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CAPACITY OF THE STATIONARY GAUSSIAN CHANNEL

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CAPACITY OF THE STATIONARY GAUSSIAN CHANNEL

Abstract

Information capacity of the stationary Gaussian channel is determined under the assumption that both the channel noise and the constraint are defined by rational spectral densities. The results complement a well-known result due to Holsinger and Gallager.

Introduction

The information capacity per unit time of the stationary Gaussian channel is a problem having solutions that date back roughly 30 years. Fano's 1961 book [1] contains a solution based on his ingenious "waterfilling" approach. However, Fano's approach was somewhat heuristic, and incomplete. Although he discussed the necessity to impose frequency constraints on the transmitted signal, appropriate constraints were not incorporated into his analysis. Thus, his result does not give a solution for physically realizable noise processes (e.g., rational spectral density).

A careful solution to the capacity problem is given in Gallager's book [2], with the basic approach attributed to Holsinger. In that work, the message process is passed through a linear filter before being transmitted: the filter's properties are such that the transmitted signal is constrained both in its total energy and the distribution of the energy over frequency. The Holsinger-Gallager solution can be formally obtained from Fano's result by replacing the noise spectral density Φ_N (in Fano's result) with the function $\Phi_N/|\hat{H}|^2$, where $\hat{H}$ is the Fourier transform of the filter.

However, the Holsinger-Gallager solution does not completely solve the capacity problem. In this paper we complete the solution. It will be seen (under mild assumptions) that the new results to be given here are complementary to those of Holsinger-Gallager (H-G); that these results and the H-G result taken together exhaust all possible situations, including infinite capacity; and that the new results enable one to transmit signals that occupy a larger portion of the effective noise bandwidth. This can lead to a larger value of capacity (for fixed signal total energy) than can be obtained under the H-G assumptions, provided that there is sufficient freedom in the selection of the constraint on the signal.

Problem Formulation

The channel is $Y = X + N$, where N represents stationary Gaussian noise, X is the transmitted signal process (independent of N), and Y is the channel output. It will be assumed that N has a rational spectral density, Φ_N. The constraint to be placed on the signal process is given by $E\|X^T\|^2_{W,T} \leq PT$, where $P > 0$ is fixed, $T > 0$, X^T is the signal process X restricted to the parameter set [0,T], and $\|\cdot\|_{W,T}$ denotes a reproducing kernel Hilbert space (RKHS) norm with parameter set [0,T]. $\|\cdot\|_{W,T}$ will be defined by a covariance function r_W

corresponding to a rational spectral density function Φ_W. H_W will denote the RKHS of r_W with parameter set $[0,\infty)$; $H_{W,T}$ will be the RKHS of r_W with parameter set $[0,T]$. The quantity sought is $\bar{C}_W(P)$, defined by $\bar{C}_W(P) = \overline{\lim_{T\to\infty}} \frac{1}{T} C_W^T(PT)$, where $C_W^T(PT)$ is the supremum of $I[X^T,Y^T]$ over all signal processes X^T that have sample paths a.s. in $H_{W,T}$ and satisfy $E\|X^T\|^2_{W,T} \leq PT$. The capacity $C_W^T(PT)$ for $T > 0$ is determined in [3].

To realize the constraint, one may proceed as follows. Let h be the L_2 Fourier transform of $\sqrt{\Phi_W}$. Let M be the message process. Set

$$E\int_0^T M^2(t)dt \leq \frac{PT}{2\pi} \qquad \text{(constraint on M)},$$

and define X^T and X_∞^T by

$$X^T(t) \equiv \int_0^T h(t-s)M(s)ds \qquad t \in [0,T]$$

$$X_\infty^T(t) \equiv \int_0^T h(t-s)M(s)ds \qquad t \geq 0.$$

Since the integral operator in $L_2[0,T]$ with h as kernel is strictly positive, $I[X^T,Y^T] = I[M,Y^T]$.

This gives $E\|X^T\|^2_{W,T} \leq E\|X_\infty^T\|^2_W = E \int_{-\infty}^{\infty} \frac{|\hat{X}_\infty^T|^2}{\Phi_W}(\lambda)d\lambda$, where $\hat{x}$ is the L_2 Fourier transform of x. Thus,

$$\frac{1}{T} E\|X^T\|^2_{W,T} \leq \frac{1}{T} E\|X_\infty^T\|^2_W = \frac{1}{T} E\int_{-\infty}^{\infty} \frac{|\hat{X}_\infty^T|^2}{\Phi_W}(\lambda)d\lambda$$

$$= \frac{1}{T} E\int_{-\infty}^{\infty} \frac{|\hat{h}\hat{M}|^2}{\Phi_W}(\lambda)d\lambda = \frac{1}{T} E\int_{-\infty}^{\infty} |\hat{M}(\lambda)|^2 d\lambda \quad (\text{since } |\hat{h}|^2 = \Phi_W)$$

$$= \frac{1}{T} E\ 2\pi\int_{-\infty}^{\infty} M^2(t)dt = \frac{1}{T}\ 2\pi E\int_0^T M^2(t)dt \leq P.$$

Moreover, $\lim_{T\to\infty} \frac{1}{T} E\|X^T\|^2_{W,T}$ exists and equals P.

The basic approach here is to compute $\bar{C}_W$ by using the results on $C_W^T(PT)$ and letting $T \to \infty$. This procedure is very complicated; the results given in [3] for $C_W^T(PT)$ are stated in terms of the spectrum of a self-adjoint operator

S_T in $L_2[0,T]$, satisfying $R_{N,T} = R_{W,T}^{\frac{1}{2}}(I_T+S_T)R_{W,T}^{\frac{1}{2}}$. Here, $R_{N,T}$ and $R_{W,T}$ denote the integral operators in $L_2[0,T]$ whose kernels are the covariance function of N and r_W, respectively. $C_W^T(PT)$ then depends on θ_T, the smallest limit point of the spectrum of S_T; the eigenvalues (if any) of S_T that are strictly less than θ_T; and the value of PT. The expression for $C_W^T(PT)$ takes several forms, depending on these quantities.

We also identify two other capacities. Let $\mathfrak{X}_W(P)$ be the set of all X (defined on $[0,\infty)$) such that for every $T > 0$, $X^T \in H_{W,T}$ with probability one, and $\overline{\lim_{T\to\infty}} \frac{1}{T} E\|X^T\|_{W,T}^2 \leq P$. Next, let $\mathcal{S} \equiv \mathcal{S}_W(P)$ be the set of all wide sense (w.s.) stationary processes X with a SDF (spectral density function), denoted by Φ_X, such that Φ_X/Φ_W is bounded and $\int_{-\infty}^{\infty} \frac{\Phi_X}{\Phi_W}(\lambda)d\lambda \leq 2\pi P$. The corresponding capacities are then

$$\overline{C}_W(P,\mathfrak{X}) = \sup\{\overline{\lim_{T\to\infty}} \frac{1}{T} I(X^T,Y^T): X \in \mathfrak{X}_W(P)\};$$

$$\overline{C}_W(P;\mathcal{S}) = \sup\{\overline{\lim_{T\to\infty}} \frac{1}{T} I(X^T,Y^T): X \in \mathcal{S}_W(P)\}.$$

Lemma 1. $\mathcal{S}_W(P) \subset \mathfrak{X}_W(P)$.

Applying Lemma 1, it is clear that $\overline{C}_W(P;\mathfrak{X}) \geq \overline{C}_W(P;\mathcal{S})$. The differences between the classes of signal processes leading to $\overline{C}_W(P)$, $\overline{C}_W(P;\mathfrak{X})$, and $\overline{C}_W(P;\mathcal{S})$ can be summarized as follows. $\overline{C}_W(P) \equiv \overline{\lim_{T\to\infty}} \frac{1}{T} C_W^T(PT)$, where $C_W^T(PT)$ is computed (separately for each T) by maximizing $I(X^T,Y^T)$ over all admissible X^T. Thus, for example, if $C_W^T(PT) = I(X^T,Y^T)$ and $C_W^{T'}(PT') = I(X^{T'},Y^{T'})$, for $T' > T$, then it is not necessary that $X^T(t) = X^{T'}(t)$ for t in $[0,T]$. However, in computing $\overline{C}_W(P;\mathfrak{X})$, the signal process X is fixed; one computes $I(X^T,Y^T)$ for this fixed process, all $T > 0$; and $\overline{C}_W(P,\mathfrak{X})$ is the supremum over all $X \in \mathfrak{X}$ of $\overline{\lim_{T\to\infty}} \frac{1}{T} I(X^T,Y^T)$. This also applies to $\overline{C}_W(P;\mathcal{S})$, with the further restriction that the signal process is w.s. stationary. However, it is not necessary that

X in $\mathcal{X}_W$ satisfy $E\|X^T\|^2_{W,T} \le PT$ for any finite T.

Let $1 + \theta = \lim_{|\lambda|\to\infty} \Phi_N(\lambda)/\Phi_W(\lambda)$, with $\theta \equiv \infty$ if the limit does not exist; $-1 \le \theta \le \infty$. θ is an important parameter for the capacity problem.

Main Results

Theorem 1. $\bar{C}_W(P) = \bar{C}_W(P;\mathcal{X}) = \bar{C}_W(P;\mathcal{S})$.

This result has the following implications for determining the capacity $\bar{C}_W(P)$. First, it is sufficient to consider only all signal processes $X \equiv \{X(t),\ t \ge 0\colon X^T \in H_{W,T}$ a.s. for all $T > 0$, and $\overline{\lim}_{T\to\infty} \frac{1}{T} E\|X^T\|^2_{W,T} \le P\}$ rather than all families of signal processes $\{X^T\colon T > 0\} = \{X^T(t),\ t \in [0,T]\colon$ $X^T \in H_{W,T}$ a.s. and $E\|X^T\|^2_{W,T} \le PT$, all $T > 0\}$. Second, these processes can be taken as w.s. stationary.

We now give the value of $\bar{C}_W(P)$. The results will be separated according to the value of the parameter θ.

Theorem 2. If $\theta = -1$, then $\bar{C}_W(P) = \infty$. $C^T_W(PT) = \infty$ can be achieved for every $T > 0$, using a signal process with finite-dimensional support.

Theorem 3. If $-1 < \theta < \infty$, let $\Lambda = \{\lambda\colon \Phi_N(\lambda) < (1+\theta)\Phi_W(\lambda)\}$.

I. If $P \ge \frac{1}{2\pi} \int_\Lambda \left[1 + \theta - \frac{\Phi_N}{\Phi_W}(\lambda)\right] d\lambda$, then

$$\bar{C}_W(P) = \frac{1}{4\pi} \int_\Lambda \log\left[(1+\theta)\frac{\Phi_W}{\Phi_N}(\lambda)\right] d\lambda + \frac{P}{2(1+\theta)} + \frac{1}{4\pi(1+\theta)} \int_\Lambda \left[\frac{\Phi_N}{\Phi_W}(\lambda) - (1+\theta)\right] d\lambda.$$

II. If $P \le \frac{1}{2\pi} \int_\Lambda \left[1 + \theta - \frac{\Phi_N}{\Phi_W}(\lambda)\right] d\lambda$, then

$$\bar{C}_W(P) = \frac{1}{4\pi} \int_{\Lambda_1} \log\left[(1+\theta)A(P)\frac{\Phi_W}{\Phi_N}(\lambda)\right] d\lambda$$

where $\Lambda_1 = \{\lambda\colon \Phi_N(\lambda) \le (1+\theta)A(P)\Phi_W(\lambda)\}$ and $A(P) \le 1$ is uniquely defined by

$$P = \frac{1}{2\pi} \int_{\Lambda_1} \left[(1+\theta)A(P) - \frac{\Phi_N}{\Phi_W}(\lambda)\right] d\lambda.$$

In Part I, $\bar{C}_W(P) \ge P/[2(1+\theta)]$. In Part II, $\bar{C}_W(P) \ge P/[2(1+\theta)A(P)] \ge P/2(1+\theta)$.

Theorem 4 (Holsinger-Gallager). If $\theta = \infty$, then

$$\bar{C}_W(P) = \frac{1}{4\pi} \int_{\Lambda_2} \log\left[B(P)\frac{\Phi_W}{\Phi_N}(\lambda)\right] d\lambda,$$

where $\Lambda_2 = \{\lambda: \Phi_N(\lambda) \leq B(P)\Phi_W(\lambda)\}$ and $B(P)$ is uniquely determined by

$$P = \frac{1}{2\pi} \int_{\Lambda_2} \left[B(P) - \frac{\Phi_N}{\Phi_W}(\lambda)\right] d\lambda.$$

Theorem 5. In Part II of Theorem 3 and in Theorem 4, capacity can be attained with a stationary Gaussian signal process X, having spectral density Φ_X. For Part II of Theorem 3, Φ_X is defined by

$$\begin{aligned} \Phi_X(\lambda) &= (1+\theta)A(P) - [\Phi_N/\Phi_W](\lambda) \qquad && \lambda \text{ in } \Lambda_1 \\ &= 0 && \text{otherwise.} \end{aligned}$$

For Theorem 4, Φ_X is defined by

$$\begin{aligned} \Phi_X(\lambda) &= B(P) - [\Phi_N/\Phi_W](\lambda) \qquad && \lambda \text{ in } \Lambda_2 \\ &= 0 && \text{otherwise.} \end{aligned}$$

In considering Theorems 2-5, it can be seen that Part II of Theorem 3 is quite similar to Theorem 4; in fact, Theorem 4 can be obtained by substituting $B(P) \equiv (1+\theta)A(P)$ in Part II of Theorem 3. This results from the fact that as $\theta \to \infty$, $A(P)$ must converge to zero (fix Φ_N and let Φ_W vary). However, Part I of Theorem 3 is quite different from Theorem 4. To gain insight into this, one can compare these results with those contained in Theorem 3 and Corollary 4 of [3]. Specifically, Part I of Theorem 3 above should be compared with Part (a) of Theorem 3 in [3]; Part II of Theorem 3 above should be compared with Part (b) of Theorem 3 in [3]; and Theorem 4 should be compared with Corollary 4 in [3].

The question of attaining capacity (in particular, by a stationary process) in Part I of Theorem 3 is still open. For any finite T, $C_W^T(PT)$ cannot be attained when the conditions of Part I are satisfied (see Theorem 3(e) of [3]).

To illustrate the difference in the effect of the constraints on the properties of the transmitted signal, suppose that $\Phi_N(\lambda) = \alpha^2/(\lambda^2+\alpha^2)$. Then $\theta = \infty$ (Theorem 4 constraint) requires that $\Phi_W(\lambda) = P_N(\lambda)/P_D(\lambda)$ have numerator polynomial P_N of degree at least 4 less than the degree of the denominator polynomial P_D. However, the model of Theorem 3 permits Φ_W to be given by P_N/P_D such that the degree of P_N is 2 less than that of P_D. This means, in particular, that the sample paths of the transmitted signal for the constraint Φ_W

satisfying Theorem 4 must be twice-differentiable, with L_2 derivative. The sample paths of the transmitted signal obeying a constraint Φ_W satisfying Theorem 3 need only be once-differentiable with L_2 derivative. Thus, the transmitted signals using the constraint of Theorem 3 need not be as smooth as those obeying a constraint satisfying Theorem 4, and consequently can utilize more of the frequency spectrum.

Conclusions

The new results given here show that

(1) Attention can be restricted to w.s. stationary signals (Theorem 1);

(2) The new results for the value of the capacity (Theorem 2 and Theorem 3) are complementary (under the assumptions used here) to the result of Holsinger and Gallager; together, these results exhaust all possible values of the capacity;

(3) The constraints permitted in obtaining Theorem 3 enable the signal process to use more of the available noise bandwidth than signal processes obeying the constraints of the Holsinger-Gallager model; if there is freedom to choose the constraint, then the capacity can be increased by using a constraint as in Theorem 3.

All of the above results are obtained under a basic assumption: both the noise and the constraint are defined by rational spectral densities. This restriction is imposed in order to obtain results that can be directly compared with the result of Holsinger and Gallager. More general results will be given in [4].

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[2] R.G. Gallager, Information Theory and Reliable Communication, Wiley, New York, 1968.

[3] C.R. Baker, Capacity of the mismatched Gaussian channel, *IEEE Trans. on Information Theory*, IT-33, 802-812 (1987).

[4] C.R. Baker and S. Ihara, Information capacity of the stationary Gaussian channel, in preparation.