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ON ESTIMATING THE DEPENDENCE BETWEEN TWO POINT PROCESSES

BY

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1. INTRODUCTION AND SUMMARY.

Let (N_A, N_B) be a stationary bivariate point process on $\mathbb{R}$. This article is concerned with statistical methods for discovering and quantifying an association between the two processes from a realization $A_1 < A_2 < \ldots < A_{n_A}$, $B_1 < B_2 < \ldots < B_{n_B}$ over a long period of time T. The paper is motivated by certain problems that arise in neurophysiology, which are very briefly described as follows (for further details see e.g. Bryant, Ruiz Marcos, and Segundo, 1973).

Two neurons, A and B, are monitored over a period of time T during which each neuron fires a sequence of impulses. The problem is to determine whether or not the impulse times are associated. An association between N_A and N_B may be construed as evidence that either the two neurons are communicating, or that they both share input from a third source.

Another problem arises in certain neurophysiological studies of learning and memory. An animal is to be taught (trained) to perform a certain task. Now consider two "connected" neurons, A and B, which are essential in the performance of this task. Record the impulse times during a period before the learning experience, obtaining a realization of $(N_A^{\text{Bef.}}, N_B^{\text{Bef.}})$, and during a period of time *well after* the learning experience, obtaining a realization of $(N_A^{\text{Aft.}}, N_B^{\text{Aft.}})$. The processes $N_A^{\text{Bef.}}$ and $N_B^{\text{Bef.}}$ may be *dependent*. The problem is to determine whether or not this dependence is "stronger" for the processes $N_A^{\text{Aft.}}$ and $N_B^{\text{Aft.}}$. A neurophysiologist may consider a change in the strength of the dependence as evidence that learning has taken place.

The two problems have very different statistical character. Let S be a statistic that "measures" the dependence between two point processes. The first problem is one of testing the hypothesis that N_A and N_B are independent, and requires only knowledge of the distribution of S under the assumption that N_A and N_B are independent. The second problem is much more difficult: to compare S across two situations we must know the distribution of S when the two point processes are dependent.

In a more general context, Ripley (1976,1977) introduced a measure K, defined on an appropriate space, that summarizes the second-order properties of the process. Before describing this measure, we need to state some assumptions and introduce some notation. Let $N_i(s,t)$ denote the number of events of type i occurring in the interval (s,t], for i = A, B. Assume that each process has no multiple occurrences, and that the intensities

$$\lambda_i = \lim_{h\to 0} \frac{1}{h} P\{N_i(t,t+h) > 0\} \qquad \text{for } i = A,\ B \tag{1}$$

are finite. (The existence of these limits was proved by Khintchine, 1960). The λ_i's then have an interpretation as mean number of occurrences per unit of time: for $t_1 < t_2$,

$$E\ N_i(t_1,t_2) = \lambda_i(t_2-t_1) \qquad \text{for } i = A,\ B \tag{2}$$

(this follows from Dobrushin's Lemma and Korolyuk's Theorem; see Leadbetter, 1968).

We now give an informal description of the measure K, adapted to the present context.

The measure K is defined on the Borel subsets of $\mathbb{R}$, and for $t_1 < t_2$, writing $K(t_1,t_2)$ for $K\{(t_1,t_2)\}$, we have

$$\begin{aligned} K(t_1,t_2) &= \frac{1}{\lambda_A} E\{N_A(t_1,t_2)\,|\,\text{a B point at } t=0\} \\ &\left(= \frac{1}{\lambda_B} E\{N_B(-t_2,-t_1)\,|\,\text{an A point at } t=0\}\right) \end{aligned} \tag{3}$$

Note that if N_A and N_B are independent, then

$$K(t_1,t_2) = t_2 - t_1\ , \tag{4}$$

regardless of the values λ_A and λ_B.

Ripley proposed the estimate of $K(t_1,t_2)$ given by

$$\hat{K}(t_1,t_2) = \frac{T}{N_A N_B} \sum_{i=1}^{n_B} \sum_{j=1}^{n_A} I\{A_j - B_i \in (t_1,t_2)\}, \tag{5}$$

where $I\{\cdot\}$ denotes the indicator function (actually, the estimate proposed by Ripley has an edge correction for points near the boundary of the period of observation; this edge correction will not concern us).

Previous work on the estimator $\hat{K}$ is concerned with spatial processes. The results center on using $\hat{K}$ to test that a single process in Poisson (Ripley, 1977; Chapter 8 of Ripley, 1981; Silverman, 1976) and on using $\hat{K}$ to test for independence of two processes (Lotwick and Silverman, 1982; Diggle and Milne, 1983).

In this paper we study the asymptotic properties of $\hat{K}(t_1,t_2)$. The main result is that under certain regularity conditions, as $n_B \to \infty$,

$$\sqrt{n_B}\,(\hat{K}(t_1,t_2) - K(t_1,t_2)) \xrightarrow{d} N(0,\sigma^2(t_1,t_2)), \tag{6}$$

where

$$\sigma^2(t_1,t_2) \text{ may be consistently estimated from the data.} \tag{7}$$

Besides providing the basis for a test of independence between N_A and N_B, (6) and (7) enable one to test whether or not $K(t_1,t_2)$ has changed in the experimental situation described earlier.

The cross-intensity function defined by

$$\lambda_{AB}(u) = \lim_{h_1,h_2 \to 0} \frac{1}{h_1 h_2} P\{N_A(u+t,u+t+h_1) > 0;\ N_B(t,t+h_2) > 0\} \tag{8}$$

is related to K by

$$K(t_1,t_2) = \frac{1}{\lambda_A \lambda_B} \int_{t_1}^{t_2} \lambda_{AB}(u)\,du\,. \tag{9}$$

Under the independence hypothesis, $\lambda_{AB} = \lambda_A \lambda_B$. Brillinger (1976) considered the random function

$$J_{AB}^T(u) = \sum_{i=1}^{n_B} \sum_{j=1}^{n_A} I\{A_j - B_i \in (u-h,u+h)\} \tag{10}$$

and showed that under suitable regularity, if $h \to 0$ and $T \to \infty$ in such a way that hT remains constant, then for $u_k^T \to u_k$, $|u_k^T - u_{k'}^T| > 2h$, $1 \le k < k' \le M$, $J_{AB}^T(u_k^T)$ are asymptotically independent Poisson random variables with means $2hT\ \lambda_{AB}(u_k)$, for $k = 1, \ldots, M$. Thus, $\hat{\lambda}_{AB}(u) = \frac{J_{AB}^T(u)}{2hT}$ can be used to estimate $\lambda_{AB}(u)$ at a finite number of points.

In practice one would graph the two functions $\hat{\lambda}_{AB}$ and $\hat{K}$ over a finite range, say $[-L,L]$ (i.e. graph $\hat{K}(-L,t)$ for $-L \le t \le L$). Although from a mathematical viewpoint λ_{AB} and K contain essentially the same information, the statistical properties of their estimates are quite different: estimation of λ_{AB} is akin to estimating a density, and from Brillinger's result the variance of $\hat{\lambda}_{AB}$ is of the order $\frac{1}{hT}$; on the other hand, estimation of K is akin to estimating a distribution function, and from (6), the variance of $\hat{K}$ is of the smaller order $\frac{1}{n_B}$. A graph of $\hat{\lambda}_{AB}$ may, however, indicate features (spikes, location of maxima and minima, etc.) that cannot be seen in the graph of $\hat{K}$. Clearly the two approaches are complementary.

2. ASYMPTOTIC DISTRIBUTION OF THE K-FUNCTION.

Let

$$U_{AB}(t_1,t_2) = E\{N_A(t_1,t_2)\,|\,\text{a } B \text{ point at } t=0\}. \tag{11}$$

We may estimate $U_{AB}(t_1,t_2)$ by

$$\hat{U}_{AB}(t_1,t_2) = \frac{1}{n_B}\sum_{i=1}^{n_B}\sum_{j=1}^{n_A} I\{A_j - B_i \in (t_1,t_2)\}\ . \tag{12}$$

Letting

$$\hat{\lambda}_i = \frac{n_i}{T} \qquad \text{for } i = A, B \tag{13}$$

we note that

$$K(t_1,t_2) = \frac{1}{\lambda_A} U_{AB}(t_1,t_2), \text{ and } \hat{K}(t_1,t_2) = \frac{1}{\hat{\lambda}_A} \hat{U}_{AB}(t_1,t_2). \tag{14}$$

To prove asymptotic normality of $\hat{K}$ (Theorem 2) we will prove joint asymptotic normality of $(\hat{U}_{AB}(t_1,t_2), \hat{\lambda}_A)$. We will in fact find it necessary to first prove joint asymptotic normality of $(\hat{U}_{AB}(t_1,t_2),\ \hat{\lambda}_A/\hat{\lambda}_B,\ 1/\hat{\lambda}_B)$. The delta-method (i.e. a first term Taylor expansion) applied to the function $f(x,y,z) = \frac{xz}{y}$ then yields the asymptotic normality of $\hat{K}(t_1,t_2)$. We also obtain the joint asymptotic normality of $(\hat{U}_{AB}(t_1,t_2),\ \hat{\lambda}_A,\ \hat{\lambda}_B)$ by applying the delta-method to the function $g(x,y,z) = (x,y/z,1/z)$.

We now need to give the statistical setting of our asymptotic investigation. The functions $U_{AB}(t_1,t_2)$ and $K(t_1,t_2)$ involve the notion of the Palm measure. That is for $\varepsilon > 0$, we consider the conditional distribution of the process (N_A,N_B) given that there is a B point in the interval $(0,\varepsilon)$, and take the limiting distribution of (N_A,N_B) as $\varepsilon \to 0$. Intuitively, this corresponds to selecting a B point "arbitrarily", and considering the process with that point labeled the origin. This notion is discussed for univariate processes by Leadbetter (1972) and for bivariate point processes by Wisniewski (1972). We will assume that the process is observed during a period of length T starting immediately after the occurrence of an "arbitrary" B point, say B_0 (thus, we will be working with the Palm measure). This mode of sampling is called semisynchronous sampling by Cox and Lewis (1972); see Wisniewski (1972) for some fundamental properties related to it. Also, for the sake of convenience, we will assume that the period of observation ends with a B point.

Let $F_{-\infty}^{B_0}$ denote the σ-field generated by the events

$$\{B_{k_1} \in (B_0+v_1,\ B_0+w_1), \ldots, B_{k_m} \in (B_0+v_m,\ B_0+w_m);$$
$$N_A(B_0+r_1,\ B_0+s_1) = h_1, \ldots, N_A(B_0+r_n,\ B_0+s_n) = h_n\}$$

for $v_i < w_i \le 0$, $k_i = -1, -2, \ldots$, $i = 1, \ldots, m$; $r_j < s_j \le 0$, $h_j = 0, 1, 2, \ldots$, $j = 1, \ldots, n$, and m and n nonnegative integers. For $u > 0$, let $F_{B_0+u}^{\infty}$ denote the σ-field generated by the events

$$\{B_{k_1} \in (B_0+v_1,\ B_0+w_1), \ldots, B_{k_m} \in (B_0+v_m,\ B_0+w_m)\ ;$$
$$N_A(B_0+r_1,\ B_0+s_1) = h_1, \ldots, N_A(B_0+r_n,\ B_0+s_n) = h_n\}$$

for $u \le v_i < w_i$, $k_i = 1, 2, \ldots$, $i = 1, \ldots, m$; $u \le r_j < s_j$, $h_j = 0, 1, 2, \ldots$, $j = 1, \ldots, n$, and m and n nonnegative integers.

Let

$$\alpha(u) = \sup\{|P(E_1 \cap E_2) - P(E_1)\, P(E_2)|\ ;\ E_1 \in F_{-\infty}^{B_0},\ E_2 \in F_{B_0+u}^{\infty}\} \tag{15}$$

If $\alpha(u) \to 0$ as $u \to \infty$, then the distant future is virtually independent of the past. We will actually need stronger conditions on $\alpha(\cdot)$.

Let $\beta > 0$, $\eta > 1$, $0 < \tau < 1$ be any constants satisfying

$$\left(\eta - \frac{\eta+1}{\beta+1}\right) \tau > 1 \tag{16}$$

Assumptions:

A1 $\displaystyle\int_0^\infty [\alpha(t)]^\tau\, t^\beta \, dt < \infty$

A2 $\displaystyle\sup_{-\infty<j<\infty} E\left\{[N_B(j,j+1)]^\eta \,\middle|\, \text{a B point at } t=0\right\} = M < \infty$

A3 $E\left\{[N_A(t_1,t_2)]^{4(1+\frac{\tau}{1-\tau})} \,\middle|\, \text{a B point at } t=0\right\} < \infty$

A4 $E\left[(B_1 - B_0)^{4(1+\frac{\tau}{1-\tau})}\right] < \infty$

A5 $E\left[N_A(B_0,B_1)\right]^{4(1+\frac{\tau}{1-\tau})} < \infty$

These assumptions are discussed towards the end of this section.

<u>THEOREM 1.</u> Assume A1 and A2. Let $U_{AB}(t_1,t_2)$, $\hat{U}_{AB}(t_1,t_2)$, and $\hat{\lambda}_i$ for $i = A, B$ be defined by (11), (12), and (13), respectively.

(i) Under A3, we have as $n_B \to \infty$

$$\sqrt{n_B}\,(\hat{U}_{AB}(t_1,t_2) - U_{AB}(t_1,t_2)) \xrightarrow{d} N(0,\gamma^2(t_1,t_2)) .$$

Furthermore, $\gamma^2(t_1,t_2)$ can be consistently estimated from the data.

(ii) Under A4 and A5 we have as $n_B \to \infty$

$$\sqrt{n_B}\,(\hat{\lambda}_A-\lambda_A,\ \hat{\lambda}_B-\lambda_B)' \xrightarrow{d} N(0,\Lambda).$$

Furthermore, Λ can be consistently estimated from the data.

(iii) Under A3 - A5 we have as $n_B \to \infty$

$$\sqrt{n_B}\,(\hat{U}_{AB}(t_1,t_2) - U_{AB}(t_1,t_2),\ \hat{\lambda}_A-\lambda_A,\ \hat{\lambda}_B-\lambda_B)' \xrightarrow{d} N(0,\Sigma(t_1,t_2)).$$

Furthermore, $\Sigma(t_1,t_2)$ can be consistently estimated from the data.

PROOF.

(i) We begin by showing asymptotic normality. Let U_i' and U_i be defined by

$$U_i' = \sum_{j=-\infty}^{\infty} I\{A_j \in (B_i+t_1, B_i+t_2)\}\ I\{B_0 \le A_j \le B_{n_B}\}$$

and

$$U_i = \sum_{j=-\infty}^{\infty} I\{A_j \in (B_i+t_1, B_i+t_2)\}.$$

Note that

$$\hat{U}_{AB}(t_1,t_2) = \frac{1}{n_B}\sum_{i=1}^{n_B} U_i' \quad . \tag{17}$$

It is clear that $\sum_{i=1}^{n_B} U_i - \sum_{i=1}^{n_B} U_i' = O_p(1)$. Thus, it suffices to prove the result with U_i's instead of U_i''s in (17). Observe that the sequence $\{U_i\}_{i=-\infty}^{\infty}$ is stationary, with mean $U_{AB}(t_1,t_2)$ and finite variance (by A3). The U_i's may be far from independent: for small k, U_i and U_{i+k} may be nearly identical. If, however, U_i and U_{i+k} are "nearly independent" for large k, then one can still hope to have a Central Limit Theorem effect. The proof consists of translating A1, the mixing condition on the point process, into a mixing condition on $\{U_i\}$ that allows the application of an appropriate central limit theorem for stationary sequences.

Let $\mu(k)$ be defined for $k = 1, 2, \ldots$ by

$$\mu(k) = \sup\{|P(E_1 \cap E_2) - P(E_1)P(E_2)|;\ E_1 \in \sigma(\ldots U_{-1}, U_0),\ E_2 \in \sigma(U_k, U_{k+1}, \ldots)\} \tag{18}$$

(Here, $\sigma(\ldots U_{-1}, U_0)$ denotes the σ-field generated by $\{\ldots, U_{-1}, U_0\}$, and similarly for $\sigma(U_k, U_{k+1}, \ldots)$). The function $\mu(\cdot)$ is called the mixing coefficient of the sequence $\{U_i\}$. Our goal is to prove that $\sum_{k=1}^{\infty} [\mu(k)]^{\tau} < \infty$. It will be more convenient however, to show instead that $\sum_{k=1}^{\infty} [\mu(2k)]^{\tau} < \infty$. The two conditions are equivalent since $\mu(\cdot)$ is nonincreasing.

Let $k \geq 1$ be fixed, let $E_1 \in \sigma(\ldots, U_{-k})$, $E_2 \in \sigma(U_k, \ldots)$, and consider $P(E_1 \cap E_2)$. Let

$$C_{-k} = \{B_{-k} - B_0 \leq -[k^{1/(\beta+1)}]\} \quad \text{and} \quad C_k = \{B_k - B_0 \geq [k^{1/(\beta+1)}]\} \quad .$$

We may write

$$P(E_1 \cap E_2) = P\{(E_1 \cap C_{-k}) \cap (E_2 \cap C_k)\} + P\{(E_1 \cap E_2) \cap (C_{-k}^c \cup C_k^c)\} \ , \tag{19}$$

where ^{c} denotes complementation.

Consider the first term on the right side of (19). For all large k, since $B_{-k} - B_0 \leq -\ k^{1/(\beta+1)}$ implies that $B_{-k} + t_2 \leq B_0$, we have $E_1 \cap C_{-k} \in F_{-\infty}^{B_0}$. Furthermore, $E_2 \cap C_k \in F^{\infty}_{B_0 + [k^{1/(\beta+1)}] + t_1}$. Therefore,

$$P\{(E_1 \cap C_{-k}) \cap (E_2 \cap C_k)\} \leq P(E_1)P(E_2) + \alpha([k^{1/(\beta+1)}] + t_1) \ . \tag{20}$$

The second term on the right side of (19) is obviously less than or equal to $P(C_{-k}^c) + P(C_k^c)$. These last two probabilities are dealt with in the same way. Consider $P(C_k^c)$. Observe that

$$P(C_k^c) \leq P\{\text{one of the intervals } (B_0+j, B_0+j+1\ ,\ j=0,1,\ldots,[k^{1/(\beta+1)}] - 1 \tag{21}$$
$$\text{has at least } [k^{\beta/(\beta+1)}] \text{ points}\}.$$

By A2, Chebyschev's inequality and Boole's inequality, the right side of (21) is less than or equal to $[k^{1/(\beta+1)}]\, M\, [k^{\beta/(\beta+1)}]^{-\eta}$. Combining this with (20) and handling the opposite inequality in s similar way, we obtain

$$\mu(2k) \le \alpha(k^{1/(\beta+1)}+t_1) + k^{1/(\beta+1)} M [k^{\beta/(\beta+1)}]^{-\eta}$$

Assumption A1 implies that

$$\sum_{k=1}^{\infty} \{\alpha(k^{1/(\beta+1)}+t_1)\}^{\tau} < \infty . \tag{22}$$

Combining (22) and (16) we obtain that $\sum_{k=1}^{\infty} [\mu(2k)]^{\tau} < \infty$, and hence that

$$\sum_{k=1}^{\infty} [\mu(k)]^{\tau} < \infty . \tag{23}$$

Assumption A3 implies in particular that

$$E\left[U_1^{2(1+\frac{\tau}{1-\tau})} \right] < \infty.$$

This, together with (23) allows us to apply Theorem 18.5.3 of Ibragimov and Linnik (1971) to conclude that for $\gamma^2(t_1,t_2)$ defined by

$$\gamma^2(t_1,t_2) = \mathrm{Var}\, U_0 + 2 \sum_{h=1}^{\infty} \mathrm{Cov}(U_0,U_h) \tag{24}$$

we have as $n_B \to \infty$

$$\sqrt{n_B}\, (\hat{U}_{AB}(t_1,t_2) - U_{AB}(t_1,t_2)) \xrightarrow{d} N(0,\gamma^2(t_1,t_2)).$$

Consider next $\gamma^2(t_1,t_2)$. Let $\nu_h = \mathrm{Cov}(U_0,U_h)$ and let $\hat{\nu}_h$ denote the sample covariance at lag h:

$$\hat{\nu}_h = \frac{\sum_{i=0}^{n_B-h-1} (U_i-\bar{U})(U_{i+h}-\bar{U})}{n_B-h} ,$$

where $\mathcal{D} = \mathcal{D}_{AB}(t_1,t_2)$. Defining the spectral density of $\{U_i\}$ by

$$f(\lambda) = \frac{1}{2\pi} \nu_0 + \frac{1}{\pi} \sum_{h=1}^{\infty} \nu_h \cos \lambda h,$$

we see that $\gamma^2(t_1,t_2) = 2\pi f(0)$. Thus, to estimate $\gamma^2(t_1,t_2)$ we have available the machinery for spectral density estimation from the time series literature. Frequently used estimates of $2\pi f(0)$ are of the form

$$\hat{\nu}_0 + 2 \sum_{h=1}^{s} c_h \hat{\nu}_h \tag{25}$$

where s and $c_1, \ldots, c_s$ are constants depending on n_B, with $s/n_B \to 0$ as $n_B \to \infty$, and $c_1 \geq c_2 \geq \ldots \geq c_s$. The choice of s and $\{c_h\}_{h=1}^{s}$ is not discussed here. For such a discussion, see any standard text on time series (e.g. Section 7.4 of Chatfield, 1980; Chapter 9 of Anderson, 1971). Consistency results for spectral density estimates have been established under conditions on $\{U_i\}$ that are not implied by Assumptions A1 - A5 (e.g. existence of all moments in Brillinger, 1975; $\{U_i\}$ is a linear process as in Anderson, 1971 and in Hannan, 1970). In the appendix, it is shown that there exists a consistent sequence of estimators of $\gamma^2(t_1,t_2)$.

(ii) Let $\xi_1, \xi_2 \in \mathbb{R}$, and let

$$X_i = \xi_1 N_A(B_{i-1},B_i) + \xi_2(B_i - B_{i-1}).$$

The sequence $\{X_i\}_{i=-\infty}^{\infty}$ is stationary, and if $\pi(\cdot)$ denotes its mixing coefficient, it is clear that (23) holds for $\pi(\cdot)$ as well. This gives a central limit theorem for $\{X_i\}$, and by the Cramér-Wold device we have that

$$\sqrt{n_B} \left(\frac{n_A}{n_B} - E\, N_A(B_0,B_1),\ \frac{T}{n_B} - E(B_1 - B_0)\right)'$$

is asymptotically normal with mean 0, and a covariance matrix which can be consistently estimated as in Part (i). Applying the delta method with the function $f(x,y) = (\frac{x}{y}, \frac{1}{y})$ we obtain that

$$\sqrt{n_B}\left\{\frac{n_A}{T}-\frac{E\,N_A(B_0,B_1)}{E(B_1-B_0)},\ \frac{n_B}{T}-\frac{1}{E(B_1-B_0)}\right\}'$$

is asymptotically normal, and it is simple to argue that $E\,N_A(B_0,B_1)/E(B_1-B_0) = \lambda_A$ and $\{E(B_1-B_0)\}^{-1} = \lambda_B$.

(iii) The proof of (iii) is similar to that of (ii) and is omitted.

<u>THEOREM 2.</u> Let $K(t_1,t_2)$ and $\hat{K}(t_1,t_2)$ be defined by (3) and (5), respectively, and assume A1 - A5. Then, as $n_B \to \infty$

$$\sqrt{n_B}\,(\hat{K}(t_1,t_2) - K(t_1,t_2) \xrightarrow{d} N(0,\sigma^2(t_1,t_2)),$$

where $\sigma^2(t_1,t_2)$ can be consistently estimated from the data.

Results giving the asymptotic normality of estimates of λ_A and λ_B (under varying sets of assumptions) already exist in the literature; see e.g. Theorem 8.6 of Daley and Vere-Jones (1972). It was necessary to establish joint asymptotic normality of $\hat{\lambda}_A$ and $\hat{U}_{AB}(t_1,t_2)$ in order to obtain asymptotic normality of $\hat{K}(t_1,t_2)$.

Any theorem giving asymptotic normality of the normalized partial sums of a stationary sequence $\{T_i\}$ must assume a moment condition on T_1 and also a mixing condition on $\{T_i\}$. In general, weakening of the moment condition must be compensated by strengthening of the mixing condition, and vice versa. Assumptions A3, A4, and A5 provide moment conditions on the sequences $\{U_i\}$, $\{N_A(B_{i-1},B_i)\}$, and $\{(B_i-B_{i-1})\}$, respectively. Assumption A2 insures that the B process "moves along" rapidly enough so that A1, the mixing condition imposed on the point process, translates into a mixing condition for the sequences $\{U_i\}$, $\{N_A(B_{i-1},B_i)\}$ and $\{(B_i-B_{i-1})\}$. Relationship (16) describes in a technical way the interplay between the mixing rate on the point process and the moment condition on the sequences $\{U_i\}$, $\{N_A(B_{i-1},B_i)\}$, and $\{(B_i-B_{i-1})\}$.

The conditions assumed by Brillinger (1976) neither imply nor are implied by A1 - A5 of the present paper. Brillinger assumes a mixing condition on the bivariate point process and also that the "second order moments" $\lambda_{ij}(\cdot)$ $(i,j = A,B)$ exist and are continuous (he also assumes existence and continuity of the "third and fourth order moments"; see equation (2.2) of his paper). This condition on $\lambda_{AB}(\cdot)$ is not satisfied by the following process: N_B is a Poisson process, and N_A is N_B shifted to the right by 1 unit. In this case, $\lambda_{AB}(1) = \infty$. This process does however satisfy A1 - A5. Conversely, it is easy to find processes (N_A, N_B) satisfying all of Brillinger's conditions, but not those of the present paper. Perhaps the simplest example is the following. Let N_A and N_B be independent, N_A being a Poisson process, and N_B being an equilibrium renewal process on $(-\infty,\infty)$ (for a definition and a construction see pp. 517-19 of Karlin and Taylor, 1975) with interarrival distribution having a first moment but no second moment. Then A4 is violated, and it is not difficult to check that this process satisfies all of Brillinger's conditions.

3. DISCUSSION.

The contributions of this paper are two-fold: proof of asymptotic normality of $\hat{K}(t_1,t_2)$ and a method for estimating the asymptotic variance $\sigma^2(t_1,t_2)$, enabling the construction of asymptotic confidence intervals for $K(t_1,t_2)$, for fixed values of t_1 and t_2.

The function $K(\cdot,\cdot)$ will usually be of interest over a continuum of values, say $-L \le t_1 < t_2 \le L$, where L is some number much smaller than T. One can plot $\hat{K}(-L,t)$ for $-L \le t \le L$ or, what is sometimes more useful, plot $K(t-\frac{d}{2}, t+\frac{d}{2})$ for $-L + \frac{d}{2} \le t \le L - \frac{d}{2}$. Here, d is some small number representing the experimenter's guess at the duration or likely duration of the effect of a B point on the A process. The function $K(t-\frac{d}{2}, t+\frac{d}{2})$ is identically equal to d if N_A and N_B are independent.

We may form the bands

$$\hat{K}(-L,t) \pm z^{(\alpha/2)}\ \hat{\sigma}(-L,t)/\sqrt{n_B} \qquad -L \le t \le L$$

and

$$\hat{K}(t-\tfrac{d}{2}, t+\tfrac{d}{2}) \pm z^{(\alpha/2)}\ \hat{\sigma}(t-\tfrac{d}{2}, t+\tfrac{d}{2})/\sqrt{n_B} \qquad -L + \tfrac{d}{2} \le t \le L - \tfrac{d}{2}$$

where $\hat{\sigma}(t_1,t_2)$ is an estimate of $\sigma(t_1,t_2)$, and $z^{(\alpha/2)}$ is the upper $\frac{\alpha}{2} \cdot 100$ percentile point of a standard normal variable. These bands of course are not simultaneous confidence bands. To form simultaneous confidence bands one would need to carry out two distinct steps:

(i) establish weak convergence of the processes

$$V_{n_B}(t) = \sqrt{n_B}(\hat{K}(-L,t) - K(-L,t))$$

and

$$W_{n_B}(t) = \sqrt{n_B}(\hat{K}(t-\tfrac{d}{2}, t+\tfrac{d}{2}) - K(t-\tfrac{d}{2}, t+\tfrac{d}{2}))$$

to Gaussian processes $V(t)$ and $W(t)$, respectively.

(ii) obtain $v^{(\alpha)}$ and $w^{(\alpha)}$, the upper $\alpha \cdot 100$ percentile points of $\sup_{-L\le t\le L} |V(t)|$ and $\sup_{-L+\frac{d}{2}\le t\le L-\frac{d}{2}} |W(t)|$, respectively.

The bands

$$\hat{K}(-L,t) \pm v^{(\alpha)}/\sqrt{n_B} \qquad -L \le t \le L$$

and

$$\hat{K}(t-\tfrac{d}{2}, t+\tfrac{d}{2}) \pm w^{(\alpha)}/\sqrt{n_B} \qquad -L + \tfrac{d}{2} \le t \le L - \tfrac{d}{2}$$

are then asymptotic simultaneous confidence bands.

A proof of weak convergence appears extremely difficult. Although desirable from a theoretical point of view, weak convergence is not useful statistically unless the distribution of the supremum of the absolute value of the limiting process can be obtained. In general this is a very difficult problem even if the Gaussian process is stationary (see Cressie and Davis, 1981). In the case of two independent Poisson processes weak convergence of $U_{n_B}(\cdot)$ and $W_{n_B}(\cdot)$ can be established. $W_{n_B}(\cdot)$ converges weakly to a stationary Gaussian process, the distribution of the supremum of which is known. Since this distribution depends only on the two rates λ_A and λ_B it can be estimated directly from the data. These results will be reported in a future paper.

APPENDIX

In this appendix it is shown that under Assumptions A1 - A5, the asymptotic variances in Theorem 1 can be consistently estimated. We discuss only $\gamma^2(t_1,t_2)$; the other asymptotic variances are handled in the same way. We emphasize that in applications simple estimates such as those given in (25) would suffice.

Consider the sequence $\{U_i\}$. For an arbitrary integer n, let

$$Y_1 = \frac{\sum_{i=1}^{n} U_i}{\sqrt{n}} . \tag{26}$$

It is well-known (and easy to see) that

$$\text{Var } Y_1 = \sum_{|\ell|<n} (1-\frac{\ell}{n})\, \nu_\ell = \gamma^2(t_1,t_2) + \varepsilon(n), \text{ where } \varepsilon(n) \to 0 \text{ as } n \to \infty. \tag{27}$$

If $Y_1, Y_2, \ldots, Y_h$ are i.i.d., then

$$E\left[\frac{\sum_{i=1}^{h} (Y_i-EY_i)^2}{h} - \gamma^2(t_1,t_2)\right]^2 = \varepsilon^2(n) + \frac{1}{h}\text{Var}(Y_1-EY_1)^2 . \tag{28}$$

By A3, $\text{Var}(Y_1-EY_1)^2$ is finite.

The idea is to divide the sequence $\{U_i\}_{i=1}^{n_B}$ into h blocks, each of size n, with the blocks separated by a distance f, and to let Y_j be the normalized sum of the U's in the j^{th} block, as in (26), for j = 1, 2, ..., h; see the diagram.

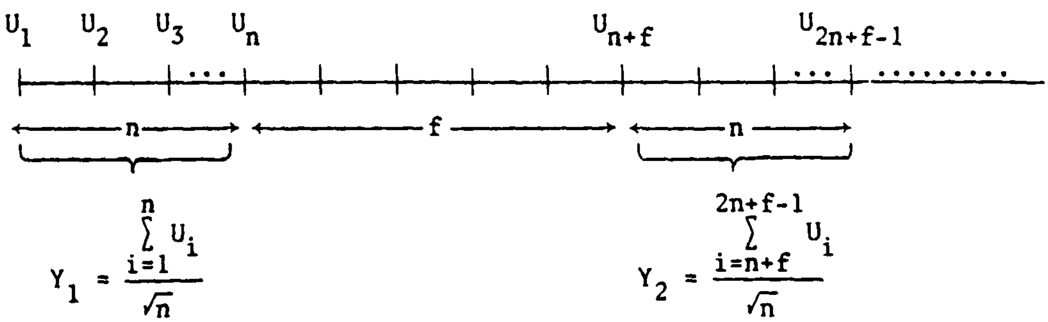

Diagram to describe $\{Y_i\}_{i=1}^{h}$.

Then

$$T = \frac{1}{h} \sum_{i=1}^{h} (Y_i - \bar{Y})^2, \quad \text{where } \bar{Y} = \frac{1}{h} \sum_{i=1}^{h} Y_i \tag{29}$$

can be used to estimate $\gamma^2(t_1,t_2)$. The proof consists basically of making the following heuristics rigorous. If f is large, then by the mixing condition on the sequence $\{U_i\}$ (see (23)) the Y_j's will be nearly independent, so that equation (28) will be approximately true; if n is large, then $\varepsilon(n)$ in (27) and (28) will be small, and if h is large then the second term on the right side of (28) will be small (note however that $\mathrm{Var}(Y_1-EY_1)^2$ depends on n). Also, the effect of substituting $\bar{Y}$ for EY_i in the left side of (28) should be negligible. Thus, T converges to $\gamma^2(t_1,t_2)$ in probability.

The numbers $n = n(n_B)$, $h = h(n_B)$ and $f = f(n_B)$ of course will depend on n_B, but in a way that is to be determined later. The dependence on n_B will be suppressed in the notation for convenience.

We begin by decomposing T given by (29) as

$$T = \frac{1}{h} \sum_{i=1}^{h} (Y_i - EY_i)^2 - (\bar{Y} - EY_1)^2 . \tag{30}$$

We will show that the first term in the right side of (30) converges to $\gamma^2(t_1,t_2)$ in quadratic mean (hence in probability), and that the second term in the right side of (30) converges to 0 in mean (hence in probability).

Considering first $\frac{1}{h} \sum_{i=1}^{h} (Y_i - EY_i)^2$, we have by (27)

$$E\left[\frac{1}{h} \sum_{i=1}^{h} (Y_i - EY_i)^2 - \gamma^2(t_1,t_2)\right]^2 = \varepsilon^2(n) + \mathrm{Var}\left\{\frac{1}{h} \sum_{i=1}^{h} (Y_i - EY_i)^2\right\} . \tag{31}$$

Writing

$$V = \frac{1}{h} \mathrm{Var}(Y_1 - EY_1)^2 \quad \text{and} \quad C = \frac{1}{h^2} \sum_{i \neq j} \sum \mathrm{Cov}((Y_i - EY_i)^2, (Y_j - EY_j)^2) \tag{32}$$

we have

$$\mathrm{Var}\left\{\frac{1}{h}\sum_{i=1}^{h}(Y_i-EY_i)^2\right\} = V + C\,, \tag{33}$$

and we examine V and C separately, beginning with V.

Bounding the variance of a random variable by its second moment and using (26), we may write

$$V \le \frac{1}{hn^2}\,E\left[\sum_{i=1}^{n} U_i\right]^4 . \tag{34}$$

The Minkowski inequality gives

$$V \le \frac{1}{hn^2}\,n^4 EU_1^4 = \frac{1}{h}\,O(n^2) \tag{35}$$

by A3.

We now consider C in (33). If $i < j$ are fixed and U_s and U_t are summands of $\sqrt{n}\,Y_i$ and $\sqrt{n}\,Y_j$, respectively, then $t \ge s + f$. Therefore, by Theorem 17.2.2 of Ibragimov and Linnik (1971)

$$|\mathrm{Cov}((Y_i-EY_i)^2,\ (Y_j-EY_j)^2)| \le [\mu(f)]^\tau \left[4 + 6\left\{E|Y_1-EY_1|^{4(1+\frac{\tau}{1-\tau})}\right\}^{\frac{1-\tau}{2}}\right], \tag{36}$$

with $\mu(\cdot)$ defined by (18). Now by the Minkowski inequality,

$$E|Y_1-EY_1|^{4(1+\frac{\tau}{1-\tau})} \le n^{2(1+\frac{\tau}{1-\tau})}\,E|R_1-ER_1|^{4(1+\frac{\tau}{1-\tau})}\quad , \tag{37}$$

and therefore, (36) can be rewritten as

$$|\mathrm{Cov}((Y_i-EY_i)^2,\ (Y_j-EY_j)^2)| = [\mu(f)]^\tau O(n). \tag{38}$$

Since the sequence $\{[\mu(f)]^\tau\}_{f=1}^{\infty}$ is nonincreasing, we may write

$$[\mu(f)]^{\tau} \le \frac{2}{f} \sum_{k=[\frac{f}{2}]} [\mu(f)]^{\tau} = o(\frac{1}{f}) \quad \text{as } f \to \infty \tag{39}$$

by (23). We conclude that for C defined by (32),

$$|C| = o(\frac{1}{f})n \quad \text{as } f \to \infty. \tag{40}$$

Consider now $(\bar{Y}-EY_1)^2$ in (30). We have

$$E(\bar{Y}-EY_1)^2 = \frac{1}{h} E(Y_1-EY_1)^2 + \frac{1}{h^2} \sum_{i \ne j}\sum E(Y_i-EY_i)(Y_j-EY_j). \tag{41}$$

Combining this with (27) and the argument used to produce (40), we obtain

$$E(\bar{Y}-EY_1)^2 = \frac{1}{h}(\gamma^2(t_1,t_2) + \varepsilon(n)) + o(\frac{1}{f})n^{\frac{1}{2}} \quad \text{as } f \to \infty. \tag{42}$$

Now let $n = n(n_B)$, $h = h(n_B)$ and $f = f(n_B)$ be sequences satisfying the conditions

$$n \to \infty, \quad \frac{n}{f} \text{ remains bounded,} \quad \text{and } \frac{n^2}{h} \to 0 \text{ as } n_B \to \infty. \tag{43}$$

Such sequences are very easy to construct. Then the right sides of equations (35), (40) and (42) converge to 0 as $n_B \to \infty$. This completes the proof.

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To assess the dependence structure in a stationary bivariate point process the second-order distribution (Ripley 1976, 1977) can be very useful. We prove that the natural estimates of this distribution, based on a realization $A_1 < A_2 < \ldots < A_{n_A}$, $B_1 < B_2 < \ldots < B_{n_B}$ are asymptotically normal, and we present a method for constructing approximate confidence intervals for this distribution.