see that it would be convenient to evaluate the *potential of the sphere* and use the results in the calculation of the potential E_r and E_θ, as is done for the case when V is given.

In equation (18) we defined the matrix $\underset{\sim}{B}$ as

$$\underset{\sim}{B} = (\underset{\sim}{1} - \lambda\underset{\sim}{G})^{-1} \ , \tag{B-5}$$

which we can expand as

$$\underset{\sim}{B} = \underset{\sim}{1} + \lambda\underset{\sim}{G} + \lambda^2\underset{\sim}{G}_2 + \cdots \lambda^p\underset{\sim}{G}_P + \cdots \ , \tag{B-6}$$

**If we are using MKS units we replace Q/a by $Q/(4\pi\varepsilon_0 a)$.*

and we need B_{0n} for n = 0 and 1. For the first few terms in (B-6) we have

$$B_{0n} = \delta_{0n} + \lambda G_{0n} + \lambda^2 (G^2)_{0n} + \lambda^3 (G^3)_{0n} + \ldots \ , \qquad \text{(B-7)}$$

and for a given term

$$(G^P)_{0n} = \sum_k G_{0k} (G^{P-1})_{kn} \ , \qquad \text{(B-8)}$$

and for $p = 2$

$$(G^2)_{0n} = \sum_k G_{0k} G_{kn} \ . \qquad \text{(B-9)}$$

Using

$$G_{nm} = (-\lambda)^{n+m} \binom{n+m}{m}$$

in (B-9),

$$(G^2)_{0n} = (-\lambda)^n \sum_k \lambda^{2k} \binom{k+n}{k} \ , \qquad \text{(B-10)}$$

since

$$G_{0k} = (-\lambda)^k \ .$$

The sum in equation (B-10) can be done and is given by

$$\sum_{k=0}^{\infty} y^k \binom{k+n}{k} = \frac{1}{(1-y)^{n+1}} \ , \qquad \text{(B-11)}$$

where $y < 1$. (This value of y is much larger than we need. In our case, the largest value of y is 1/4.)

Using (B-11) in equation (B-10) we obtain

$$(G^2)_{0n} = \frac{(-\lambda)^n}{(1-\lambda^2)^{n+1}} \ . \tag{B-12}$$

Extending this result to the next term gives

$$\begin{aligned}(G^3)_{0n} &= \sum_k (G^2)_{0k} G_{kn} \\ &= \frac{(-\lambda)^n}{1-\lambda^2} \sum_k \frac{\lambda^{2k}}{(1-\lambda^2)^k} \binom{k+n}{k} \ ,\end{aligned} \tag{B-13}$$

and from equation (B-11) we get

$$G^3_{0n} = \frac{(-\lambda)^n (1-\lambda^2)^n}{(1-\lambda^2-\lambda^2)^{n+1}} \ . \tag{B-14}$$

We have written the denominators in (B-14) in the particular form so that if we let the denominator in (B-13) be A_2 and the denominator in (B-14) be A_3, we have

$$A_3 = A_2 - \lambda^2 \ . \tag{B-15}$$

For

$$(G^4)_{0n} = \sum_k (G^3)_{0k} G_{kn} \ , \tag{B-16}$$

we have

$$(G^4)_{0n} = \frac{(-\lambda)^n}{A_3} \sum \lambda^2 \frac{A_2{}^k}{A_3{}^k} \binom{k+n}{k} \ , \tag{B-17}$$

and from equation (B-11),

$$(G^4)_{0n} = (-\lambda)^n \frac{A_3^n}{(A_3 - \lambda^2 A_2)^{n+1}} \ . \tag{B-18}$$

If we now let

$$A_4 = A_3 - \lambda^2 A_2 \ , \qquad \text{(B-19)}$$

we get $(G^4)_{0n} = (-\lambda)^n \dfrac{A_3^n}{A_4^{n+1}}$.

The result in (B-19) suggests the form

$$A_p = A_{p-1} - \lambda^2 A_{p-2} \ , \qquad \text{(B-20)}$$

with

$$A_0 = 1$$

and (B-21)

$$A_1 = 1 \ ,$$

an assumption consistent with A_2 and A_3 given above. To check the result, we calculate $(G^p)_{0n}$ using (B-20). That is,

$$(G^p)_{0n} = \sum_k (G^{p-1})_{0k} G_{kn}$$

$$(G^p)_{0n} = (-\lambda)^n \frac{(A_{p-1})^n}{(A_{p-1} - \lambda^2 A_{p-2})^{n+1}} \ , \qquad \text{(B-22)}$$

and with the result given in (B-20) gives

$$G^p_{0n} = (-\lambda)^n \frac{(A_{p-1})^n}{(A_p)^{n+1}} \qquad \text{(B-23)}$$

Then with the result of (B-23) substituted into equation (B-7) we obtain

$$B_{0n} = \delta_{0n} + (-\lambda)^n \sum_{p=1}^{\infty} \lambda_p \frac{(A_{p-1})^n}{(A_p)^{n+1}} \,, \tag{B-24}$$

and this result with the A_p given by the recursion relation in (B-20) gives a convenient form for computation of B_{00} and B_{01} for use in equation (B-3) to determine the potential of the sphere.

It would be nice, from an analytic point of view, to be able to calculate B_{nm} from (B-6) by calculating each power of $\underset{\sim}{G}$ directly. However for $(G^2)_{nm}$ we have

$$(G^2)_{nm} = (-\lambda)^{n+m} \sum_{k=0}^{\infty} \lambda^{2k} \binom{n+k}{k}\binom{k+m}{k} \,, \tag{B-25}$$

but we have not been able to do the sum in closed form. Karayianis[1] has shown that

$$\sum_{k=0}^{\infty} \lambda^{2k} \binom{n+k}{k}\binom{k+m}{k} = \frac{1}{(1-\lambda^2)^{n+m+1}} \sum_{k} \lambda^{2k} \binom{n}{k}\binom{m}{k} \,,$$

which is Euler's transformation for the hypergeometric function $F(a, b; c; \lambda^2)$ given by Rainville[2] and does not appear to be capable of further simplification.

[1] *Nick Karayianis, Certain Summations Involving Binomial Coefficients and Their Relation to Dyson's Conjecture, Harry Diamond Laboratories, HDL-TR-1217 (April 1964), 7.*

[2] *E. D. Rainville, Special Functions, MacMillan, New York (1960), Ch 10.*

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