AD-A207 881

CONDITIONAL DEPENDENCE

Final Report

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May 1989

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No 0704-0188

1a REPORT SECURITY CLASSIFICATION: Unclassified	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S): Biometric Lab Report #89-1	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION: University of Illinois; 6b OFFICE SYMBOL (If applicable)	7a NAME OF MONITORING ORGANIZATION: Personnel & Training Research Programs Office of Naval Research (Code 1142PT)
6c ADDRESS (City, State, and ZIP Code): Illinois State Psychiatric Institute 529W, 1601 West Taylor Street, Chicago, Illinois 60612	7b ADDRESS (City, State, and ZIP Code): 800 N. Quincy Street, Arlington, Virginia 22217-5000
8a NAME OF FUNDING/SPONSORING ORGANIZATION; 8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: N 00014-85-K-0586
8c ADDRESS (City, State, and ZIP Code)	10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO 61153N; PROJECT NO RR04206; TASK NO RR04204-01; WORK UNIT ACCESSION NO NR 150-538

11 TITLE (Include Security Classification)

Conditional Dependence

12 PERSONAL AUTHOR(S)

Robert D. Gibbons, R. Darrell Bock and Donald Hedeker

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Final	FROM 8/01/85 TO 8/01/88	March 15, 1989	21

16 SUPPLEMENTARY NOTATION

Submitted for publication in Psychometrika

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	
05	09		

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

The probability of m correlated random variables drawn from a multivariate normal distribution being non-negative is:

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \Phi_n(x_1, x_2, \ldots, x_n)\, \partial x_1, \partial x_2, \ldots, \partial x_n.$$

Exact results for this probability integral are unavailable for $m > 3$. Approximations for higher dimensional problems have generally yielded poor results except for *special cases*, such as compound symmetry, which is of limited value in practice. The purpose of this paper is to present a general approximation of this probability integral. The algorithm developed here is computationally tractable for $m = 50$ and accurate for very general correlational structures. The performance of this algorithm is compared to results based on Clark's (1961) original approximation, Gaussian quadrature formulae, and Monte Carlo simulation methods. Application of this approximation to problems of conditional dependence in IRT estimation problems is discussed.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION: Unclassified
22a NAME OF RESPONSIBLE INDIVIDUAL: Dr. Charles Davis	22b TELEPHONE (Include Area Code): 202/696-4046; 22c OFFICE SYMBOL: ONR 1142PT

1 INTRODUCTION

To varying degrees, all tests of ability contain items that violate the assumption of conditional independence on which much of present item response theory (IRT) is based. If the association between pairs of items cannot be explained entirely by their relationship with the underlying ability, conventional methods for evaluating the likelihood of latent structure models become inappropriate and may produce biased estimates of item-parameters and corresponding estimates of ability. Under the assumption of conditional independence, the probability of subject i responding in pattern $\boldsymbol{x}_i = [x_{i1}, x_{i2}, \ldots, x_{in}]$, conditional on ability θ_i is

$$P(\boldsymbol{x} = \boldsymbol{x}_i \mid \theta_i) = \prod_{j=1}^{n} [p_j(\theta_i)]^{x_{ij}} [1 - p_j(\theta_i)]^{1-x_{ij}}, \tag{1}$$

where x_{ij} is the jth item score for subject i ($x_{ij} = 1$ if correct, otherwise $x_{ij} = 0$), and $p_j(\theta)$ is the item response function (IRF) that expresses the probability that a subject with ability θ_i will respond correctly to item j. The IRF may be obtained from any one of several item-response models (*e.g.*, one, two, or three parameter logistic or normal ogive models). If the ability θ under study does not account for all of the association between items, the assumption of conditional independence is not valid and the conditional probability cannot be expressed simply as a continued product of the individual item-response probabilities.

Three general approaches to the problem of failure of conditional independence are available. First, we can simply ignore the dependence and risk obtaining bias in our estimates of item parameters and ability and corresponding estimates of precision. Stout (1987) has shown that if, for a fixed level of ability θ, the average item covariance is "negligible" as test length grows, that is,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{cov}(x_i, x_j \mid \theta) \to 0, \tag{2}$$

then the test is "essentially independent" and the usual results for the locally independent case apply (*i.e.*, equation 1). As rigorous tests for essential independence become available, this will provide the means for judging one's confidence in selecting this alternative.

Second, if we saturate the latent space with additional dimensions, we can at some point achieve a conditionally independent solution. If the required dimensionality is small, say 2 or 3, this may be feasible using the approach described by Bock and Aitkin (1981), where the integration of the K-dimensional ability distribution $g(\boldsymbol{\theta}) \sim N(0, 1)$ is approximated using Gaussian quadrature formulae. There are two potential drawbacks to this approach. First, if there are numerous "method" related factors and a single primary ability factor, the dimensionality required to bring about conditional independence may be too high for practical purposes. Second, if there is a single method related factor and a single primary ability, it may be difficult to find a two-dimensional solution that preserves this structure. Indeed, the orthogonal solution will often divide the items based on the method related factor.

The third approach to this problem is to formulate a generalized IRT model of dependence in which the simplicity of a unidimensional model is retained. In the generalized model the conditional response-pattern probability is of the form $P(\boldsymbol{x} = \boldsymbol{x}_i \mid \theta_i, \Sigma)$, where Σ is an $n \times n$ symmetric inter-item covariance matrix. It should be clear that $P(\boldsymbol{x} = \boldsymbol{x}_i \mid \theta_i, \Sigma)$ is equal to the right side of (1) only when Σ is diagonal, which is only true under conditional independence. When Σ has nonzero off-diagonal elements, $P(\boldsymbol{x} = \boldsymbol{x}_i \mid \theta_i, \Sigma)$ becomes difficult to evaluate, and, in fact, no closed form expression has been obtained for general Σ beyond $n = 3$. The result of our preliminary studies has been the development of an approximation of $P(\boldsymbol{x} = \boldsymbol{x}_i \mid \theta_i, \Sigma)$ for general Σ and $n \leq 50$. The focus of our current work is on the development of a unidimensional IRT model of dependence that incorporates these generalized probability estimates.

2 THEORETICAL BASIS FOR MODELS OF DEPENDENCE

Experimentally, failure of conditional independence can result from two general conditions: item content and item presentation.

2.1 Item Content

If different items require different abilities, aptitudes or cognitive processes, then the IRF must depend not only on the primary ability, but on the secondary cognitive skills or method related factors as well. As a result, association among the residuals is introduced and the assumption of conditional independence of a unidimensional IRT model is no longer tenable.

For example, assume that the ability space is two-dimensional, where θ_1 represents a primary ability and θ_2 a secondary skill that is required for success on some of the items. With respect to item j, the underlying response process (response-strength variable) is:

$$y_j = \lambda_{j1}\theta_1 + \lambda_{j2}\theta_2 + \varepsilon_j \tag{3}$$

Conditional on θ_1, the variance of y_j is:

$$\begin{aligned} \mathcal{V}(y_j \mid \theta_1) &= \mathcal{V}(\varepsilon_j + \lambda_{j2}\theta_2) \\ &= \mathcal{V}(\varepsilon_j) + \mathcal{V}(\lambda_{j2}\theta_2) \\ &= 1 - \lambda_{j1}^2 - \lambda_{j2}^2 + \lambda_{j2}^2\mathcal{V}(\theta_2) \\ &= 1 - \lambda_{j1}^2 \end{aligned} \tag{4}$$

This result follows because $\mathcal{V}(\theta_1) = \mathcal{V}(\theta_2) = 1$. Conditional on θ_1, the residual inter-item covariances are:

$$\text{cov}(y_i, y_j \mid \theta_1) = \lambda_{i2}\lambda_{j2} \tag{5}$$

or expressed as a residual correlation,

$$r_{(y_i, y_j \mid \theta_1)} = \frac{\lambda_{i2}\lambda_{j2}}{\sqrt{1-\lambda_{i1}^2}\sqrt{1-\lambda_{j1}^2}} \tag{6}$$

2.2 Item Presentation

In terms of item presentation, conditional independence implies, that for a fixed level of ability, the probability of a correct response to item j is independent of the examinee's performance on all other items. However, in the context of adaptive testing, the presentation of item $j+1$, is in fact, conditional on the success or failure on item j, where the items $1, 2, \ldots, j+1$ are ordered in terms of difficulty, presumably along a single ability dimension of interest. In practical terms, this assumption implies that the individual item response probabilities for subjects with the same ability would be identical for all possible orderings of the items on the test.

When item presentation is the result of a random process, the assumption of order invariance may well be reasonable. However, when item presentation is systematic and based on either progressive linear increases or decreases in difficulty (conditional on a provisional estimate of examinee ability), it seems likely that the conditional item-response probabilities would be affected. For example, if the initial ability estimate is low, several items of increasing difficulty may be presented. If a sequence of similar problems is presented, the probability of a correct response may be both a function of the examinee's primary ability and in addition the subject's skill at recognizing the sequence, level of expectation, and perhaps even learning.

At the very least, we must admit the possibility that random orderings and systematic orderings of items have some impact on the response process and corresponding probability estimates. As an analogy, consider an auditory stimulation experiment in which the lower bound of the subject's hearing level is to be established. If the volume of the tone is presented in uniform steps from loud to soft, the resulting estimate of hearing level and corresponding estimate of precision (if replicated) will clearly be different than if the volume of the tone is presented in a random order. Clearly, the knowledge of the presentation pattern and expectation of the degree of difficulty of the next discrimination task (*i.e.*, tone versus noise) will contribute to the likelihood that the subject will make the correct response. As will be demonstrated, the consequence of ignoring this residual association will primarily be an underestimate of the posterior standard deviation. Since the convergence of an adaptive testing session is often based on the precision of the estimated ability, on average, the result will be a premature conclusion of the testing session, and false sense of certainty.

3 Approximating Multivariate Normal Orthant Probabilities

In this section, we describe a general approximation for multivariate normal orthant probabilities. Using a threshold argument, we show how the approximation can be used to estimate the probability of any of the 2^n possible binary response patterns realized in a testing session.

3.1 The Clark Algorithm

Based on earlier work by Clark (1961), Gibbons, Bock and Hedeker (1987) developed a very general approximation to the probability that n correlated random variables drawn from a multivariate normal distribution are jointly non-negative; that is;

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \Phi_n(x_1, x_2, \ldots, x_n)\, dx_1, dx_2, \ldots, dx_n. \tag{7}$$

Assuming that the underlying response process is multivariate normal and that the correct responses to a series of items on a test are the result of exceeding a threshold on one or more "latent" continua $\boldsymbol{\theta}$, our modified Clark algorithm can provide probability estimates of any of the 2^n possible binary response patterns without any restriction on the form of the inter-item covariances after conditioning on θ. An overview of Clark's original formulae and our modification of the algorithm is now presented.

Designating positive directions 1 and negative directions 0, we may represent the probability of the positive orthant of an n-variate distribution by $P(1,1,\ldots,1)$, that of the negative orthant by $P(0,0,\ldots,0)$, and that of any one of the other $2^n - 2$ orthants by inserting the appropriate pattern of 1's and 0's. The Clark algorithm provides a computing approximate for any orthant of a multivariate normal distribution with arbitrary vector mean and covariance matrix. Clark (1961) derives the following formulas.

Let any three successive components from an n-variate vector, $\boldsymbol{y}_1$, be distributed:

$$\begin{bmatrix} y_i \\ y_{i+1} \\ y_{i+2} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_i \\ \mu_{i+1} \\ \mu_{i+2} \end{bmatrix}, \begin{bmatrix} \sigma_i^2 & & \\ \sigma_i \sigma_{i+1} \rho_{i,i+1} & \sigma_{i+1}^2 & \\ \sigma_i \sigma_{i+2} \rho_{i,i+2} & \sigma_{i+1} \sigma_{i+2} \rho_{i+1,i+2} & \sigma_{i+2}^2 \end{bmatrix} \right) \tag{8}$$

Let $\bar{y}_i = \max(y_i) = y_i$, and compute the probability that $y_{i+1} > \bar{y}_i$ as follows:

$$\begin{aligned} \text{set} \qquad & z_{i+1} = (\mu_i - \mu_{i+1})/\zeta_{i+1}, \\ \text{where} \qquad & \zeta_{i+1}^2 = \sigma_i^2 + \sigma_{i+1}^2 - 2\sigma_i \sigma_{i+1} \rho_{i,i+1}. \\ \text{Then} \quad P(y_{i+1} > \bar{y}) & = P(y_{i+1} - \bar{y} > 0) \\ & = \Phi(-z_{i+1}) \end{aligned} \tag{9}$$

the value of the univariate normal distribution function at the standard deviate $-z_{i+1}$.

Now let $\tilde{y}_{i+1} = \max(y_i, y_{i+1})$ and assume (as an approximation) that $(y_{i+2}, \tilde{y}_{i+1})$ is bivariate normal with means,

$$\begin{aligned} \mu(y_{i+2}) &= \mathcal{E}(y_{i+2}) = \mu_{i+2} \\ \mu(\tilde{y}_{i+1}) &= \mathcal{E}(\tilde{y}_{i+1}) = \mu_i\Phi(z_{i+1}) + \mu_{i+1}\Phi(-z_{i+1}) + \zeta_{i+1}\phi(z_{i+1}), \end{aligned} \tag{10}$$

variances

$$\begin{aligned} \sigma^2(y_{i+2}) &= \mathcal{E}(y_{i+2}^2) - \mathcal{E}^2(y_{i+2}) = \sigma_{i+2}^2, \\ \sigma^2(\tilde{y}_{i+1}) &= \mathcal{E}(\tilde{y}_{i+1}^2) - \mathcal{E}^2(\tilde{y}_{i+1}), \end{aligned}$$

where

$$\mathcal{E}(\tilde{y}_{i+1}^2) = (\mu_i^2+\sigma_i^2)\Phi(z_{i+1})+(\mu_{i+1}^2+\sigma_{i+1}^2)\Phi(-z_{i+1})+(\mu_i+\mu_{i+1})\zeta_{i+1}\phi(z_{i+1}), \tag{11}$$

and correlation

$$\rho(\tilde{y}_{i+1}, y_{i+2}) = \frac{\sigma_i\rho_{i,i+2}\Phi(z_{i+1}) + \sigma_{i+1}\rho_{i+1,i+2}\Phi(-z_{i+1})}{\sigma(\tilde{y}_{i+1})}. \tag{12}$$

Then,

$$P(y_{i+2} = \max(y_i, y_{i+1}, y_{i+2})) = P((y_{i+2} - y_{i+1} > 0) \cap (y_{i+2} - y_i > 0)) \tag{13}$$

is approximated by

$$\begin{aligned} P(y_{i+2} > \tilde{y}_{i+1}) &= P(y_{i+2} - \tilde{y}_{i+1} > 0) \\ &= \Phi\left(\frac{\mu_{i+2} - \mu(\tilde{y}_{i+1})}{\sqrt{\sigma_{i+2}^2 + \sigma^2(\tilde{y}_{i+1}) - 2\sigma_{i+2}\sigma(\tilde{y}_{i+1})\rho(\tilde{y}_{i+1}, y_{i+2})}}\right) \end{aligned}$$

Assuming as a working approximation that $\tilde{y}_{i+1}$ is normally distributed with the above mean and variance, we may therefore proceed, recursively from $i = 1$ to $i = n - 1$, where y_{n+1} is an independent dummy variate with mean zero and variance zero (i.e. $y_{n+1} = 0$). Then,

$$\begin{aligned} & P[y_{n+1} = \max(y_1, y_2, \ldots, y_{n+1})] \\ &= P[(y_{n+1} - y_1 > 0) \cap (y_{n+1} - y_2 > 0) \cap \ldots \cap (y_{n+1} - y_n > 0)] \\ &= P[(-y_1 > 0) \cap (-y_2 > 0) \cap \ldots \cap (-y_n > 0)] \end{aligned}$$

approximates the probability of the negative orthant of the specified multivariate normal distribution. The probability of any other orthant can be obtained by reversing the signs of the variates corresponding to 1's in the orthant pattern.

3.2 The Modified Clark Algorithm

In an earlier paper Gibbons and Bock (1987) noted that the accuracy of the Clark approximation diminishes with increasing magnitude of the correlations. If we apply the Clark approximation directly to estimates of inter-item correlations, it will generally yield biased results due to the size of correlations. This is true regardless of whether the correlation matrix exhibits the property of conditional independence. Alternatively, if we examine the residual inter-item correlation matrix at fixed points on the ability scale, we will observe the identity matrix for conditionally independent solutions or small residual correlations for those item pairs that are conditionally dependent. In light of this, we evaluate the response function at several fixed points on the ability scale using Gauss-Hermite quadrature, and correct these estimates using the Clark algorithm. These corrections depend only on the residual inter-item correlations, which in practice should be quite small. The modified Clark algorithm proceeds as follows.

Step 1 Obtain a factor solution of dimension K, using full information factor analysis for binary data (Bock and Aitkin, 1981; Bock, Gibbons and Muraki, 1988).

Step 2 Using the estimated factor loadings for dimensions $2 \ldots K$, compute the estimated residual correlation matrix $\mathbf{R}^*$; that is, the residual correlations among the n items conditioning on the primary ability dimension θ_1. For example, when $K = 2$,

$$r_{ij} = \frac{\lambda_{i2}\lambda_{j2}}{\sqrt{1-\lambda_{i1}^2}\sqrt{1-\lambda_{j1}^2}} \tag{14}$$

Step 3 Given the previous values of item thresholds γ_j and primary item factor loadings λ_{j1} compute the invariant item parameters a_j (slope) and c_j (intercept) as:

$$a_j = \lambda_{j1}/\sqrt{1-\lambda_{j1}^2}$$

and

$$c_j = -\gamma_j/\sqrt{1-\lambda_{j1}^2}$$

Step 5 At each point on the ability dimension (i.e. at each quadrature node X_k) compute the value of the response function for each item as:

$$z_{jk} = c_j + a_j X_k \tag{15}$$

where X_k are the nodes of the Gauss-Hermite polynomial (see Stroud and Sechrest, 1966).

Step 6 At each quadrature point, substitute the values of z_{jk} for the mean vector $\boldsymbol{\mu}$ and $\mathbf{R}^*$ for the covariance matrix Σ and compute the Clark approximated probability

$C_l(X_k)$. Accumulating these probabilities over all quadrature nodes for a given response pattern ($\boldsymbol{x}_l$) yields the desired marginal probability estimate

$$\begin{aligned} h(\boldsymbol{x}_l) &= \int_{-\infty}^{\infty} C_l(\theta)\phi(\theta)\partial(\theta) \\ &= \sum_{k=1}^{q} C_l(X_k)A(X_k) \end{aligned}$$

where $A(X_k)$ is the corresponding weight at quadrature node X_k.

We note that, in practice, the effect of assuming normality of the maximum of two jointly normal variables, overestimates the probability in the tail of the distribution. To correct this, we apply an empirically based correction factor to probability estimates that are less than 0.04;

$$C_l^*(X_k) = C_l(X_k)^{.88-.17[\log_{10}(C_l(X_k))]} \tag{16}$$

This correction factor has been found to provide the necessary adjustment across the entire quadrature space as tested on tests of length 5 to 40 items.

3.3 Simulation Studies

To simulate the desired response process, we began by constructing a two-dimensional factor structure that might be encountered in a typical testing situation. For the five-item case, we selected the factor matrix:

$$\boldsymbol{\Lambda} = \begin{bmatrix} .7 & 0 \\ .6 & 0 \\ .5 & 0 \\ .4 & 0 \\ .3 & 0 \end{bmatrix} \tag{17}$$

Assuming a correlation between the two ability dimensions of .5, the orthogonal projection of this matrix (*i.e.*, $\Lambda(\mathbf{T}^{-1})'$ where $\mathbf{T}$ is the Cholesky factor of the $2x2$ correlation matrix), is:

$$\Lambda(\mathbf{T}^{-1})' = \begin{bmatrix} .70 & -.40 \\ .60 & -.35 \\ .50 & -.29 \\ .40 & -.23 \\ .30 & -.17 \end{bmatrix} \tag{18}$$

For $n = 10, 20$, and 40, the matrix was replicated 2, 4 and 8 times respectively. Given this factor structure, the item-responses were simulated as follows.

Step 1: Obtain $n + 2$ random normal deviates. Call the first two z_1 and z_2 and the remaining n, $\epsilon_1, \ldots, \epsilon_n$.

Step 2: Simulate the primary ability of subject i (θ_1) as $z_1 = (z_1 + \bar{\theta})\sigma_\theta$, where $\bar{\theta}$ and σ_θ are the mean and standard deviation of the generating ability distribution [*i.e.*, $N(\bar{\theta}, \sigma_\theta^2)$]. For studies of EAP ability estimates, z_1 is generally fixed at 0, 1, or 2.

Step 3: Determine the secondary ability of subject i (θ_2) as $z_2 \sim N(0, 1)$.

Step 4: Compute the residual disturbance for each item as $\epsilon_j = \epsilon_j \sigma_j$, where $\sigma_j = \sqrt{1 - \lambda_{j1}^2 - \lambda_{j2}^2}$.

Step 5: Compute the response process for item j as:

$$y_j = \lambda_{j1} z_1 + \lambda_{j2} z_2 + \epsilon_j \tag{19}$$

(Note: for conditional independence $\lambda_{j2} = 0$).

Step 6: If $y_j > \gamma_j$ then the response to item j for subject i is correct ($x_{ij} = 1$) otherwise it is incorrect ($x_{ij} = 0$).

3.4 Method for Study No. 1

Using the previously described method of simulating item-response patterns, we computed the accuracy of several methods of computing item-response pattern probabilities. These methods included the original Clark algorithm, the modified Clark algorithm, the modified Clark algorithm with $\mathbf{R}^* = \mathbf{I}$, one-dimensional Gaussian quadrature (40 points), and two-dimensional Gaussian quadrature (1600 points). We would expect similar results for the modified Clark algorithm and the two-dimensional Gaussian quadrature, and similar results for the modified Clark algorithm with $\mathbf{R}^* = \mathbf{I}$ and one-dimensional quadrature. The original Clark algorithm should perform poorly throughout. The simulation study consisted of four conditions; 5 and 10 items each with conditional independence and dependence. The accuracy of these five probability estimators was evaluated by comparing them to the corresponding Monte Carlo estimates obtained from one million simulates for 5 items and ten million for 10 items.

3.5 Results

The results of the first simulation study are displayed in Tables 1-4. Tables 1 and 3 display the probability estimates for the 5 and 10 item conditional independence case and Tables 2 and 4 display results for the case of dependence. For independence, one and two-dimensional quadrature results are identical as are the two modified Clark algorithms. For five items and independence (Table 1), the results are virtually identical for modified Clark and quadrature methods; whereas the original Clark performed poorly as expected. For 10 items and independence (Table 3), the modified Clark algorithm actually out performed the quadrature for the 14 selected patterns and did slightly worse for the total number of patterns realized in the sample of ten million. Again, the original Clark algorithm performed poorly relative to the other estimators.

For conditional dependence, on five items (Table 2), the modified Clark and two-dimensional quadrature estimates were virtually identical. Similarly, when the residuals are incorrectly assumed to be zero (*i.e.*, $\mathbf{R}^* = \mathbf{I}$), the modified Clark and one-dimensional quadrature yielded virtually identical results as expected. Again, the original Clark performed poorly. For 10 items (Table 4) similar results were obtained.

4 Estimating Ability in the Presence of Conditional Dependence

Our next study was designed to determine the impact of conditional dependence on estimates of ability. Again, we were concerned with the case in which both a primary ability dimension and a method related dimension were both present in tests of variable length, administered to samples of varying ability.

4.1 EAP estimates of Ability

Let $x_{ij} = 1$ if item j is answered correctly by respondent i, otherwise $x_{ij} = 0$. The probability that item j is answered correctly by a respondent with ability θ is:

$$P(x_j = 1 \mid \theta) = \Phi_j(\theta) \tag{20}$$

The likelihood of θ, given the response vector

$$\boldsymbol{x}_i = [x_1, x_2, \ldots x_n], \tag{21}$$

is

$$L_i(\theta) = \prod_{j=1}^{n} [\Phi_j(\theta)]^{x_j} [1 - \Phi_j(\theta)]^{1-x_j}. \tag{22}$$

The EAP estimate of the ability of subject i, $\hat{\theta}_i$, given the item responses $\boldsymbol{x}_i$, is approximated as

$$\hat{\theta}_i = \frac{\sum_{k=1}^{q} X_k L_i(X_k) W(X_k)}{\sum_{k=1}^{q} L_i(X_k) W(X_k)}. \tag{23}$$

and the posterior standard deviation is

$$PSD(\hat{\theta}_i) = \sqrt{\frac{\sum_{k=1}^{q} (X_k - \hat{\theta}_i)^2 L_i(X_k) W(X_k)}{\sum_{k=1}^{q} L_i(X_k) W(X_k)}}, \tag{24}$$

where X_k is one of the q quadrature nodes and $W(X_k)$ is the corresponding normalized quadrature weight.

4.2 EAP Ability Estimates via the Clark Algorithm

When the assumption of conditional independence is violated, the previous estimator is no longer valid, because the continued-product probability requires all inter-item residual covariances to be zero. If responses to a particular test are determined both by a primary ability and by a method related dimension, the residual inter-item covariances will be nonzero even if the primary ability and method related dimensions are uncorrelated. In this case, the modified Clark algorithm may be used to obtain the correct likelihood, $C_i^*(X_k)$, which can be substituted for $L_i(X_k)$ in the previous equations, and correct estimates of the mean and variance of the posterior distribution can be obtained.

4.3 Method for Study No. 2

The conditions of the experiment included:

number of items (10, 20 and 40)

level of ability - $\theta = 0, 1, 2$

and conditional dependence

In all cases, 40 quadrature points were selected to fill a ± 2 range about the generating ability value. For the two-dimensional quadrature solution, the second dimension was evaluated using 40 quadrature points between -2 and 2, since $\theta_2 \sim N(0, 1)$. Item thresholds were selected as equally spaced points between -2 and 2. For each condition, 500 examinees were simulated. EAP estimates were computed using the modified Clark algorithm with $\mathbf{R} = \mathbf{R}^*$ and $\mathbf{R} = \mathbf{I}$, and by one and two-dimensional quadrature solutions. The results were summarized in terms of the mean and standard deviation of $\hat{\theta}_1$, the root mean square error (MSE), and average PSD.

4.4 Results

The results for tests of length 10, 20, and 40 items are displayed in Tables 5-7. The results are summarized as follows:

1. Both two-dimensional quadrature and the modified Clark algorithm produce extremely similar average ability estimates for 10 and 20 item tests, at all three ability levels, with and without conditional dependence. At 40 items, however, the modified Clark algorithm produces a slight downward bias in the average estimated ability, which is in turn responsible for increased MSEs relative to the two-dimensional quadrature solution.

2. As expected, the variability in the uncorrected ability estimates (*i.e.*, one-dimensional quadrature and Clark with $\mathbf{R} = \mathbf{I}$) was consistently increased over the modified Clark and two-dimensional quadrature solutions, but within each set were virtually identical.

3. As expected, the MSE of the uncorrected ability estimates (*i.e.*, one-dimensional quadrature and Clark with $\mathbf{R} = \mathbf{I}$) was consistently increased over the modified Clark and two-dimensional quadrature solutions, but within each set were virtually identical. The exception to this was the previously noted result for 40 items.

4. As expected, the average PSD of the uncorrected ability estimates (*i.e.*, one-dimensional quadrature and Clark with $\mathbf{R} = \mathbf{I}$) was consistently underestimated relative to the modified Clark and two-dimensional quadrature solutions, but within each set were virtually identical. This result was consistent for all simulated conditions.

5 Estimating Item Parameters in the Presence of Conditional Dependence

By replacing $P(\boldsymbol{x} = \boldsymbol{x}_i \mid \theta_i)$ with $P(\boldsymbol{x} = \boldsymbol{x}_i \mid \theta_i, \Sigma)$, we may, in theory, obtain consistent estimates of item parameters and corresponding interval estimates, regardless of the level of residual dependence that remains after we condition on θ_1. To do this requires some estimate of Σ. Three general approaches are available. First, as in the previous section, we may fit a K-dimensional model using the methods described by Bock and Aitkin (1981) and Bock, Gibbons and Muraki (1988), then compute Σ based on the factor loading coefficients obtained from factors 2 through K. Second, we may use the tetrachoric correlation matrix to approximate the total association in the population and express the residual association as the difference between the elements of the tetrachoric matrix and the expected correlation matrix based on provisional estimates of the one-factor model on each iteration. In this way, residual covariances are computed from the estimated item parameters on each iteration. Third, we may directly model the elements of Σ, by assuming that, once we condition on θ_1, all that remains is a first-order autocorrelation among the residual errors of measurement. This type of structure might be plausible in the adaptive testing situation in which the presentation of future items is based on the success of previous items. As an illustration of this approach, we applied the second alternative to the well known 5-item LSAT section 7 dataset. Bock and Lieberman (1970) showed that these data did not fit a unidimensional normal model ($G^2 = 31.59, df = 21, p < .05$); however, Bock and Aitkin were able to fit a two-factor normal model to these data with reasonable success ($G^2 = 21.23, df = 17, p < .22$). In contrast, the unidimensional model of dependence, also fit these data reasonably well ($G^2 = 21.83, df = 16, p < .15$).

Inspection of the parameters estimates in Table 8, for all three models, reveal several interesting results.

1. The assumption of conditional independence is a convenience, but *not* a requirement for estimating the parameters of IRT models.

2. The model of dependence identified a primary ability dimension represented by all of the items and a few modest residual covariances:

$$\Sigma = \begin{bmatrix} 0.71 & & & & \\ -.14 & 0.55 & & & \\ -.13 & -.09 & 0.39 & & \\ +.07 & -.08 & -.05 & 0.83 & \\ +.06 & -.14 & -.06 & -.01 & 0.83 \end{bmatrix} \tag{25}$$

In contrast, the 2-factor model, was dominated by only 2 items, making the dependence model a more parsimonious solution.

3. The improvement in fit of the model of dependence over the model of independence was substantial.

4. Standard errors for the models of dependence are slightly larger than for the model of independence, but considerably smaller than the 2-factor model. This result may suggest that the 2-factor model is not completely identified for this 5-item test.

5. The iterative solution proposed here converged to the same solution from a variety of starting values, which suggests that estimation of item parameters, from which the residual correlations are computed, does not lead to an indeterminate solution. This is useful because it avoids "two-stage" solutions.

6. In the present example, the sample tetrachoric solution performed quite well. Alternatively, a model of higher dimension could be used to estimate the overall sample correlation matrix **R**. In the present example, this leads to a fit statistic of $G^2_{17} = 23.96, p < .12$ as compared to $G^2_{16} = 21.83, p < .15$ for the tetrachoric solution, which suggests that the tetrachoric solution is quite adequate for this purpose.

6 Summary and Conclusions

The results of this research reveal that it is quite possible to estimate accurately the multivariate normal orthant probabilities for high-dimensional normal integrals with no restriction on the form of the covariance matrix or mean vector. Prior to these results, closed form solutions were only available for the trivariate normal distribution and approximations of even quadrivariate normal integrals were available only for special cases such as equa-correlation and "band matrices" (Kendall, 1941; Moran, 1948; McFadden, 1960; Abrahamson, 1964; Childs, 1967; Dutt, 1973; and Dutt and Lin, 1975).

Application of the Clark based method to problems in item-response theory seems promising. Our results on Bayes estimation of ability, estimating item parameters, and testing goodness of fit, clearly reveal that the modified Clark algorithm resolves typical cases of conditional dependence and provides a statistical solution that preserves the intended focus of the test.

The results of this work have application in other areas of statistics as well. As an example, the modified Clark algorithm can be used to solve the problem of multivariate generalizations of probit analysis (Ashford and Sowden, 1970). Similarly, the likelihood of multinomial probit models used for evaluating discrete choice problems with correlated choice alternatives (Daganzo, 1979), can also now be evaluated. Furthermore, as demonstrated by Gibbons and Bock (1987), the modified Clark algorithm can also be used to estimate the parameters of a random effects probit model in which the errors of measurement exhibit first-order autocorrelation.

Acknowledgement

We would like to thank Ming-mei Wang for her very helpful suggestions.

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Table 1

Conditional Independence
5 Items
Probability Estimates

pattern	Monte Carlo	Clark	Mod. Clark	Mod. Clark R=I	Quadrature 1-dimensional	Quadrature 2-dimensional
00000	.0092690	.0016769	.0058844	.0058844	.0049920	.0049920
00001	.0000310	.0000358	.0000166	.0000166	.0000270	.0000270
00010	.0002670	.0003293	.0001777	.0001777	.0002229	.0002229
00011	.0000010	.0000079	.0000003	.0000003	.0000014	.0000014
00100	.0015670	.0010638	.0013925	.0013925	.0012793	.0012793
00101	.0000050	.0000229	.0000049	.0000049	.0000084	.0000084
00110	.0000840	.0002130	.0000427	.0000427	.0000701	.0000701
01000	.0082570	.0022187	.0080540	.0080540	.0069502	.0069502
01001	.0000480	.0000484	.0000217	.0000217	.0000486	.0000486
01010	.0004120	.0004384	.0003050	.0003050	.0004077	.0004077
01011	.0000030	.0000108	.0000006	.0000006	.0000036	.0000036
01100	.0023920	.0014105	.0024988	.0024988	.0023838	.0023838
01101	.0000250	.0000311	.0000077	.0000077	.0000217	.0000217
01110	.0001780	.0002845	.0001162	.0001162	.0001846	.0001846
01111	.0000030	.0000072	.0000003	.0000003	.0000022	.0000022
10000	.0908200	.0708713	.0869044	.0869044	.0858275	.0858275
10001	.0008220	.0015482	.0007681	.0007681	.0008440	.0008440
10010	.0069930	.0071212	.0081681	.0081681	.0072004	.0072004
10011	.0001080	.0004378	.0000584	.0000584	.0000990	.0000990
10100	.0429120	.0384821	.0488560	.0488560	.0437355	.0437355
10101	.0005770	.0013366	.0005361	.0005361	.0006260	.0006260
10110	.0051440	.0048801	.0056860	.0056860	.0054192	.0054192
10111	.0001000	.0004327	.0000593	.0000593	.0001080	.0001080
11000	.3252630	.3454604	.3215043	.3215043	.3339557	.3339557
11001	.0058970	.0103496	.0068320	.0068320	.0059707	.0059707
11010	.0505540	.0553977	.0589594	.0589594	.0521171	.0521171
11011	.0013330	.0034446	.0013598	.0013598	.0013324	.0013324
11100	.3440720	.3509381	.3384136	.3384136	.3489993	.3489993
11101	.0099590	.0128232	.0118505	.0118505	.0095529	.0095529
11110	.0891580	.0840372	.0879184	.0879184	.0845265	.0845265
11111	.0037460	.0046399	.0036022	.0036022	.0030822	.0030822
average difference		.0028954	.0012275	.0012275	.0010744	.0010744
maximum difference		.0201974	.0084054	.0084054	.0086927	.0086927

Table 2

Conditional Dependence
5 Items
Probability Estimates

pattern	Monte Carlo	Clark	Mod. Clark	Mod. Clark R=I	Quadrature 1-dimensional	Quadrature 2-dimensional
00000	.0132430	.0003325	.0086805	.0058537	.0049920	.0070033
00001	.0000230	.0000102	.0000117	.0000160	.0000270	.0000210
00010	.0001820	.0000824	.0001344	.0001729	.0002229	.0001637
00100	.0012020	.0002234	.0012086	.0013689	.0012793	.0009430
00101	.0000030	.0000068	.0000028	.0000046	.0000084	.0000038
00110	.0000310	.0000562	.0000229	.0000411	.0000701	.0000307
01000	.0065430	.0003425	.0071056	.0080097	.0069502	.0052413
01001	.0000270	.0000106	.0000121	.0000208	.0000486	.0000228
01010	.0002020	.0000850	.0001486	.0002965	.0004077	.0001849
01011	.0000010	.0000027	.0000002	.0000006	.0000036	.0000011
01100	.0011580	.0002304	.0013144	.0024581	.0023838	.0010983
01101	.0000060	.0000070	.0000034	.0000073	.0000217	.0000066
01110	.0000430	.0000581	.0000376	.0001120	.0001846	.0000547
10000	.0981450	.0643565	.0939096	.0869296	.0858278	.0921583
10001	.0006680	.0009775	.0006281	.0007487	.0008440	.0006757
10010	.0056640	.0047805	.0066756	.0080588	.0072004	.0057416
10011	.0000580	.0002493	.0000434	.0000560	.0000990	.0000635
10100	.0355610	.0282723	.0421209	.0488300	.0437357	.0362899
10101	.0003900	.0007558	.0004003	.0005209	.0006260	.0004223
10110	.0034520	.0026826	.0038756	.0055940	.0054193	.0036949
10111	.0000660	.0002323	.0000378	.0000568	.0001080	.0000630
11000	.3227360	.3646366	.3181484	.3218323	.3339569	.3342406
11001	.0051340	.0094979	.0062500	.0067287	.0059707	.0053714
11010	.0462420	.0509119	.0549946	.0589751	.0521173	.0479139
11011	.0011850	.0032962	.0012766	.0013298	.0013324	.0011967
11100	.3468730	.3639200	.3413597	.3387589	.3490006	.3532162
11101	.0103010	.0124808	.0120872	.0117273	.0095529	.0098653
11110	.0963190	.0867242	.0953072	.0879407	.0845268	.0906198
11111	.0045420	.0047778	.0042022	.0035504	.0030822	.0036917
average difference		.0050750	.0014122	.0025535	.0023840	.0014398
maximum difference		.0419006	.0087526	.0132690	.0123172	.0115046

Table 3

Conditional Independence
10 Items
Probability Estimates

pattern	Monte Carlo	Clark	Mod. Clark	Mod. Clark R=I	Quadrature 1-dimensional	Quadrature 2-dimensional
0000001100	.0000001	.0000114	.0000000	.0000000	.0000001	.0000001
0000100001	.0000011	.0000082	.0000010	.0000010	.0000009	.0000009
0001010000	.0000118	.0001839	.0000053	.0000053	.0000112	.0000112
0110010000	.0001329	.0006086	.0000856	.0000856	.0001406	.0001406
0110100000	.0016666	.0013821	.0012122	.0012122	.0014266	.0014266
1001100000	.0027151	.0042885	.0019134	.0019134	.0024568	.0024568
1010000000	.0067213	.0079889	.0055955	.0055955	.0058672	.0058672
1011000000	.0089667	.0111335	.0085417	.0085417	.0085844	.0085844
1100000000	.0146445	.0186212	.0131084	.0131084	.0134876	.0134876
1101000000	.0220419	.0296640	.0227618	.0227618	.0218673	.0218673
1110000000	.0462991	.0617006	.0463947	.0463947	.0466949	.0466949
1110100000	.0477081	.0625243	.0501873	.0501873	.0490546	.0490546
1111000000	.0968781	.1280707	.0959419	.0959419	.0998421	.0998421
1111100000	.1177844	.1480838	.1172587	.1172587	.1223589	.1223589
average difference		.0078048	.0006538	.0006538	.0008825	.0008825
all patterns		.0007246	.0001281	.0001281	.0000643	.0000643
maximum difference		.0311926	.0024792	.0024792	.0045745	.0045745
all patterns		.0311926	.0066958	.0066958	.0045745	.0045745

Table 4

Conditional Dependence
10 Items
Probability Estimates

pattern	Monte Carlo	Clark	Mod. Clark	Mod. Clark R=I	Quadrature 1-dimensional	Quadrature 2-dimensional
0000001100	.0000001	.0000159	.0000000	.0000000	.0000003	.0000000
0000100001	.0000009	.0000046	.0000008	.0000010	.0000009	.0000009
0001010000	.0000023	.0000667	.000021	.0000053	.0000112	.0000026
0110010000	.0000328	.0001205	.0000301	.0000856	.0001406	.0000337
0110100000	.0013382	.0002668	.0010167	.0012122	.0014266	.0011223
1001100000	.0028129	.0033192	.0020223	.0019134	.0024568	.0025560
1010000000	.0077306	.0065970	.0064644	.0055955	.0058672	.0068002
1011000000	.0088950	.0078580	.0077004	.0085417	.0085844	.0084590
1100000000	.0176867	.0209829	.0162748	.0131084	.0134876	.0164043
1101000000	.0240336	.0305718	.0246059	.0227618	.0218673	.0238656
1110000000	.0517423	.0713755	.0521337	.0463947	.0466949	.0523689
1110100000	.0506626	.0678296	.0527083	.0501873	.0490546	.0523895
1111000000	.1033206	.1465492	.1029978	.0959419	.0998421	.1073350
1111100000	.1212253	.1645829	.1208567	.1172587	.1223589	.1274467
average difference		.0097958	.0006206	.0018991	.0014549	.0011343
all patterns		.0008165	.0001337	.0002247	.0002245	.0000912
maximum difference		.0433576	.0020457	.0073787	.0050474	.0062214
all patterns		.0433576	.0059526	.0117328	.0116041	.0062214

Table 5

Results of Simulation
EAP Ability Estimates
10 Items

Condition	Mean	SD	MSE	PSD
Independence				
$\theta = 0$				
Quadrature	-.0123	.2694	.2697	.4670
Clark	-.0165	.2768	.2772	.4624
$\theta = 1$				
Quadrature	.7876	.2415	.3218	.4557
Clark	.7831	.2498	.3309	.4483
$\theta = 2$				
Quadrature	1.5720	.1582	.4566	.4179
Clark	1.5692	.1652	.4618	.4150
Dependence				
$\theta = 0$				
Quadrature 1D	-.0189	.3044	.3050	.4586
Clark R=I	-.0239	.3148	.3157	.4525
Quadrature 2D	-.0154	.2638	.2643	.4775
Modified Clark	-.0191	.2674	.2681	.4765
$\theta = 1$				
Quadrature 1D	.7974	.2665	.3365	.4516
Clark R=I	.7915	.2769	.3467	.4437
Quadrature 2D	.7753	.2255	.3185	.4648
Modified Clark	.7703	.2291	.3246	.4609
$\theta = 2$				
Quadrature 1D	1.5754	.1699	.4578	.4174
Clark R=I	1.5735	.1785	.4628	.4143
Quadrature 2D	1.5560	.1432	.4669	.4185
Modified Clark	1.5568	.1507	.4685	.4173

Table 6

Results of Simulation
EAP Ability Estimates
20 Items

Condition	Mean	SD	MSE	PSD
Independence				
$\theta = 0$				
Quadrature	.0125	.2881	.2883	.4200
Clark	-.0244	.2953	.2963	.4231
$\theta = 1$				
Quadrature	.8285	.2784	.3271	.4147
Clark	.7934	.2791	.3474	.4132
$\theta = 2$				
Quadrature	1.6600	.2414	.4172	.3992
Clark	1.6434	.2543	.4383	.3926
Dependence				
$\theta = 0$				
Quadrature 1D	-.0104	.3588	.3589	.4036
Clark R=I	-.0366	.3641	.3659	.4052
Quadrature 2D	-.0070	.2766	.2767	.4551
Modified Clark	-.0359	.2799	.2822	.4642
$\theta = 1$				
Quadrature 1D	.8222	.3385	.3824	.4004
Clark R=I	.7879	.3374	.3986	.3966
Quadrature 2D	.7819	.2522	.3336	.4418
Modified Clark	.7445	.2525	.3594	.4410
$\theta = 2$				
Quadrature 1D	1.6745	.2879	.4348	.3933
Clark R=I	1.6585	.2971	.4529	.3876
Quadrature 2D	1.6077	.2092	.4449	.4095
Modified Clark	1.5932	.2193	.4625	.4095

Table 7

Results of Simulation
EAP Ability Estimates
40 Items

Condition	Mean	SD	MSE	PSD
Independence				
$\theta = 0$				
Quadrature	.0025	.2985	.2985	.3515
Clark	-.0845	.3336	.3442	.3595
$\theta = 1$				
Quadrature	.8730	.2927	.3191	.3521
Clark	.7809	.3375	.4024	.3520
$\theta = 2$				
Quadrature	1.7406	.2664	.3720	.3565
Clark	1.6729	.2990	.4434	.3525
Dependence				
$\theta = 0$				
Quadrature 1D	**-.0354**	**.4224**	**.4239**	**.3285**
Clark R=I	**-.1038**	**.4123**	**.4252**	**.3392**
Quadrature 2D	**-.0254**	**.2906**	**.2917**	**.4329**
Modified Clark	**-.1318**	**.3230**	**.3489**	**.4351**
$\theta = 1$				
Quadrature 1D	.8612	.4229	.4452	.3266
Clark R=I	.7786	.4195	.4744	.3273
Quadrature 2D	.7909	.2851	.3537	.4175
Modified Clark	.6602	.3129	.4622	.3991
$\theta = 2$				
Quadrature 1D	1.7585	.3850	.4546	.3369
Clark R=I	1.6973	.3973	.4997	.3336
Quadrature 2D	1.6272	.2428	.4453	.3917
Modified Clark	1.5462	.2671	.5269	.3883

Table 8

LSAT Section 7
Models of Independence and Dependence

Item	Independence					Dependence	
	1-Factor		2-Factor			1-Factor	
	γ_j	λ_j	γ_j	λ_{1j}	λ_{2j}	γ_j	λ_j
1	-.95 (.07)	.49 (.10)	-.94 (.72)	.77 (1.07)	.14 (.40)	-1.08 (.09)	.54 (.11)
2	-.41 (.05)	.55 (.10)	-.41 (.06)	.21 (.15)	.48 (.19)	-.45 (.07)	.67 (.14)
3	-.74 (.11)	.70 (.18)	-.73 (.42)	.23 (.40)	.78 (.87)	-.82 (.18)	.78 (.26)
4	-.27 (.05)	.42 (.08)	-.27 (.05)	.32 (.15)	.26 (.11)	-.30 (.05)	.42 (.07)
5	-1.01 (.06)	.38 (.09)	-1.01 (.07)	.32 (.14)	.22 (.09)	-1.17 (.08)	.41 (.10)
χ^2	31.71		21.17			21.83	
df	21		17			16	
$p <$	.06		.22			.15	

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