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Bethesda, Maryland 20084–5000

Differential Forms of Euler's Turbo-Machinery Equation

by

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release;
2b DECLASSIFICATION / DOWNGRADING SCHEDULE	distribution unlimited.

4 PERFORMING ORGANIZATION REPORT NUMBER(S) DTRC/TM-27-88-56	5 MONITORING ORGANIZATION REPORT NUMBER(S)

6a NAME OF PERFORMING ORGANIZATION David Taylor Research Center	6b OFFICE SYMBOL (If applicable)	7a NAME OF MONITORING ORGANIZATION

6c ADDRESS (City, State, and ZIP Code) Annapolis, MD 21402-5067	7b ADDRESS (City, State, and ZIP Code)

8a NAME OF FUNDING / SPONSORING ORGANIZATION	8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER

8c ADDRESS (City, State, and ZIP Code)	10 SOURCE OF FUNDING NUMBERS			
	PROGRAM ELEMENT NO	PROJECT NO	TASK NO	WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)
Differential Forms of Euler's Turbomachinery Equation

12 PERSONAL AUTHOR(S)
Dr. Herman B. Urbach

13a TYPE OF REPORT Final R&D	13b TIME COVERED FROM 1987 TO 1988	14 DATE OF REPORT (Year, Month, Day) 1988 December 14	15 PAGE COUNT

16 SUPPLEMENTARY NOTATION

17	COSATI CODES		18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Turborotor Energy Transfer;
			Differential Forms of Euler's Equation;
			The Unsteadiness Paradox .

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Galilean transformations between the absolute and moving frame impose a crypto-steady-state relation between time derivations of a thermodynamic function in the absolute frame and their gradients in the moving frame. These crypto-steady relationships are inherently contained within the Navier-Stokes Equations for the absolute and moving frames.

The substantial total enthalpy derivative coupled with the substantial entropic energy derivative may be written solely in terms of the flow field of the moving frame. In the moving frame the relative total enthalpy, known as the rothalpy, yields a vanishing substantial derivative. Therefore, the substantial entropic energy derivative is uncoupled and explicit in the moving frame. This explicit substantial entropic energy derivative is invariant in all frames and may be used to obtain an uncoupled explicit substantial total enthalpy derivative. This latter derivative was hypothesized to be a (continued on reverse side)

20 DISTRIBUTION / AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED/UNLIMITED
22a NAME OF RESPONSIBLE INDIVIDUAL Dr. Herman B. Urbach	22b TELEPHONE (Include Area Code) 301-267-2864 22c OFFICE SYMBOL

DD Form 1473, JUN 86 Previous editions are obsolete. SECURITY CLASSIFICATION OF THIS PAGE

S/N 0102-LF-014-6603

Block 19   (continued)

differential form of Euler's Turbomachinery Equations corrected for real viscous losses.

The fact that integration of the differential form indeed yields the classical integral form of Euler's Turbomachinery Equation validates the aforementioned hypothesis. Also, it is demonstrated that energy transfer like lift is a purely potential process involving kinetic energy only.

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NOMENCLATURE

<u>Roman</u>

$\bar{e}$	a unit vector
E_k	the Ekman Number
f	any continuous function
h	enthalpy per unit mass of fluid
h_o	total enthalpy per unit mass fluid
h_{ow}	relative total specific enthalpy or rothalpy
H_o	total enthalpy of a fluid; in linear systems, the total enthalpy of a fluid per unit length of blade
L	lift per unit length of blade
m	system mass
m	mass flow rate
n	a constant
p	local static pressure
p_o	total pressure
q	a generalized curvilinear coordinate
r	radial coordinate of a cylindrical coordinate system
$\bar{R}$	position vector for the point of interest in a fluid
R_e	the Reynolds Number
s	entropy per unit mass of fluid
S	entropy
t	time

T	absolute temperature
$\bar{U}$	velocity of the blade and a function of the radius
$\bar{V}$	velocity
w	specific work
$\bar{W}$	relative fluid velocity in a moving rotor frame
z	axial coordinate of a cylindrical coordinate system

Greek

α	flow angle
Γ	the circulation
$\bar{\zeta}$	the observed vorticity in any frame of reference
θ	tangential angle coordinate in cylindrical coordinates
μ	the absolute viscosity
ν	the kinematic viscosity
ρ	local fluid density
τ	volume
ψ	a stream function
ω	angular velocity

Subscripts

i	input conditions
o	indicates inclusion of the kinetic energy or total conditions
o	outlet conditions
r	indicates the radial component
s	ideal state of constant entropy

v	indicates the absolute frame
vi	indicates the viscous contribution
w	indicates the moving frame
z	indicates the axial component of velocity
θ	indicates the tangential component of velocity
ω	indicates a rotational component

ABSTRACT

Galilean transformations between the absolute and moving frame impose a crypto-steady-state relation between time derivatives of a thermodynamic function in the absolute frame and their gradients in the moving frame. These crypto-steady relationships are inherently contained within the Navier-Stokes Equations relating the absolute and moving frames.

The substantial total enthalpy derivative coupled with the substantial entropic energy derivative may be written solely in terms of the flow field of the moving frame. In the moving frame the relative total enthalpy, known as the rothalpy, yields a vanishing substantial derivative. Therefore, the substantial entropic energy derivative is uncoupled and explicit in the moving frame. This explicit substantial entropic energy derivative is invariant in all frames and may be used to obtain an uncoupled explicit substantial total enthalpy derivative. This latter derivative was hypothesized to be a differential form of Euler's Turbomachinery Equations corrected for real viscous losses.

The fact that integration of the differential form indeed yields the classical form of Euler's Turbomachfinery Equation validates the aforementioned hypothesis. Also, it is demonstrated that energy transfer, like lift, is a purely *potential* process involving kinetic energy only.

ADMINISTRATIVE INFORMATION

ACKNOWLEDGEMENT

INTRODUCTION

Modern computational fluid dynamic solutions of the Navier-Stokes Equations for incompressible turbomachinery domains appear to be on the threshold of yielding details of the entire flow field of real machines. It is therefore of great interest to determine the thermodynamic energy transfer from the purely fluid dynamic aspects of the flow. The computational device which serves this purpose is Crocco's Equations [1,2].

A monument of turbomachinery technology is Euler's Turbomachinery Equation which is based upon thermodynamic definitions of work and Newton's Laws. Since the Navier-Stokes Equations and Crocco's Equation in a rotating (moving) frame are also based upon thermodynamics and Newton's Laws, they must in principle contain Euler's Turbomachinery Equation in differential form and, on integration, in integral form. Also contained within the moving-frame statement of Crocco's Equation is a vector expression of the Galilean transformation that relates the frames.

Integration of Crocco's Equation in the absolute frame has given rise to the "Unsteadiness Paradox [3-7]" to explain rotor energy transfer. An aspect of the paradox which must be rationalized (vide infra) is Vavra's statement (page 209 of Reference 8) that if the relative velocity and the rotor velocity are constant with time, then

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the absolute velocity must be time independent.

Preston has invoked the use of a time-dependent potential function to obtain energy transfer in systems of potential vortices. He did not concern himself with the practical problem of fluid boundaries because, in effect his analysis of the point potential vortices did not raise such an issue. Such a system of calculation cannot, of course, treat non-ideal viscous systems with vorticity and with real moving physical boundaries. The objectives in this study were to make no assumptions with regard to ideality in the three-dimensional domain. However, viscosity and vorticity were excluded in a two-dimensional study of the linear rotor.

The Galilean transformation that connects the moving rotor frame and the absolute or laboratory frame provides a relationship between the frames so that integration of the energy rate may be conveniently performed in a time-independent frame with a time-independent set of coordinates. Time-dependence is largely ignored in the design of marine propellers. The fact that energy transfer is routinely calculated in a steady-state moving frame provides philosophical questions concerning the proper interpretations of the Unsteadiness Paradox.

The Galilean transformation leads to simplified expressions for the substantial total-enthalpy transfer rate which is uncoupled from the expressions for the substantial entropic energy rate. The fact that the total enthalpy and the entropic energy rates are uncoupled makes for simplified integration of the total enthalpy transfer and leads to a differential form of the Euler Turbomachinery Equation and

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the anticipated form of the integral Turbomachinery Equation corrected for non-ideal flow. The uncoupled expressions were applied to a two-dimensional linear turbine and yield the expected result. These analyses and questions pertaining to the proper interpretation of time-dependent terms in the absolute frame are addressed in the body of this paper.

2.0 The Galilean Transformation

The moving frame and the absolute frame of a turborotor are connected by a Galilean transformation, which imposes relationships between the coordinates of the frames. From these relationships the vector operations in the two frames may be derived.

In the following discussion the subscripts v, and w will represent the absolute and the moving frame coordinate and vector values. (See Figure 1 and the Nomenclature for definitions of quantities.) Time, t, will be invariant in both frames so that

$$t_v = t_w = t .$$ (2.0.01)

The coordinate values r and z will exhibit similar properties

$$r_v = r_w = r ,$$ (2.0.02)

and

$$z_v = z_w + z_0 = z_w = z ,$$ (2.0.03)

where we have set z_0 to 0.

Thus, for these coordinates and the time, a subscript is redundant. However, for the angular coordinate, θ,

$$\theta_v = \theta_w + \omega t ,$$ (2.0.04)

where ω is the angular velocity.

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Now from the calculus, the total derivative of a function f is

$$Df = \left[\frac{\partial f}{\partial t}\right]_{q_i} dt + \left[\frac{\partial f}{\partial q_1}\right]_{t,q_i \neq q_1} dq_1 + \left[\frac{\partial f}{\partial q_2}\right]_{t,q_i \neq q_2} dq_2$$

$$+ \left[\frac{\partial f}{\partial q_3}\right]_{t,q_i \neq q_3} dq_3 \qquad (2.0.05a)$$

or

$$Df = \left[\frac{\partial f}{\partial t}\right]_{q_i} dt + d\bar{R} \cdot \nabla f . \qquad (2.0.05b)$$

The Eulerian substantial derivative is contained within (2.0.05). The first term of the derivative requires that the coordinates be fixed and only time is varied. The second gradient term demands that time be fixed.

Employing (2.0.04) and assuming that the angular velocity of the rotor is constant,

$$d\theta_v = d\theta_w + \omega dt . \qquad (2.0.06)$$

Also,

$$\frac{1}{r}\left[\frac{\partial}{\partial \theta_w}\right]_{\theta_v} = \frac{1}{r\omega}\left[\frac{\partial}{\partial t}\right]_{\theta_v} = -\frac{1}{U}\left[\frac{\partial}{\partial t}\right]_{\theta_v} . \qquad (2.0.07)$$

Similar results are obtained with cartesian coordinates. If U is the velocity of the moving frame given by

$$y_v = y_w + Ut, \qquad (2.0.08)$$

then

$$\left[\frac{\partial}{\partial y_w}\right]_{y_v} = -\frac{1}{U}\left[\frac{\partial}{\partial t}\right]_{y_v} . \qquad (2.0.09)$$

Equations (2.0.07) and (2.0.09) define, in fact, the "crypto-steady criterion," for determining whether a frame exists in which the flow regime is truly steady state.

The vector operator ∇ in (2.0.05b) is independent of time. In the absolute cartesian frame

$$\nabla_v \equiv \bar{i}\left[\frac{\partial}{\partial x}\right]_{y_v,z,t} + \bar{j}\left[\frac{\partial}{\partial y_v}\right]_{x,z,t} + \bar{k}\left[\frac{\partial}{\partial z}\right]_{x,y_v,t} . \qquad (2.0.10)$$

The vector operator ∇_w in the moving cartesian frame is defined:

$$\nabla_w \equiv \bar{i}\left[\frac{\partial}{\partial x}\right]_{y_w,z,t} + \bar{j}\left[\frac{\partial}{\partial y_w}\right]_{x,z,t} + \bar{k}\left[\frac{\partial}{\partial z}\right]_{x,y_w,t} . \qquad (2.0.11)$$

From (2.0.08) with or without the constraint that time is fixed

$$\nabla_v = \nabla_w = \nabla , \qquad (2.0.12)$$

which applies wherever a moving device which can effect energy transfer exists in the fluid. Since (2.0.12) represents a vector equation it is applicable to all coordinate systems even though the unit vectors may not be invariant under imposition of the transformation (2.0.04).

Now from (2.0.07) for any function f,

$$\left[\frac{\partial f}{\partial t}\right]_{q_i} = - U \frac{\partial f}{\partial q_u} = - \bar{U}\cdot\nabla_w f , \qquad (2.0.13)$$

where q_i represents all the position coordinates and q_u is the moving-frame coordinate of a generalized curvilinear orthogonal coordinate system which completely defines the position and velocity of the energy-transferring device such as a sail or rotor. Equation (2.0.13) is an extension of the usual crypto-steady relation.

Finally, the Galilean transformation relates the velocity $\bar{V}$, the

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relative velocity $\bar{W}$, and the velocity $\bar{U}$ of the device or rotor engaged in energy transfer, i.e.,

$$\bar{V} = \bar{W} + \bar{U} .\qquad(2.0.14)$$

Equation (2.0.14) is the vector derivative of Equations (2.0.02) through (2.0.04). The converse arguments starting from (2.0.14) and (2.0.01) would lead to (2.0.02) through (2.0.04).

2.1 The Interpretation of Time Derivatives

The time derivative of the static blade to blade pressure is obtained from Equation 2.0.13 by substituting p for the function f. Using cylindrical coordinates

$$\left[\frac{\partial p}{\partial t}\right]_{r,\theta_v,z} = -\frac{U}{r}\left[\frac{\partial p}{\partial \theta_w}\right]_{r,t,z} = -\frac{\partial p}{\partial(\theta_w/\omega)_t} .\qquad(2.1.01)$$

Figure 2a shows a point P(r,z,t) fixed in the absolute frame between blades of a rotor rotating into decreasing values of θ_v. The thermodynamic properties of the point P at the suction side of blade 1 change as time advances and the pressure side of blade 2 approaches P. When the wall passes through point P fluid properties cease to exist at P. Thermodynamic information about the fluid at that point ceases. The time interval Δt for n blades and angular velocity ω is

$$t_2 - t_1 = \Delta t(r,z) = (2\pi/n - \delta(r,z))/\omega ,\qquad(2.1.02)$$

where $\delta(r,z)$ is the blade thickness in radians.

Although time is fixed in the moving spatial derivatives, the

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point $P(r,z_o,t_o)$ of Figure 2b must span the entire blade space in that fixed instant of time. The angular change $\Delta\theta_w$ and the distance $\Delta\sigma$ is in a direction negative to the blade motion $r\omega$.

$$\Delta\sigma = r\Delta\theta_w = r(2\pi/n - \delta) . \qquad (2.1.03)$$

Figure 2c shows a curve of the intra blade pressure distribution changing with time in the absolute frame. In the moving frame, it represents the spatial distribution in a fixed instant of time.

2.2 Time Dependence and Frame of Reference

Vavra has noted that Equation (2.0.14) suggests that if $\overline{W}$ is independent of time, then $\overline{V}$ is also independent of time. This point is an overlooked aspect of the Unsteadiness Paradox [3-7]. The converse statement is also true and the Unsteadiness Paradox would imply that flow must be unsteady in the moving frame. With a slight rearrangement (2.0.14) becomes

$$\overline{W} = \overline{V} - \overline{U} . \qquad (2.2.01)$$

If $\overline{V}$ is time dependent, how can $\overline{W}$ not be time dependent? The observer who sits on an ideal rotor in an ideal infinite fluid sees no time dependence in measured thermodynamic properties at a point. However, when the same observer passes to the absolute frame, he measures time-dependent thermodynamic properties each time a blade passes by that point.

The problem is resolved by noting that the coordinates used in

(2.0.14) or (2.2.01) determine whether the observer perceives time dependence. If the velocity vectors are written in terms of r, θ_w, and z, the measurements are in the moving frame and both $\overline{V}(r,\theta_w,z)$ and $\overline{W}(r,\theta_w,z)$ are time independent. Writing $\overline{V}$ and $\overline{W}$ in the absolute coordinates requires that θ_w be written

$$\theta_w = \theta_v - \omega t . \tag{2.2.02}$$

Therefore, $\overline{V}(r,\theta_v-\omega t,z)$ and $\overline{W}(r,\theta_v-\omega t,z)$ are both time dependent in the absolute frame. To demonstrate note that

$$\left[\frac{\partial \overline{V}}{\partial t}\right]_{absolute} \equiv \left[\frac{\partial \overline{V}}{\partial t}\right]_{r,\theta_v,z} = \left[\frac{\partial \overline{W}}{\partial t}\right]_{r,\theta_v,z} . \tag{2.2.03}$$

Now dropping all subscripts but θ_v and θ_w,

$$\left[\frac{\partial \overline{V}}{\partial t}\right]_{\theta_v} = \left[\frac{\partial \overline{W}(r,\theta_v,z)}{\partial t}\right]_{\theta_v} + \frac{\partial \overline{W}(r,\theta_v,z)}{\partial \theta_w}\left[\frac{\partial \theta_w}{\partial t}\right]_{\theta_v} . \tag{2.2.04}$$

Finally

$$\left[\frac{\partial \overline{V}}{\partial t}\right]_{\theta_v} = 0 - \omega \frac{\partial \overline{W}}{\partial \theta_w} \neq 0 . \tag{2.2.05}$$

The reader may now prove that $\left[\frac{\partial \overline{V}}{\partial t}\right]_{\theta_w}$ does in fact vanish.

Before proceding with developments below, it is useful to restate a set of rules implied above which should simplify analysis for the reader.

1. For operations in the absolute frame use absolute frame variables. This implies that

$$\overline{W}(\theta_w) \Rightarrow \overline{W}(\theta_v,t) . \tag{2.2.06}$$

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2. For operations in the moving frame of the rotor use moving frame variables to obtain steady state. This implies that using (2.0.04)

$$\bar{V}(\theta_v, t) \Rightarrow \bar{V}(\theta_w) . \tag{2.2.07}$$

If the variables θ_v and t are not entirely constrained according to (2.0.04) then the flow is not truly crypto-steady.

3. The partial derivatives with respect to time in the moving and absolute frames cannot in general, be equal, i.e.,

$$\left[\frac{\partial}{\partial t}\right]_{\theta_v} \neq \left[\frac{\partial}{\partial t}\right]_{\theta_w} . \tag{2.2.08}$$

3.0 Derivation of the Differential and Integral Forms

3.1 Aspects of Time-Dependence in the Absolute and Moving Frames

The Navier-Stokes Equations exhibit a modified formalism including Coriolis terms in a unique steady-state rotating frame which develops from the Galilean transformation (2.0.14). Even in the frames of a device without rotation, the definition of the moving frame inherently contains and always invokes the crypto-steady relation. Provided there is uniform upstream flow, an observer moving with the device (for example a sail) may observe crypto steady-state behavior in the fluid. Nevertheless, the time-dependent terms must be recovered on imposition of the Galilean transformation to the absolute frame.

The Navier-Stokes Equations may be formalized to exhibit explicitly the rotational motion of the energy-transferring rotor device in incompressible flow thus:

$$\frac{\partial \bar{V}}{\partial t} + \bar{V} \cdot \nabla \bar{V} = \frac{\partial \bar{W}}{\partial t} + \bar{W} \cdot \nabla \bar{W} + 2\bar{\omega} \times \bar{W} - \nabla U^2/2$$

$$= - \frac{\nabla p}{\rho} + \nu \nabla^2 \bar{V} . \tag{3.1.01}$$

If the moving frame is in fact a steady-state frame and if the angular velocity vanishes then explicit rotational terms must drop out and

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$$\frac{\partial \bar{V}}{\partial t} + \bar{V} \cdot \nabla \bar{V} = \bar{W} \cdot \nabla \bar{W} = -\frac{\nabla p}{\rho} + \nu \bar{\nabla}^2 V \ . \qquad (3.1.02)$$

Preston[6] has shown that time dependence and a non-vanishing acceleration can be recovered in the absolute frame with an analysis based upon potential functions to describe the flow. Resorting to relations of the type of Equation (2.0.08) he obtained the required time dependence. Vavra[8] has also shown that time dependence may be recovered by use of a viscous model. Nevertheless, as mentioned above, Vavra asserts (pages 110, 111 of Reference 8) that if in Equation (2.0.14) the relative velocity, $\bar{W}$, is steady state, so must $\bar{V}$ be steady state. Likewise, if $\bar{V}$ is unsteady then $\bar{W}$ must be unsteady. Following the discussion in Section 2.2, the paradox is resolved by noting that the coordinate frame must be specified to determine whether time-dependence may be observed. The steady-state equations involving θ_w or y_w may be rewritten with substitution of these variables using Equations (2.0.04) and (2.0.08) respectively. Then, time dependence is captured in the absolute frame.

Considering the linear rotor it is also possible to demonstrate recovery of the acceleration term using (2.0.12) and (2.0.14). Using (2.0.14) to replace $\bar{W}$ in (3.1.02) the result is:

$$\frac{\partial \bar{V}}{\partial t} + \bar{V} \cdot \nabla \bar{V} = (\bar{V} - \bar{U}) \cdot \nabla (\bar{V} - \bar{U}) \ . \qquad (3.1.03)$$

Remembering that

$$\bar{U} \cdot \nabla \bar{U} = 0 \ , \qquad (3.1.04)$$

the result obtained is

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$$\frac{\partial \overline{V}}{\partial t} + \overline{V} \cdot \nabla \overline{V} = \overline{V} \cdot \nabla \overline{V} - \overline{U} \cdot \nabla \overline{V} \, . \tag{3.1.05}$$

Therefore

$$\frac{\partial \overline{V}}{\partial t} = \overline{U} \cdot \nabla \overline{V} = - U \frac{\partial \overline{V}}{\partial q_u} \, . \tag{3.1.06}$$

Equation (3.1.06) follows from Equation (2.0.14) and the fact that the Navier-Stokes Equations are invariant with respect to the substitution (2.0.14). However, (3.1.06) is anticipated by (2.0.13) and provides an indication that the argument used for (2.0.13) is consistent.

Following the above arguments it is affirmed that time-dependence and acceleration are recovered in the absolute frame even though these are not present in the moving frame.

3.2 The Coupled Substantial Total Enthalpy and Entropic Energy Rate with Viscous Terms in the Moving and Absolute Frames

It will be assumed that time dependence in the moving frame is negligible. Therefore in (3.1.01) the acceleration term in the moving frame will be dropped.

$$\frac{\partial \overline{V}}{\partial t} + \nabla V^2/2 - \overline{V} \times (\nabla \times \overline{V}) = \overline{W} \cdot \nabla \overline{W} + 2 \, \overline{\omega} \times \overline{W} - \nabla U^2/2$$

$$= \frac{-\nabla p}{\rho} + \nu \nabla^2 \overline{V} \, . \tag{3.2.01}$$

If the gradient form of the second law, and the definitions of the

total enthalpy namely

$$\frac{\nabla p}{\rho} = \nabla h - T\nabla s \ , \tag{3.2.02}$$

and

$$\nabla h_o = \nabla h + \nabla V^2/2 \ , \tag{3.2.03}$$

are combined with (3.2.01), Crocco's Equation with viscosity is obtained which for the absolute frame is

$$\frac{\partial \overline{V}}{\partial t} - \overline{V} \times (\nabla \times \overline{V}) = -\nabla h_o + T\nabla s + \nu \nabla^2 \overline{V} \ . \tag{3.2.04}$$

Taking the dot product of the velocity on (3.2.04) and rearranging,

$$\overline{V} \cdot \nabla h_o = -\frac{\partial V^2/2}{\partial t} + T\overline{V} \cdot \nabla s + \nu \overline{V} \cdot \nabla^2 \overline{V} \ . \tag{3.2.05}$$

The substantial derivative of the total enthalpy is

$$\frac{Dh_o}{Dt} = \left[\frac{\partial h_o}{\partial t}\right]_{x_i} + \overline{V} \cdot \nabla h_o \ . \tag{3.2.06}$$

Combining (3.2.05) and (3.2.06) the substantial derivative with non-ideal viscous terms is obtained.

$$\frac{Dh_o}{Dt} = \left[\frac{\partial h}{\partial t}\right]_{x_i} + T\overline{V} \cdot \nabla s + \nu \overline{V} \cdot \nabla^2 \overline{V} \ . \tag{3.2.07}$$

Using the time derivative of the second law, i.e.,

$$\left[\frac{\partial h}{\partial t}\right]_{x_i} = \frac{1}{\rho}\left[\frac{\partial p}{\partial t}\right]_{x_i} + T\left[\frac{\partial s}{\partial t}\right]_{x_i} \ , \tag{3.2.08}$$

and substituting (3.2.08) in (3.2.07),

$$\frac{Dh_o}{Dt} = \frac{1}{\rho}\left[\frac{\partial p}{\partial t}\right]_{x_i} + T\frac{Ds}{Dt} + \nu \overline{V} \cdot \nabla^2 \overline{V} \ . \tag{3.2.09}$$

Employing Equation (2.0.13) with the static pressure as the arbitrary

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function

$$\frac{Dh_o}{Dt} = -\frac{U}{\rho}\frac{\partial p}{\partial x_u} + T\frac{Ds}{Dt} + \nu\bar{V}\cdot\nabla^2\bar{V} \ , \tag{3.2.10}$$

where x_u is the velocity of device in the moving frame.

Since the gradient of static quantities is invariant in all frames

$$\frac{U}{\rho}\frac{\partial p}{\partial x_u} = \frac{U}{\rho r}\left[\frac{\partial p}{\partial \theta_w}\right]_t = \frac{U}{\rho r}\left[\frac{\partial p}{\partial \theta_v}\right]_t = \frac{\bar{U}}{\rho}\cdot\nabla p \ . \tag{3.2.11}$$

Combining (3.2.10) and (3.2.11) the equation of the substantial total enthalpy derivative is

$$\frac{Dh_o}{Dt} = -\frac{\bar{U}}{\rho}\cdot\nabla p + T\frac{Ds}{Dt} + \nu\bar{V}\cdot\nabla^2\bar{V} \ . \tag{3.2.12}$$

It is not obvious in (3.2.12) how to decouple the enthalpic energy rate from the entropic energy rate. The value of (3.2.12) derives from the fact that Equation (3.2.01) may be used to replace the pressure derivative. From the dot product of $\bar{U}$ with the moving frame equality in (3.2.01) the pressure gradient term becomes

$$-\frac{U}{\rho r}\frac{\partial p}{\partial \theta_w} = -\bar{U}\cdot\frac{\nabla p}{\rho} = \bar{U}\cdot(\bar{W}\cdot\nabla)\bar{W} + 2\bar{U}\cdot\bar{\omega}\times\bar{W}$$

$$-\bar{U}\cdot\nabla U^2/2 - \nu\bar{U}\cdot\nabla^2\bar{V} \ . \tag{3.2.13}$$

Equations (3.2.12) and (3.2.13) may be added to eliminate the pressure gradient term. This operation yields the coupled substantial total enthalpy and entropic energy rates in the moving frame.

$$\frac{Dh_o}{Dt} - T\frac{Ds}{Dt} = \bar{U}\cdot(\bar{W}\cdot\nabla)\bar{W} + 2\bar{U}\cdot\bar{\omega}\times\bar{W} - \bar{J}\cdot\nabla U^2/2$$

$$+ \nu\bar{W}\cdot\nabla^2(\bar{W}+\bar{U}) \ . \tag{3.2.14}$$

Note that the pressure term may also have been eliminated by employing the dot product of $\overline{U}$ on the acceleration term in the absolute frame. Since, the difficulty of integration with time-dependent boundaries must first be addressed, there may be little advantage in this. Nevertheless, for completeness, the equivalent form in the absolute frame is appropriate here.

$$\frac{Dh_o}{Dt} - T \frac{Ds}{Dt} = \overline{U} \cdot \frac{\partial \overline{V}}{\partial t} + \overline{U} \cdot (\overline{V} \cdot \nabla) \overline{V} + \nu (\overline{V} - \overline{U}) \cdot \nabla^2 \overline{V} \ . \qquad (3.2.15)$$

Although static quantities are invariant in all frames, the total quantities are peculiar to their frame because they contain kinetic energies which are a function of the frame. In Equation (3.2.14) the substantial total energy rate in the absolute frame is defined in terms of quantities and coordinates in the moving frame.

It is now useful to inquire into the substantial relative total energy rate of the moving frame.

3.3 Uncoupling the Substantial Total Enthalpy Rates and the Entropic Energy Rates

Equation (3.2.01) is the starting point for developing the objective. The gradient of the absolute kinetic energy is subtracted in the moving frame.

$$\frac{\partial \overline{V}}{\partial t} - \overline{V} \times (\nabla \times \overline{V}) = -\nabla V^2/2 + \nabla W^2/2 + (\nabla \times \overline{W}) \times \overline{W}$$

$$+ 2\overline{\omega} \times \overline{W} - \nabla U^2/2 = -\nabla(h + V^2/2) + T\nabla s + \nu \nabla^2 \overline{V} \ . \qquad (3.3.01)$$

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Using (2.0.14) to eliminate V in the right member of (3.3.01)

$$-\nabla W_u U - \nabla U^2 + (\nabla \times \overline{W} + 2\overline{\omega}) \times \overline{W}$$

$$= -\nabla h_o + T\nabla s + \nu\nabla^2(\overline{W} + \overline{U}) \ . \tag{3.3.02}$$

Before substituting the terms, an intermediate expression is useful.

$$h + \frac{V^2}{2} - UW_u - U^2 = h + \frac{W^2}{2} - \frac{U^2}{2} \equiv h_{ow} \ , \tag{3.3.03}$$

where the quantity h_{ow} in (3.3.03) is the total relative enthalpy[7] or the rothalpy. Combining (3.3.02) and (3.3.03) an expression for Crocco's Equation in the moving frame is derived.

$$\nabla h_{ow} - T\nabla s = \overline{W} \times (\nabla \times \overline{W} + 2\overline{\omega}) + \nu\nabla^2(\overline{W} + \overline{U}) \ . \tag{3.3.04}$$

Now, if the upstream flow is thoroughly mixed and without energy gradients

$$\frac{Dh_{ow}}{Dt} = \frac{\partial h_{ow}}{\partial t} + \overline{W} \cdot \nabla h_{ow} = \overline{W} \cdot \nabla h_{ow} \ , \tag{3.3.05}$$

and

$$T\left[\frac{Ds}{Dt}\right]_w = T\frac{\partial s}{\partial t} + T\overline{W} \cdot \nabla s = T\overline{W} \cdot \nabla s = T\left[\frac{Ds}{Dt}\right]_v \ . \tag{3.3.06}$$

where the last equality in (3.3.06) indicates that the entropic energy rate is invariant in all frames. Taking the dot product of $\overline{W}$ on (3.3.04) yields

$$\frac{Dh_{ow}}{Dt} = T\frac{Ds}{Dt} + \nu\overline{W} \cdot \nabla^2(\overline{W} + \overline{U}) \ . \tag{3.3.07}$$

Now Vavra notes (Reference 8, page 124) that in the moving frame following a particle of fluid

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$$Tds = dq_o - dt\overline{W} \cdot \overline{f}_r \quad , \tag{3.3.08}$$

where q_o is the specific external heat rate and $dt\overline{W} \cdot \overline{f}_r$ is the specific thermal equivalent of frictional work. Since it is assumed that there are no external heat sources or sinks, and that crypto-steady flow prevails, it may be stated that

$$T\frac{Ds}{Dt} = - \overline{W} \cdot \overline{f}_r = - \nu\overline{W} \cdot \nabla^2(\overline{W} + \overline{U}) \quad . \tag{3.3.09}$$

Combining (3.3.07) with (3.3.09) the desired result is

$$\frac{Dh_{ow}}{Dt} = 0 \quad . \tag{3.3.10}$$

Therefore, using (3.3.05) it is asserted that for arbitrary real flows without external heat sinks and with crypto-steady characteristics

$$\nabla h_{ow} = \nabla \left(h + \frac{W^2}{2} - \frac{U^2}{2} \right) = \nabla(h_o - UV_u) = 0 \quad . \tag{3.3.11}$$

The right member of (3.3.11) is one of the differential or gradient forms of Euler's Turbomachinery Equations in the absolute frame. Integration over a stream tube in the absolute frame yields the classic Euler Equation.

$$\Delta h_o = \Delta(UV_u) \quad . \tag{3.3.12}$$

Note that (3.3.12) is contained within Crocco's Equation (3.2.14).

Combining (3.3.04) and (3.3.11), the entropic energy gradient may be defined thus:

$$T\nabla s = (\nabla \times \overline{W} + 2\overline{\omega}) \times \overline{W} - \nu\nabla^2(\overline{W} + \overline{U}) \quad , \tag{3.3.13}$$

and the substantial entropic energy rate is

$$T\frac{Ds}{Dt} = -\nu\bar{W}\cdot\nabla^2(\bar{W}+\bar{U}) = \nu\bar{W}\cdot\nabla\times\nabla\times\bar{W} \ . \tag{3.3.14}$$

For real fluids, dissipation is at least positive and

$$\bar{W}\cdot\nabla\times\nabla\times\bar{W} \geq 0 \geq \bar{W}\cdot\nabla^2(\bar{W}+\bar{U}) \ . \tag{3.3.15}$$

The substantial total enthalpy rate in the absolute frame is now obtained by eliminating the entropic energy rate in (3.2.14) with (3.3.14).

$$\frac{Dh_o}{Dt} = \bar{U}\cdot(\bar{W}\cdot\nabla)\bar{W} + 2\bar{U}\cdot\bar{\omega}\times\bar{W} \ . \tag{3.3.16}$$

Equation (3.3.16), expressing the power transfer, is another of the differential forms of Euler's Turbomachinery Equation.

With (3.3.14) it is now possible to resolve the problem proposed following Equation (3.2.12) of how to uncouple the total-enthalpy, pressure relationship. Equation (3.3.14) can be used to eliminate the substantial entropic energy derivative.

$$\frac{Dh_o}{Dt} = -\frac{\bar{U}}{\rho}\cdot\nabla p + \nu\bar{U}\cdot\nabla^2(\bar{W} + \bar{U}) \ . \tag{3.3.17}$$

From both (3.3.16) and (3.3.17) it may be concluded that only the component of flow paralleling the blade velocity $\bar{U}$ contributes to the energy transfer between the fluid and rotor. Also, from (3.3.15) the viscous term is greater than zero, and it may be deduced that for turbines which have positive $\bar{U}$ energy transfer is less than ideal. For compressors, with negative $\bar{U}$, energy transfer is greater than ideal.

One of several significant issues raised by Equations (3.3.16) and

(3.3.17) is that (3.3.16) is a valid expression for both ideal and non-ideal flows, since the viscous correction has already been made. However, in (3.3.17) the second term of the right member vanishes for ideal flows. Also (3.3.17) indicates that neither axial or radial pressure gradients are germane to the calculation of specific total enthalpy transfer. (Mass flow rates are of course a function of axial or radial pressure gradients.) Note, in constrast with some views (see Reference 9 pages 7 and 8), only transverse pressure gradients parallel to $\bar{U}$ contribute to total enthalpy transfer. The impulse stages of turbomachines prove that axial or radial pressure gradients play no role in energy transfer. Moreover, from (3.3.16) for ideal flows which have no vorticity only kinetic energy gradients paralleling the blade motion $\bar{U}$ contribute to total enthalpy transfer. Thus total enthalpy transfer like lift is an ideal flow phenomenon (see below) and is a linear rather than a non-linear property of the flow.

Moreover, since the effects of compressibility may arise explicitly only in the viscous terms (the argument has thus far ignored compressibility although extension to the compressible domain demands addition of a second viscous term), it is concluded that compressibility effects do not contribute to the ideal mechanism of total enthalpy transfer. It is restated again that specific total enthalpy transfer arises only from kinetic energy gradients which lie parallel to the blade velocity $\bar{U}$.

The substantial total enthalpy rate given by (3.3.16) is a differential form of Euler's Turbomachinery Equation in the time-independent coordinates of the moving frame. A proper test of

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(3.3.16) would be the applicability of the equation to integration over the rotor blade-to-blade flow. Moving-frame integration should predict a total enthalpy transfer compatible with that of the Euler Turbomachinery Equation. Therefore, the integration of (3.3.16) will be performed in two tests, one in the three-dimensional domain and the second in a two-dimensional linear turbine. Compatibility of the results with Euler's Equation or the consequences thereof will lend credence to the logic of analysis employed in the derivation of (3.3.13) through (3.3.17).

4.0 Integration of the Total Enthalpy Rate

In the integration process it will be assumed that the flow may be divided into streams which pass between a given pair of blades. In the rotating frame the streamtube walls are fixed steady-state walls associated with a steady-state mass flow rate m which may consist of radial and axial mass flow components.

4.1 Derivation of the Integral Form from the Differential Form

The differential form of Euler's Turbomachinery Equation (3.3.16) is reproduced here for clarity and ease of discussion:

$$\frac{Dh_o}{Dt} = \bar{U} \cdot (\bar{W} \cdot \nabla)\bar{W} + 2\bar{U} \cdot \bar{\omega} \times \bar{W} \ . \tag{4.1.01}$$

On integrating it will be necessary to multiply by the density.

$$\iiint \rho \ \frac{Dh_o}{Dt} \ d\tau$$

$$= \iiint \rho [\ \bar{U} \cdot (\bar{W} \cdot \nabla)\bar{W} + 2\bar{U} \cdot \bar{\omega} \times \bar{W} \ ] r dr d\theta dz \tag{4.1.02}$$

The first term of the right member of (4.1.02) is the tangential component of the convective term obtained on dot multiplication with $\bar{U}$, i.e.;

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$$\rho \bar{U} \cdot (\bar{W} \cdot \nabla) \bar{W}$$

$$= \rho U \left[W_r \frac{\partial W_\theta}{\partial r} + \frac{W_\theta}{r} \frac{\partial W_\theta}{\partial \theta} + W_z \frac{\partial W_\theta}{\partial z} + \frac{W_r W_\theta}{r} \right] . \quad (4.1.03)$$

The units in (4.1.03) are power per unit volume of space. The first and last terms of (4.1.03) will be combined in an integral indentified by $I_{1,4}$ thus:

$$I_{1,4} = \iiint \rho r \omega \frac{W_r}{r} \frac{\partial r W_\theta}{\partial r} \, r dr d\theta dz . \quad (4.1.04)$$

The factors in (4.1.04) may be rearranged to express the radial mass flow. First notice that

$$m_r(r) = \rho W_r A_r = m_r \quad (4.1.05)$$

where A_r, the normal radial area, is given by

$$A_r(r) = \iint r d\theta dz = r \Delta \theta \Delta z . \quad (4.1.06)$$

If $\rho \Delta \theta \Delta z$ is nearly constant in the blade space for steady-state radial flow then

$$r W_r \approx constant . \quad (4.1.07)$$

It will be assumed that an average $r W_r$ allowing for blade thickness variations, etc. has been defined through

$$r \widetilde{W}_r(r) = \frac{1}{\Delta \theta \Delta z} \iint_{\theta_{min}}^{\theta_{max}} r W_r \, d\theta dz . \quad (4.1.08)$$

Then

$$\tilde{\tilde{m}}_r(r) = \iint \rho r \tilde{W}_r \frac{dA_r}{r} = \iint \rho r \tilde{W}_r(r) d\theta dz = m_r . \qquad (4.1.09)$$

The last equality arises because the radial mass flow must be constant in the steady state. We may write:

$$I_{1,4} = \int \left(\iint \rho r \tilde{W}_r d\theta dz \right) \frac{\partial r \omega \tilde{W}_\theta}{\partial r} dr , \qquad (4.1.10)$$

where

$$\tilde{W}_\theta(r) = \frac{1}{\Delta\theta\Delta z} \iint_{\theta_{min}}^{\theta_{max}} W_\theta \, d\theta dz . \qquad (4.1.11)$$

Combining (4.1.09) through (4.1.11), the resultant integral is

$$I_{1,4} = m_r \int d(U\tilde{W}_\theta) = m_r[(U\tilde{W}_\theta)_{r2} - (U\tilde{W}_\theta)_{r1}] . \qquad (4.1.12)$$

The third term of the right member of (4.1.03) may be written to show the axial mass flow rate m_z explicitly.

$$I_3 = \int \left(\iiint \rho \tilde{V}_z r dr d\theta \right) \frac{\partial \tilde{U} \tilde{W}_\theta}{\partial z} dz , \qquad (4.1.13)$$

where the axial velocity $\tilde{V}_z$ has been averaged over Δr and $\Delta\theta$ and

$$\tilde{U}\tilde{W}_\theta(z) = \frac{1}{\Delta r \Delta\theta} \iint UW_\theta dr d\theta . \qquad (4.1.14)$$

Substituting the axial mass rate for the parenthesis in (4.1.13),

$$I_3 = m_z \int d(\tilde{U}\tilde{W}_\theta) = m_z[(\tilde{U}\tilde{W}_\theta)_{z2} - (\tilde{U}\tilde{W}_\theta)_{z1}] . \qquad (4.1.15)$$

The second term of the right member provides an integral which

contains the tangential kinetic energy.

$$I_2 = \iint \frac{\rho \omega r}{2} \left[\int \frac{\partial W_\theta^2}{\partial \theta} d\theta \right] drdz$$

$$= \iint \frac{\rho \omega r}{2} \left[W_\theta^2(\theta_2) - W_\theta^2(\theta_1) \right] drdz = 0 . \qquad (4.1.16)$$

Since the tangential velocities at the blade walls is the blade velocity, the integral vanishes.

Now identifying the second term of the right member of (4.1.02) as, I_5, we may write

$$I_5 = \iiint 2\rho \bar{U} \cdot \bar{\omega} \times \bar{W} r drd\theta dz, \qquad (4.1.17)$$

and

$$I_5 = \iiint 2\rho \omega^2 r W_r r drd\theta dz . \qquad (4.1.18)$$

Again we display the terms representing radial mass flow.

$$I_5 = 2\omega^2 \int r \left[\iint \rho r W_r \, d\theta dz \right] dr = 2\omega^2 m_r \int r dr . \qquad (4.1.19)$$

Finally, following the arguments above,

$$I_5 \approx m_r \left[(U^2)_{r2} - (U^2)_{r1} \right] . \qquad (4.1.20)$$

Summing the components of integration, $I_{1,4}$ through I_5,

$$\iiint \rho \frac{Dh_0}{Dt} d\tau = m_r \left[\Delta_r (U \tilde{W}_\theta) + \Delta_r U^2 \right]$$

$$+ m_z \Delta_z (U \tilde{W}_\theta) . \qquad (4.1.21)$$

where Δ_r and Δ_z represent the change along r and z respectively.

Now adding $m_z \Delta_z \tilde{U}^2$ which is zero to (4.1.21),

$$\iiint \rho \frac{Dh_o}{Dt} d\tau = m_r \Delta_r \left[(U + \tilde{W}_\theta) U \right]$$

$$+ m_z \Delta_z \left[(\tilde{U} + \tilde{W}_\theta) \tilde{U} \right] . \qquad (4.1.22)$$

In (4.1.22) the terms $\tilde{W}_\theta$ are averaged over θ and z in the first term and over r and θ in the second term. If the total steady-state mass rate m between a pair of blades is

$$m = m_r + m_z , \qquad (4.1.23)$$

then,

$$\Delta h_o = f_r \Delta_r (\tilde{U} \tilde{V}_\theta) + f_z \Delta_z (U \tilde{V}_\theta) , \qquad (4.1.24)$$

Equation (4.1.24) represents Euler's Turbomachinery Equation with mixed flows and the terms $\tilde{V}_\theta$ and $\tilde{U}$ are averaged over the blade space where necessary. The coefficients f_r and f_z represent the radial and axial fractions of the mass flow.

4.2 Closure in Three Dimensions

The integral expression Equation (4.1.24) exhibits a formal similarity and compatibility with Euler's Turbomachinery Equation. The derivation lends credence to the hypothesis that Equation (3.3.16) is indeed a differential form of Euler's Turbomachinery Equation, (3.3.12)

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in time-independent frames.

A two-dimensional test of the differential form (3.3.16) on an ideal linear device where the solution is known precisely will now be examined.

5.0 The Substantial Total Enthalpy Rate in a Two-Dimensional Device

An infinite circular cylinder with bound circulation, as shown in Figure 3, is an elemental linear turbine. It may be considered as an infinite sail on a sailboat or an infinite wing on a sailplane. The device extracts energy from the ideal inviscid working fluid. Work is performed on the sailplane (fixed to a vertical rail) by raising its height at uniform speed U against gravity. Work on the sailboat is performed by moving the boat at uniform speed U which elevates a weight attached at minus infinity by an infinite tether. In the moving frame the apparent velocity of the ideal working fluid at infinite distance is W_θ. The relationship between the absolute and moving coordinate system and the velocities is given by the Galilean transformation following (2.0.08) and (2.0.14).

5.1 The Total Enthalpy Transfer Rate

The flow field will exhibit no relative enthalpy gradient in accordance with (3.3.11) and in the absence of vorticity and viscosity no entropic energy gradient in accordance with (3.3.13) and is therefore ideal.

Since the flow field is ideal, the flow domain may be described by

a potential function or its conjugate stream function. The lift is therefore the ideal lifting force, L, of the Kutta-Joukowski Equation given by

$$L = \rho W_o \Gamma , \qquad (5.1.01)$$

where Γ is the scalar circulation. The units are force per unit length of cylinder. In the absolute and moving frame the lift component L_y directed parallel to the y axis of Figure 3 is given by

$$L_y = \rho W_{ox} \Gamma = \rho V_{ox} \Gamma , \qquad (5.1.02)$$

where the subscript x represents the x component. Recalling that U is the velocity of motion of the device (sail or wing or rotating cylinder) as perceived in the absolute frame, the power is the product of U and L_y.

$$\text{Power unit length} = \rho U W_{oz} \Gamma . \qquad (5.1.03)$$

Since we assume that there is no heat rate,

$$\frac{D H_o}{D t} = -\rho U W_{ox} \Gamma , \qquad (5.1.04)$$

where H_o is the total enthalpy of the system per unit length. Equation (5.1.04) is the anticipated relationship which should ultimately be developed from the differential form (3.3.15). Successful derivation of (5.1.04) from the differential form (3.3.15) should provide further confirmation of its validity.

5.2 The Stream Function, Velocity and Relative Enthalpy in the Frame of the Blade

Since ideal flow has been assumed in the moving frame of the blade, the stream function, ψ, is the usual function modified for motion along the y axis.

$$\Psi = -W_{oy}(1-a^2/r^2)r\cos\theta + W_{ox}(1-a^2/r^2)r\sin\theta$$

$$+ (\Gamma/2\pi)\ln(r/a) \ . \tag{5.2.01}$$

The constant a is the radius of the cylinder.

The velocity components in cylindrical coordinates are

$$W_r = W_{oy}(1-a^2/r^2)\sin\theta + W_{ox}(1-a^2/r^2)\cos\theta \ , \tag{5.2.02}$$

and

$$W_\theta = W_{oy}(1+a^2/r^2)\cos\theta - W_{ox}(1+a^2/r^2)\sin\theta - \Gamma/2\pi r \ . \tag{5.2.03}$$

The cartesian velocity components are obtained by the usual transformation as follows:

$$W_x = W_{ox} + \frac{a^2 W_{ox}(y^2-x^2)}{(x^2+y^2)^2} - \frac{2a^2 W_{oy} xy}{(x^2+y^2)^2}$$

$$+ \frac{\Gamma}{2\pi} \frac{y}{(x^2+y^2)} \ , \tag{5.2.04}$$

$$W_y = W_{oy} - \frac{2a^2 W_{ox} xy}{(x^2+y^2)^2} + \frac{a^2 W_{oy}(x^2-y^2)}{(x^2+y^2)^2}$$

$$- \frac{\Gamma}{2\pi} \frac{x}{(x^2+y^2)} \ . \tag{5.2.05}$$

Now the relative vorticity must vanish because potential flow cannot have vorticity. A check of the vorticity in the relative frame shows that indeed it vanishes.

$$\nabla \times \overline{W} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x_w} & \dfrac{\partial}{\partial y_w} & \dfrac{\partial}{\partial z_w} \\ W_x & W_y & 0 \end{vmatrix} = 0 \ . \tag{5.2.06}$$

If the time-dependent portion of the total enthalpy change does not vanish, the local time derivative of the relative velocity must be examined. From (5.2.04) and (5.2.05)

$$\partial \overline{W}/\partial t = 0 \ . \tag{5.2.07}$$

Thus, as expected, from (3.1.01) with vanishing entropy gradient, angular velocity, viscosity and fixed U,

$$\nabla h_{ow} = -\partial \overline{W}/\partial t + \overline{W} \times (\nabla \times \overline{W}) = 0 \ . \tag{5.2.08}$$

5.3 The Substantial Total Enthalpy Derivative Without Rotation

In the linear two-dimensional system, the differential form of the turbomachinery equations (3.3.16) is simplified because the rotation vanishes.

$$\frac{Dh_o}{Dt} = \overline{U} \cdot (\overline{W} \cdot \nabla) \overline{W} \ . \tag{5.3.01}$$

Since the vorticity vanishes

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$$\bar{W}\cdot\nabla\bar{W} = \nabla W^2/2 \ , \tag{5.3.02}$$

and employing (2.0.13)

$$\frac{Dh_o}{Dt} = \bar{U}\cdot\nabla W^2/2 = \frac{U}{2}\frac{\partial W^2}{\partial y_w} \ . \tag{5.3.03}$$

The integrated substantial total enthalpy rate per unit length is

$$\frac{DH_o}{Dt} = \iint \rho \frac{Dh_o}{Dt} dydx = \iint \rho \frac{U}{2}\frac{\partial W^2}{\partial y} dydx \ . \tag{5.3.04}$$

The subscript on y in (5.3.04) has been dropped. In terms of the velocity components

$$\frac{DH_o}{Dt} = \iint \frac{\rho U}{2}\frac{\partial (W_x^2 + W_y^2)}{\partial y} dydx \ . \tag{5.3.05}$$

Integration of (5.3.04) will be performed over all space per unit length z of the blade. The choice of time is immaterial since the fluid dynamics are steady state in the moving frame and the thermodynamic rates over all space are invariant with time. It is understood that the integration applies only to the fluid domain and that boundaries at solid walls are observed.

$$\frac{DH_o}{Dt} = \rho U \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial W^2/2}{\partial y} dydx$$

$$= \rho U \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\,(W^2/2)dx \ . \tag{5.3.06}$$

See Figure 4 for a definition of the boundary conditions of y on the blade wall. Since the kinetic energy in a conservative system, like thermodynamic quantities, is a function of state, the integration in

(5.3.06) is a function of the endpoints only which is verified by the fact that the integrand is a total differential.

$$\frac{DH_o}{Dt} = \rho U \int_{-\infty}^{\infty} \left[\int_{-\infty}^{-(a^2-x^2)^{1/2}} d(W^2/2) \right.$$

$$\left. + \int_{(a^2-x^2)^{1/2}}^{\infty} d(W^2/2) \right] dx \ . \tag{5.3.07}$$

Since the velocity is uniform at infinity (and all time and space derivatives vanish at infinity), the contributions at infinity will cancel in (5.3.07). Therefore

$$\frac{DH_o}{Dt} = -\rho U \int_{-\infty}^{\infty} \int_{-(a^2-x^2)^{1/2}}^{(a^2-x^2)^{1/2}} dW^2/2)dx \ . \tag{5.3.08}$$

According to Figure 4 integration of (5.3.08) with respect to x yields contributions from $-a \leq x \leq a$ only.

$$\frac{DH_o}{Dt} = -\rho U \int_{-a}^{a} \int_{-(a^2-x^2)^{1/2}}^{(a^2-x^2)^{1/2}} d(W_x^2 + W_y^2)/2dx \ . \tag{5.3.09}$$

Now from Equation (5.2.04) and (5.2.05) and noting from Figure 3 that

$$-U = W_{oy}, \tag{5.3.10}$$

we obtain,

$$(W_x^2/2)_{t=0} = \frac{W_{ox}^2}{2} + \frac{a^4 W_{ox}^2 (y^2-x^2)^2}{2(x^2+y^2)^4} + \frac{2a^4 U^2 x^2 y^2}{(x^2+y^2)^4}$$

$$+ \frac{\Gamma^2 y^2}{8\pi^2(x^2+y^2)^2} + \frac{a^2 W_{ox}^2 (y^2-x^2)}{(x^2+y^2)^2} + \frac{2a^2 W_{ox} U xy}{(x^2+y^2)^2}$$

$$+ \frac{\Gamma W_{ox} y}{2\pi(x^2+y^2)} + \frac{2a^4 W_{ox} U x y(y^2-x^2)}{(x^2+y^2)^4}$$

$$+ \frac{a^2 W_{ox} \Gamma y (y^2-x^2)}{2\pi(x^2+y^2)^3} + \frac{a^2 U \Gamma x y^2}{\pi(x^2+y^2)^3} , \qquad (5.3.11)$$

and

$$(W_y^2/2)_{t=0} = \frac{U^2}{2} + \frac{a^4 U^2 (x^2-y^2)}{2(x^2+y^2)^4} + \frac{2a^4 U W_{ox} x y(x^2-y^2)}{(x^2+y^2)^4}$$

$$+ \frac{2a^2 U W_{ox} xy}{(x^2+y^2)^2} + \frac{a^2 U^2 (x^2-y^2)}{(x^2+y^2)^2} + \frac{2a^4 W_{ox}^2 x^2 y^2}{(x^2+y^2)^4}$$

$$+ \frac{a^2 W_{ox} \Gamma x^2 y}{\pi(x^2+y^2)^3} + \frac{\Gamma^2 x^2}{8\pi^2(x^2+y^2)^2} + \frac{\Gamma U x}{2\pi(x^2+y^2)}$$

$$+ \frac{a^2 U \Gamma x(x^2-y^2)}{2\pi(x^2+y^2)^3} . \qquad (5.3.12)$$

Now the integration of the total derivative in (5.3.07) is performed using Equations (5.3.11) and (5.3.12).

$$\frac{DH_o}{Dt} = -\rho U \int_{-a}^{a} \left[\frac{4UW_{ox}x(a^2-x^2)^{1/2}(2x^2-a^2)}{a^4} \right.$$

$$+ \frac{2W_{ox}\Gamma x(a^2-x^2)^{1/2}}{\pi a^4} + \frac{8W_{ox}Ux(a^2-x^2)^{1/2}}{a^2}$$

$$+ \frac{W_{ox}\Gamma(a^2-x^2)^{1/2}}{\pi a^4} + \frac{4W_{ox}Ux(a^2-x^2)^{1/2}(a^2-2x^2)}{a^4}$$

$$+ \left. \frac{W_{ox}\Gamma(a^2-x^2)^{1/2}(a^2-2x^2)}{\pi a^4} \right] dx \ . \tag{5.3.13}$$

Note that only terms of (5.3.11) and (5.3.12) which are odd terms in y make any contribution to (5.3.13). Since the first and fifth terms cancel, only four terms remain. The integration with respect to x is performed through a transformation employing

$$x = a \cos\theta \ , \tag{5.3.14}$$

and

$$dx = -a \sin\theta \ d\theta \ . \tag{5.3.15}$$

The integration limits are given by

$$\theta = \pi \text{ when } x = -a \ ,$$

$$\theta = 0 \text{ when } x = a \ . \tag{5.3.16}$$

Making the substitutions

$$\frac{DH_o}{Dt} = -\rho U\left[\frac{2W_{ox}\Gamma}{\pi}\int_{\pi}^{0}\cos^2\theta\sin^2\theta d\theta\right.$$

$$- 8W_{ox}Ua\int_{\pi}^{0}\cos\theta\sin^2\theta d\theta \quad \frac{W_{ox}\Gamma}{\pi}\int_{\pi}^{0}\sin^2\theta d\theta$$

$$\left.- \frac{W_{ox}\Gamma}{\pi}\int_{\pi}^{0}\sin^2\theta(1-2\cos^2\theta)d\theta\right] \, . \qquad (5.3.17)$$

In (5.3.17) the second integral makes no contribution because it is antisymmetric. The first integral cancels the second term in the last integral to yield from the surviving terms

$$\frac{DH_o}{Dt} = -\rho W_{ox}\Gamma U = -\rho V_{ox}\Gamma U \, . \qquad (5.3.18)$$

Equation (5.3.18) is identical with (5.1.04), and this result consititutes confirmation of the validity of the differential form of the turbomachinery Equation (3.3.15). For the linear case, the energy transfer rate of the rotor is proportional to the component of the kinetic energy gradient parallel to the moving rotor (or sail).

Recall that energy transfer occurs in the narrow domain of integration indicated in Figure 4 which ranges to infinity. Thus the velocity of sound must be infinite in the potential system in agreement with the assumption of incompressibility.

6.0 The Integral and Differential Isentropic Efficiency

The usual expression for the isentropic efficiency of a turborotor is integral in form. The information used to generate the isentropic efficiency is obtained from the measured thermodynamic flows, the pressure, and the temperature at the inlet and outlet frames of the device or if possible of the rotor and stator sections. The recorded information represents the integral end points of the thermodynamic conditions and the integral form of the efficiency is the useful form.

6.1 The Integral Expression for the Isentropic Efficiency

The classical isentropic turbine and isentropic compressor efficiencies, η_t and η_c are given by

$$\eta_t = \left[(\Delta h_o)_{ideal} / \Delta h_o \right]^{-1} \tag{6.1.01}$$

and

$$\eta_c = (\Delta h_o)_{ideal} / \Delta h_o \ . \tag{6.1.02}$$

The ideal integral total enthalpy change includes the entropic loss term.

$$(\Delta h_o)_{ideal} = \Delta h_o - \int T ds \ . \tag{6.1.03}$$

The absolute value of the ideal total enthalpy change is larger than $|\Delta h_o|$ for the turbine rotor and smaller than $|\Delta h_o|$ for the compressor

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rotor.

The rotor entropic energy rate is obtained from the integration of (3.3.14) over the rotor space.

$$\iiint \rho T \frac{Ds}{Dt} \, d\tau = -\iiint \bar{\mu} \mathbf{W} \cdot \nabla^2 (\bar{\mathbf{W}} + \bar{\mathbf{U}}) r \, dr \, d\theta \, dz \ . \qquad (6.1.04)$$

Finally,

$$\iiint \rho T \frac{Ds}{Dt} \, d\tau = m \int T ds \ . \qquad (6.1.05)$$

Combining (6.1.04), (6.1.05) and (6.1.03) with (6.1.01), the isentropic device efficiency becomes

$$\eta_{\pm} = (1 - \int T ds / \Delta h_o)^{\pm 1} \ . \qquad (6.1.06)$$

The positive subscripted exponent is used for the compressor rotor efficiency where Δh_o is positive, whereas the negative sign is used for turbine rotors wherein Δh_o is negative. The integral form of the efficiency may be written

$$\eta_{\pm} = (1 + \mu \iiint \bar{\mathbf{W}} \cdot \nabla^2 (\bar{\mathbf{W}} + \bar{\mathbf{U}}) d\tau / \iiint \rho \bar{\mathbf{U}} \cdot (\bar{\mathbf{W}} \cdot \nabla \bar{\mathbf{W}} + 2\bar{\omega} \times \bar{\mathbf{W}}) d\tau)^{\pm 1} . (6.1.07)$$

6.2 The Differential Form of the Efficiency

A differential form η_D is obtained from Equation (6.1.07) by dropping the integrating operations.

$$\eta_{D\pm} = (1 + \nu \bar{\mathbf{W}} \cdot \nabla^2 (\bar{\mathbf{W}} + \bar{\mathbf{U}}) / \bar{\mathbf{U}} \cdot (\bar{\mathbf{W}} \cdot \nabla \bar{\mathbf{W}} + 2\bar{\omega} \times \bar{\mathbf{W}}))^{\pm 1} \ . \qquad (6.2.01)$$

This differential form is not very fruitful for calculations of the integral device performance. Nor is it very sensitive as a measure of the local level of loss in the design of blade shapes. A parameter, β, more sensitive by an order of magnitude should be the ratio of the entropic energy rate to the substantial total enthalpy rate or

$$\beta = \frac{\nu \overline{W} \cdot \nabla^2 (\overline{W} + \overline{U})}{\overline{U} \cdot [\nabla W^2 / 2 + \overline{\xi} \times \overline{W}]} \ . \tag{6.2.02}$$

It is interesting to separate the potential and rotational components of the total enthalpy transfer. Moreover, it is noteworthy that great success has been achieved in the design of marine screws, etc., using the assumption of pure potential flow. Therefore, the rotational component of the flow will be written thus:

$$\nabla \times \overline{W} + 2\overline{\omega} = (\nabla \times \overline{W}_s + 2\overline{\omega}) + \nabla \times \overline{W}_{vi} = \overline{\xi} \ , \tag{6.2.03}$$

where $\nabla \times \overline{W}_{vi}$ is the non-ideal vorticity, $\nabla \times \overline{W}_s$ is the relative vorticity in ideal flow, the parenthetical ideal term vanishes, and $\overline{\xi}$ is a residual vorticity arising from non-ideal viscous mechanisms. Then

$$\beta = \frac{\nu \overline{W} \cdot \nabla^2 (\overline{W} + \overline{U})}{\overline{U} \cdot [\nabla W^2 / 2 + \overline{\xi} \times \overline{W}]} \ . \tag{6.2.04}$$

Also the ratio may be written with non-dimensional starred variables including the Reynolds and Ekman numbers.

$$\beta = \left[R_o \frac{\overline{U}^* \cdot \nabla^* \overline{W}^{*2} / 2}{\nu \overline{W}^* \cdot \nabla^{*2} (\overline{W}^* + \overline{U}^*)} \right.$$

$$\left. + E_k \frac{\overline{U}^* \cdot \overline{\zeta}^* \times \overline{W}^*}{\nu^* \overline{W}^* \cdot \nabla^{*2} (\overline{W}^* + \overline{U}^*)} \right]^{-1} . \tag{6.2.05}$$

In bulk axial flows, the residual vorticity will be small compared with the linear velocity. Therefore, in the bulk flow the Ekman number will be small and the rotational contribution will be small. In the boundary layers the Ekman term will be the more significant term. In any case, the beta ratio can be examined immediately to study the consequence of modifying blade geometry and fluid flow field.

7.0 A Practical Application of the Differential Form

Let it be assumed that the differential form, Equation (3.3.15), is employed in a situation where the Coriolis term may be neglected. This will be the case when the ratio of forces from linear kinetic energy gradients to Coriolis force is large (as measured by the Rossby number) in a linear cascade of rotor blades. For this system the geometry may be cartesian. Then from (3.3.15)

$$\frac{Dh_o}{Dt} = \bar{U}\cdot\nabla W^2/2 + \bar{U}\cdot(\nabla x \bar{W} + 2\bar{\omega})x\bar{W} \approx \frac{U}{2}\frac{\partial W^2}{\partial y_w} . \qquad (7.0.01)$$

Note that in an ideal system the vorticity term would vanish in any case.

In a turbine rotor, it is expected that most of the flow will exhibit a negative substantial total enthalpy derivative so that net work is performed by the rotor outside the system. Figure 5 shows the streamlines of a linear cascade observed by the observer moving with the blades. Velocity vectors just above the suction surface of the blades are significantly larger than the velocity vectors on the pressure surface of the blade. On the pressure surface there is a sudden decrease in the velocity. In contrast, the velocity increases rapidly just above and beyond the stagnation point over the suction side of the blade. Thus for this region the kinetic energy gradient must be positive with total enthalpy.

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Figure 6 shows lines of equal total pressure for a two dimensional moving cascade of turbine blades[10]. Although the analytical results displayed in the Figure are based upon a viscous Navier-Stokes treatment of the flow field, the viscous losses are only of the order of 6%. Thus, the total enthalpy gradient is approximately very similar to the total pressure gradient. The results in the Figure show that positive increases in the total enthalpy occur just above and beyond the stagnation point on the suction side of the blade as predicted in the previous paragraph.

8.0 Conclusions

Galilean transformations between the absolute and moving frame impose a crypto-steady-state relation between time derivatives of a thermodynamic function in the absolute frame and their gradients in the moving frame. These crypto-steady relationships are inherently contained within the Navier-Stokes Equations for the absolute and moving frames.

The substantial total enthalpy derivative coupled with the substantial entropic energy derivative may be written solely in terms of the flow field of the moving frame. In the moving frame the relative total enthalpy, known as the rothalpy yields a vanishing substantial derivative. Therefore the substantial entropic energy derivative is uncoupled and explicit in the moving frame. This explicit substantial entropic energy derivative is invariant in all frames and may be used to obtain an uncoupled explicit substantial total enthalpy derivative. This latter derivative was hypothesized to be a differential form of Euler's Turbomachinery Equations corrected for real viscous losses.

The fact that integration of the differential form indeed yields the classical integral form of Euler's Turbomachinery Equation validates the aforementioned hypothesis. Applied to a two-dimensional linear turbine such as a sailplane with infinite sail, the significant contribution derives from the gradient of the kinetic energy in the

direction of blade motion.

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The Galilean Transformation:

$$t_v = t_w = t$$

$$z_v = z_w + z_0 = z_w = z$$

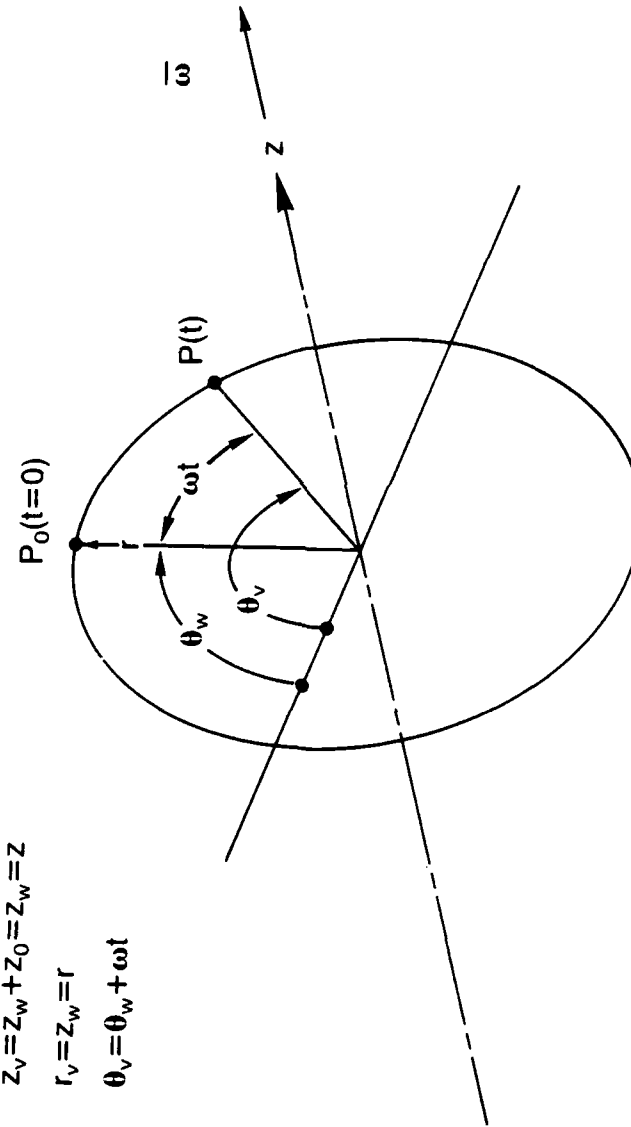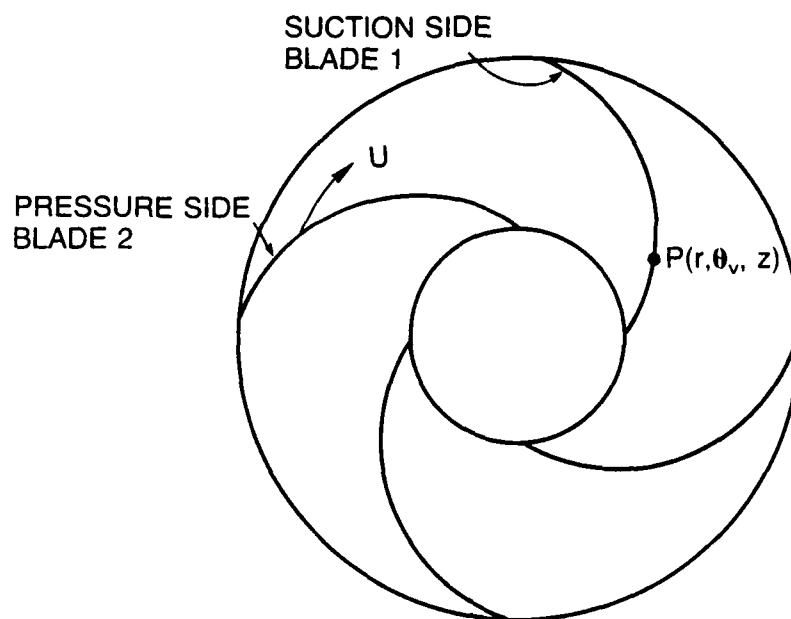$$r_v = z_w = r$$

$$\theta_v = \theta_w + \omega t$$

Fig. 1. Configurational relationships between the absolute and moving coordinate systems

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a) Absolute frame
P(r, θ_v, z) is fixed in space. Time changes. Blades rotate.

$$\left(\frac{\partial p}{\partial t}\right)_{r,\theta_v, z} = \frac{-u}{r}\left(\frac{\partial p}{\partial \theta_w}\right)_{r, z, t}$$

OVER TIME, Δt, THE PRESSURE SIDE OF THE NEXT BLADE
MOVES TO THE POINT P(r, θ_v, z).

Fig. 2.   Interpretation of the time-dependent pressure term in rotor flow

b) Moving frame
P(r, z, t_0) moves towards increasing θ_w in a fixed instant, t_0. Blades appear stationary.
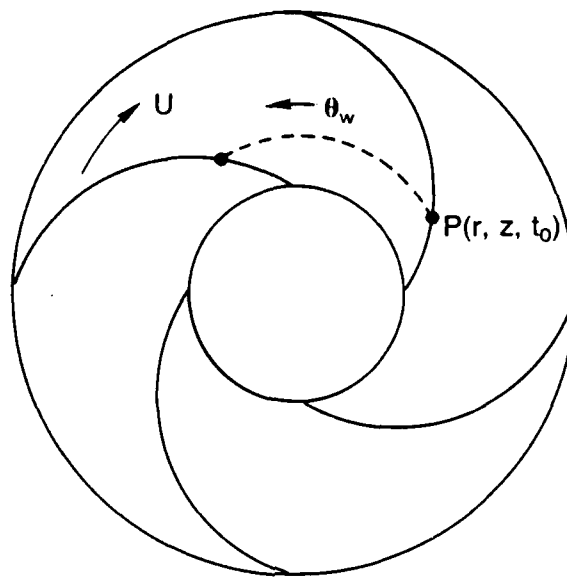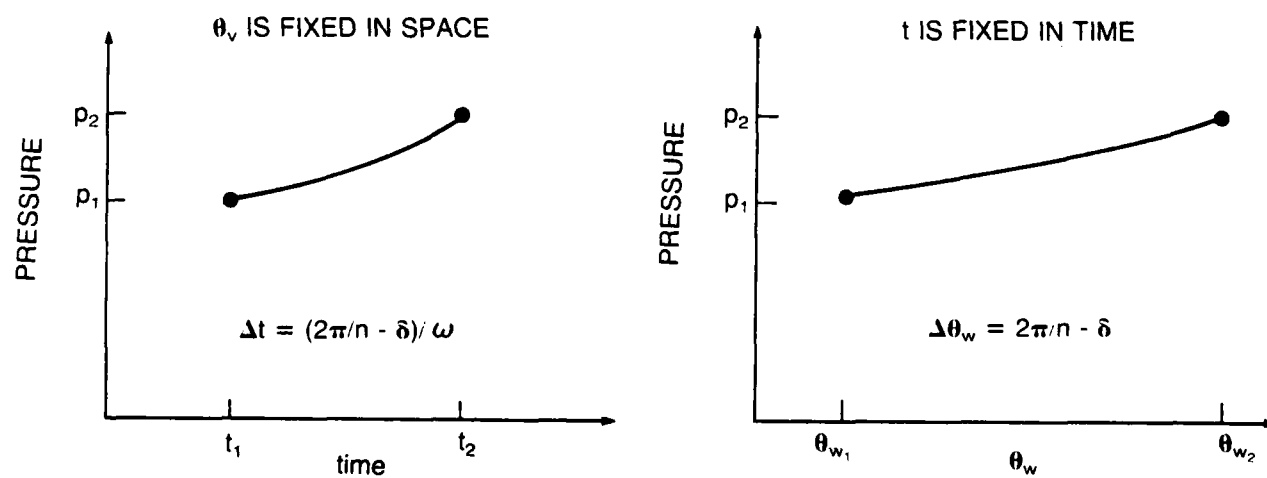

c)

Fig. 2.   Interpretation of the time-dependent pressure term in rotor flow (cont.)
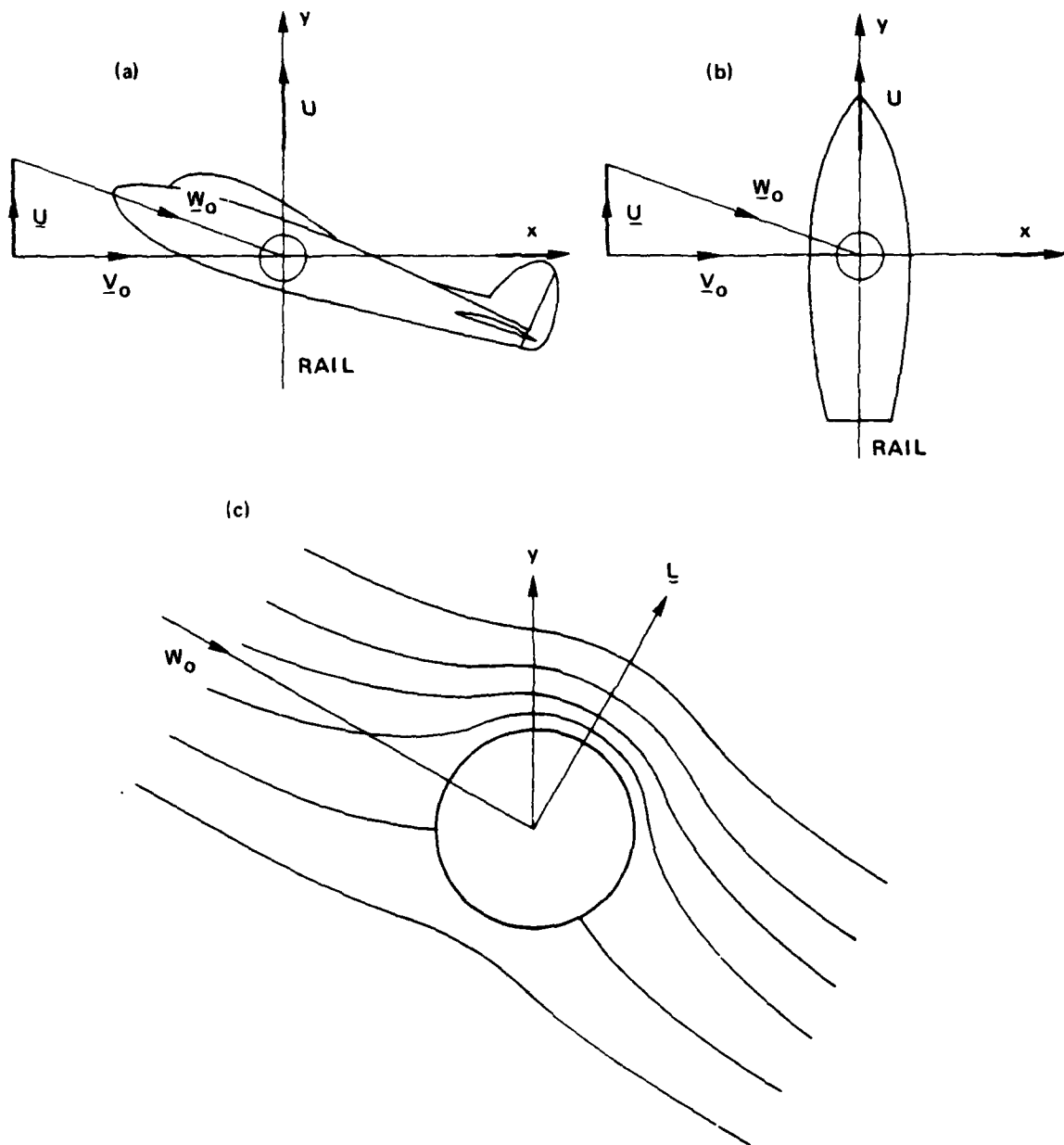
Fig. 3. The single-blade linear turbine: (a) Sailplane version,
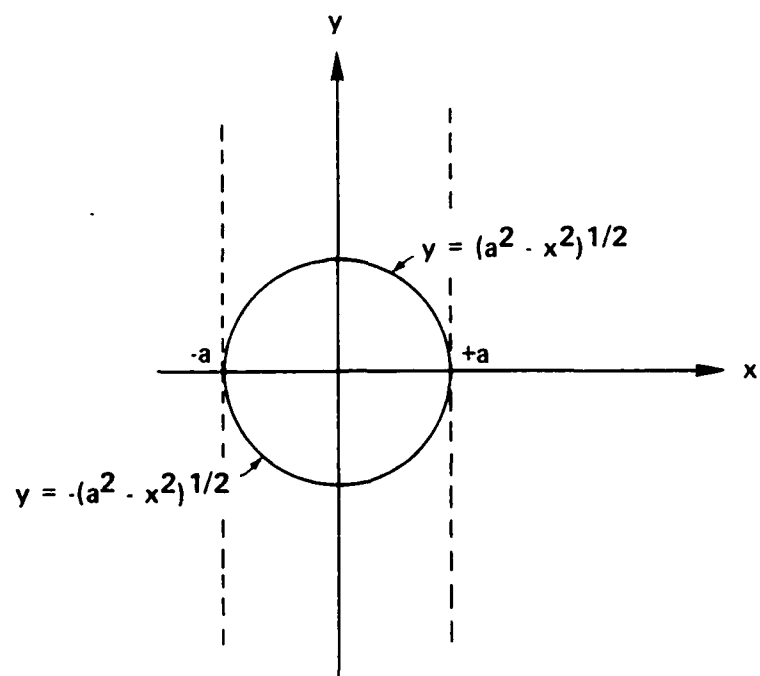(b) sailboat version, and (c) the rotating cylinder blade.

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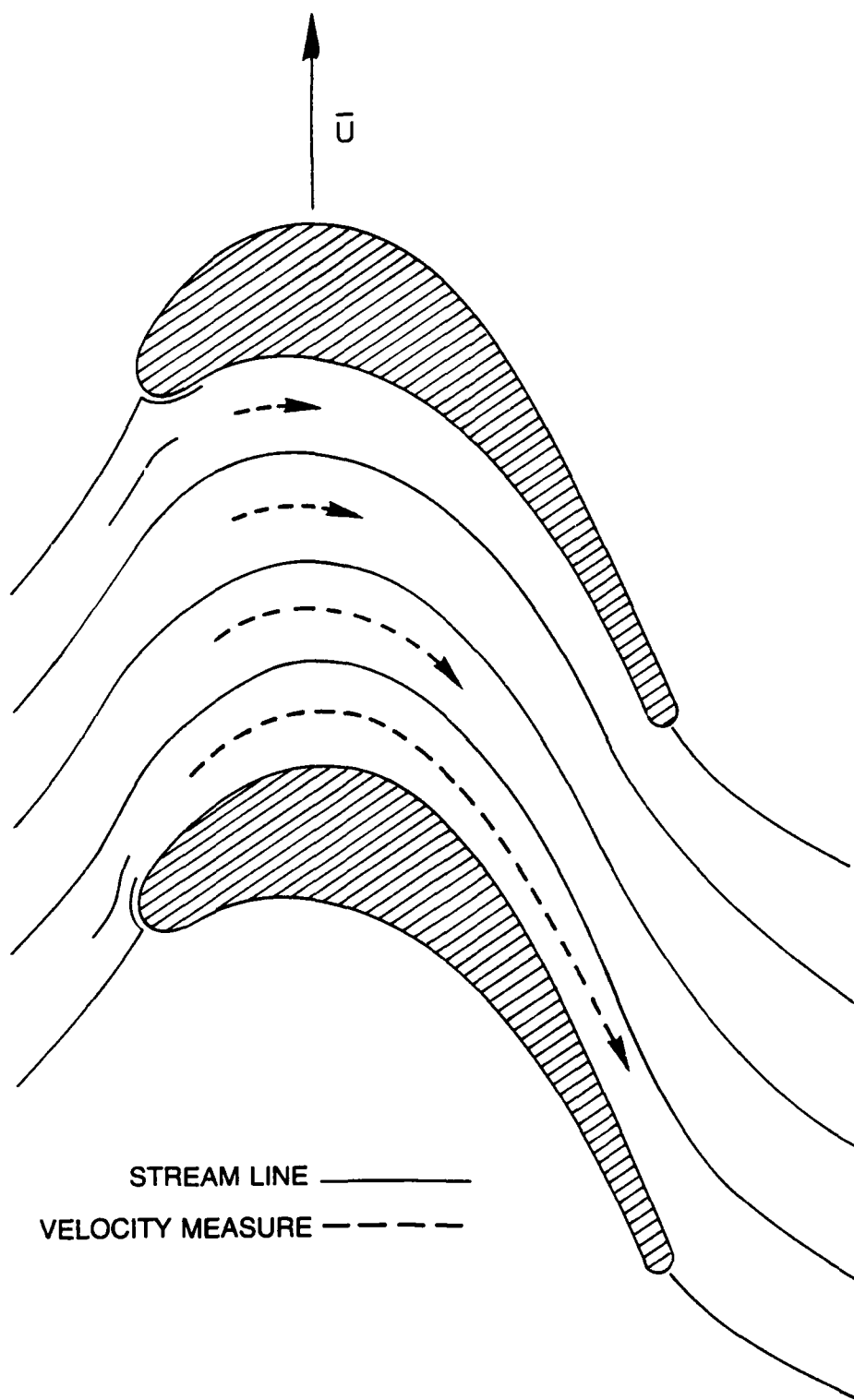Fig. 4. Zone of interaction between the fluid and the blade.

$\bar{U}$

STREAM LINE ————————

VELOCITY MEASURE — — — —

Fig. 5.   Fluid field behavior in a moving cascade of blades
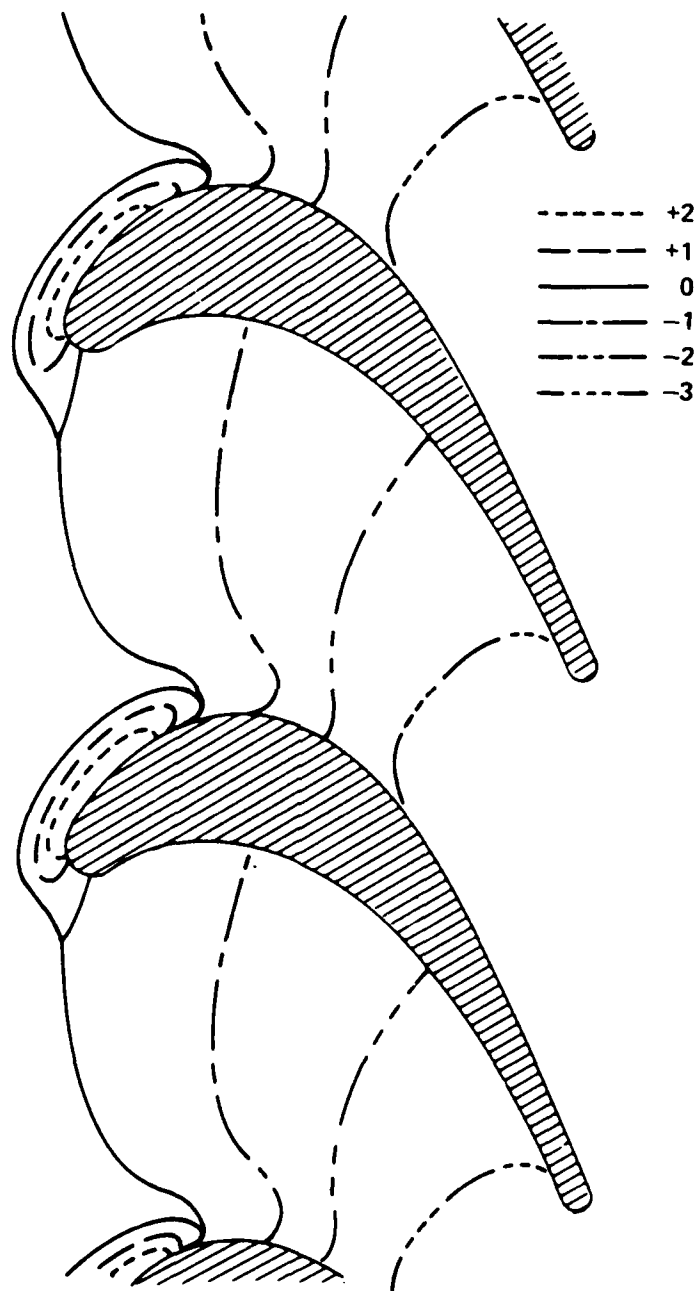
Fig. 6. Lines of constant total pressure (not to scale) (After Rai).

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