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SOME ~~REMARKS~~ ON REACHABILITY OF

INFINITE-DIMENSIONAL LINEAR SYSTEMS

by

Yutaka YAMAMOTO*

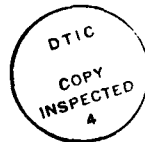Department of Applied Mathematics and Physics

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Proposed running head: REACHABILITY OF INFINITE-DIM. SYSTEMS

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ABSTRACT.

In this paper we shall be concerned with the question of reachability when allowing distribution inputs. We show that a certain class of systems accept distribution inputs, but, in general, they cannot be exactly reachable. We shall also consider the problem of the uniqueness of canonical realizations in relation to exact reachability, and show that Matsuo's result on uniqueness ([5]) does not apply to the example given in Baras, Brockett, and Fuhrmann [1].

1. INTRODUCTION.

There have been a number of interesting investigations on continuous-time constant (infinite-dimensional) linear systems. For example, Kalman and Hautus [3] proposed a framework for treating such systems with the following setup: Ω (the input space) := $E'_{(-\infty,0]}$ (the space of distributions with compact support contained in $(-\infty, 0]$), Γ (the output space) := $E_{[0,\infty)}$ (the space of C^∞-functions on $[0, \infty)$). The (zero-initial state) input/output map (external behavior of the system) is then represented by convolution of an input ω and a fixed C^∞-function (<u>impulse response function</u>, or <u>weighting pattern</u>) A. Using this framework, they successfully derived a differential equation description of an internal model associated with such external behavior. For this internal model they took the state space to be the quotient space $E'_{(-\infty,0]}/\ker f$, where f denotes the input/output map. Note that their realization (internal model) is always canonical in the classical sense (i.e., (exactly) reachable and observable). But the discussion on the character of the state space $E'_{(-\infty,0]}/\ker f$ was left somewhat open.

Later Matsuo [5] showed that every canonical realization in this framework is isomorphic to the Kalman and Hautus realization $E'_{(-\infty,0]}/\ker f$ <u>if we demand that the state space be barreled.</u>

Somewhat independently from this line, there have been a great deal of approaches to constant linear systems described by functional differential equations in a Banach space X:

$$\frac{dx}{dt} = Fx + Gu, \quad x \in X, \quad u \in U, \tag{1.1}$$

where U is also a Banach space.

In this context several authors (for example, Baras, Brockett, and Fuhrmann [1]; Fuhrmann [2]; Triggiani [9, 10]) noted that under some mild assumptions the system (1.1) cannot be exactly reachable with L^1- (or L^∞-) inputs with bounded support.

It seems that there is no detailed account on the precise relationship between these two types of approaches at present. For example, realizations given by Kalman and Hautus [3] are always exactly reachable (but with distribution inputs) by construction. So this leads to the following interesting question: Can the system (1.1) be made exactly reachable by enlarging the input space to the space of distributions with compact support?

We shall start by showing that a certain subclass of systems of type (1.1) indeed "accepts" E'-inputs, i.e., there exists a continuous linear map $g: E'_{(-\infty,0]} \to X$ which extends the usual reachability map. One of the objectives of this paper is then to show (first, abstractly for general systems, then concretely for an example) that they still cannot be exactly reachable. This, however, leads to yet another interesting question. Suppose we restrict our attention to the reachable set of (1.1). Then we obtain an exactly reachable system with the topology induced from the whole space X. Now according to Matsuo [5], if it is further observable, it must be topologically isomorphic to the Kalman and Hautus realization $E'_{(-\infty,0]}/\ker f$, if thus constructed state space is barreled. On the other hand, nonuniqueness of such realizations is clearly shown by an example given by Baras, Brockett, and Fuhrmann [1]. In the last section we show that the reachable set of such a system, with the induced topology from the whole space X, is not barreled.

2. MATHEMATICAL PRELIMINARIES.

Throughout the discussion k denotes a fixed field, either $\underline{\underline{R}}$ or $\underline{\underline{C}}$, with the usual topology. Every space is a locally convex Hausdorff space over k. Here we list some results on locally convex spaces and distributions that are needed later. We shall however omit proofs since they are available in the following standard references: Köthe [4]; Schaefer [6]; L. Schwartz [7]; Treves [8].

Let $E_{(-\infty,0]}$ denote the set of all k-valued C^{∞}-functions on $(-\infty, 0]$. The space $E_{(-\infty,0]}$ is a Fréchet space with its topology generated by the following countable family of seminorms:

$$p_{m,\alpha}(\psi) := \sup\ \{|(\tfrac{d}{dt})^{j}\psi(t)|;\ -\alpha \leq t \leq 0,\ \ 0 \leq j \leq m\}, \qquad (2.1)$$

where m and α are positive integers.

DEFINITION 2.2. <u>A subset</u> B <u>of</u> $E_{(-\infty,0]}$ <u>is bounded iff for every</u> $m > 0,\ \alpha > 0$ <u>there exists</u> $C_{m,\alpha} > 0$ <u>such that</u>

$$p_{m,\alpha}(\psi) \leq C_{m,\alpha} \quad \underline{\text{for all}}\ \ \psi \in B. \qquad (2.3)$$

Let $E'_{(-\infty,0]}$ be the dual space of $E_{(-\infty,0]}$, i.e., the set of all continuous linear forms on $E_{(-\infty,0]}$. It is well known that $E'_{(-\infty,0]}$ consists of distributions with compact support contained in $(-\infty, 0]$. Let $\langle\phi, \omega\rangle$ denote the value of $\omega \in E'_{(-\infty,0]}$ evaluated at $\phi \in E_{(-\infty,0]}$. The dual space $E'_{(-\infty,0]}$ is equipped with the <u>strong dual topology</u> defined by the seminorms

$$p_B(\omega) := \sup_{\phi \in B} |\langle \phi, \omega \rangle|, \quad \omega \in E'_{(-\infty,0]}, \tag{2.4}$$

where B runs over all bounded subsets of $E_{(-\infty,0]}$.

We shall make use of the following lemmas later.

LEMMA 2.5. <u>Let</u> $\omega \in E'_{(-\infty,0]}$ <u>and</u> B <u>a bounded subset of</u> $E_{(-\infty,0]}$. <u>Then there exists a constant</u> $C > 0$ <u>such that</u>

$$|\langle \phi, \omega \rangle| \le C \quad \underline{\text{for all}} \quad \phi \in B. \tag{2.6}$$

LEMMA 2.7. <u>Let</u> X <u>be a normed linear space with the norm</u> $\| \cdot \|$ <u>and</u> $g: E'_{(-\infty,0]} \to X$ <u>a linear map. Then</u> g <u>is continuous iff there exists a bounded set</u> B <u>in</u> $E_{(-\infty,0]}$, <u>and a constant</u> $C > 0$ <u>such that</u>

$$\| g(\omega) \| \le C p_B(\omega) \quad \underline{\text{for all}} \quad \omega \in E_{(-\infty,0]}. \tag{2.8}$$

The following definition will be needed in Section 5.

DEFINITION 2.9. <u>Let</u> X <u>be a locally convex Hausdorff space. A subset</u> T <u>of</u> X <u>is a barrel iff it satisfies the following conditions:</u>

(a) T <u>is convex;</u>

(b) T <u>is balanced in the sense that</u> $\alpha x \in T$ <u>for all</u> $|\alpha| \le 1$ <u>and</u> $x \in T$;

(c) T <u>is absorbing, i.e., for every</u> $x \in X$ <u>there exists a scalar</u> $\alpha \ne 0$ <u>such that</u> $\alpha x \in T$;

(d) T <u>is closed</u>.

<u>The space</u> X <u>is barreled iff every barrel is a neighborhood of</u> 0.

Remark. For a locally convex space one can choose a neighborhood base consisting of barrels. But the converse of this statement is, in general, false.

3. EXTENSION OF INPUTS.

Consider a constant linear system Σ defined by the following functional-differential equation in a reflexive Banach space X:

$$\frac{dx}{dt} = Fx + Gu, \tag{3.1}$$

where F is a continuous linear operator in X, G a fixed element of X, and u (input) a scalar-valued function. Suppose that the initial state of the system is zero at $t = -\infty$ and an input u of bounded support is applied to the system until $t = 0$. Then the resulting state at time 0 is given by

$$x(0) = \int_{-\infty}^{0} \exp(-Ft)Gu(t)dt. \tag{3.2}$$

Note that for every $x^* \in X'$ (= the dual space of X) the following equalities hold:

$$\langle x(0), x^* \rangle = \int_{-\infty}^{0} \langle \exp(-Ft)Gu(t), x^* \rangle dt \tag{3.3}$$

$$= \int_{-\infty}^{0} \langle \exp(-Ft)G, x^* \rangle u(t)dt$$

at least for sufficiently smooth u. Note also that the right-hand side of (3.3) has the form that u "acts" on the function $\langle \exp(-Ft)G, x^* \rangle$. This observation suggests the following definition:

$$\langle g^{\Sigma}(u), x^* \rangle := \langle\langle \exp(-Ft)G, x^* \rangle, u \rangle, \tag{3.4}$$

where $g^{\Sigma}(u)$ denotes the "state" resulting at time 0 under the action of an input $u \in E'_{(-\infty,0]}$. Of course, in order that (3.4) makes sense, $\langle \exp(-Ft)G, x^* \rangle$ must be C^{∞}. Indeed, we have

LEMMA 3.5. <u>For every</u> $x^* \in X'$, $\langle \exp(- Ft)G, x^* \rangle$ <u>is a</u> C^∞-<u>function</u> <u>on</u> $(- \infty, 0]$.

  <u>Proof</u>. Immediate from the assumption that F is continuous. □

  Clearly (3.4) gives a linear form on X'. But it is still not guaranteed that $g^\Sigma(u)$ belongs to X. In order to show this, we shall prove that $g^\Sigma(u)$ belongs to X'', which is equal to X by our hypothesis that X is reflexive. (This is the only place where this assumption comes into play. Further, one can indeed remove this hypothesis by using the Mackey-Arens theorem, but the proof would become a little more involved.) Let B_1 denote the unit ball in X'. We start with the following

LEMMA 3.6. <u>Let</u> $K := \{\langle \exp(- Ft)G, x^* \rangle; x^* \in B_1\}$. <u>Then</u> K <u>is a</u> <u>bounded set in</u> $E_{(-\infty,0]}$.

  <u>Proof</u>. For every $m, \alpha > 0$, we have the following estimate:

$$\sup_{\substack{0 \le j \le m \\ -\alpha \le t \le 0}} |(\frac{d}{dt})^j \langle \exp(- Ft)G, x^* \rangle|$$

$$= \sup_{\substack{0 \le j \le m \\ -\alpha \le t \le 0}} |\langle (- F)^j \exp(- Ft)G, x^* \rangle|$$

$$\le \sup_{\substack{0 \le j \le m \\ -\alpha \le t \le 0}} \{\| F \|^j \exp(- \| F \|t) \| G \| \| x^* \|\}$$

$$\le \exp(\| F \|\alpha)\beta^m \| G \| \quad (\| x^* \| \le 1),$$

where $\beta := \max \{\| F \|, 1\}$. Thus K is bounded by Definition (2.2). □

PROPOSITION 3.7. <u>For every</u> $u \in E'_{(-\infty,0]}$, $g^{\Sigma}(u)$ <u>belongs to</u> X''.

<u>Proof</u>. Let us calculate $\| g^{\Sigma}(u) \|_{X''}$:

$$\| g^{\Sigma}(u) \|_{X''} = \sup_{x^* \in B_1} |\langle g^{\Sigma}(u), x^* \rangle|$$

$$= \sup_{x^* \in B_1} |\langle \langle \exp(-Ft)G, x^* \rangle, u \rangle|$$

$$= \sup_{\phi \in K} |\langle \phi, u \rangle| < +\infty$$

because of Lemma (2.5) (note K is bounded). Hence $g^{\Sigma}(u) \in X''$. $\square$

THEOREM 3.8. <u>The correspondence</u> $g^{\Sigma}: E'_{(-\infty,0]} \to X: u \mapsto g^{\Sigma}(u)$ <u>is continuous</u>.

<u>Proof</u>. Let $V_{K,\epsilon}$ be a neighborhood of 0 in $E'_{(-\infty,0]}$ given by

$$V_{K,\epsilon} := \{u \in E'_{(-\infty,0]}; \sup_{\phi \in K} |\langle \phi, u \rangle| \le \epsilon\}.$$

Let u belong to $V_{K,\epsilon}$. Then we have

$$\| g^{\Sigma}(u) \|_X = \| g^{\Sigma}(u) \|_{X''} \quad (X \text{ is reflexive})$$

$$= \sup_{x^* \in B_1} |\langle g^{\Sigma}(u), x^* \rangle| \quad (B_1 = \text{the unit ball of } X')$$

$$= \sup_{x^* \in B_1} |\langle \langle \exp(-Ft)G, x^* \rangle, u \rangle|$$

$$= \sup_{\phi \in K} |\langle \phi, u \rangle| \le \epsilon. \quad \square$$

<u>Remark</u>. It is easy to see that

$$g^{\Sigma}(u) = \int_{-\infty}^{0} \exp(-Ft)Gu(t)dt$$

for sufficiently smooth u. Hence thus defined g^{Σ} is indeed a continuous extension of the usual reachability map (see (3.2)).

4. LACK OF EXACT REACHABILITY.

We have seen that the input space can be extended to $E'_{(-\infty,0]}$ for systems of type (3.1). We pose the question: Can the reachable set X_R $(= g^\Sigma(E_{(-\infty,0]}))$ be the whole space X? If it were, we would have that the mapping g^Σ is an open mapping by Matsuo's uniqueness theorem (see also Pták's open mapping theorem — Schaefer [6, IV.8.3, Corollary 1]). So X must be isomorphic to the quotient space $E'/\ker g^\Sigma$. (In the sequel we denote $E'_{(-\infty,0]}$ and $E_{(-\infty,0]}$ simply by E', E, respectively.) Thus our question may be rephrased as "<u>Can</u> $E'/\ker g^\Sigma$ <u>be a Banach space?</u>" The following Proposition (4.1) claims this is not the case unless $E'/\ker g^\Sigma$ is finite-dimensional.

PROPOSITION 4.1. <u>The space</u> $E'/\ker g^\Sigma$ <u>is a Banach space if and only if it is of finite-dimension.</u>

<u>Proof</u>. We note from Schaefer [6, IV.9.7, Example 3] that E' is a nuclear space, and hence $E'/\ker g^\Sigma$ itself is a nuclear space (Schaefer [6, III.7.4]). But a nuclear space can be a Banach space iff it is of finite-dimension (Schaefer [6, III.7.1]). □

Hence, in general, the systems of type (3.1) cannot be exactly reachable. Let us further note the following additional result on the structure of $E'/\ker g^\Sigma$.

PROPOSITION 4.2. <u>The space</u> $E'/\ker g^\Sigma$ <u>is complete but not metrizable unless it is finite-dimensional.</u>

<u>Proof</u>. First we note that E' is B-complete (Schaefer [6, IV.8, Example 2]) so that its quotient $E'/\ker g^\Sigma$ is also B-complete (Schaefer [6, IV.8.3, Corollary 3]). Since every B-complete space is complete (Schaefer [6, IV.8.1]), $E'/\ker g^\Sigma$ is complete.

Secondly, since $E'/\ker g^{\Sigma}$ is a complete locally convex space, it is metrizable if and only if it is a Fréchet space. On the other hand, $E'/\ker g^{\Sigma}$ is a DF-space (for a definition, see Köthe [4, 29.3]) as a separated quotient of a DF-space E' (Köthe [4, 29.5]). But a DF-space is a Fréchet space if and only if it is a Banach space; see, for example, Köthe [4, 29.1]. By Proposition (4.1) this is impossible unless it is finite-dimensional. $\square$

5. EXAMPLE.

In this section we confine ourselves to the example given in Baras, Brockett, and Fuhrmann [1]. The example is the system Σ defined by

$$\frac{d}{dt}x_n = \lambda_n x_n + g_n u, \quad n = 1, 2, 3, \ldots,$$
$$y(t) = \sum_{n=1}^{\infty} h_n x_n, \tag{5.1}$$

where the state space $X = \ell^2$, and $\{g_n\}$, $\{h_n\} \in \ell^2$. They showed that if $\lambda_n \neq \lambda_m$ for $n \neq m$ and $g_n \neq 0$, $h_n \neq 0$ for all n, this type of system is weakly canonical (i.e., the reachable set is dense in ℓ^2, and observable), but there can be still many nonisomorphic systems of type (5.1) that have the same external behavior.

Let us assume $|\lambda_n| \leq \lambda$. Note that this is equivalent to assuming that the operator: $(x_1, x_2, \ldots, x_n, \ldots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots)$ is bounded. We already know that Σ accepts E'-inputs by Theorem (3.8), for Σ is a special case of (3.1) by putting $X := \ell^2$, $F := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$, $G := \{g_n\}$. Our objective in this section is to explicitly see that

(i) $X_R := g^{\Sigma}(E'_{(-\infty,0]})$ is _not_ equal to the whole space ℓ^2, i.e., Σ cannot be exactly reachable even with E'-inputs;

(ii) the reachable subspace X_R, with the topology induced from ℓ^2, is _not_ barreled.

The second statement explains why Matsuo's result on the uniqueness of canonical realizations does not apply to this example.

Given an input $u \in E'$, we can express $g^\Sigma(u)$ (see (3.4)) as

$$g^\Sigma(u)|_n = \langle \exp(-\lambda_n t)g_n, u \rangle, \quad n = 1, 2, \ldots,$$

where $g^\Sigma(u)|_n$ denotes the n-th coordinate of $g^\Sigma(u)$. Let $M := \{\exp(-\lambda_n t); n = 1, 2, \ldots\}$; it is easy to see that M is a bounded subset of E. Then we have

PROPOSITION 5.2. <u>The reachable set</u> X_R <u>is not equal to</u> ℓ^2, <u>i.e.,</u> <u>the system</u> Σ <u>is not exactly reachable even with</u> E'<u>-inputs</u>.

Proof. First, we have the following estimate:

$$g^\Sigma(u)|_n = |\langle \exp(-\lambda_n t)g_n, u \rangle| \qquad (5.3)$$

$$= |\langle \exp(-\lambda_n t), u \rangle| |g_n|$$

$$= \sup_{\psi \in M} |\langle \psi, u \rangle| |g_n|$$

$$= p_M(u)|g_n| \quad \text{for all } n.$$

Clearly $p_M(u)$ is finite by Lemmas (2.5) and (2.6). On the other hand, the following Lemma (5.4) shows that there always exists an ℓ^2-sequence $\{y_n\}$ such that no positive constant C satisfies the inequality $|y_n| \leq C|g_n|$. This proves the assertion. $\square$

LEMMA 5.4. <u>For every</u> ℓ^2<u>-seuqence</u> $\{g_n\}$ <u>such that</u> $g_n \neq 0$ <u>for all</u> n, <u>there exists</u> $\{y_n\} \in \ell^2$ <u>such that</u> $|y_n/g_n|$ <u>is unbounded</u>.

Proof. Assume the contrary. For each positive integer m, define K_m by

$$K_m := \{\{x_n\} \in \ell^2; |x_n| \leq m|g_n| \text{ for all } n\}.$$

It is easy to verify that K_m is a closed symmetric convex set for each m. By the assumption we have $\ell^2 = \bigcup_{m=1}^{\infty} K_m$. Then by the Baire category theorem, at least one of the K_m's, in fact K_1 itself, contains an open ball V. Since K_1 is symmetric and convex, we may assume V is of the type: $V = \{x \in \ell^2; \| x \| \leq \epsilon\}$. Choose a number k such that $2|g_k| < \epsilon$. Let $x^{(k)}$ be the element of ℓ^2 given by $x_k^{(k)} = 2g_k$ and $x_j^{(k)} = 0$ if $j \neq k$. Then $\| x^{(k)} \| = 2|g_k| < \epsilon$, so $x^{(k)} \in V$. But $|x_k^{(k)}| = 2|g_k| > |g_k|$. Thus $x^{(k)} \notin K_1$, contrary to the assertion $V \subset K_1$. $\square$

Now given $\{x_n\} \in X_R$, we define $\| \{x_n\} \|^{\infty}$ by

$$\| \{x_n\} \|^{\infty} := \sup \{|x_n/g_n|; n = 1, 2, \ldots\}.$$

In view of (5.3), $\| \{x_n\} \|^{\infty}$ is well defined for every $\{x_n\} \in X_R$, but not necessarily so for $\{x_n\} \notin X_R$.

PROPOSITION 5.5. <u>Let</u> $T := \{x \in X_R; \| x \|^{\infty} \leq 1\}$. <u>Then</u> T <u>is a barrel of</u> X_R.

Proof. It is easy to see that T is convex, balanced and absorbing (cf. Definition (2.9)). We must show that T is closed. Let x^p be a sequence in T converging to $x^0 \in X_R$. Since $x^p \to x^0$ in ℓ^2, each component x_j^p converges to x_j^0 for every j. Since $|x_j^p| \leq 1$ for all j, $|x_j^0| \leq 1$ follows. Thus x^0 belongs to T. $\square$

The next Proposition (5.6) claims that the above defined T is not a neighborhood of X_R, thereby establishing the fact that X_R is not barreled.

PROPOSITION 5.6. The barrel T defined above is not a neighborhood of 0 in X_R.

Proof. Assume the contrary. Define a linear map $\Psi: X_R \to \ell^\infty$ by

$$\Psi(\{x_n\}) := \{x_n/g_n\}.$$

According to (5.3) this is a well-defined map. By the definition of Ψ, $\|\Psi(\{x_n\})\| \leq 1$ if and only if $\{x_n\} \in T$. In other words $T = \Psi^{-1}(B_1^\infty)$, where B_1^∞ denotes the unit ball of ℓ^∞. This means that Ψ is continuous by our assumption. Since X_R is known to be dense in ℓ^2 (cf. Baras, Brockett, and Fuhrmann [1]), there exists a unique continouos extension $\overline{\Psi}: \ell^2 \to \ell^\infty$ such that $\overline{\Psi}(\{x_n\}) = \{x_n/g_n\}$ is well defined on the whole space ℓ^2. But this contradicts the conclusion of Lemma (5.4). Hence T cannot be a neighborhood of X_R. $\square$