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# SYMMETRY IN THE BASIC SCIENCES

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APRIL 1989

FINAL REPORT

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**REPORT DOCUMENTATION PAGE**

<b>1a. REPORT SECURITY CLASSIFICATION</b> Unclassified		<b>1b. RESTRICTIVE MARKINGS</b>													
<b>2a. SECURITY CLASSIFICATION AUTHORITY</b>		<b>3. DISTRIBUTION/AVAILABILITY OF REPORT</b> Approved for public release. Unlimited distribution.													
<b>2b. DECLASSIFICATION/DOWNGRADING SCHEDULE</b>															
<b>4. PERFORMING ORGANIZATION REPORT NUMBER(S)</b>		<b>5. MONITORING ORGANIZATION REPORT NUMBER(S)</b>													
<b>6a. NAME OF PERFORMING ORGANIZATION</b> Depts of Biology, Physics, Mathematics, & Chemistry		<b>6b. OFFICE SYMBOL</b> <i>(If applicable)</i> DFB, DFP, DFMS, DFC	<b>7a. NAME OF MONITORING ORGANIZATION</b>												
<b>6c. ADDRESS (City, State and ZIP Code)</b> US Air Force Academy Colorado 80840-5701		<b>7b. ADDRESS (City, State and ZIP Code)</b>													
<b>8a. NAME OF FUNDING/SPONSORING ORGANIZATION</b>	<b>8b. OFFICE SYMBOL</b> <i>(If applicable)</i>	<b>9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER</b>													
<b>8c. ADDRESS (City, State and ZIP Code)</b>		<b>10. SOURCE OF FUNDING NOS.</b> <table border="1" style="width:100%; border-collapse: collapse; margin-top: 5px;"> <tr> <td style="width:25%; text-align: center;">PROGRAM ELEMENT NO.</td> <td style="width:25%; text-align: center;">PROJECT NO.</td> <td style="width:25%; text-align: center;">TASK NO.</td> <td style="width:25%; text-align: center;">WORK UNIT NO.</td> </tr> <tr> <td> </td> <td> </td> <td> </td> <td> </td> </tr> </table>		PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.								
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<b>11. TITLE (Include Security Classification)</b> Symmetry in the Basic Sciences		<table border="1" style="width:100%; border-collapse: collapse;"> <tr> <td style="width:33%;"><b>13a. TYPE OF REPORT</b> Final Report</td> <td style="width:33%;"><b>13b. TIME COVERED</b> FROM Aug 88 TO Apr 89</td> <td style="width:33%;"><b>14. DATE OF REPORT (Yr., Mo., Day)</b></td> </tr> <tr> <td colspan="2"><b>15. PAGE COUNT</b> 71</td> <td> </td> </tr> </table>		<b>13a. TYPE OF REPORT</b> Final Report	<b>13b. TIME COVERED</b> FROM Aug 88 TO Apr 89	<b>14. DATE OF REPORT (Yr., Mo., Day)</b>	<b>15. PAGE COUNT</b> 71								
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<b>15. PAGE COUNT</b> 71															
<b>12. PERSONAL AUTHOR(S):</b> Joseph E. Toole, LTC, USAF; David W. Jensen, Maj, USAF; Mark E. Rogers, Maj, USAF; Paul J. Chernek, Capt, USAF; Thomas E. Erstfeld, Capt, USA		<b>16. SUPPLEMENTARY NOTATION</b> <i>by USAF Academy faculty members -                  ↳ This report summarizes talks given at an inter-disciplinary colloquium</i>													
<table border="1" style="width:100%; border-collapse: collapse;"> <tr> <th colspan="3" style="text-align: left;">17. COSATI CODES</th> </tr> <tr> <td style="width:33%;">FIELD</td> <td style="width:33%;">GROUP</td> <td style="width:33%;">SUB. GR.</td> </tr> <tr> <td> </td> <td> </td> <td> </td> </tr> <tr> <td> </td> <td> </td> <td> </td> </tr> </table>		17. COSATI CODES			FIELD	GROUP	SUB. GR.							<b>18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)</b> Held 7-10 Nov 88,	
17. COSATI CODES															
FIELD	GROUP	SUB. GR.													
<b>19. ABSTRACT (Continue on reverse if necessary and identify by block number)</b> <p>From 7-10 November 1988, an interdisciplinary colloquium entitled "Symmetry in the Basic Sciences" was held at the U. S. Air Force Academy, Colorado. Participating in the colloquium were the Academy's Departments of Mathematical Sciences, Chemistry, Biology, and Physics. Each day a faculty member or a team of faculty members from one of the departments gave a presentation on the role of symmetry in their discipline. This report is a summary of those talks.</p> <p>Chapter 1 presents the basic mathematical theory behind plane symmetry groups. This theory is then applied in classifying the symmetry of bounded figures, frieze patterns and wallpaper patterns. Recently developed algorithms are included to help analyze complex designs.</p> <p align="right"><i>(continued on reverse)</i></p>															
<b>20. DISTRIBUTION/AVAILABILITY OF ABSTRACT</b> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>		<b>21. ABSTRACT SECURITY CLASSIFICATION</b> Unclassified													
<b>22a. NAME OF RESPONSIBLE INDIVIDUAL</b> David W. Jensen, Major, USAF		<b>22b. TELEPHONE NUMBER</b> <i>(Include Area Code)</i> (719) 472-4470	<b>22c. OFFICE SYMBOL</b> DFMS												

19. ABSTRACT (continued)

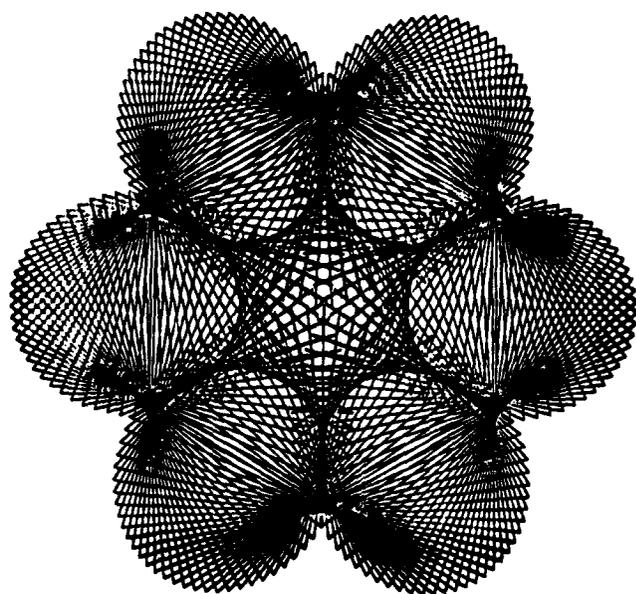
Chapter 2 discusses symmetry operations relevant to three-dimensional crystallography. In particular, the seven crystal systems that classify the thirty-two crystallographic point groups are described. These are then used to construct the Bravais lattices.

Chapter 3 investigates the role of symmetry in biological forms. Specifically, D'Arcy Thompson's work on growth and form of molluscan shells is reviewed with an attempt to explain the consequences of that growth and form to the natural history of the Chambered Nautilus and its ancestors.

Chapter 4 looks at the central role symmetry has increasingly played in physics by examining the Principle of Least Action and the invariance of the Lagrangian under a transformation. Noether's Theorem guarantees that a conservation law is associated with each of these symmetries. Examples include the conservation of energy, linear momentum, and angular momentum, as well as the purely quantum mechanical symmetry of invariance under an exchange operation. A brief look at gauge theories is the final example of how symmetry has become a guiding principle in the formulation of new theories. (eds) ←



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**SYMMETRY  
IN THE  
BASIC SCIENCES**

**An Interdisciplinary View of the Subject of Symmetry**

**Proceedings of the Symmetry Colloquium**

**U.S. Air Force Academy, Colorado, 7-10 November 1988**

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## INTRODUCTION

From 7-10 November 1988, an interdisciplinary colloquium entitled "Symmetry in the Basic Sciences" was held at the U. S. Air Force Academy, Colorado. Participating in the colloquium were the Academy's Departments of Mathematical Sciences, Chemistry, Biology, and Physics. Each day a faculty member or a team of faculty members from one of the departments gave a presentation on the role of symmetry in their discipline. This report is a summary of those talks. The order of the chapters reflects the order in which the department presentations were given. The chairman and organizer of the colloquium was Major David Jensen from the Academy's Department of Mathematical Sciences.

## PLANE SYMMETRY GROUPS

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This paper presents the basic mathematical theory behind plane symmetry groups. This theory is then applied in classifying the symmetry of bounded figures, frieze patterns and wallpaper patterns. Recently developed algorithms are included to help analyze complex designs. Numerous examples are presented to clarify concepts and to illustrate the various symmetry types.

### INTRODUCTION

Plane symmetry groups are a mathematical classification system for describing the symmetry of two-dimensional figures and patterns. Here mathematics truly comes alive, as abstract symbols used to describe symmetry can be immediately visualized as the rotation of a snowflake, the pattern on a vase, or the ceramic tile of an ornate kitchen floor. The goal of this paper: to introduce the basic theory behind plane symmetry groups, and to present some simple algorithms one may use to analyze complex designs.

To make the theory accessible to as broad an audience as possible, the mathematics has been deliberately downplayed. Mathematical proofs are relegated to the bibliography. Since mosaics and patterns are visual creations, examples and illustrations are used frequently in the text. The prerequisites to understanding the contents of this report are modest. The author assumes that the reader is comfortable with the basic notation and theory for sets, functions, and the composition of functions. This report is an abbreviated version of an earlier paper written jointly by Major Jensen and Cadet Firstclass Gary Harvey [5].

## Section 1

### BASIC THEORY

A binary operation is a function that takes two elements in a nonempty set  $G$  and assigns to them a unique element also in the set  $G$ . Using standard function notation, if  $*$  is a binary operation on  $G$  we write  $*$  :  $G \times G \rightarrow G$ . Familiar examples of binary operations abound, with one of the easiest being the real numbers under normal addition ( $+ : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ).

With this notion of a binary operation we can define a group--one of the most important algebraic structures in the world of mathematics.

#### Definition 1.1 Group $(G,*)$

A group  $(G,*)$  is any nonempty set  $G$  together with a binary operation  $*$  :  $G \times G \rightarrow G$  that satisfies the following three properties:

Associative Property:  $a * (b * c) = (a * b) * c$ , for all  $a, b$ , and  $c \in G$ .

Identity Property: There exists  $e \in G$  such that  $a * e = e * a = a$ , for all  $a \in G$ . We call  $e$  the identity element of the group.

Inverse Property: For every  $a \in G$ , there exists  $b \in G$ , such that  $a * b = b * a = e$ .

We call  $b$  the inverse of  $a$  and write  $b = a^{-1}$

We say  $(H,*)$  is a subgroup of a given group  $(G,*)$  when  $H$  is a nonempty subset of  $G$ , and  $(H,*)$  is itself a group. Also, a group  $(G,*)$  is called Abelian if all its elements commute, that is  $a * b = b * a$  for all  $a$  and  $b$  in  $G$ .

The definition of a group is important because the properties listed above give it enough structure to be useful, while at the same time the definition is not too restrictive. There are lots of groups. The real numbers under normal addition  $(\mathbb{R}, +)$  form a group with identity  $e = 0$  and inverses of the form  $a^{-1} = -a$ . Rather than  $\mathbb{R}$ , we could have just as easily chosen the rational numbers  $\mathbb{Q}$  or the integers  $\mathbb{I}$  and formed the groups  $(\mathbb{Q}, +)$  and  $(\mathbb{I}, +)$ . For another example, the real numbers (excluding zero) under normal multiplication,  $(\mathbb{R} - \{0\}, \cdot)$ , form a group with identity  $e = 1$  and inverses given by  $a^{-1} = 1/a$ .

It is also easy to define groups of matrices or groups of functions under various binary operations. In particular, consider  $\mathbb{R}_2$ , the set of all points in the plane, and let  $G$  be the set of all one-to-one functions from  $\mathbb{R}_2$  onto  $\mathbb{R}_2$ . Then consider  $(G, \circ)$  where  $\circ$  represents the composition of functions. First, note that all one-to-one, onto functions from  $\mathbb{R}_2$  to  $\mathbb{R}_2$  are invertible. Therefore, elements in  $G$  have inverses. Moreover,  $\circ$  is a binary operation on  $G$  since the composition of two invertible functions is again an invertible function. The other two properties needed to establish that  $(G, \circ)$  is a group follow readily from the fact that all invertible functions from  $\mathbb{R}_2$  to  $\mathbb{R}_2$  are associative under composition, and the identity function  $i : \mathbb{R}_2 \rightarrow \mathbb{R}_2$  defined by  $i(p) = p$  for all  $p \in \mathbb{R}_2$  is the logical choice for the group identity element.

Our first real progress in applying group theory to questions of symmetry comes when we consider a special subset  $H$  of  $G$ . We let  $H$  be precisely the invertible maps from  $\mathbb{R}_2$  to  $\mathbb{R}_2$  that also preserve distance between points. Using the usual notation of vertical lines for distance, we have

$$H = \{ \alpha \in G : |p - q| = |\alpha(p) - \alpha(q)| \text{ for all } p, q \in R_2 \},$$

where  $|p - q|$  is the straight line distance from point  $p$  to point  $q$  and  $|\alpha(p) - \alpha(q)|$  is the straight line distance from point  $\alpha(p)$  to point  $\alpha(q)$ .

It is easy to show that  $H$  is a subgroup of  $G$ , and we call the elements of  $H$  the motions, or isometries, of the plane [5]. Moreover, it can be shown that there are only four types of motions possible [1]:

1. Translation

A mapping  $\alpha$  that sends all points in  $R_2$  the same distance  $d$  in the same direction  $\theta$ . To illustrate, consider  $p_1, q_1 \in R_2$  with  $\alpha(p_1) = q_1$ ,  $i = 1, 2, 3$ :

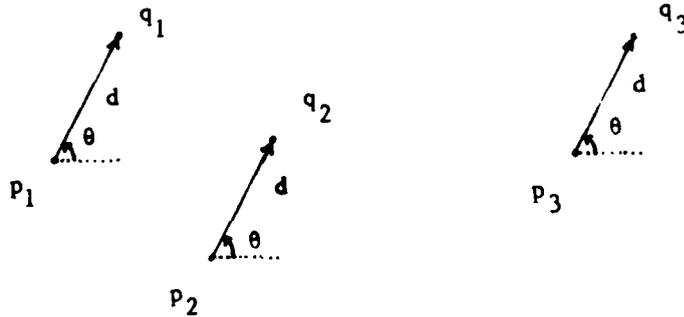


Figure 1.1

2. Rotation

A mapping  $\alpha$  obtained by rotating the plane clockwise a fixed amount  $\phi$  about a fixed point  $p$ . To illustrate, consider  $p_1, q_1 \in R_2$  with  $\alpha(p_1) = q_1$ :

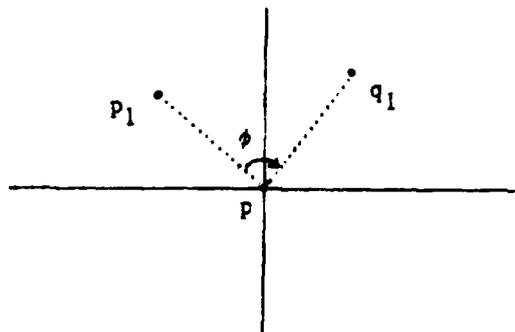


Figure 1.2

3. Mirror

A mapping  $\alpha$  obtained by reflecting the plane through a fixed line  $L$  (that is, a mapping that sends each point  $p$  to a point  $q$  such that  $L$  is the perpendicular bisector of the straight line between  $p$  and  $q$ ). To illustrate consider  $p_i, q_i \in R_2$  with  $\alpha(p_i) = q_i, i = 1, 2, 3, 4$ :

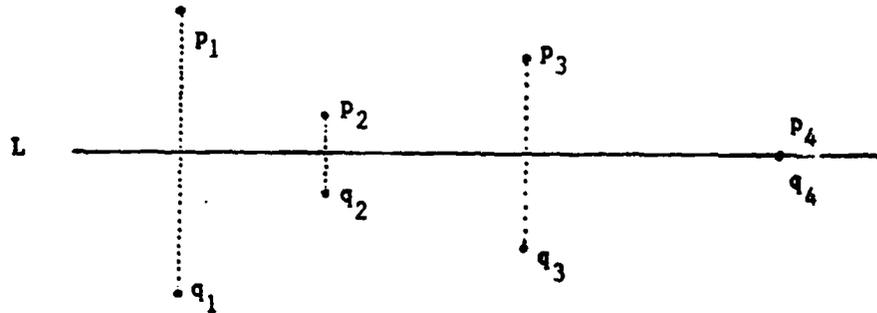


Figure 1.3

4. Glide

A mapping  $\alpha$  composed of a translation in the direction of a fixed line  $L$ , followed by a mirror through  $L$ . To illustrate let  $p_i, q_i \in R_2$  with  $\alpha(p_i) = q_i, i = 1, 2$ :

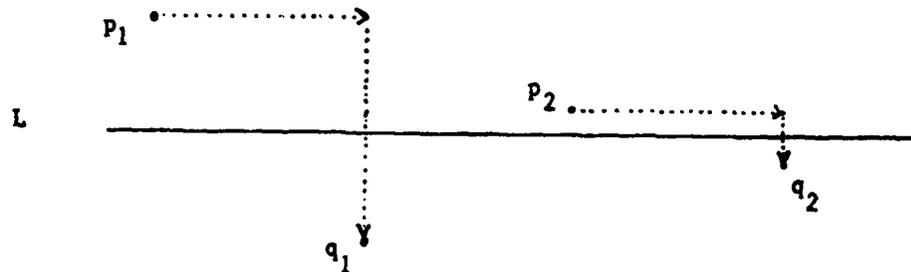


Figure 1.4

Having defined the subgroup  $(H, \circ)$  of  $(G, \circ)$  we are nearing our goal of being able to use group theory to analyze the symmetry of figures and designs in the

Euclidean plane. The next definition restricts  $(H, \circ)$  in a very natural way, leaving us with precisely the motions we need to describe the symmetry of a given figure.

Definition 1.2      Symmetry Group of  $T$

Let  $T$  be any nonempty set of points in the plane,  $T \subseteq \mathbb{R}_2$ . Define a subset  $H_T$  of  $H$  by  $H_T = \{\alpha \in H : \alpha(T) = T\}$ . Here  $\alpha(T) = T$  denotes set invariance, that is  $\alpha(p) \in T$  for every  $p \in T$ . It can be shown that  $H_T$  is itself a subgroup of  $G$  [5]. We call  $H_T$  the symmetry group of  $T$ .

The way to view the definition of  $H_T$  is as follows:

- Start with a given figure in the plane. For a simple example, take a circle of radius  $r$  centered at the origin of the Cartesian coordinate system.
- Consider the points that makeup the figure to be  $T$ . Therefore, for our example, the set  $T$  is the locus of points satisfying  $x^2 + y^2 = r^2$ .

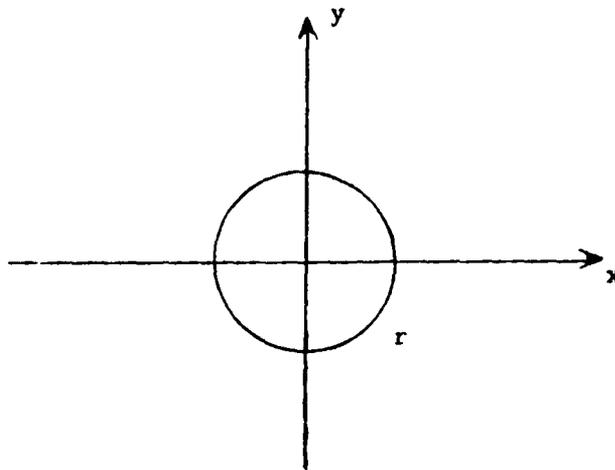


Figure 1.5

- Then  $H_T$  is exactly those translations, rotations, mirrors and glides that map  $T$  back onto itself. When  $T$  is the circle shown in Figure 1.5 (in fact, when  $T$  is any bounded figure) we will see in Section 2 that translations and glides cannot be elements of  $H_T$ . If we consider rotations, there are obviously an infinite number of possibilities, since any rotation about the origin will leave  $T$  invariant. In addition, any mirror through a line passing through the origin will also map  $T$  onto itself.

Example 2.1 Find the symmetry group of  $T$  where  $T$  contains only two distinct points, say  $T = \{p_1, p_2\}$ . Note first that the identity map  $i$  is in  $H_T$ . For if  $i(p) = p$  for every  $p$  in the plane, then certainly  $i(T) = T$ . To determine the other motions in  $H_T$ , let  $L_1$  be the straight line through points  $p_1$  and  $p_2$ , and let  $L_2$  be the perpendicular bisector of the line segment from  $p_1$  to  $p_2$ .

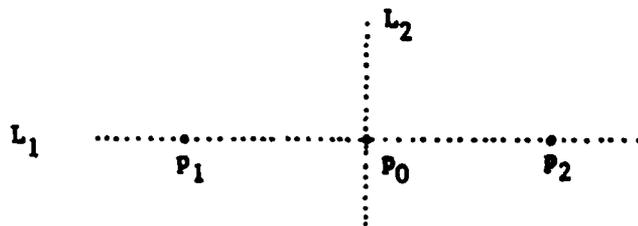


Figure 1.6

Using the definitions of a translation and a glide, it is easy to see that as in the case of the circle, translations and glides cannot be elements of  $H_T$ . The only rotation in  $H_T$  is the rotation of  $180^\circ$  about the point  $p_0$ . (We don't count the case where we pick a point, say  $p_1$ , and rotate everything  $360^\circ$  about that point. After all, this just yields the identity map, which we have already acknowledged as being in  $H_T$ .) The only two mirrors possible are reflections through the lines  $L_1$  and  $L_2$ . Note that although reflection through  $L_1$  leaves points  $p_1$  and  $p_2$  unchanged, it certainly changes other points in the plane and

is therefore distinct from the identity map. We conclude that  $H_T$  contains exactly four elements: the identity map,  $180^\circ$  rotation about  $p_0$ , reflection through  $L_1$ , and reflection through  $L_2$ .

Example 2.2 Find the symmetry group of  $T$  where  $T$  is the set of points that makeup the footsteps depicted below [2]:

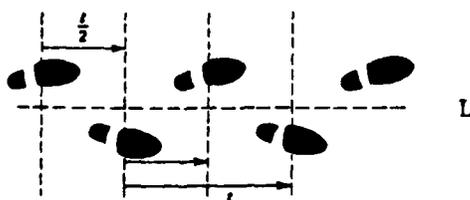


Figure 1.7

The footprints are assumed to continue infinitely to the right and to the left. There are no rotations or mirrors in  $H_T$ . However, this is the first example we have encountered where translations play a part. A translation of length  $t$  (or any integer multiple of  $t$ ) in the direction of  $L$  will map  $T$  onto itself. There is also a glide in this case consisting of a  $t/2$  translation (or any integer multiple of  $t/2$ ) in the direction of  $L$  followed by a reflection through  $L$ . Note that because different integer multiples of the period  $t$  ( $t/2$ ) give rise to different translations (glides),  $H_T$  has an infinite number of elements. A symmetry group that has an infinite number of elements is called an infinite symmetry group. Likewise, a finite symmetry group is one with only a finite number of elements.

In this section we have introduced the basic theory behind plane symmetry groups. In the next three sections we will see how to use this theory to classify the symmetry of different figures and designs in the plane.

## Section 2

### SYMMETRY GROUPS FOR PLANE BOUNDED FIGURES

A bounded figure in the plane is one which can be encompassed by a circle of finite radius. In this section we classify the types of symmetry groups, that is the sets of motions  $H_T$ , that are possible for plane bounded figures. The task is easier than it might first appear. Translations (and glides) cannot be motions in the symmetry group of a set  $T$  which represents a bounded figure. A simple proof of this fact can be found in [5]. Therefore, in dealing with plane bounded figures, we need only consider rotations and mirrors.

Consider the symmetry group of an equilateral triangle:

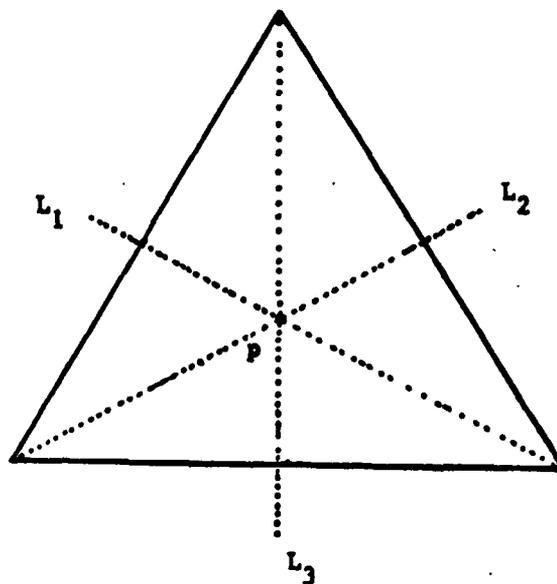


Figure 2.1

$H_T = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$  where

$\alpha_1$  = identity map

$\alpha_2$  = rotation  $120^\circ$  clockwise around  $p$

$\alpha_3$  = rotation  $240^\circ$  clockwise around  $p$

$\alpha_4$  = reflection through  $L_1$

$\alpha_5$  = reflection through  $L_2$

$\alpha_6$  = reflection through  $L_3$

We call this symmetry group a dihedral group and write  $H_T = D_3$ . The order of a group is simply the number of elements in the group. Therefore, the order of  $D_3$  is 6. A part of  $D_3$  that we are especially interested in is the subgroup  $C_3 = \{\alpha_1, \alpha_2, \alpha_3\}$ . Note that  $C_3$  can be generated by repeated compositions of the single rotation  $\alpha_2$ :

$$C_3 = \{\alpha_2, \alpha_3, \alpha_1\} = \{\alpha_2, \alpha_2 \circ \alpha_2, \alpha_2 \circ \alpha_2 \circ \alpha_2\} = \{\alpha_2, \alpha_2^2, \alpha_2^3\}$$

A group which is generated by a single element in the group is called a cyclic group. Therefore  $C_3$  is a cyclic subgroup of order 3.

From the development of  $D_3$ , the symmetry group of an equilateral triangle, it is easy to envision a similar development for the symmetry group of a square. We would obtain a dihedral group  $D_4$  with eight elements, (4 reflections and 4 rotations). Once again, the rotations would form a cyclic subgroup, in this case  $C_4$ . More generally the symmetry group of any regular  $n$ -sided polygon is  $D_n$  (with subgroup  $C_n$ ) where [2]:

$C_n$  is a cyclic group of order  $n$ , consisting of clockwise rotations through  $k \left(\frac{360^\circ}{n}\right)$ ,  $0 \leq k < n$ , around a fixed point  $p$ .

$D_n$  is a dihedral group of order  $2n$  and consists of  $C_n$  together with reflections through  $n$  axes that intersect at  $p$  and divide the plane into  $2n$  equal angular regions.

With these definitions of  $C_n$  and  $D_n$  we can now classify all possible finite symmetry groups for plane bounded figures. Specifically, we have the following powerful result:

Theorem 2.1 A finite symmetry group of a plane bounded figure must be either a cyclic group  $C_n$  or a dihedral group  $D_n$ .

An especially well-written proof of Theorem 2.1 is provided by Durbin [2]. We will not discuss the proof here except to note that the word "finite" is important. The circle we discussed in Section 1 is certainly a plane bounded figure, but it cannot be classified as either  $C_n$  or  $D_n$  for finite  $n$ . As we saw earlier, the symmetry group of a circle contains an infinite number of rotations and reflections. The symmetry groups for circular figures are a special case and are called continuous symmetry groups. They are often denoted  $C_\infty$ . Except for circular figures, all plane bounded figures have finite symmetry groups and Theorem 2.1 applies.

While  $C_n$  and  $D_n$  (and  $C_\infty$  for circular figures) classify the symmetry for bounded figures, we still need to address the more difficult unbounded case. That's the next challenge to be taken up in Sections 3 and 4 where we will look at patterns (figures that repeat themselves at regular intervals in the plane). Before moving on however, let's consider some especially beautiful examples of plane bounded figures. The cardioids and roses that follow were derived from

Dr. Peter M. Mauer's recent work in computer graphics [9]. A special thanks to Lt Colonels William J. Riley and Robert L. James, Office of the Dean of the Faculty, U. S. Air Force Academy, Colorado, for programming Dr. Mauer's algorithm and actually generating the illustrations.

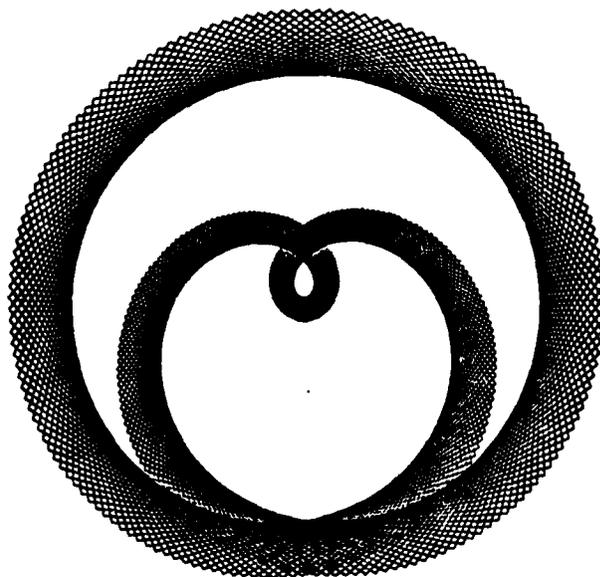


Figure 2.2

"Spiral of Archimedes"

Symmetry:  $D_1$  (Bipolar Symmetry)

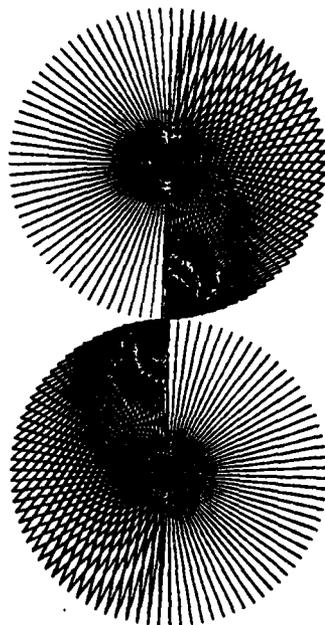


Figure 2.3

Symmetry:  $C_2$

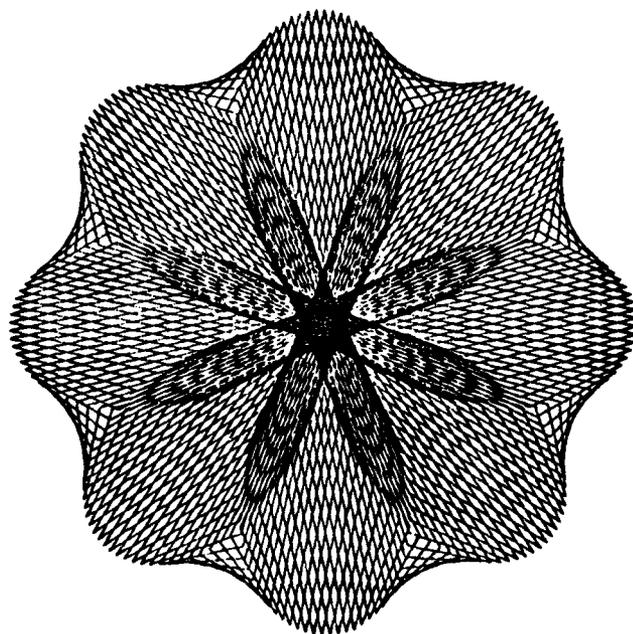


Figure 2.4  
Symmetry:  $D_8$

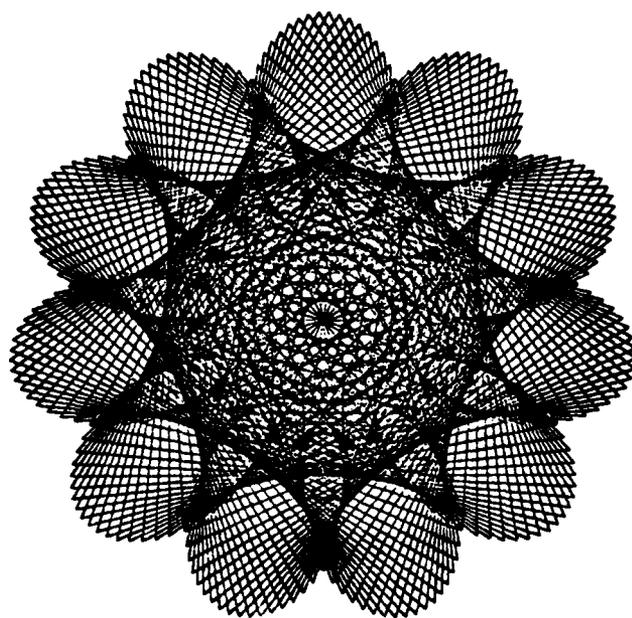


Figure 2.5  
Symmetry:  $D_{11}$

### Section 3

#### FRIEZE GROUPS

A frieze is any decorative strip or border that contains lettering, sculpture, pictures, etc. (In classical architecture, the frieze is that part of the entablature between the architrave and the cornice.) From our group symmetry point of view we are interested in those two-dimensional designs located in a frieze that repeat themselves at regular intervals. We assume these designs continue infinitely in both directions along a straight line. The footsteps we encountered in example 1.2 are a good example of a frieze pattern.

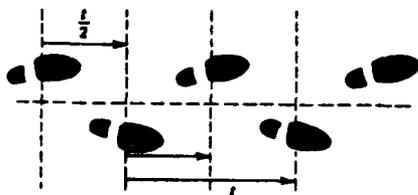


Figure 3.1

Like all frieze patterns, the symmetry group of the footprints is an infinite symmetry group. However, note that the footprints do have a minimum translation period, in this case  $t$ . The existence of a minimum translation period identifies

the pattern as having what is called a discrete symmetry group. This is not always the situation, as when we consider the stripe pattern depicted below:

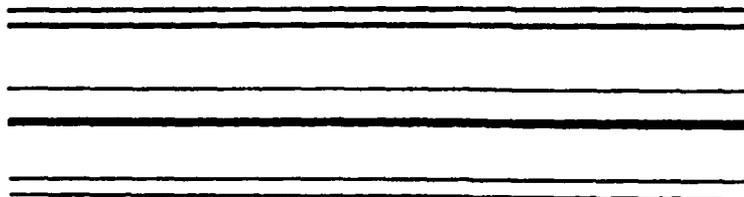


Figure 3.2

A stripe pattern has no minimum translation period and we say that its symmetry group is continuous (this is really the same idea we encountered with circles when dealing with bounded figures--there no minimum rotation existed). We will assume for the rest of this paper that we are dealing with only discrete symmetry groups.

The symmetry group of a frieze pattern is called a frieze group, and there are exactly seven types of frieze groups [8]. This classification is based on the fact that the only motions possible for a frieze pattern are:

- translations along a fixed line  $L$
- $180^\circ$  rotations about points on  $L$
- a horizontal mirror through  $L$
- vertical mirrors perpendicular to  $L$
- glides with respect to  $L$

Every frieze group must have translations, but it is the presence or absence of the other motions that defines the symmetry. The seven types of frieze groups are depicted in the following illustrations taken from John R. Durbin's book [2], "Modern Algebra: An Introduction."

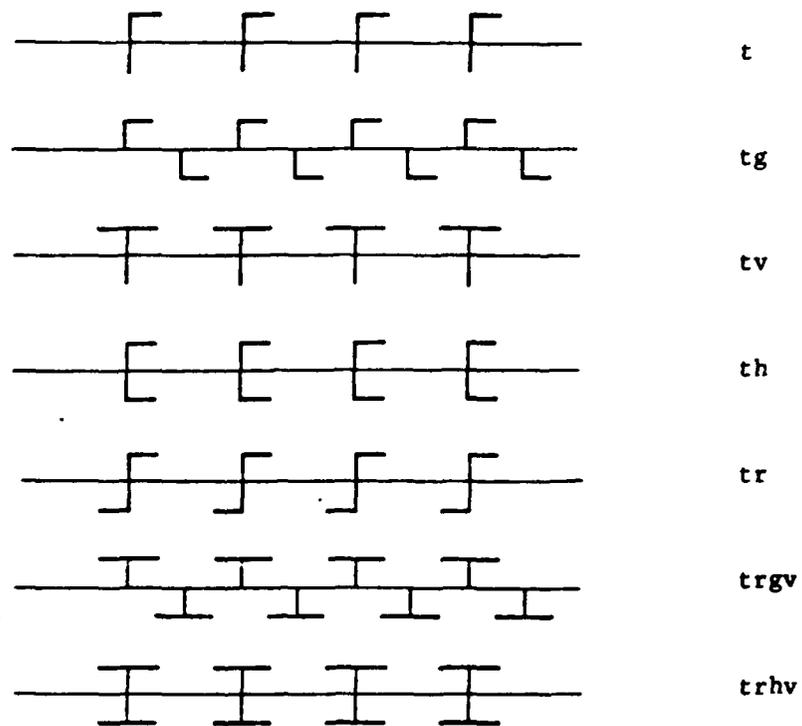


Figure 3.3

Key: t - translations  
 g - glides  
 v - vertical mirrors  
 h - horizontal mirror  
 r - rotations

Doctors Bruce Rose and Robert Stafford have recently created a simple algorithm to aid in classifying frieze patterns [10]. With slight modification, the algorithm is as follows:

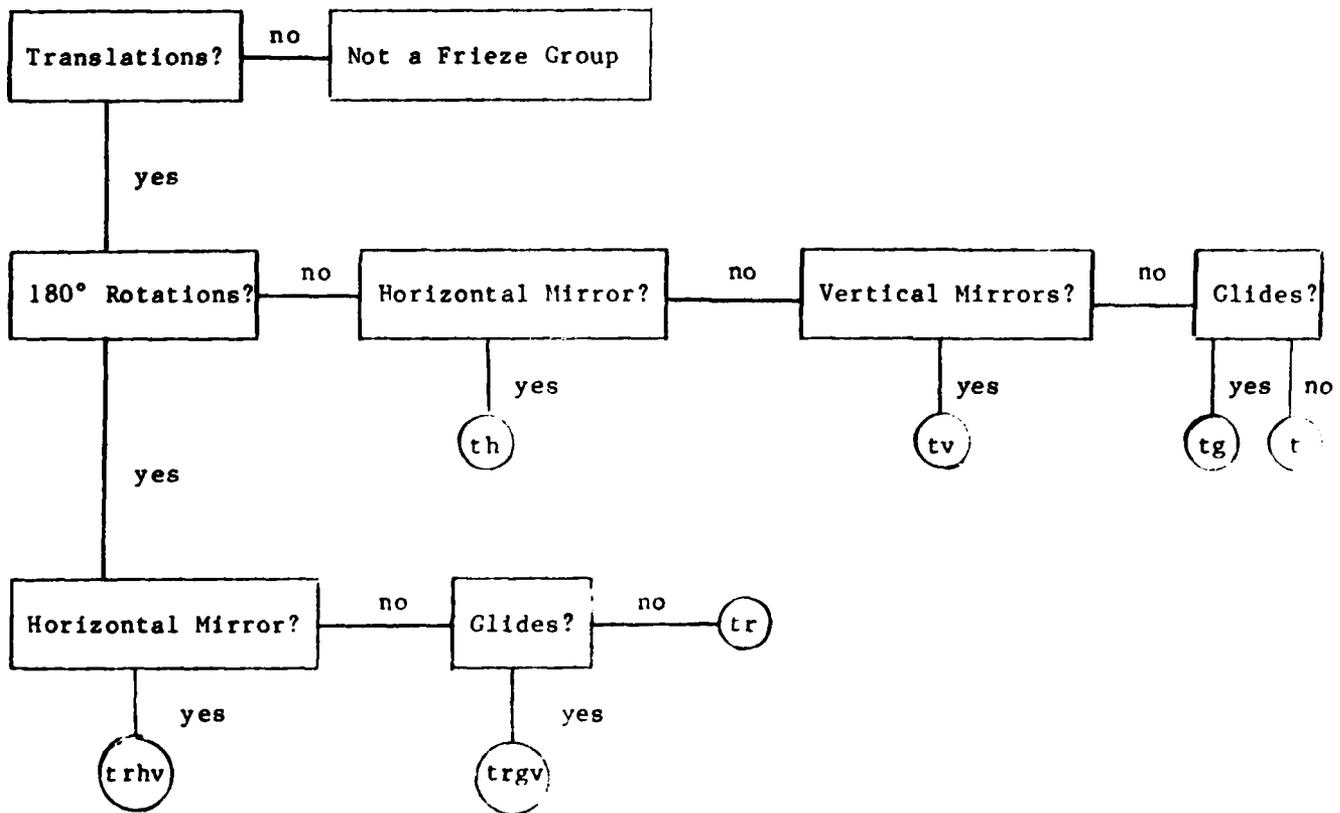


Figure 3.4

Example 3.1

Classify the symmetry group  $H_T$  for the graph of  $f(x) = \sin x$ .

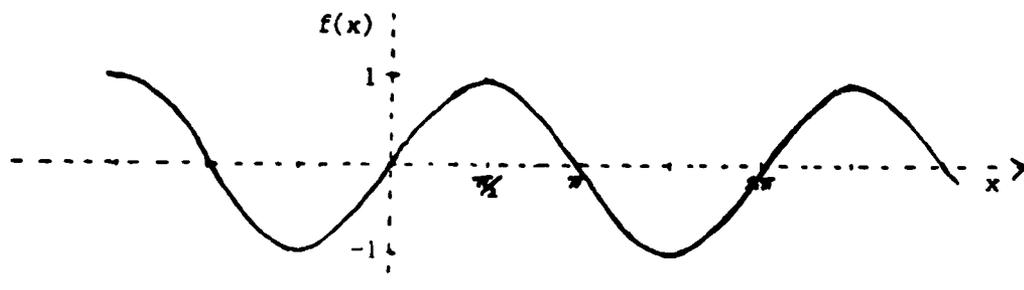


Figure 3.5

While translations along the  $x$ -axis (minimum period of  $2\pi$ ) are obvious, there are also  $180^\circ$  rotation points along the  $x$ -axis at  $\pm n\pi$ ,  $n$  an integer. Using the algorithm in Figure 3.4 we would next ask if  $H_T$  contains a horizontal mirror through the  $x$ -axis. It doesn't, but we do observe that a translation of  $\pi$  units along the  $x$ -axis followed by a reflection through that axis is a member of the symmetry group. Therefore, glides are elements of  $H_T$  and we conclude that the symmetry type is "trgv".

We conclude this section with seven illustrations taken from Owen Jones' classic "The Grammar of Ornament," first published in 1856 [6]. As an example of the impact color has on symmetry, notice that the shading in the "trhv" illustration doubles the minimum translation period. Polychromatic Symmetry is a fascinating field. For those interested in learning about this subject, one of the most enjoyable places to start is Caroline MacGillavry's book [7], "Fantasy and Symmetry: The Periodic Drawings of M. C. Escher".

Key to Figure 3.6:

- t - Medieval Stained Glass - Cathedral of Bourges
- tv - Medieval Stained Glass - Cathedral of Bourges
- tr - Persian Manuscript - British Museum
- trhv - Persian Manuscript - British Museum
- th - Medieval Stained Glass - Cathedral of Bourges
- tg - Persian Manuscript - British Museum
- trgv - Greek Vase - British Museum or the Louvre

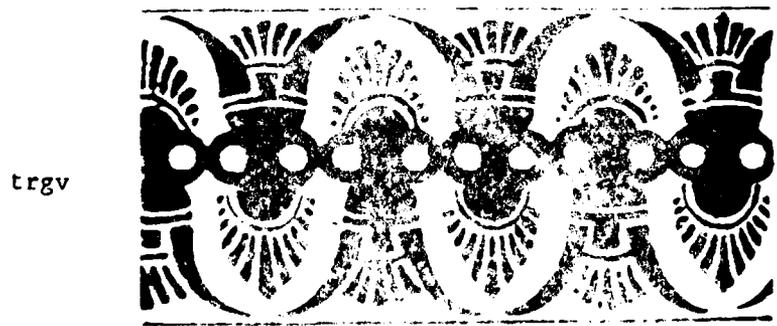
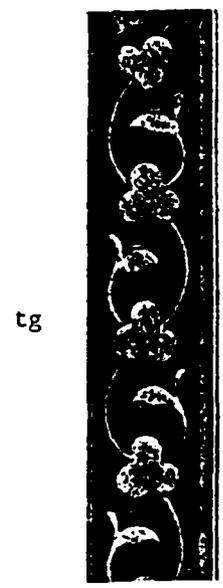
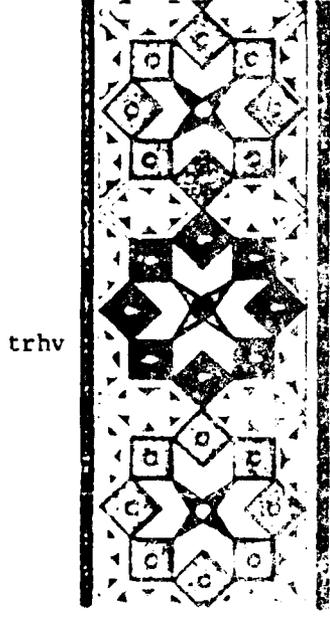
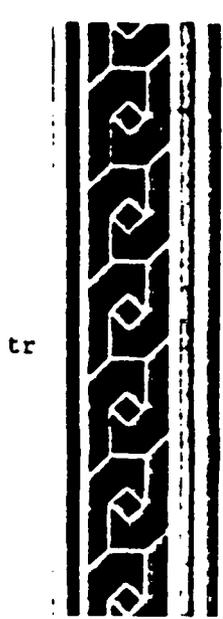


Figure 3.6  
-21-

Section 4

WALLPAPER GROUPS

Wallpaper patterns are those patterns in the plane that repeat themselves at regular intervals in two non-parallel directions.

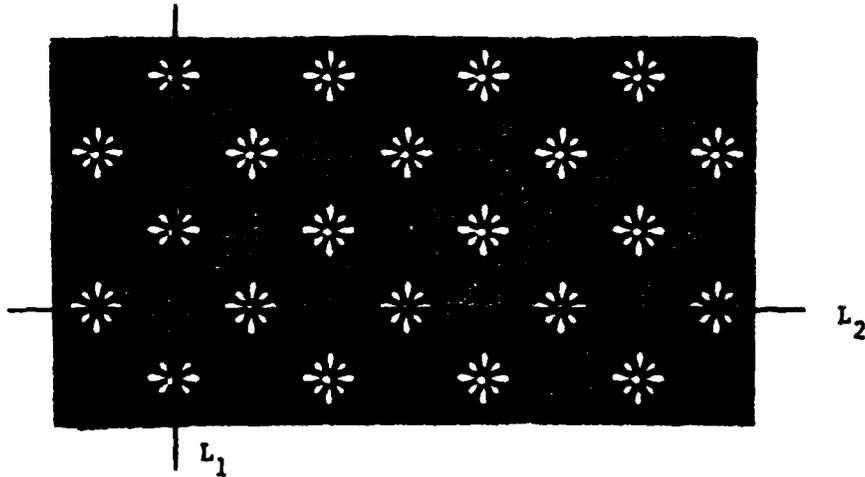


Figure 4.1

The above pattern is a reduced copy of an actual wallpaper sample. Notice that we have independent translations along the two lines  $L_1$  and  $L_2$ . We assume the wallpaper design repeats itself infinitely, filling the entire plane. We call the symmetry group of a wallpaper pattern a wallpaper group. As with a frieze group, a wallpaper group is an infinite symmetry group. The key to classifying wallpaper groups was unlocked in the 1890's by the Russian crystallographer E. S. Fedorov: he determined there are only 17 types of wallpaper groups. We will not take up the proof of Fedorov's assertion, except to say that at the heart of the proof lies one of the most elegant and useful tools found in any branch of mathematics, the Crystallographic Restriction. The Crystallographic Restriction tells us that the only nontrivial rotations possible in a wallpaper group are rotations of  $60^\circ$ ,  $90^\circ$ ,  $120^\circ$ , and  $180^\circ$ .

The best informal discussions that explain the Crystallographic Restriction and why there are 17 types of wallpaper groups are given by Durbin [2] and Schattschneider [11]. For those with a hearty background in mathematics, a full group theory development is given by Schwarzenberger [12].

In theory, determining the symmetry type of a given wallpaper pattern should be easy and straightforward. In reality, however, the symmetry type can often be devilishly obscure. Therein lies the challenge and fun. Fortunately, there are some marvelous aids to help us analyze complex designs. Schattschneider has compiled a useful table for classifying wallpaper patterns [11]. Virtually the same table has been put into algorithm form by Drs Rose and Stafford [10] and is reproduced from Durbin's book [2] in Figure 4.2.

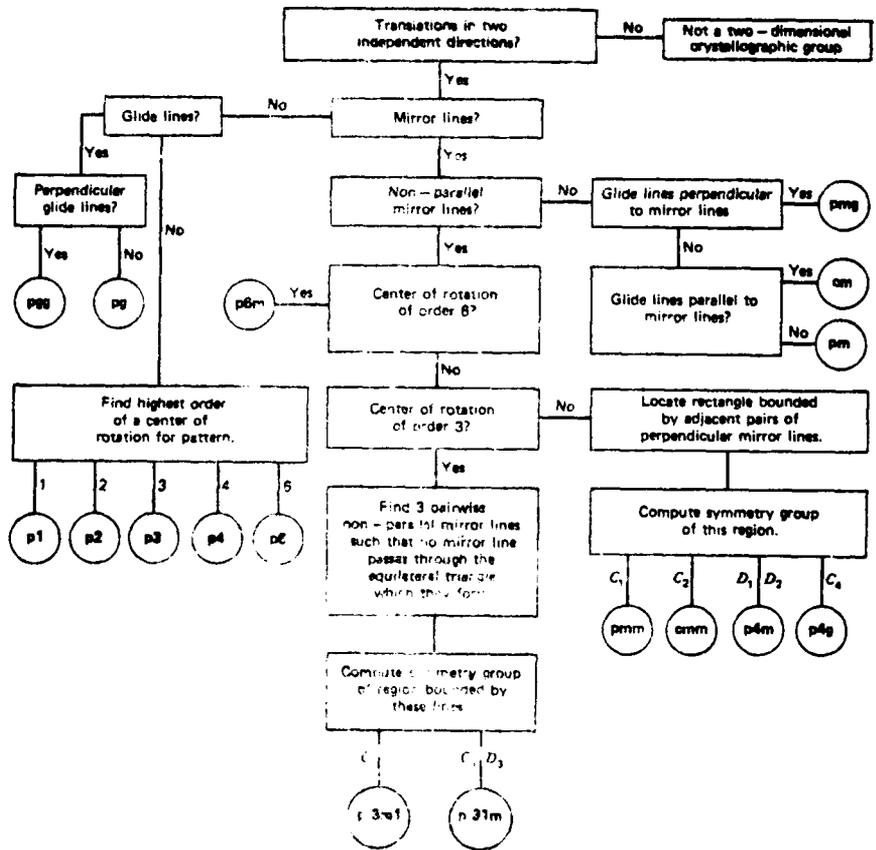


Figure 4.2

In Figure 4.2 the symbols used for the 17 types of wallpaper groups (also called two-dimensional crystallographic groups) are those most commonly accepted and come from a coding system designed by crystallographers. A full explanation of the symbols is given by Schattschneider [11].

Example 4.1 Classify the symmetry group  $H_T$  for the following illustration taken from Owen Jones' book [6], "The Grammar of Ornament."

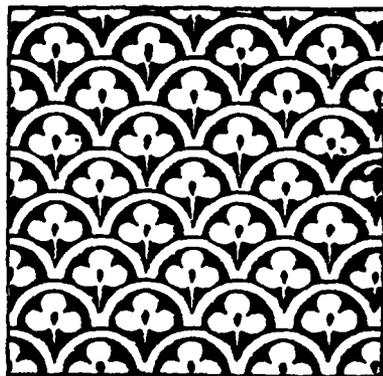


Figure 4.3

This pattern is easy to analyze using the algorithm in Figure 4.2. First, note that there are vertical mirror lines through the center of each leaf. By observation, these are the only mirror lines for this pattern. Therefore, non-parallel mirror lines do not exist. We next ask if there are horizontal glide lines (perpendicular to the vertical mirror lines). The answer is no, since any horizontal reflection would have to change the "arches" from being concave down to being concave up. This brings us to the final question: Are there vertical glide lines? Careful observation tells us that there are if we shift the pattern vertically half a period and then reflect it through lines like the one depicted in figure 4.4. Figure 4.5 traces the decision process we have followed and we conclude that the pattern has symmetry "cm".



WALLPAPER GROUPS

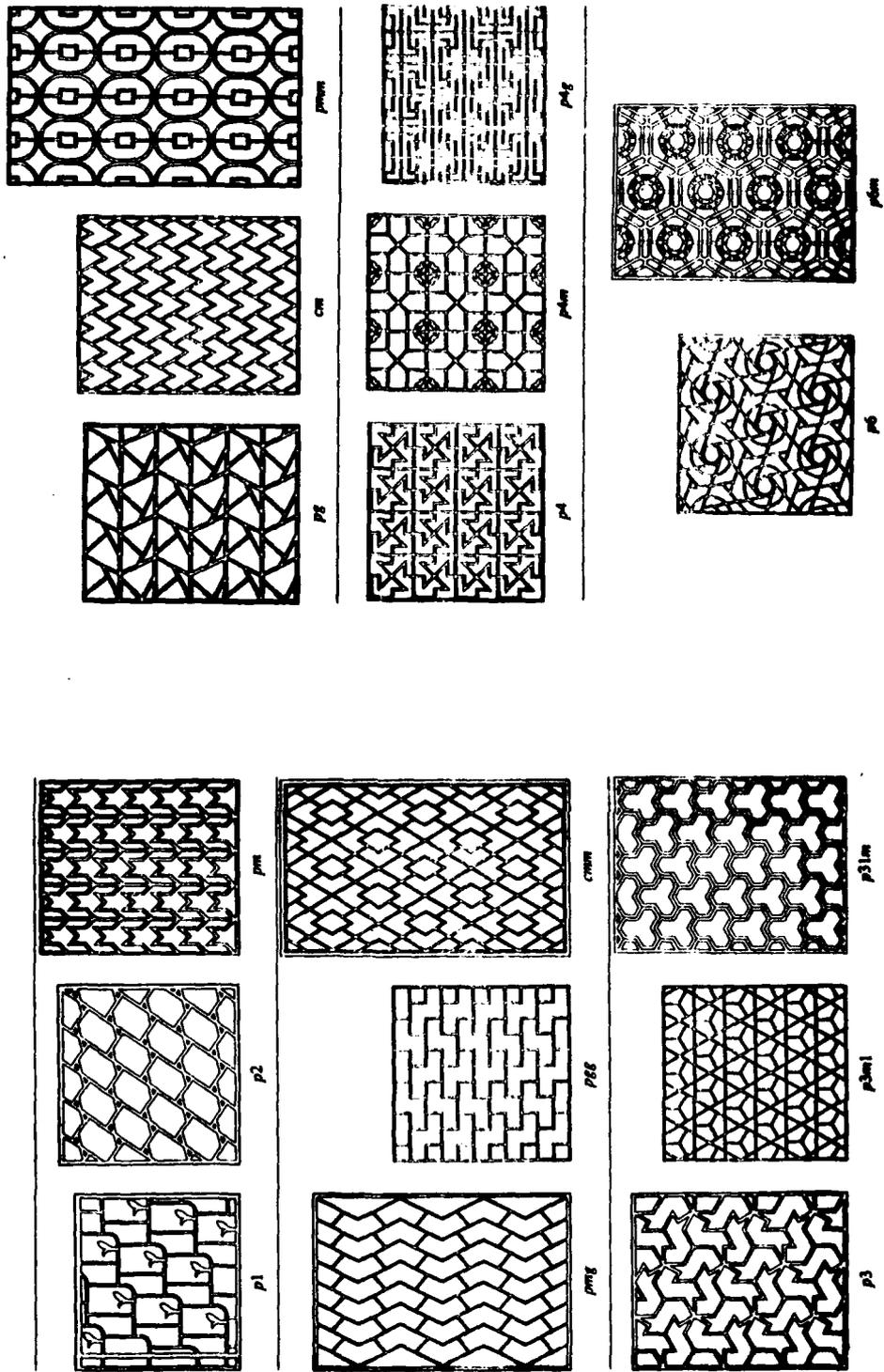


Figure 4.6

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## SYMMETRY AND CRYSTALLOGRAPHY

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The symmetry operations relevant to crystallography are discussed. In particular, the seven crystal systems that classify the thirty-two crystallographic point groups are described. These are then used to construct the Bravais lattices.

Many are fascinated by the beauty of crystals, those objects of polyhedral shape bounded by flat faces. One observes several elements of symmetry when studying the surfaces of large crystals, but most are surprised to learn that it is not that a crystal possesses certain symmetry elements, but that symmetry defines the crystal system being studied. The purposes of this paper are: 1) to define crystallinity and some reasons for studying it; 2) to describe the symmetry elements which define crystals; and 3) to describe the different crystal lattice systems that symmetry does allow.

In defining a crystal, one must be aware that the much-admired large, beautiful single crystals found naturally are not the only substances that can be classified as crystals. Erosion or intentional destruction can obliterate the flat faces and polyhedral shapes of these crystals, yet what remains is still crystalline. In addition, many materials do not exist as single crystals, but exist as an aggregate of thousands of microscopic crystals; these materials are called polycrystalline. A definition of crystallinity which includes the different types of materials listed above was stated simply by Barrett: A crystal consists of atoms arranged in a pattern that repeats periodically in three

dimensions [1]. This periodic arrangement of atoms is what distinguishes crystalline solids from amorphous solids. The "pattern" in the definition can consist of a single atom, a group of atoms, a single molecule, or a group of molecules.

The reason crystals are formed can be explained by energetics. Consider the cooling of a liquid. When the temperature of the liquid drops below its freezing point, the molecules of the liquid no longer have sufficient kinetic energy to overcome the intermolecular forces among the molecules. The molecules assume fixed orientations and positions with respect to one another, and solidification occurs. As each molecule joins the growing solid particle, it is oriented so as to minimize the forces acting upon it. Each molecule entering the solid phase is influenced in almost exactly the same way as the preceding molecule; therefore, the solid particle consists of a three-dimensional ordered array of molecules, i.e., it is a crystal.

Because the three-dimensional array of molecules in a crystal is ordered, one can consider any point within the crystal. Moving one's reference in a straight line from this point will eventually take one to a new point with an environment identical to the original point. This can be repeated until one reaches the surface of the crystal. This array of points is actually periodic in the direction defined by any line connecting two identical points, but all of the directions can be described by taking vector sums of two arbitrary nonparallel vectors.

To aid discussions and calculations it is convenient to choose some points and axes of reference. This may be done by choosing one point at random, and then considering all points identical

with this point. In three dimensions, this set of identical points is called a "lattice". If the lattice points are connected by straight lines, the three-dimensional space can be divided into equal parallelepipeds. Note that choosing any one of these parallelepipeds and repeatedly translating it generates the whole crystal. The generating parallelepiped is called a "unit cell". A unit cell is always a parallelepiped, and it is a template for the structure. By knowing the arrangement of atoms within one unit cell, one knows the atomic arrangement for the entire crystal.

The size and shape of a unit cell may be specified by means of the lengths  $a$ ,  $b$ , and  $c$  of the three independent edges and the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  between these edges. The angle  $\alpha$  is between  $b$  and  $c$ ,  $\beta$  is between  $a$  and  $c$ , and  $\gamma$  is between  $a$  and  $b$ . These axes define a coordinate system appropriate to the crystal. In some respects it would be simpler to always use a Cartesian coordinate system, but the advantages of a coordinate system based on the lattice vectors outweigh the simplicity of Cartesian geometry.

Any parallelepiped whose edges connect lattice points is a valid unit cell, and there are an infinite number of such possibilities. It is even permissible to have lattice points inside a unit cell. A unit cell with lattice points only at the corners is called "primitive"; otherwise the cell is said to be "centered". As with the case of the selection of coordinate axes, the decision to select a centered or primitive unit cell depends on which would provide a greater advantage in defining a particular crystal.

The study of crystals provides one with chemical information about a substance. As stated earlier, it would in some respects be

simpler to define crystalline axes based on Cartesian coordinates. However, this would be useful only if the crystal's environment remained constant. For example, if a crystal's axes were defined by Cartesian coordinates, could one be assured that the lengths  $a$ ,  $b$ , and  $c$  would change proportionately upon heating? Or, would the angles remain at  $90^\circ$ ? Because crystals are based on symmetry rather than Cartesian coordinates, this problem does not arise. If two directions in a crystal are equivalent by symmetry, they necessarily have the same thermal expansion coefficients, and will remain equivalent with changing temperature. Besides thermal expansion, other physical properties of crystals which depend on direction include electrical conductivity, magnetic susceptibility, elasticity, and optical properties [2]. Determination of these physical properties can only be performed on crystalline substances. In contrast, an amorphous solid, such as glass, which has its atoms arranged randomly, cannot have the same forces interacting between atoms. Therefore, glass shatters rather than cleaves along crystalline planes. Glass softens and eventually liquifies rather than possesses a sharp melting point.

Because crystals are defined by symmetry, it is best at this point to define the term "symmetry" and describe the symmetry operations which pertain to crystals. Sands described symmetry as follows: An object or figure is said to have symmetry if some movement of the figure or operation on the figure leaves it in a position indistinguishable from its original position [2]. Table I summarizes the symmetry operations that leave at least one lattice point unchanged. Detailed descriptions of Table I can be found elsewhere [2-4]. The Schoenflies symbol for symmetry elements is

generally used by spectroscopists, whereas the Hermann-Mauguin symbol is used by crystallographers. Because crystalline arrays are considered to be infinite, crystals also possess translational symmetry which leaves no points unchanged.

Because the symmetry elements which are important to the study of crystals have been defined, it is now possible to describe the different crystal systems allowed by symmetry. Consider the development of a lattice for a crystal having a twofold, i.e.,  $180^\circ$ , rotation axis. The rotation axis can intersect a plane at a point arbitrarily labelled  $(0, 0, 0)$ .

The direction of the twofold axis will be called the  $y$  direction. Any point on this axis will have the coordinates  $(0, y, 0)$ . Consider also two other lattice points  $(x', y', z')$  and  $(x'', y'', z'')$ ; the  $x$  and  $z$  coordinates refer to arbitrary axes, which are not necessarily lattice vectors. They are chosen, for convenience, normal to the  $y$  axis. It should be obvious that if lattice point  $(x', y', z')$  exists, the twofold axis through the origin will generate lattice point  $(-x', y', -z')$ . Because lattice points are equivalent, one must consider the twofold axis also passing through these points, which further generates other lattice points. New lattice points can be generated by taking sums or differences of the coordinates of lattice points. For example, the sum of the coordinates of the two lattice points cited above yields the lattice point  $(0, 2y', 0)$ . This coordinate shows there exist an infinite number of lattice points along the  $y$  axis, at integral multiples of  $2y'$ . Scaling things as nicely as possible, let one be the distance between adjacent lattice points in the  $y$  direction, and choose the  $b$  axis as the vector between these adjacent lattice

points; the coordinates of successive points become  $(0, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 2, 0)$ , etc. If  $n$  is an even number,  $y'$  is an integer, and the lattice point  $(x', y', z')$  may be written  $(x', m, z')$  where  $m$  is an integer. A new lattice point may be generated by the difference  $(x', m, z') - (0, m, 0) = (x', 0, z')$ . If  $n$  is an odd number,  $y'$  is a half-integer, i.e.,  $1/2, 3/2$ , etc. In this case, a lattice point  $(2x', 2y', 2z')$  could be rewritten  $(2x', m, 2z')$ , and the difference  $(2x', m, 2z') - (0, m, 0) = (2x', 0, 2z')$  generates another lattice point. Since either  $(x', 0, z')$  or  $(2x', 0, 2z')$  is a lattice point, and since any point with a zero  $y$  coordinate is in the plane referred to originally in this discussion, the vector from the origin to this point is a lattice vector perpendicular to the  $b$  axis. One can set  $z' = 0$ , and  $x'$  is either an integer or a half-integer. By similar reasoning, either  $(x'', 0, z'')$  or  $(2x'', 0, 2z'')$  is a lattice point. One therefore discovers two lattice vectors perpendicular to  $b$ , which can be called  $a$  and  $c$ . As a consequence of the presence of a twofold rotation axis, it is possible to choose unit cell edges so that  $\alpha = 90^\circ$  and  $\gamma = 90^\circ$ . This choice of axes is also possible if the symmetry operation is a mirror plane rather than a twofold axis. In this case the unique axis is perpendicular to the mirror plane. Furthermore, point group  $C_{2h}$ , which contains a  $C_2$  axis and a perpendicular mirror plane, also requires the existence of two lattice vectors normal to the unique axis.

The requirement of two lattice vectors normal to the unique axis characterizes the monoclinic system. A crystal is said to be monoclinic if symmetry elements are present such that it is possible to pick a unit cell that has  $\alpha = 90^\circ$  and  $\gamma = 90^\circ$ , with no

other conditions on the dimensions and shape of the cell. The point groups that impose these, and only these, restrictions on the lattice vectors are  $C_2$ ,  $C_s$ , and  $C_{2h}$ , and these are the monoclinic point groups. It is not sufficient to define the monoclinic system by stating  $a \neq b \neq c$ ,  $\alpha = \gamma = 90^\circ \neq \beta$ . It is the fact that  $\alpha = \gamma = 90^\circ$  by virtue of symmetry that characterizes the system as monoclinic.

Two point groups impose no restrictions on the lattice symmetry. These are  $C_1$  and  $C_i$ , which characterize the triclinic system.

Point groups  $D_2$ ,  $C_{2v}$ , and  $D_{2h}$  require there exist three mutually perpendicular lattice vectors. Therefore, it is possible to choose a unit cell with  $\alpha = \beta = \gamma = 90^\circ$  -- the orthorhombic system.

If the point group includes one (and only one) 4 or  $\bar{4}$  axis there exist vectors so that it is possible to choose  $a = b$ ,  $\alpha = \beta = \gamma = 90^\circ$  with  $c$  parallel to the 4 or  $\bar{4}$  axis. This is the tetragonal system.

The presence of a 6 or  $\bar{6}$  axis characterizes the hexagonal system. If  $c$  is parallel to the sixfold axis, the unit cell has  $a = b$ ,  $\alpha = \beta = 90^\circ$ ,  $\gamma = 120^\circ$ .

The presence of one (and only one) 3 or  $\bar{3}$  axis denotes the trigonal system. Two types of lattices occur in the trigonal system, one being identical with the hexagonal system. The other trigonal lattice has  $a = b = c$ ,  $\alpha = \beta = \gamma$ . The lattice is called rhombohedral, and the threefold axis is along the cell body diagonal.

If the point group includes four threefold axes, the system is cubic. It is possible to choose three equal axes at right angles to each other, and the four body diagonals of the unit cell cube will correspond to the threefold axes.

A summary of the crystallographic point groups defining these systems is given in Table II. This is also given in stereographic projection form, as shown in Figure I [4]. For these projections, the following must be noted: filled points represent atoms in the upper hemisphere, open circles represent atoms in the lower hemisphere, vertical planes of symmetry are represented by straight lines, and tilted planes are represented by curved lines. In addition, rotation axes are represented by the following symbols: 2 -  , 3 -  , 4 -  , 6 -  ,  $\bar{3}$  -  ,  $\bar{4}$  -  ,  $\bar{6}$  - .

The last few paragraphs discussed seven crystal systems allowed by symmetry. Although individual molecules may have, for example,  $C_5$  symmetry, this is not possible in a crystal due to translational symmetry. (A proof of this is given by Sands [2].) There are only thirty-two combinations of symmetry elements possible in a crystal, as was shown in Figure I.

In selecting a unit cell based on symmetry elements, it may turn out that a nonprimitive, or centered, cell is obtained. In the triclinic system no symmetry restrictions occur, so a primitive cell can always be chosen. In other crystal systems, however, centered cells are frequently encountered. In the development of the monoclinic system, by starting with a lattice point  $(x', y', z')$ , it was proved the lattice point  $(0, 2y', 0)$  exists, with  $2y' = n$ . It was also shown that either  $(x', 0, z')$  or  $(2x', 0, 2z')$  is a lattice point, and that the vector from the origin to this point is

a lattice vector perpendicular to  $b$ . If both  $x'$  and  $y'$  are half-integers, the point  $(1/2, 1/2, 0)$  is a lattice point, and the unit cell defined by  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$  is not primitive. A primitive cell may be selected, but it would not be possible in this case to have  $b$  lie on the unique axis and  $\alpha = \gamma = 90^\circ$ . To preserve the advantages of a unit cell chosen on the basis of symmetry, a centered cell is chosen. This unit cell is called C centered; the centering is on the C face - the face of the unit cell bounded by the  $a$  and  $b$  axes. There are lattice points at  $(0, 0, 0)$  and at  $(1/2, 1/2, 0)$ . Points differing from these by  $l, m, n$ , where  $l, m$ , and  $n$  are integers, are also lattice points.

In considering nonprimitive cells, one must be aware of the number of lattice points per unit cell. The primitive unit cell has eight lattice points at each of its vertices. Each lattice point is shared by seven other unit cells; therefore,  $8 \times 1/8 = 1$  lattice point is in a primitive cell. The C centered unit cell has two lattice points in a plane shared by one other cell, in addition to the eight points at the vertices; therefore, the C centered cell has  $(8 \times 1/8) + (2 \times 1/2) = 2$  lattice points. In general, a unit cell containing  $n$  lattice points has a volume  $n$  times the volume of a primitive cell in the same lattice. The volume of a C centered cell is, therefore, twice the volume of the corresponding primitive cell. In addition, a unit cell could be body centered, i.e., it has a central lattice point unshared by other cells in addition to its eight lattice points at the vertices. The body centered cell has  $(8 \times 1/8) + 1 = 2$  lattice points, and twice the volume of the corresponding unit cell. Finally, a unit cell could have lattice

points in the center of all its faces shared by one other cell, in addition to its eight points at the vertices. This face centered cell has  $(8 \times 1/8) + (6 \times 1/2) = 4$  lattice points, and four times the volume of the corresponding unit cell.

It was shown earlier that a monoclinic cell may be primitive or C centered. Are other distinctive types of centering possible? There could be A centering, but this differs from C centering only in the choice of names for the a and c axes, so this is not distinct. It is possible to construct a body centered monoclinic cell, but when two adjacent unit cells are considered, one may draw different axes which produce a C centered monoclinic cell. With a lattice of monoclinic symmetry, one will always be able to select either a primitive or C centered cell satisfying the monoclinic condition  $\alpha = \gamma = 90^\circ$ . These are the only distinct lattice types consistent with monoclinic symmetry. These considerations may be extended to the other crystal systems. The result is that there are just fourteen of these space lattices. These were first deduced by M.A. Bravais in 1848, and are referred to as Bravais lattices [2]. The fourteen Bravais lattices are shown in Figure II [5].

This report merely touches the surface of the fascinating field of crystallography. In addition to using symmetry to define Bravais lattices, symmetry also defines the 230 space groups that are allowed by translations of the Bravais lattice points, symmetry is used in x-ray crystallographic determination, and other areas of crystallographic study as well. It is hoped the reader will be stimulated to learn more about crystals.

TABLE I. ELEMENTS OF POINT SYMMETRY

Type of element	Description of operation	Examples	
		Schoenflies symbol	Hermann-Mauguin symbol
Rotation axis	Counterclockwise rotation of $360^\circ/n$ about axis	$C_1, C_2, C_3, C_4$	1, 2, 3, 4
Mirror plane	Reflection through a plane	$\sigma$	m
Identity	Rotation of $360^\circ$ about any axis. All objects and geometric figures possess this element	$E = C_1$	1
Center of inversion (center of symmetry)	All points inverted through a center of symmetry	i	$\bar{1}$
Improper rotation axis (rotatory reflection axis)	Rotation of $360^\circ/n$ followed by reflection in a plane perpendicular to the axis	$S_1, S_2, S_3, S_4$	
Improper rotation axis (rotatory inversion axis)	Rotation of $360^\circ/n$ followed by inversion through a point on the axis		$\bar{1}, \bar{2}, \bar{3}, \bar{4}$

TABLE II. CRYSTALLOGRAPHIC POINT GROUPS

Crystal system	Schoenflies symbol	Hermann-Mauguin symbol	Order of group
Triclinic	$C_1$	1	1
	$C_i$	$\bar{1}$	2
Monoclinic	$C_2$	2	2
	$C_s$	m	2
	$C_{2h}$	2/m	4
Orthorhombic	$D_2$	222	4
	$C_{2v}$	mm2	4
	$D_{2h}$	mmm	8
Tetragonal	$C_4$	4	4
	$S_4$	$\bar{4}$	4
	$C_{4h}$	4/m	8
	$D_4$	422	8
	$C_{4v}$	4mm	8
	$D_{2d}$	$\bar{4}2m$	8
	$D_{4h}$	4/mmm	16
Trigonal	$C_3$	3	3
	$C_{3i}$ (or $S_6$ )	$\bar{3}$	6
	$D_3$	32	6
	$C_{3v}$	3m	6
	$D_{3d}$	$\bar{3}m$	12
Hexagonal	$C_6$	6	6
	$C_{3h}$	$\bar{6}$	6
	$C_{6h}$	6/m	12
	$D_6$	622	12
	$C_{6v}$	6mm	12
	$D_{3h}$	$\bar{6}m2$	12
	$D_{6h}$	6/mmm	24
Cubic	T	23	12
	$T_h$	m3	24
	O	432	24
	$T_d$	$\bar{4}3m$	24
	$O_h$	m3m	48

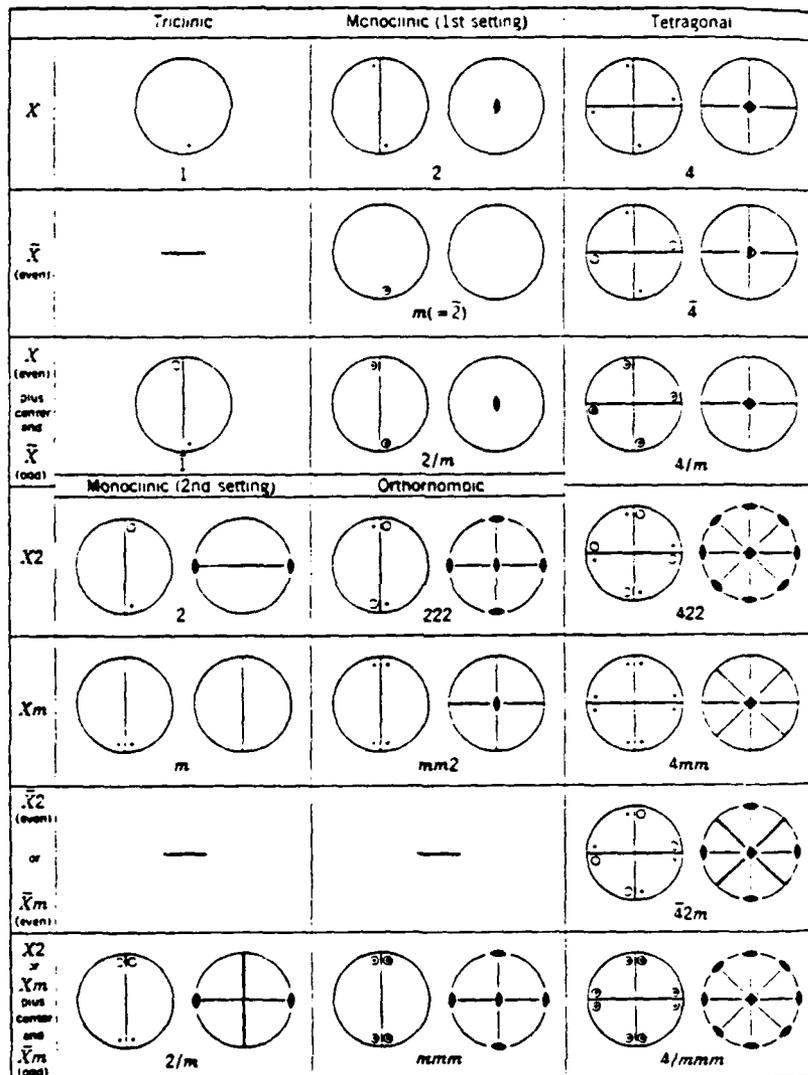
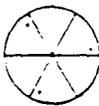
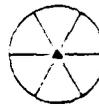
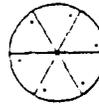
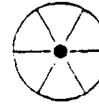
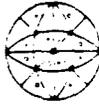
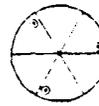
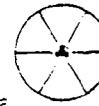
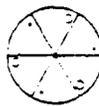
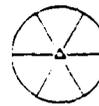
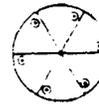
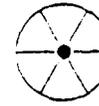
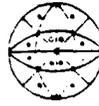
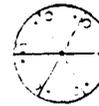
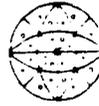
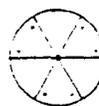
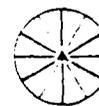
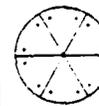
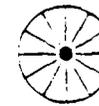
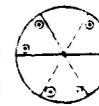
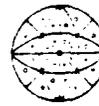
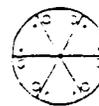
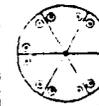
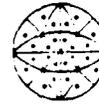


FIGURE I. Stereographic Projections for the Thirty-two Crystallographic Point Groups.

Trigonal		Hexagonal		Cubic		
						X
3		6		23		
—				—		$\bar{X}$ (even)
		6				
						X (even) plus center and $\bar{X}$ (odd)
3		6/m		m3		
						X <sub>2</sub>
32		622		432		
				—		X <sub>m</sub>
3m		6mm				
—						$\bar{X}_2$ (even) or $\bar{X}_m$ (even)
		6m2		43m		
						X <sub>2</sub> or X <sub>m</sub> plus center and $\bar{X}_m$ (odd)
3m		6/mmm		m3m		

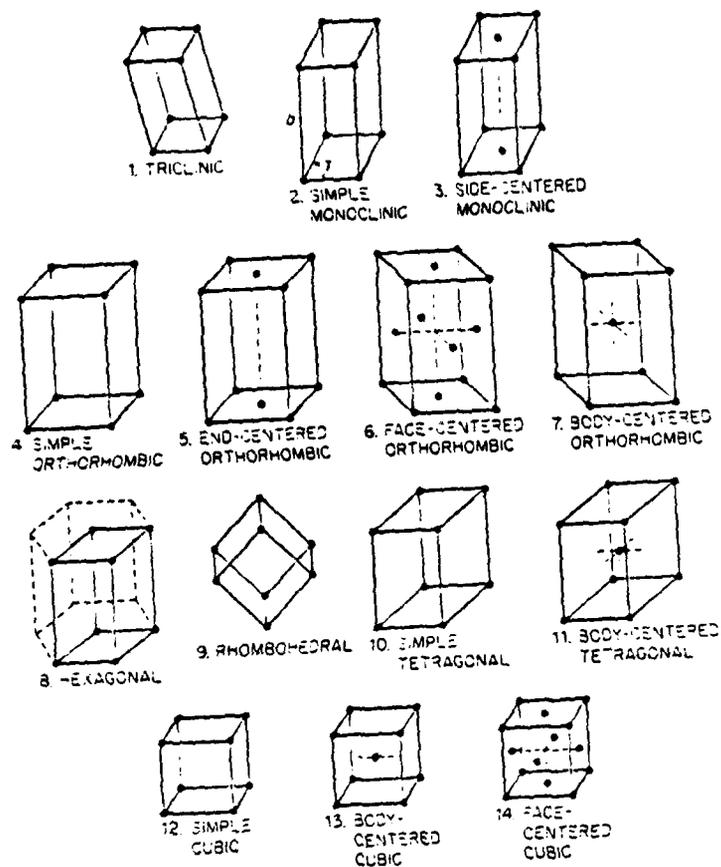


FIGURE II. The Fourteen Bravais Lattices

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## SYMMETRY IN BIOLOGICAL FORMS: THE CHAMBERED NAUTILUS

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Biologists are sometimes reluctant to describe phenomena in terms of known physical laws. D'Arcy Thompson, however, is an exception to that generality. His work on growth and form of molluscan shells is reviewed with an attempt to explain the consequences of that growth and form to the natural history of the Chambered Nautilus and its ancestors.

### 1. INTRODUCTION

As part of scientific endeavor, there is a need to quantify phenomena--weight, mass, time, that is, to conceptualize through numbers. Unfortunately, the biologist, though not in all sub-disciplines of biology, has been slow to call upon mathematics and physical laws to deal with seemingly complex phenomena. More often than not descriptive life scientists invoke ingenious design--a supernatural creator--to explain these phenomena. Enter D'Arcy Wentworth Thompson, whose two volume treatise [4] on growth and form has transformed the way we look at life forms.

Thompson [4, p. 15] contends that growth and form are understood only in terms of physical constraints operating on the expression of the genome of organisms, and one must realize that no living thing survives unless it conforms to the laws of physics and mathematics. Hand in hand with this idea is the realization that growth, in itself an exceptionally complex affair, must be understood before one can

understand form. Growth brings about form in living things--an increase in size, simply, or the development of complex structures from undifferentiated beginnings.

The inducer of growth and its consequent form is force. If we can recognize or deduce the forces acting on organisms, or their parts, during growth, we may not only understand more about these resultant forms, but predict changes in form as the forces vary.

Form is explained in terms of magnitude and direction, while growth involves these same concepts, plus the concept of time. In attempting to explain form one also must consider that as the surface area increases as the square of a linear dimension, the volume increases by a power of three. One can show that growth in volume is almost directly related to an increase in weight or mass, provided the form and specific gravity are unchanged [4, p. 23]. This relationship between volume and weight will allow us to make inferences about growth and form.

Symmetry abounds in the biological world, from the bilateral symmetry of the flatworms, earthworms, and humans, to the radial symmetry of starfishes and hydroids. Another way of looking at symmetry in biological forms takes its roots from the mathematicians' and physicists' definition of symmetry. In this sense, symmetry exists in systems if an operation is performed on that system and the system essentially remains unchanged [3]. This paper will show that, within the broader definition of symmetry described above, the equiangular spiral growth of molluscan shells is such that the symmetry of the shell is maintained and only the scalar dimensions change.

## 2. MATHEMATICS OF SPIRALS

There are two types of spirals described in mathematics--the equable spiral or spiral of Archimedes, and the equiangular spiral, first described by Descartes. The equable spiral is described with a point of origin (O) from which generates a radius vector (R) and a point (P) traveling along the radius vector at a constant velocity [4, p. 752] (Figure 1). What results is akin to coiling a rope on a hard surface. Equiangular spirals, on the other hand, are described if our point (P) moving along the radius vector increases velocity as it moves from the pole (Figure 2). In other words, there is a geometric progression in length of the radius vector not an arithmetic increase.

The equiangular spiral may be described in a number of mathematical ways. First (see Figure 2), the distances measured along the curve from its origin (O) intercepted by various radii (B, C, etc.) are proportional to the lengths of the radii (OB, OC) [4, p. 754]. Another way of mathematically conceiving equiangular spirals is seen in Figure 3. If we resolve growth at any point (P) into a force F acting along the line joining (P) to the origin (O) and a force T acting perpendicular to (OP), and the magnitude of these forces remains constant, then the resultant of the two forces (PQ) will make a constant angle with the radius vector--the very property Descartes discovered [4, p. 757], and thus the name, equiangular spirals.

### 3. SHELL GROWTH IN THE CHAMBERED NAUTILUS

Figure 4 shows a cross-examine of a Chambered Nautilus shell. Unlike other spiral-shelled molluscs whose body mass extends throughout the spiral shell, the Nautilus and its allies (nautiloids and ammonites) reside only in the most recently laid down chamber of the shell. In

nautiloids, there is a small hole which connects the chambers into which extends a living tissue, the siphuncle. The function of the siphuncle putatively is to balance fluids and gas in the chambers to influence buoyancy. Growth in Nautilus occurs as shell material (a proteinaceous matrix embedded with calcium and other ions) is secreted by living tissue. Each increment of growth is similar to the preceding one. Thus it appears that the shape of the organism is not altered with growth. But does this growth truly result in an equiangular spiral, as defined previously?

Aristotle noted that certain things are not altered in shape when growth occurs. He referred to such growth as gnomonic growth. A gnomon is defined as that which remains after a figure has been removed from a similar but larger figure. Figure 5 helps to illustrate this idea. The isosceles triangle ABC, when either of the two base angles is bisected by BD, results in a new triangle, BCD, similar to the original and a gnomon, ABD [4, p. 761]. If we take an isosceles triangle (Figure 6) and add or subtract a series of gnomons, then the apices (A, B, C, etc.) of all the triangles have their loci on an equiangular spiral [4, p. 762, 763]. Returning to the Nautilus, if we see that the growth of the shell is such that each successive addition is similar in shape to the previous (i.e. gnomonic), then the shell must be an equiangular spiral, provided the force driving the rate of growth remains relatively constant. Mosely [1], in fact, confirmed gnomonic growth in Nautilus by taking median transverse sections through the shells and found that the distance of any two whorls measured on a radius vector is 1/3 of the next two whorls, and is thus an equiangular spiral.

#### 4. CONSEQUENCES AND CONSIDERATIONS

At the end of the Cretaceous period of the Mesozoic era 65 million years ago, many groups of plants and animals died out. One of these groups was the ammonites. The genus *Nautilus* is the only remaining group of relatives of those once abundant molluscs. Several hypotheses attempt to account for this large extinction. Since hard parts of organisms tend to fossilize better than soft parts, the ammonites and their nautiloid relatives lend themselves to study. One researcher, Peter Ward [5], has advanced the hypothesis that the extinction of the ammonites (and conversely, the ability of the nautiloids to have survived) is due in part to their inability to adapt to selection pressure in their marine ecosystem. Such pressures would have influenced the form of these groups of organisms (and hence growth) through differential reproductive success.

Ammonoid shells come in a variety of shapes and sizes (most conform to equiangular spirals). Since ammonoids and nautiloids were predators, so long as food sources were abundant in their environment, and they had no significant predators themselves, there was no selective pressure for efficient motility. If, however, food sources became depleted in an area, or if there were increased competition, then pressures may have driven changes in groups of these organisms toward more efficient motility. Investigations [5, p. 139] have uncovered just such a situation in the late Mesozoic era. At that time, more efficient predators (such as marine reptiles, bony fishes, and the like) appear in the fossil record which may have preyed on the ammonoids directly or competed with them for prey. In either case, the selection pressure was

there. If the trend in ammonoid shells can be shown to move toward a more efficient motile structure, then we may have indirect evidence to support Ward's extinction hypothesis.

David Raup [2] has investigated the swimming efficiencies of various shell shapes based on computer modeling (See Figure 7). He considered three parameters: shape of the closed curve generating the shell, given as a ratio of breadth to height of opening (S), rate at which the generating curve expands (W), and rate at which the generating curve moves away from coiling axis (D). To analyze the effects these three parameters have on swimming ability, one must consider attitude control [5, p. 139]. Swimming in ammonoids and nautiloids is accomplished by forcing water through a nozzle of tissue, much like squids and octopuses. In ammonoids (and nautiloids), the center of mass and buoyancy are separated. The greater this separation, the greater the dynamic stability. High W, low D shells (Figure 7) have centers of mass and buoyancy widely separated and are good swimmers (*Nautilus*, e.g.). Low W, high D shells would have centers of mass and buoyancy close together and the water-jet propulsion exhibited by ammonoids and nautiloids in this situation would result in spinning rather than efficient forward movement [5, p. 141]. The fossil records show us that many ammonoids had the low W, high D type shells possibly rendering them unfit to compete in the late Mesozoic. But, a trend is seen in the ammonoid fossils recovered from near the end of the Mesozoic toward the high W, low D type shells suggesting selective pressure at work.

Unfortunately, the adaptation to more efficient swimming (translated into the ability to compete successfully) came too late to stave off the extinction of the ammonoids.

Consequently, the growth and form of ammonoids and nautiloids (the equiangular spiral) influenced their ability to survive in their marine ecosystems. Selective pressures on growth resulted in equiangular spirals in ammonoids that ultimately may have led to their demise. *Nautilus* remains as a relic. As Ward concludes [5, p. 147]: ". . . the time for chambered shells, no matter how well engineered, is past." Thus, the developmental program (genotype) of these organisms that dictated the equiangular spiral pattern of growth resulted in a form which allowed their successful radiation for millions of years, but may have also ultimately led to their downfall.

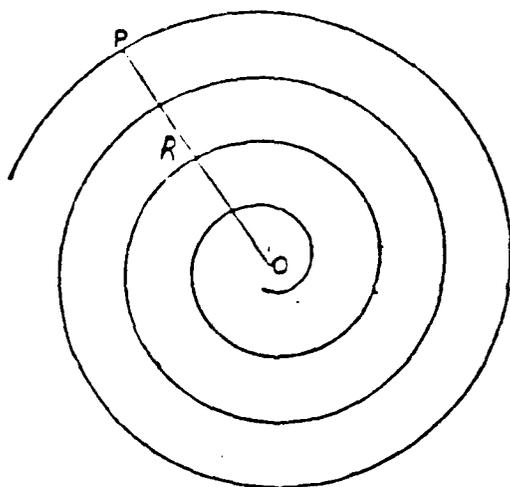


Figure 1. Spiral of Archimedes. [4, p. 753]

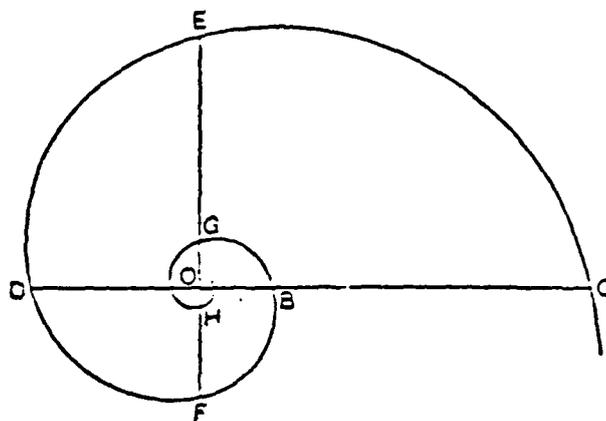


Figure 2. Equiangular spiral. [4, p. 754]

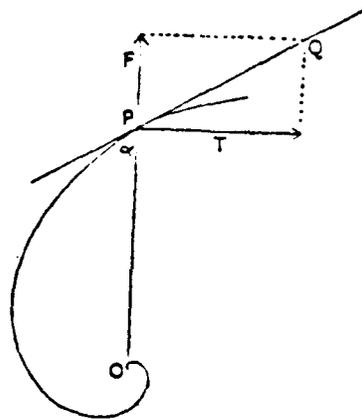


Figure 3. Dynamical aspect of the equiangular spiral. [4, p. 756]

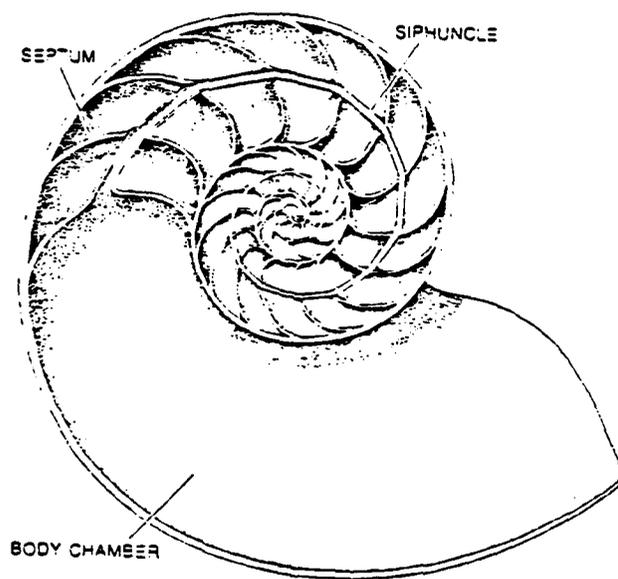


Figure 4. The Chambered Nautilus. [2, p. 138]

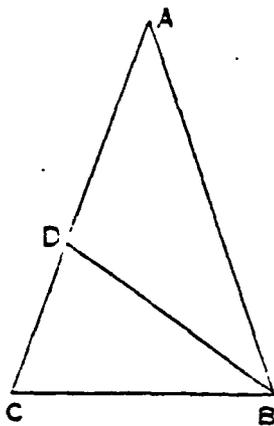


Figure 5. Isosceles triangle with its gnomon. [4, p. 762]

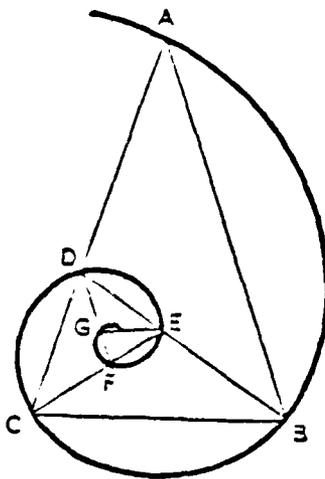


Figure 6. Isosceles triangle with gnomon generated equiangular spiral. [4, p. 762]

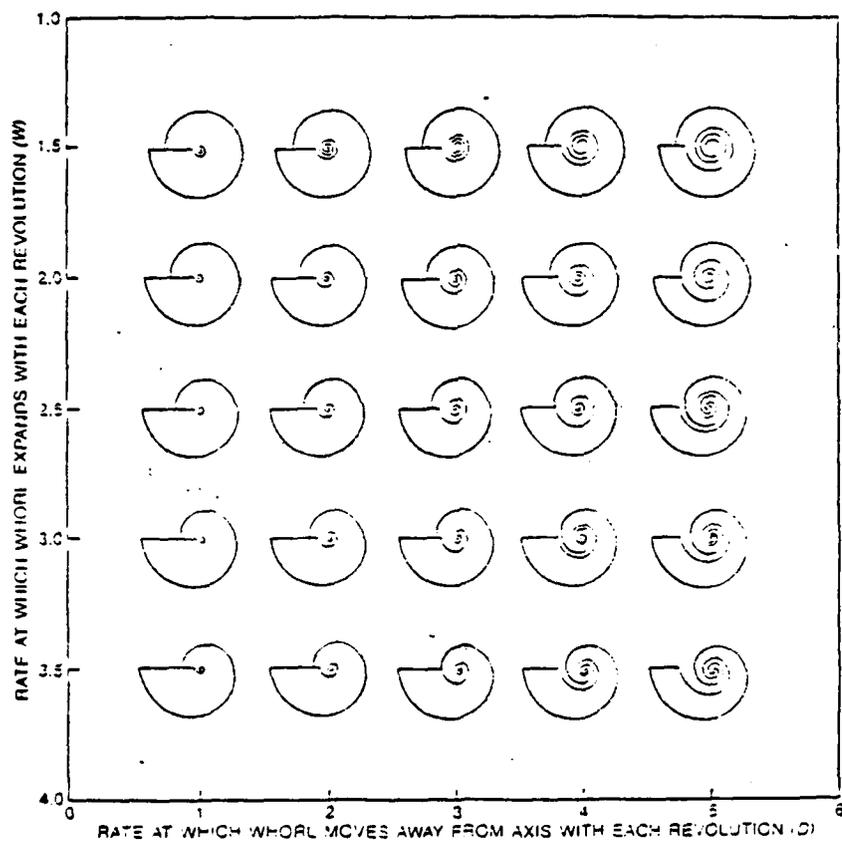


Figure 7. Computer generated shapes of shells.  
 (See text for explanation.) [2, p. 141]

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## SYMMETRY IN PHYSICS

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Symmetry has increasingly played a central role in physics. We examine that role by examining the Principle of Least Action and the invariance of the Lagrangian under a transformation. Noether's Theorem guarantees that a conservation law is associated with each of these symmetries. Examples include the conservation of energy, linear momentum, and angular momentum as well as the purely quantum mechanical symmetry of invariance under an exchange operation. A brief look at gauge theories is the final example of how symmetry has become a guiding principle in the formulation of new theories.

### I. INTRODUCTION.

The noted physicist Richard Feynman stated, "A thing is symmetrical if there is something we can do to it, so that after we have done it, it looks the same as it did before." [1, p. 52-1] The earlier discussions have examined "things" such as wallpaper, seashells, and crystals. We will deal with more abstract quantities like Lagrangians, Hamiltonians and wave functions that model physical systems. Theoretical physicists construct these mathematical models of the physical system so that they can explore the properties of the system without actually having to do experiments on the system. The theoretical physicists predict results that the experimental physicists try to measure to test the theories.

Physical theories have always had symmetries in them, but they were not a central result or fundamental concept until recently. As Eugene P. Wigner, one of the key players in the early development of group theory and quantum mechanics, said, "Symmetry considerations were not thought to be particularly important before this century, and were not well articulated." [2, p. 4] However, modern theories, in developing a model of reality, frequently start with a particular symmetry. The most grandiose theory, Grand Unification

Theory, begins with the assumption that nature is symmetric under the exchange of bosons and fermions in an attempt to unify all known forces into one force.

Noether's Theorem states that for every continuous symmetry transformation under which the Lagrangian is invariant in form, there is a conservation law [3]. Conservation laws are important because they help unravel the behavior of a system. For example, if we know linear momentum is conserved in a collision of two particles, we can relate quantities before and after the collision. A continuous symmetry transformation is a mathematical operation such as a rotation about some point, or a translation in space or time, that is performed on the physical system (or the mathematical quantity that models the system). (The mathematics of continuous transformations lead into the study of Lie groups, which is beyond the scope of our discussion.) What is meant by a "Lagrangian" will become clear later but for the moment we just take it to mean a particular mathematical function. "Invariance in form" means that the mathematical function doesn't change its form when we operate on it. If we begin with a constant, say  $C$ , times a variable, say  $x$ , we will end with a constant, say  $C'$ , times a variable, say  $x'$ . The constants and the variables may have changed but the form has not. The invariance of form means that the laws of physics that govern the behavior of the system remain unchanged by the transformation.

Some of the quantities that are conserved include energy, linear momentum, angular momentum, electric charge, baryon number, lepton family number, isospin, and strangeness. However, not all the symmetries that generate these conservation laws are known. These are some of the mysteries that are yet to be solved.

## II. THE PRINCIPLE OF LEAST ACTION.

At first, the theories that describe how nature behaves seem diverse.

Physicists study nature by examining the interactions between two distinctly different types of things, particles and fields. Newtonian mechanics describes the motion of individual particles while classical statistical physics describes the dynamics of a large ensemble of distinguishable particles. Maxwell's equations describe the dynamics of electromagnetic fields. Einstein's general relativity describes the behavior of gravitational fields. Quantum mechanics retains the distinction between classical particles and fields, calling the particles that make up matter fermions and quantizing the fields into particles called bosons. Yet a common thread in all this diversity is the Principle of Least Action. At the core of the Principle of Least Action is symmetry.

Following Hill [4], we can define an integral,  $S$ , that we call the "action":

$$S = \int_{t_a}^{t_b} \mathcal{L}(x_k, \dot{x}_k, t) dt \quad (1)$$

Here,  $x_k$  represents independent variables that describe the system, while the  $\dot{x}_k$  are dependent variables that, with their first derivatives, describe the physical system. The integrand is the function we call the Lagrangian. It has units of energy, and in a simplistic way can be thought of as the difference between a system's kinetic energy and its potential energy. That difference is the amount of "action" left in the system. Nature appears to dislike higher derivatives, as they are not needed to adequately describe a physical system. The fundamental laws of classical physics result from finding the extremum of this action integral:

$$\delta S = 0 \quad (2)$$

This is usually a minimum condition, hence the name Principle of Least Action, although sometimes we are surprised to find that nature maximizes the action integral. Finding the extremum is the realm of calculus of variations; our problem is finding the system's Lagrangian. Once we have the Lagrangian, we can obtain equations of motion that tell how the physical system modeled by the Lagrangian behaves.

To obtain the Lagrangian, the first approach is to work backwards from a known law of physics. This is how the principle was first discovered. The Principle of Least Action arises in many fields of physics: Hamilton's formulation of classical mechanics, Fermat's principle in optics, and Feynman's path integral formulation of quantum mechanics are three examples. This approach showed the generality of the Lagrangian formulation but failed to show the underlying symmetries.

A more useful approach is to obtain the Lagrangian by restricting the form of the Lagrangian with known symmetries. We then exhaust all possible restricted forms of the Lagrangian to see which form agrees with experiments. We can also postulate new symmetries to further refine the Lagrangian. For example, some particle physicists are proposing a Lagrangian based on the Lie group  $SU(5)$  to model the Grand Unified Theory.

Let's illustrate this approach by constructing the classical Lagrangian of a free particle (a point particle of mass  $m$  whose potential energy is zero) by simply using known symmetries. We know that the laws of physics should be invariant under a translation in space-time, i.e., the laws of physics don't change anywhere in the universe at any time. This is one of the central postulates of Einstein's theory of special relativity. If we restrict our study to low speeds, then we can treat space and time separately. We will need to study the Lagrangian under various transformations, such as a change of coordinate systems. The space and time intervals in two different

coordinate systems can be related by

$$d\vec{r}^2 = d\vec{r}'^2 \quad ; \quad dt = dt' \quad (3)$$

Here, time is the independent variable and the position,  $\vec{r}$ , is the dependent variable. The first derivative of  $\vec{r}$  is the velocity of the particle. Thus the Lagrangian has the form

$$\mathcal{L}(x_k, \dot{x}_k, t) = \mathcal{L}(\vec{r}, \vec{v}, t) \quad (4)$$

The first symmetry is a translation symmetry due to the assumed homogeneity of space. For example, a pendulum should behave the same whether it is in New York or Paris. So we require the Lagrangian to be invariant under an infinitesimal translation of axis which implies that it cannot depend on  $\vec{r}$ :

$$\mathcal{L}(\vec{r}, \vec{v}, t) = \mathcal{L}(\vec{r} + \delta\vec{r}, \vec{v}, t) = \mathcal{L}(\vec{v}, t) \quad (5)$$

We also assume that space is isotropic, i.e., the Lagrangian should be invariant under an infinitesimal rotation of the axes. This is a rotational symmetry,  $C_\infty$ . Thus the Lagrangian cannot depend on a vector, but only on the magnitude of the velocity:

$$\mathcal{L}(\vec{r}, \vec{v}, t) = \mathcal{L}(\vec{r} + \delta\vec{\phi} \times \vec{r}, \vec{v}, t) = \mathcal{L}(\vec{v} \cdot \vec{v}, t) \quad (6)$$

It also shouldn't matter when we perform our experiment. In other words, we assume that time is homogeneous, so that the Lagrangian is invariant under an infinitesimal translation in time and thus cannot depend explicitly on

time:

$$\mathcal{L}(\vec{r}, \vec{v}, t) = \mathcal{L}(\vec{r}, \vec{v}, t + dt) = \mathcal{L}(v^2) \quad (7)$$

We need the Lagrangian to have units of energy, so we "guess" at a form:

$$\mathcal{L}(\vec{r}, \vec{v}, t) = \frac{1}{2} m v^2 \quad (8)$$

This is the classical expression for kinetic energy of a free particle. The key point is that the result was obtained solely from symmetry constraints on the Lagrangian.

Translation invariance (the fact that space is homogeneous) implies that total linear momentum is conserved. If a force is applied to the particle, space is no longer homogeneous. In one direction, the particle "feels" a force. The symmetry is said to be broken and the conservation law is no longer valid. So even the breaking of the symmetry tells us something -- the particle accelerates! The symmetries we find in nature are usually broken symmetries; the manner in which the symmetries are broken provides clues to developing correct theories. Rotational invariance leads to conservation of total angular momentum, a symmetry that is broken by applying a torque to the system. Invariance under time translation tells us that the total energy of the system is conserved. However, in the fuzzy world of quantum mechanics, other symmetries and conservation laws exist that are not so well connected. We will discuss these after a short diversion into the Lagrangians of quantum mechanics.

### III. QUANTUM LAGRANGIANS.

In quantum mechanics, the wave function is a complex-valued function that

contains all that is physically knowable about the system. The Schrödinger wave equation is the nonrelativistic equation that tells how the wave function, and thus the physical system, evolves in time and space. Consider a particle of mass  $m$  that is in a region of space where a potential  $V(\vec{r})$  exists. Let  $\psi(\vec{r}, t)$  be the wave function that describes the particle. The Lagrangian is

$$\mathcal{L}(\vec{r}, \psi, \nabla\psi, \frac{\partial\psi}{\partial t}, t) = i\hbar\psi^*\psi - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - V(\vec{r})\psi^*\psi \quad (9)$$

where  $i$  is the imaginary unit,  $\sqrt{-1}$ , and  $\hbar = 1.0546 \times 10^{-34}$  J-sec. The independent variables are  $\vec{r}$  and  $t$ , while the dependent variables are the wave function and its first partial derivatives in space and time. If we seek the extremum by applying the variational rule

$$\frac{\partial\mathcal{L}}{\partial\psi} - \sum_{x,y,z} \frac{\partial}{\partial r_i} \left[ \frac{\partial\mathcal{L}}{\partial(\frac{\partial\psi}{\partial r_i})} \right] - \frac{\partial}{\partial t} \left[ \frac{\partial\mathcal{L}}{\partial\dot{\psi}} \right] = 0 \quad (10)$$

we obtain the Schrödinger wave equation (SWE):

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2\psi + V(\vec{r})\psi \quad (11)$$

The Lagrangian given in Eq(9) lacks symmetry under a space-time inversion, i.e.,

$$\vec{r} \rightarrow -\vec{r} ; t \rightarrow -t$$

Notice that the spatial derivatives in the Laplacian are invariant under the spatial inversion, but the temporal derivative changes sign. We can construct a Lagrangian that has symmetry under a space-time inversion. If we try

$$\mathcal{L} = -\frac{c^2}{2} \left[ (\nabla\psi)^2 - \left(\frac{\partial\psi}{\partial t}\right)^2 + \left(\frac{mc}{\hbar}\right)^2 \psi^2 \right] \quad (12)$$

we obtain the Klein-Gordon equation, which is the relativistically correct version of the SWE:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi \quad (13)$$

Notice that the derivatives in Eq(12) are all squared, giving rise to a symmetry under the space-time inversion.

If we solve the SWE, we only get positive energy eigenvalues. The Klein-Gordon equation predicts negative energy states, which initially seemed to be extraneous. We now know that the negative energy states predicted by the Klein-Gordon equation represent antimatter, a prediction that is justified by the increased symmetry of this equation over the SWE.

#### IV. PARTICLE EXCHANGE SYMMETRY.

Consider two identical, indistinguishable particles, such as atoms of the same element, that are non-interacting. Then we can represent the total wave function of the system of two particles by the product of the wave functions of each particle:

$$\psi_{12}(\vec{r}_1, \vec{r}_2, t) = \psi_1(\vec{r}_1, t) \psi_2(\vec{r}_2, t) \quad (14)$$

This looks oddly like the probability of two independent events, and indeed the wave function has a probabilistic interpretation (if it is suitably normalized). The square of the magnitude of the wave function represents the probability of finding the two particles at locations  $\vec{r}_1$  and  $\vec{r}_2$  at some time  $t$ . The symmetry is that the probability cannot change if we exchange

these two indistinguishable particles:

$$|\psi_{12}|^2 = |\psi_{21}|^2 \quad (15)$$

Taking the square root, we obtain two different cases. If

$$\psi_{12} = +\psi_{21} \quad (16)$$

the total wave function is said to be symmetric and describes bosons, the particles that quantum mechanics uses to describe fields. An example is the photon that is the quantum of the electromagnetic field. If

$$\psi_{12} = -\psi_{21} \quad (17)$$

the total wave function is said to be antisymmetric and describes fermions. A deeper development shows that fermions have a unique property: only one of them can be in a given quantum state. This is called the Pauli exclusion principle and accounts for the fact that matter takes up space. Examples of fermions include protons, neutrons, and electrons that make up ordinary matter. This is another fundamental result that is due solely to a symmetry.

## V. GAUGE THEORIES

Our final topic is a brief discussion of gauge theories [5]. A gauge is simply a measure like a yardstick or stopwatch. It makes sense that changing the length of our gauge shouldn't change the physical system being measured. This is called a global gauge transformation. For example, we can change the electric potential at every point in space by the same amount and not change the physical laws, because it is only the potential difference that matters.

Thus the system is invariant under such a global gauge transformation. This symmetry leads to conservation of electric charge.

A global transformation on the wave function has the form

$$\psi' = e^{iq\Lambda} \psi \quad (18)$$

Here,  $q$  and  $\Lambda$  are constants. Note that the probability represented by the square of the amplitude of the wave function is unaffected by the transformation, again illustrating the invariance to the global gauge transformation.

The real interest starts when we look at invariance under a local gauge transformation wherein we change the value of the wave function at every point by an amount that depends on the location in space and time:

$$\psi' = e^{iq \frac{\Delta(\vec{r}, t)}{c\hbar}} \psi \quad (19)$$

Here, the gauge  $\Delta(\vec{r}, t)$  is a scalar function of space and time. If we require that the Schrödinger wave equation be invariant in form, we must introduce a field represented by a vector potential  $\vec{A}$  and a scalar potential  $A_0$  into the equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left[ -i\hbar \nabla - q\vec{A} \right]^2 \psi + V(\vec{r})\psi + qA_0\psi \quad (20)$$

This new field is the familiar electromagnetic field, which we are forced to include in quantum mechanics just on the basis of symmetry [6].

The transformation given in Eq(19) is an example of a unitary transformation of dimension one,  $U(1)$ . Yang and Mills extended the concept of local gauge invariance to two dimensions where the gauge function is now a

matrix and the wave function is a vector [7]. Although Yang and Mills attempted to obtain a theory for the strong nuclear force and failed, Steve Weinberg and others later showed this model to be correct for the weak force [5, p.132]. Based solely on this symmetry, they predicted new elementary particles, the intermediate vector bosons, that were found exactly as predicted. It now appears that all four fundamental forces (the weak force, gravitational force, electromagnetic force, and strong force) are described by models that are invariant under a local gauge transformation, another example of how symmetry plays a central role in the description of nature [5].

#### VI. CONCLUSION.

The symmetries that a physicist sees are more subtle than those we observe in everyday life. Symmetries represent transformations on a physical system that, although they have the power to alter the system, leave the system unchanged. Such invariance leads to conservation laws that govern the physical system. The profound role of symmetry in the laws of physics has finally come into its own through the Principle of Least Action and gauge theories. The study of nature, at the fundamental level, is the study of symmetries and the breaking of those symmetries.

Michael Faraday wrote, "We come into this world, we live and depart from it, without our thoughts being called specifically to consider how all this takes place; and were it not for the exertions of some few enquiring minds, who have looked into these things, and ascertained the very beautiful laws and conditions by which we do live and stand upon this earth, we should hardly be aware that there was anything wonderful in it." So it is with symmetry in physics.

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