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This report treats the case of a long ring-stiffened circular cylindrical shell made of rigid-perfectly plastic material. The approximate Tresca square is used as yield surface with the corresponding associated flow rule. The shell is subjected to an external axisymmetric dynamic overpressure of rectangular form. Maximum permanent deformations are obtained.

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FOREWORD

This report presents the first phase of an ongoing program to establish the permanent deflections attained by a circular cylindrical ring-stiffened shell, when subjected to dynamic overpressure.

The study employs the ideas developed by Professor P. G. Hodge 30 years ago. It will serve as a starting platform for more complicated analysis in the near future.

This work was sponsored by the Office of Naval Technology through the Naval Surface Warfare Center's Block Program, "Explosives and Undersea Warheads" (D. E. Phillips).

Approved by:

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INTRODUCTION

This report is the first step towards an ongoing effort to obtain a closed form or a relatively simple numerical solution to the problem of a ring-stiffened, circular cylindrical shell subjected to a dynamic pressure load. Maximum permanent deflections after load removal are obtained.

The analytical solution of the dynamic problem, even under substantial simplifying assumptions, can be very complex and cumbersome. In this report, we rederive and extend the analytical solutions obtained by Hodge.[1,2] Considerable simplification can be achieved if we are content with numerical treatment of the governing equations on the computer.

This study employs the assumptions listed below.

1. (Geometrical and Boundary Conditions Assumptions) Rotational and axial symmetries are assumed. For ring-stiffened shells only a typical half-frame spacing $(L_T/2)$ is considered. (This assumption implies a relatively long shell, away from supports which may affect behavior.)

2. (Loading Assumptions) The dynamic loading is axisymmetric in the form of a step function applied as a pressure over a finite length of time and then removed (i.e., of rectangular type). The duration of the load is very short. Axial compression is not accounted for.

3. (Initial Conditions Assumptions) The structure starts from rest with zero initial velocity and displacements.

4. (Material Assumptions) Material is rigid perfectly plastic. The elastic strain energy is much smaller than the energy dissipated through plastic deformation. The simplest form of the Tresca yield surface is employed, and no residual strains are considered. No strain rate effects are included.

5. (Geometrical Nonlinearity Assumptions) There is no geometrical nonlinearity. Small deformations, strains, and rotations are considered. Transverse shear deformation is neglected. The employed strain-displacement equations are linear, and the equilibrium equations are based on the undeformed configuration.

The method is not new; it was first used by P. Hodge, Jr.[1,2] 30 years ago. What is new is that we have been able to rework most of its details, point out any differences with Hodge's results, and build the foundations on which more complicated analyses, such as by Jones (Reference 3) and Duszek (Reference 4), can be completed. Work similar to Hodge's original work[1,2] was also presented by Kuzin and Shapiro,[5] and Sankaranarayanan[6] and Hodge.[7] Unlike the work by Hodge[8] and Sankaranarayanan,[9] in which the static collapse load was established under rigid plastic conditions, the work by Klement[10] was an elastoplastic treatment of the static collapse load, while References 1 through 7 were concerned with the dynamic case. Onat,[11] Duszek,[4,12] and Lance[13] present analyses of load-displacement predictions for post-yield behavior under static loads, accounting for changes in geometry. These analyses were based on moderately large displacement theory and assumed rigid perfectly plastic material. Furthermore, Reference 13 presented a bounding principle for finite deflections and static loadings, and compared it with a solution by the "rate formulation," first proposed by Onat (Reference 11). Finally, in all fairness, the work by Lintholm and Bessey[14] must be mentioned, where clamped and axially restrained beams were loaded impulsively, and it was concluded that the rigid perfectly plastic model was inadequate due to the strong influence of elastic effects.

This report is organized in the following way. After a brief introduction there is a section of problem statement with its proposed solution technique. The actual analysis pertaining to the obtained equations shows up in Appendices A (preliminary analysis) through E. The analytical results of this analysis are summarized in Tables 1 through 19 for the convenience of the reader. This is followed by a section on results for five cylindrical shells and a brief discussion of these computations. There is a Nomenclature section defining the terms used and a relevant list of references.

PROBLEM STATEMENT AND SOLUTION METHOD

Consider a ring-stiffened, circular cylindrical shell (Figure 1) with constant frame spacing (L_T). The shell has multiple frame spaces. However, only a typical spacing will be analyzed. The body as well as the externally applied dynamic loading are assumed fully axisymmetric (Figure 2). The loading is a constant overpressure of magnitude P and form $P = P(t)$ acting over a time interval $0 \leq t \leq t_0$ (Figure 3). The time t_0 is relatively short.

The material is assumed "rigid-perfectly plastic," [(Figure 4) References 15 through 21]. Therefore there are no deformations up to a critical stress. When the critical stress, pressure, or load is reached, however, there will be unrestricted plastic flow. In actuality, strains and deformations will not increase without bounds because buckling or fracture of the material will instead take place. In the dynamic case accelerated motion will be resisted by the inertia of the body. The rigid-perfectly plastic assumption is reasonable, where elastic deformations are very small compared to plastic ones.

To solve the problem, it is required that we employ some of the ideas from plasticity theory.[7,15,24,25] Therefore, we briefly mention that, when a metal goes beyond its yield point, the point representing the state of stress in stress space lies on a surface. Typically, we have the Von Mises ellipse, the Tresca hexagon, and the simplification of the Tresca hexagon, or the Tresca square. Through a change of variables, the stress space can be written in terms of moment and force resultants. When more plastic flow occurs, the direction in which this happens is given by the flow rule. The corresponding strain rate vector (in strain space) is such that it is normal to the yield surface and is directed outwards. The employed yield surface (References 7, 15, and 19) is the simplified "approximate" Tresca square. The associated flow rule and normality conditions (References 7, 15, and 19) are also used. The equations of equilibrium (References 1, 2, 7, and 22) are obtained in the undeformed configuration. The strain-displacement expressions (Reference 22) used employ only linear terms. Therefore, the theory is infinitesimal and the analysis is simplified considerably.

The boundary, initial, and jump conditions (Reference 23) are discussed in the actual solution.

The method of solution is as follows:

1. Assume a "kinematically admissible" velocity profile. That is a velocity (or displacement) distribution that satisfies velocity (or displacement) constraints; to ensure material stability the total external work by the loads on these displacements is positive.*

2. Assume the portion of the yield surface on which we are on. We employ "an associated flow rule." The "outwards" normality condition of the strain rate vector (written in terms of displacement rates) on the yield surface leads to certain conditions being satisfied. These make the mechanism of instantaneous motion admissible.

3. Satisfy initial, boundary, and "jump" conditions (References 7, 23, 27, and 28). These last conditions are at "hinge circles" for shells. (In the case of beams we have just hinges.)

4. Satisfy the equations of equilibrium in the undeformed state.

*Drucker's postulate on material stability.[15]

5. Verify that the obtained stress profiles (moment and force distributions) do not violate the yield surface at any point. If they do, then another "kinematically admissible" velocity profile must be assumed, and steps 1 through 4 repeated.

The maximum "residual" deflections are not bounds* of any kind. They can, however, provide useful information on damage (deflections, not strains) by dynamic overpressures and, therefore, serve as a guide for the estimation of deformations.

*See upper and lower bound theorems[15] for static analysis. Because our problem is dynamic, estimates of this kind do not produce bounds.

RESULTS AND DISCUSSION

We summarize our analytical results in Tables 1 through 19. All non-dimensional quantities that appear in all the Tables (1 through 25) are defined in the Nomenclature as well as the Appendices. Tables 1 through 19 have been organized with respect to two non-dimensional parameters. The first one is a pressure parameter, controlling whether the loading is low or high. The second one is a non-dimensional parameter c^2 that combines shell radius (a), shell thickness (h) and half frame spacing (L). The loading is termed "low" if $1 + \frac{2}{c^2} \leq p \leq 1 + \frac{6}{c^2}$ and "high" if $p > 1 + \frac{6}{c^2}$.

The distinction of whether a shell is termed as either short or long depends on whether the non-dimensional constant $c^2 = \frac{L^2_T}{2ah} = \frac{2L^2}{ah}$ is less or greater than 6. Tables 1 and 2 pertain to short shells and low loading, tables 3 through 5 to long shells and low loading. Tables 6 through 12 to short shells and high loading, and Tables 13 through 19 to long shells and high loading.

This method was applied to five cases of ring-stiffened shells, not necessarily representative of marine structures. Table 20 gives the geometrical and material characteristics of these models. Model No. 1 was subjected to an overpressure of 600 psi. This model was analyzed by using the results of the case of short shells and low loading. The applied overpressure exceeded the critical overpressure (524.569 psi). Therefore, plastic flow took place. Table 21 gives results for this situation. The final permanent deformations are small in comparison to the shell thickness. At a non-dimensional half length of 1.0 (i.e., at middle between two stiffeners) residual displacement is only 0.645×10^{-2} inches. Table 21 also gives the time for the shell to come to rest.

Table 22 gives results for Model No. 2. In this case, the length between stiffeners was increased from 3.543 to 10.630 in. The analysis falls in the long shells, high loading case, since c^2 (= 28.2546) exceeds 6. In this case, the applied overpressure (600 psi) also was in excess of the critical value (343.089 psi). The residual displacements are an order of magnitude larger than for Model No. 1. In this case, the deflection profile is shown both as a function of non-dimensional time (vertically down) and as a function of distance from the left support (middle point is represented by 1.000). Table 23 displays the results of Model No. 3, identical to Model No. 2, but with lower applied overpressure, P = 525 psi. The residual displacements are smaller than for Model No. 2.

Table 24 gives results for Model No. 4, which differs in geometry from Models 1, 2 and 3. Table 24 shows that although the applied overpressure (1200 psi) is not much larger than the critical overpressure (986.432 psi), this method of analysis can lead to unrealistic displacement. This is due to the omission of the geometrically nonlinear terms in the strain-displacement relations [see Appendix A, Equations (A-12) through(A-15)] and because the equilibrium equation [Equation (A-1) of Appendix A] was not obtained in the deformed state.

Table 25 gives results for Model No. 5. In this case, the overpressure was reduced to 987.0 psi and its period of duration increased from 0.10×10^{-2} sec. to 0.1×10^{-1} sec. (with respect to Model No. 4). We observe that although the exerted overpressure (987.0 psi) is slightly larger than the critical value

(986.432 psi), the permanent deformations obtained are larger than 1 diameter, and there is no need to compute deformations to the middle of the shell. The same table also gives the position and the velocity of the hinge circle as a function of non-dimensional time. Furthermore, we observe that all models come to rest, as is demonstrated by the velocity of the final point in time. (See Tables 21 through 25.)

In conclusion, this study has shown that it is possible to obtain useful expressions for permanent deformations. However, we must extend the method to, at least, include the effect of end load [n_x term in Equation (A-30) which will result in a three-dimensional Tresca cube for the approximate yield surface], and account for geometric nonlinearities in the strain-displacement relations to be able to obtain useful expressions of residual deformations. Unlike finite element methods, this technique can provide very useful estimates of residual deformations with minimal computational effort, provided a proper analysis has been developed.

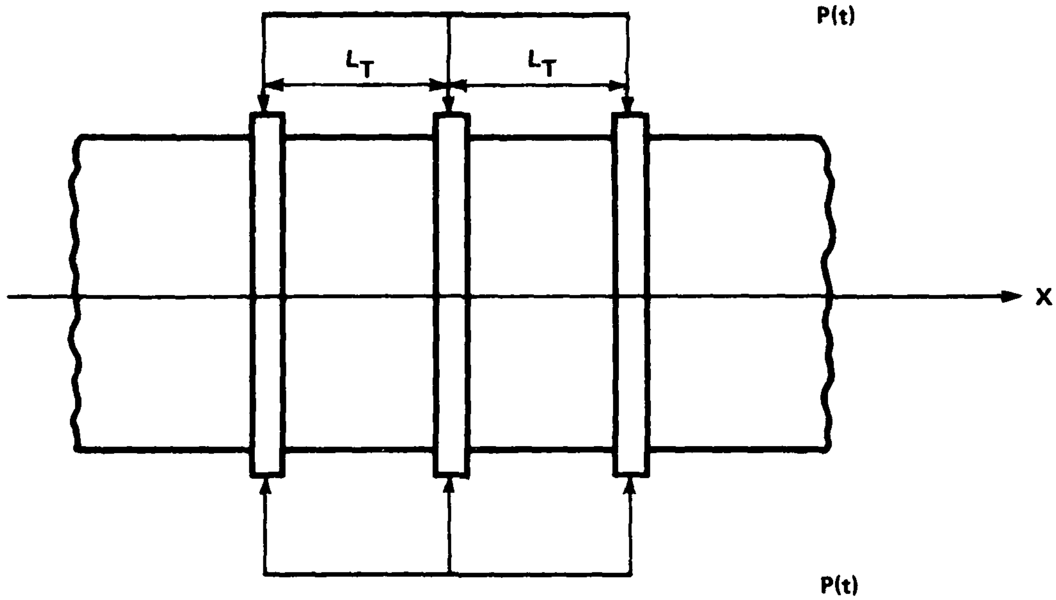
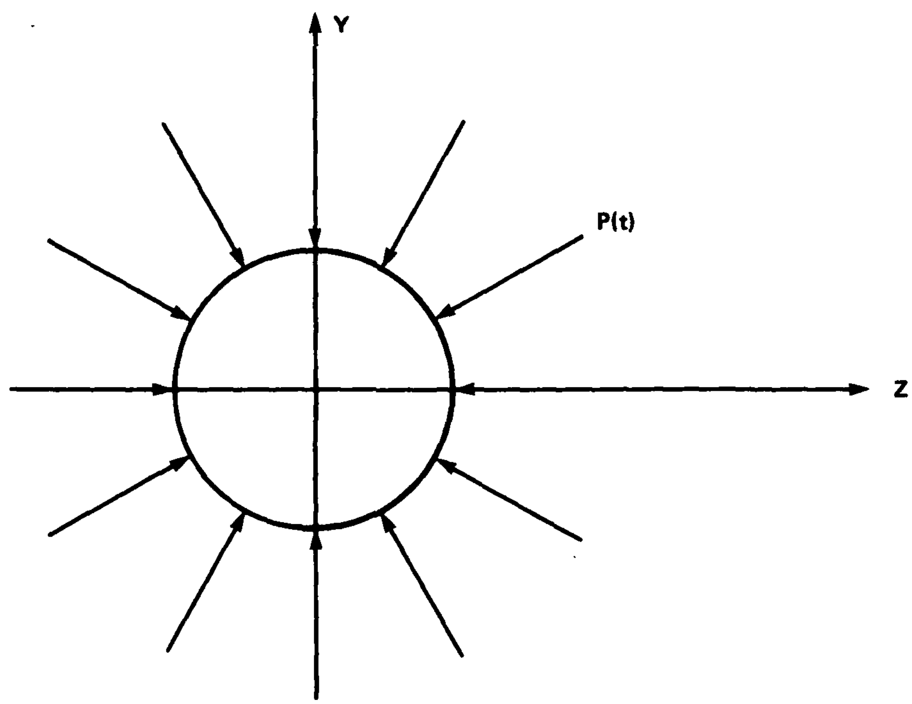


FIGURE 1. RING-STIFFENED CIRCULAR CYLINDRICAL SHELL (FRAME SPRING L_T)



NOTE: THIS FIGURE DISPLAYS AN AXISYMMETRIC STRUCTURE AND LOADING IN THE FORM OF INWARD PRESSURE P(t)

FIGURE 2. CROSS-SECTION OF FIGURE 1

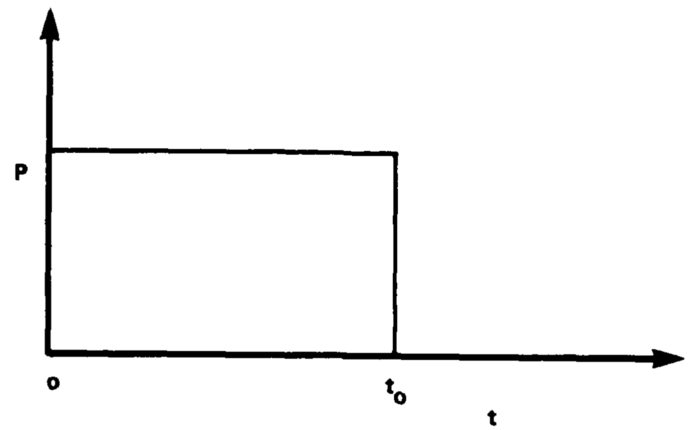


FIGURE 3. RECTANGULAR PRESSURE DISTRIBUTION ACTING OVER TIME t_o

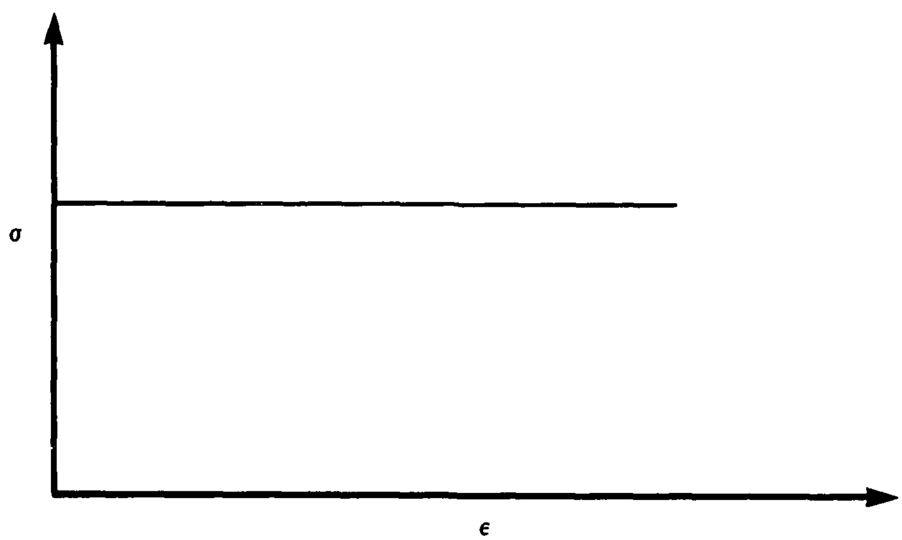


FIGURE 4. STRESS-STRAIN CURVE FOR RIGID, PERFECTLY PLASTIC MATERIAL

TABLE 1. SUMMARY, SHORT SHELLS, LOW LOADING, 1

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(0 < C^2 < 6\right)$	LOW $\left(1 + \frac{2}{C^2} \leqslant p \leqslant 1 + \frac{6}{C^2}\right)$
CONDITIONS	$0 \leqslant \tau \leqslant 1$	
MOMENT RESULTANT	$m_x(x,\tau) = \left[\frac{C^2}{2}(p-1) - 1\right]x^3 - C^2(p-1)x^2 + \left[\frac{C^2}{2}(p-1) + 3\right]x - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \frac{3}{2C^2}\left[\frac{C^2}{2}(p-1) - 1\right]x\tau^2$	
VELOCITY	$\dot{w}(x,\tau) = \frac{3}{2C^2}\left[C^2(p-1) - 2\right]x\tau$	
ACCELERATION	$\ddot{w}(x,\tau) = \frac{3}{2C^2}\left[C^2(p-1) - 2\right]x$	
TIME τ_o	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 2. SUMMARY, SHORT SHELLS, LOW LOADING, 2

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(0 < C^2 \leqslant 6\right)$	LOW $\left(1 + \frac{2}{C^2} \leqslant p \leqslant 1 + \frac{6}{C^2}\right)$
CONDITIONS	$1 \leqslant \tau \leqslant \frac{p}{p_o}$	
MOMENT RESULTANT	$m_x(x,\tau) = -\left[\frac{C^2}{2} + 1\right]x^3 + C^2 x^2 + \left[3 - \frac{C^2}{2}\right]x - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \left[-\frac{3}{2C^2}\left(\frac{C^2}{2} + 1\right)\tau^2 + \frac{3}{2}p\tau - \frac{3}{4}p\right]x$	
VELOCITY	$\dot{w}(x,\tau) = \left[-\frac{3}{C^2}\left(\frac{C^2}{2} + 1\right)\tau + \frac{3}{2}p\right]x$	
ACCELERATION	$\ddot{w}(x,\tau) = -\frac{3}{C^2}\left(\frac{C^2}{2} + 1\right)x$	
TIME τ_o	$\tau_o = \frac{p}{p_o} = \frac{C^2}{\left(C^2 + 2\right)}p$	
DISPLACEMENT AT REST	$w(x,\tau_o) = \frac{px}{\left(1 + \frac{2}{C^2}\right)^2}\left[\frac{3}{2}\left(\frac{1}{2} + \frac{1}{C^2}\right)p - \frac{3}{4}\left(1 + \frac{2}{C^2}\right)^2\right]$	

TABLE 3. SUMMARY, LONG SHELLS, LOW LOADING, 1

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $\left(C^2 > 6\right)$	LOW $\left(1 + \frac{2}{C^2} \leqslant p \leqslant 1 + \frac{6}{C^2}\right)$
CONDITIONS	$0 \leqslant \tau \leqslant 1$	
MOMENT RESULTANT	$m_x\ (x, \tau) = \left[\frac{C^2}{2}(p-1) - 1\right] x^3 - C^2\,(p-1)\,x^2 + \left[\frac{C^2}{2}(p-1) + 3\right] x - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w\,(x, \tau) = \frac{3}{2C^2}\left[\frac{C^2}{2}(p-1) - 1\right] x\,\tau^2$	
VELOCITY	$\dot{w}\,(x, \tau) = \frac{3}{2C^2}\left[C^2\,(p-1) - 2\right] x\tau$	
ACCELERATION	$\ddot{w}\,(x, \tau) = \frac{3}{2C^2}\left[C^2\,(p-1) - 2\right] x$	
TIME τ_0	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 4. SUMMARY, LONG SHELLS, LOW LOADING, 2

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $\left(C^2 > 6\right)$	LOW $\left(1 + \frac{2}{C^2} \leq p \leq 1 + \frac{6}{C^2}\right)$
CONDITIONS	$0 \leqslant x < u \quad u = \alpha(\tau - 1) \quad \dot{u} = \alpha \quad \alpha = \dfrac{(C^2 - 6)}{3\left[\frac{C^2}{2}(p-1) - 1\right]}, 1 \leqslant \tau \leqslant \tau_o \quad \left(u \leqslant 1 - \frac{\sqrt{6}}{C}\right)$	
MOMENT RESULTANT	$m_x = -1$	
MEMBRANE RESULTANT	$n_\varphi = 0$	
DISPLACEMENT	$w = \dfrac{3}{2C^2}\left[\dfrac{C^2}{2}(p-1) - 1\right] x$	
VELOCITY	$\dot{w} = 0$	
ACCELERATION	$\ddot{w} = 0$	
TIME τ_o	$\tau_o = 1 + \dfrac{1}{\alpha}\left(1 - \dfrac{\sqrt{6}}{C}\right)$	
DISPLACEMENT AT REST	$w(x, \tau_o) = \dfrac{3}{2C^2}\left[\dfrac{C^2}{2}(p-1) - 1\right] x$ FOR $0 \leqslant x < \alpha(\tau_o - 1)$	

TABLE 5. SUMMARY, LONG SHELLS, LOW LOADING, 3

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $\left(C^2 > 6\right)$	LOW $\left(1 + \frac{2}{C^2} \leqslant p \leqslant 1 + \frac{6}{C^2}\right)$
CONDITIONS	$\alpha = \dfrac{\left(C^2 - 6\right)}{3\left[\frac{C^2}{2}(p-1) - 1\right]}, \quad 1 \leqslant \tau \leqslant \tau_o \qquad u \leqslant \tau \leqslant 1$ $u \leqslant 1 - \dfrac{\sqrt{6}}{C}$	
MOMENT RESULTANT	$m_x(x,\tau) = 1 + \dfrac{1}{(1-u)^3}\left[-4x^3 + 6\left(1+u\right)x^2 - 12ux - 2(1-3u)\right]$ where $u = \alpha(\tau - 1)$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \dfrac{3}{2C^2}\left[\dfrac{C^2}{2}(p-1) - 1\right]x + \dfrac{6}{C^2}\left(\dfrac{1}{\alpha}\right)\left\{\dfrac{C^2}{6}\left[(x+\alpha)(\tau-1) - \dfrac{\alpha}{2}(\tau^2 - 1)\right] + \right.$ $\left.\left[\dfrac{1}{\alpha}\log_e\lvert 1 - \alpha(\tau-1)\rvert + \dfrac{(1-x)}{\alpha} \cdot \dfrac{\alpha(\tau-1)}{(1+\alpha-\alpha\tau)}\right]\right\}$	
VELOCITY	$\dot{w}(x,\tau) = \dfrac{6}{C^2}\left(\dfrac{1}{\alpha}\right)\left[\dfrac{C^2}{6} - \dfrac{1}{(1-u)^2}\right](x-u)$	
ACCELERATION	$\ddot{w}(x,\tau) = \left(\dfrac{6}{C^2}\right)\dfrac{[1 + \alpha(\tau-1) - 2x]}{[1+\alpha-\alpha\tau]^3} - 1$	
TIME τ_o	$\tau_o = 1 + \dfrac{3\left[\frac{C^2}{2}(p-1) - 1\right]}{C\left(C+\sqrt{6}\right)} = 1 + \dfrac{1}{\alpha}\left(1 - \dfrac{\sqrt{6}}{C}\right)$	
DISPLACEMENT AT REST	$w(x,\tau_o) = \dfrac{3}{2C^2}\left[\dfrac{C^2}{2}(p-1) - 1\right]x + \dfrac{1}{\alpha^2}\left\{(x+\alpha)\left(1 - \dfrac{\sqrt{6}}{C}\right) - \dfrac{1}{2}\left[\left\{\left(1 - \dfrac{\sqrt{6}}{C} + \alpha\right\}^2 - \alpha^2\right]\right\}$ $+ \dfrac{1}{\alpha^2}\left\{\dfrac{6}{C^2}\left[\dfrac{1}{2}\log_e\lvert C\rvert\right] + (1-x)\left(\dfrac{\sqrt{6}}{C} - \dfrac{6}{C^2}\right)\right\}$	

TABLE 6. SUMMARY, SHORT SHELLS, HIGH LOADING, 1

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(C^2 < 6\right)$	HIGH LOAD $\left(p > 1 + \frac{6}{C^2}\right)$
CONDITIONS	$u_o^2 = \frac{6}{C^2(p-1)}$ $\quad 0 \leqslant \tau \leqslant 1$	$0 \leqslant x < u_o$
MOMENT RESULTANT	$m_x(x,\tau) = 2\left[\frac{x}{u_o} - 1\right]^3 + 1$ OR $= 2\left[\frac{x}{u_o}\right]^3 - 6\left[\frac{x}{u_o}\right]^2 + 6\left[\frac{x}{u_o}\right] - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \frac{1}{2}\,\frac{(p-1)}{u_o}\,\tau^2 x$	
VELOCITY	$\dot{w} = \frac{(p-1)}{u_o}\,\tau\, x$	
ACCELERATION	$\ddot{w} = \frac{(p-1)}{u_o}\, x$	
TIME τ_o	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 7. SUMMARY, SHORT SHELLS, HIGH LOADING, 2

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(c^2 < 6\right)$	HIGH LOAD $\left(p > 1 + \frac{6}{c^2}\right)$
CONDITIONS	$u_o^2 = \frac{6}{c^2\left(p-1\right)}$	$0 \leqslant \tau \leqslant 1$ $u_o < x \leqslant 1$
MOMENT RESULTANT	$m_x = 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w = \frac{1}{2}\left(p-1\right)\tau^2$	
VELOCITY	$\dot{w} = (p-1)\tau$	
ACCELERATION	$\ddot{w} = p-1$	
TIME τ_o	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 8. SUMMARY, SHORT SHELLS, HIGH LOADING, 3

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(C^2 < 6\right)$	HIGH LOAD $\left(p > 1 + \frac{6}{C^2}\right)$ $\quad p > \tau$
CONDITIONS	$u^2 = \frac{6\tau}{C^2(p-\tau)}$ $\quad 0 \leqslant x < u$, $\quad 1 \leqslant \tau \leqslant \tau'$ $\quad \tau' = \frac{p}{1+\frac{6}{C^2}} = \frac{p}{p_1}$	
MOMENT RESULTANT	$m_x(x,\tau) = -\frac{1}{2}\left(2 + C^2u^2\right)\left(\frac{x}{u}\right)^3 + C^2u^2\left(\frac{x}{u}\right)^2 + \left(3 - \frac{C^2u^2}{2}\right)\left(\frac{x}{u}\right) - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$u_o^2 = \frac{6}{C^2(p-1)}$ $w(x,\tau) = \frac{1}{2}(p-1)\left(\frac{x}{u_o}\right) + x\left[\frac{C}{2\sqrt{6}}\left\{\sqrt{\tau}\,(p-\tau)^{3/2} - (p-1)^{3/2}\right\} + \frac{3C}{4\sqrt{6}}\,p\left\{\sqrt{\tau(p-\tau)} - \sqrt{(p-1)}\right\} + \frac{\sqrt{6}}{8}C\,p^2\left\{\tan^{-1}\left(\sqrt{\frac{\tau}{p-\tau}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\}\right]$ FOR $0 \leqslant x \leqslant u_o$ $w(x,\tau) = -\frac{p}{2}\left[1 - 2p\,\frac{C^2x^2}{\left(6+C^2x^2\right)} + p\,\frac{C^4x^4}{\left(6+C^2x^2\right)^2}\right] + \frac{\sqrt{6}}{8}C\,p^2\,x\left[\tan^{-1}\left(\frac{Cu}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{Cx}{\sqrt{6}}\right)\right]$ $+ x\,p^2\left[\frac{3C^2u}{\left(6+C^2u^2\right)^2} - \frac{3C^2x}{\left(6+C^2x^2\right)^2} + \frac{3C^2u}{4\left(6+C^2u^2\right)} - \frac{3C^2x}{4\left(6+C^2x^2\right)}\right]$ FOR $u_o \leqslant x \leqslant u$	
VELOCITY	$\dot{w}(\tau) = (p-\tau)\left(\frac{x}{u}\right) = \frac{C}{\sqrt{6}}\,\frac{(p-\tau)^{3/2}}{\sqrt{\tau}}\,x$	
ACCELERATION	$\ddot{w}(\tau) = -\frac{x}{u}\left[1 + \frac{(p-\tau)\,\dot{u}}{u}\right] = \frac{1}{2}\left(\frac{x}{u}\right)\left[2 + \frac{p}{\tau}\right]$	
TIME τ_o	AT TIME $\tau' = \frac{p}{p_1}$, TRAVELING HINGE u MOVES TO MIDLENGTH $x = u = 1$	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 9. SUMMARY, SHORT SHELLS, HIGH LOADING, 4

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $C^2 < 6$	HIGH LOAD $p > 1 + \frac{6}{C^2}$
CONDITIONS	$u^2 = \frac{6\tau}{C^2(p-\tau)}$ $\quad u < x \leqslant 1$, $\; p > \tau$, $\; 1 \leqslant \tau \leqslant \tau'$	$\tau' = \frac{p}{\left(1 + \frac{6}{C^2}\right)} = \frac{p}{p_1}$
MOMENT RESULTANT	$m_x = -1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(\tau) = \frac{1}{2}\left[2p\tau - \tau^2 - p\right]$	
VELOCITY	$\dot{w}(\tau) = p - \tau$	
ACCELERATION	$\ddot{w}(\tau) = -1$	
TIME τ_o	AT TIME $\tau' = \frac{p}{p_1}$, TRAVELING HINGE u MOVES TO MIDLENGTH $x = u = 1$ AND THIS REGIME SHRINKS TO ZERO	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 10. SUMMARY, SHORT SHELLS, HIGH LOADING, 5

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $c^2 < 6$	HIGH LOAD $p > 1 + \frac{6}{c^2}$
CONDITIONS	$p > \tau \quad \tau' \leq \tau \leq \tau_o \quad \tau' = \frac{c^2}{c^2+6} p$	
MOMENT RESULTANT	$m_x = -\left[1 + \frac{c^2}{2}\right] x^3 + c^2 x^2 + \left[3 - \frac{c^2}{2}\right] x - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(\tau) = \frac{3}{2} x \left[p\tau - \frac{1}{2} \frac{(c^2+2)}{c^2} \tau^2\right] + E_1(x)$ WHERE FOR $0 \leq x \leq u_o$ E_1 IS GIVEN IN TABLE 11 $u_o < x \leq 1$ E_1 IS GIVEN IN TABLE 12	
VELOCITY	$\dot{w} = \left(\frac{3}{2}\right) x \left[p - \frac{(c^2+2)}{c^2} \tau\right]$	
ACCELERATION	$\ddot{w} = -\frac{3}{2} x \left[\frac{c^2+2}{c^2}\right]$	
TIME τ_o	$\tau_o = \left[\frac{c^2}{c^2+2}\right] p$	
DISPLACEMENT AT REST	$w(\tau_o) = \frac{3}{4} \left(\frac{c^2}{c^2+2}\right) p^2 x + E_1(x) \Big]_{\tau = \tau_o}$	WHERE E_1 WILL BE CALCULATED EITHER BY TABLE 11 OR TABLE 12

TABLE 11. $E_1(x)$ FOR $0 < x < u_o$

$$E_1(x) = \frac{1}{2}(p-1)\,\frac{x}{u_o} - \frac{3}{4}\,x\,p^2\,\frac{\left(C^2+10\right)C^2}{\left(C^2+6\right)^2} + x\left[\frac{3C^2}{\left(C^2+6\right)^2}\,p^2 - \frac{C}{2\sqrt{6}}(p-1)^{3/2} + \frac{3C^2p^2}{4\left(C^2+6\right)} - \frac{3C}{4\sqrt{6}}\,p\sqrt{p-1} + \frac{\sqrt{6}}{8}\,C\,p^2\left\{\tan^{-1}\left(\frac{C}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\}\right]$$

TABLE 12. $E_1(x)$ FOR $u_o \leqslant x \leqslant 1$

$$E_1(x) = -\frac{3}{4} x\, p^2 \frac{(C^2+p)\, C^2}{(C^2+6)^2} - \frac{p}{2}\left[1 - 2p\frac{C^2x^2}{(6+C^2x^2)} + p\frac{C^4x^4}{(6+C^2x^2)^2}\right]$$

$$+ \frac{\sqrt{6}}{8} C\, p^2 x \left\{\tan^{-1}\left(\frac{C}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{C\,x}{\sqrt{6}}\right)\right\} + x\, p^2 \left\{\frac{3C^2}{(6+C^2)^2} - \frac{3C^2x}{(6+C^2x^2)^2}\right.$$

$$\left. + \frac{3C^2}{4(6+C^2)} - \frac{3C^2x}{4(6+C^2x^2)}\right\}$$

TABLE 13. SUMMARY, LONG SHELLS, HIGH LOADING, 1

TYPE	SHELL TYPE	AB PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH LOAD $p > 1 + \frac{6}{C^2}$
CONDITIONS	$u_o^2 = \frac{6}{C^2(p-1)}$	$0 \leqslant \tau \leqslant 1$ $0 \leqslant x \leqslant u_o$
MOMENT RESULTANT	$m_x(x,\tau) = 2\left(\frac{x}{u_o} - 1\right)^3 + 1$ or $= 2\left(\frac{x}{u_o}\right)^3 - 6\left(\frac{x}{u_o}\right)^2 + 6\left(\frac{x}{u_o}\right) - 1$	POINTS ALONG AB ON TRESCA SQUARE
MEMBRANE RESULTANT	$n_\varphi = -1$	POINTS ALONG AB ON TRESCA SQUARE
DISPLACEMENT	$w(x,\tau) = \frac{1}{2}\frac{(p-1)}{u_o}\tau^2 x$	
VELOCITY	$\dot{w} = \frac{(p-1)}{u_o}\tau x$	
ACCELERATION	$\ddot{w} = \frac{(p-1)}{u_o} x$	
TIME τ_O	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 14. SUMMARY, LONG SHELLS, HIGH LOADING, 2

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGHLOAD $p > 1 + \frac{6}{C^2}$
CONDITIONS	$u_o^2 = \frac{6}{C^2(p-1)}$	$0 \leqslant \tau \leqslant 1$ $u_0 < x \leqslant 1$
MOMENT RESULTANT	$m_x = 1$	POINT B ON TRESCA SQUARE
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w = \frac{1}{2}(p-1)\tau^2$	
VELOCITY	$\dot{w} = (p-1)\tau$	
ACCELERATION	$\ddot{w} = p-1$	
TIME τ_0	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 15. SUMMARY, LONG SHELLS, HIGH LOADING, 3

TYPE	SHELL TYPE	AB PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH $p > 1 + \frac{6}{C^2}$ $(p \neq \frac{3}{2})$
CONDITIONS	$o \leqslant x \leqslant y$ $\quad 1 \leqslant \tau \leqslant \tau_1$ $u_o = \sqrt{\frac{6}{C^2(p-1)}}$ $\quad \tau_1 = p - \sqrt{\frac{(p-1)^3}{(2p-3)}} \left\{ 2 - \frac{C^2}{6} \Theta_1^2 \right\}$ Θ_1 defined in TABLE 16, ATTACHMENT 1	
MOMENT RESULTANT	$m_x = -1$	
MEMBRANE RESULTANT	$n_\varphi = 0$	
DISPLACEMENT	$w = \frac{1}{2} \frac{(p-1)}{u_o} x$	
VELOCITY	$\dot{w} = 0$	
ACCELERATION	$\ddot{w} = 0$	
TIME τ_O	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 16. SUMMARY, LONG SHELLS, HIGH LOADING, 4

TYPE	SHELL TYPE	AB PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH LOAD $p > 1 + \frac{6}{C^2}$ $(p \neq \frac{3}{2})$
CONDITIONS	SEE ATTACHMENT 1. $y \leqslant x \leqslant u \qquad 1 \leqslant \tau \leqslant \tau_1$ $\Theta^2 = \frac{6}{C^2}\left[2 - \frac{(2p-3)}{(p-1)^3}(p-\tau)^2\right],$ $\tau_1 = p - \sqrt{\frac{(p-1)^3}{(2p-3)}\left\{2 - \frac{C^2}{6}\Theta_1^2\right\}}$	$\Theta = \left(\frac{2\sqrt{3}}{C}\right)\frac{\left[\coth\left(\frac{C(u-u_o)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{Cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{Cu_o}\coth\left(\frac{C(u-u_o)}{\sqrt{3}}\right)\right]}$ $\theta_1 < \min\left(1, \frac{2\sqrt{3}}{C}\right)$, if $p > \frac{3}{2}$ $\frac{2\sqrt{3}}{C} < \theta_1 < 1$, if $p < \frac{3}{2}$
MOMENT RESULTANT	$m_x(x,\tau) = \frac{1}{(u-y)^3}\left[-4x^3 + 6(u+y)x^2 - 12uyx - (y+u)(y^2 - 4yu + u^2)\right]$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	SEE ATTACHMENT 2.	
VELOCITY	$\dot{w} = (p-\tau)\frac{(x-y)}{\Theta}$	
ACCELERATION	$\ddot{w} = \left(\frac{6}{C^2}\right)\frac{1}{(u-y^3)}\left[u+y-2x\right] - 1$	
TIME τ_O	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 16. ATTACHMENT 1

$$\Theta_1 = \frac{2\sqrt{3}}{C} \frac{\left[\coth\left(\frac{C(1-u_o)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{Cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{Cu_o}\coth\left(\frac{C(1-u_o)}{\sqrt{3}}\right)\right]}$$

$$u = u_o + \frac{2\sqrt{3}}{C} \log_e \left[\left|\frac{\left(\Theta + \frac{2\sqrt{3}}{C}\right)\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(\Theta - \frac{2\sqrt{3}}{C}\right)\left(u_o + \frac{2\sqrt{3}}{C}\right)}\right|\right]$$

$$y = u_o - \Theta + \frac{\sqrt{3}}{2C} \log_e \left[\left|\frac{\left(\Theta + \frac{2\sqrt{3}}{C}\right)\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(\Theta - \frac{2\sqrt{3}}{C}\right)\left(u_o + \frac{2\sqrt{3}}{C}\right)}\right|\right]$$

TABLE 16. ATTACHMENT 2

$y < x \leqslant u_o$

$$w(x,\tau) = \frac{1}{2}\,\frac{(p-1)}{u_o}\,x +$$

$$+ \frac{C^2}{6}\,\frac{(p-1)^3}{(2p-3)}\left[(x-u_o)(\Theta-u_o) + \frac{1}{2}\,(\Theta^2-u_o{}^2) - \frac{\sqrt{3}}{2C}\,(\Theta-u_o)\,\log_e\left|\frac{\left(u_o-\dfrac{2\sqrt{3}}{C}\right)}{\left(u_o+\dfrac{2\sqrt{3}}{C}\right)}\right| -\right.$$

$$-\frac{\sqrt{3}}{2C}\left\{\left(\Theta+\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta+\frac{2\sqrt{3}}{C}\right)\right| - \left(u_o+\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o+\frac{2\sqrt{3}}{C}\right)\right| -\right.$$

$$\left.\left.-\left(\Theta-\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta-\frac{2\sqrt{3}}{C}\right)\right| + \left(u_o-\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o-\frac{2\sqrt{3}}{C}\right)\right|\right\}\right]$$

$u_o < x \leqslant u$

$$w(x,\tau) = \frac{1}{2}\,(p-1) +$$

$$+ \frac{C^2}{6}\,\frac{(p-1)^3}{(2p-3)}\left[(x-u_o)(\Theta-u_o) + \frac{1}{2}\,(\Theta^2-u_o{}^2) - \frac{\sqrt{3}}{2C}\,(\Theta-u_o)\,\log_e\left|\frac{\left(u_o-\dfrac{2\sqrt{3}}{C}\right)}{\left(u_o+\dfrac{2\sqrt{3}}{C}\right)}\right| -\right.$$

$$-\frac{\sqrt{3}}{2C}\left\{\left(\Theta+\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta+\frac{2\sqrt{3}}{C}\right)\right| - \left(u_o+\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o+\frac{2\sqrt{3}}{C}\right)\right| -\right.$$

$$\left.\left.-\left(\Theta-\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta-\frac{2\sqrt{3}}{C}\right)\right| + \left(u_o-\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o-\frac{2\sqrt{3}}{C}\right)\right|\right\}\right]$$

TABLE 17. SUMMARY, LONG SHELLS, HIGH LOADING, 5

TYPE	SHELL TYPE	AB PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH $p > 1 + \frac{6}{C^2}$ $\left(p \neq \frac{3}{2}\right)$
CONDITIONS	$u \leqslant x \leqslant 1$	
MOMENT RESULTANT	$m_x = 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	SEE ATTACHMENT 1.	
VELOCITY	$\dot{w} = p - \tau$	
ACCELERATION	$\ddot{w} = -1$	
TIME τ_0	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE 17. ATTACHMENT 1

$$u < x \leqslant 1 \quad u = u^* \text{ at } \tau = \tau^*, \ \Theta = \Theta^*$$

$$w(x,\tau) = \frac{1}{2}(p-1) + p\,[\tau - \tau^*] - \frac{1}{2}[\tau^2 - \tau^{*2}] +$$

$$+ \frac{C^2}{6}\,\frac{(p-1)^3}{(2p-3)}\left[(u^* - u_o)(\Theta^* - u_o) + \frac{1}{2}(\Theta^{*2} - u_o^{\,2})\right.$$

$$- \frac{\sqrt{3}}{2C}(\Theta^* - u_o)\log_e\left|\frac{\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(u_o + \frac{2\sqrt{3}}{C}\right)}\right| -$$

$$- \frac{\sqrt{3}}{2C}\left\{\left(\Theta^* + \frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta^* + \frac{2\sqrt{3}}{C}\right)\right| - \left(u_o + \frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o + \frac{2\sqrt{3}}{C}\right)\right| -\right.$$

$$\left.\left. - \left(\Theta^* - \frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta^* - \frac{2\sqrt{3}}{C}\right)\right| + \left(u_o - \frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o - \frac{2\sqrt{3}}{C}\right)\right|\right\}\right]$$

TABLE 18. SUMMARY, LONG SHELLS, HIGH LOADING, 6

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH $p > 1 + \frac{6}{C^2}$ $\quad p \neq \frac{3}{2}$
CONDITIONS	$y_1 = u_o - \theta_1 + \frac{\sqrt{3}}{2C} \log_e \left[\left\lvert \frac{\left(\theta_1 + \frac{2\sqrt{3}}{C}\right)\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(\theta_1 - \frac{2\sqrt{3}}{C}\right)\left(u_o + \frac{2\sqrt{3}}{C}\right)} \right\rvert\right]$ $\quad u_o = \frac{\sqrt{6}}{C\sqrt{p-1}}$ $\tau_1 \leqslant \tau \leqslant \tau_o \quad \theta_1 = \frac{2\sqrt{3}}{C} \frac{\left[\coth(A) + \frac{2\sqrt{3}}{Cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{Cu_o}\coth(A)\right]}, A = \frac{C\left(1 - u_o\right)}{\sqrt{3}}$ $\tau_1 = p - \left\{\frac{(p-1)^3}{(2p-3)}\left[2 - \frac{C^2}{6}\theta^2\right]\right\}^{1/2},$ $0 \leqslant x \leqslant y \quad y \leqslant 1 - \frac{\sqrt{6}}{C}$	
MOMENT RESULTANT	$m_x = -1$	
MEMBRANE RESULTANT	$n_\varphi = 0$	
DISPLACEMENT	$w(x, \tau) = \frac{1}{2}\frac{(p-1)}{u_o} x$	
VELOCITY	$\dot{w} = 0$	
ACCELERATION	$\ddot{w} = 0$	
TIME τ_o	$\tau_o = \tau_1 + \frac{\left(p - \tau_1\right)\left(1 - y_1\right)}{\left[1 - y_1 + \frac{\sqrt{6}}{C}\right]}$	
DISPLACEMENT AT REST	$w(x, \tau_o)$	

TABLE 19. SUMMARY, LONG SHELLS, HIGH LOADING, 7

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH $p > 1 + \frac{6}{C^2} \quad p \neq \frac{3}{2}$
CONDITIONS	$y_1 = u_o - \theta_1 + \frac{\sqrt{3}}{2C} \log_e \left[\left\| \frac{(\theta_1 + \frac{2\sqrt{3}}{C})(u_o - \frac{2\sqrt{3}}{C})}{(\theta_1 - \frac{2\sqrt{3}}{C})(u_o + \frac{2\sqrt{3}}{C})} \right\| \right] \quad u_o = \frac{\sqrt{6}}{C\sqrt{p-1}}$ $\tau_1 \leqslant \tau \leqslant \tau_o \quad \theta_1 = \frac{2\sqrt{3}}{C} \frac{\left[\coth(A) + \frac{2\sqrt{3}}{Cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{Cu_o}\coth(A)\right]}, \quad A = \frac{C(1-u_o)}{\sqrt{3}}$ $\tau_1 = p - \left\{ \frac{(p-1)^3}{(2p-3)} \left[2 - \frac{C^2}{6}\theta^2\right] \right\}^{1/2},$ $y \leqslant x \leqslant 1 \quad y \leqslant 1 - \frac{\sqrt{6}}{C} \quad y = \left[1 - \frac{6}{C^2} \frac{1}{(1-y_1)^2}\right] \frac{(1-y_1)}{(p-\tau_1)} (\tau - \tau_1) + y_1$ $\dot{y}(\tau) = \left[1 - \frac{6}{C^2} \frac{1}{(1-y_1)^2}\right] \cdot \frac{(1-y_1)}{(p-\tau_1)}$	
MOMENT AND MEMBRANE RESULTANT	$m_x = 1 + \frac{1}{(1-y)^3} \left[-4x^3 + 6(1+y)x^2 - 12yx - 2(1-3y)\right]$ $n_\varphi = -1$	
DISPLACEMENT	$w(y,\tau) = \frac{1}{2} \frac{(p-1)}{u_o} y(\tau_1) + D_1 \left[C^2 \left\{ A_1(\tau - \tau_1) - \frac{1}{2} B_1 (\tau^2 - \tau_1^2) \right\} + 6 \left\{ \frac{1}{B_1} \log_e \left\| \frac{(C_1 - B_1\tau)}{(C_1 - B_1\tau_1)} \right\| + \frac{(1-x)}{B_1} \left\{ \frac{1}{(C_1 - B_1\tau)} - \frac{1}{(C_1 - B_1\tau_1)} \right\} \right\} \right]$	(SEE NEXT PAGE FOR MORE INFORMATION.)
VELOCITY	$\dot{w} = \frac{\left[1 - \frac{6}{C^2} \frac{1}{(1-y)^2}\right] (p-\tau_1)}{\left[1 - \frac{6}{C^2} \frac{1}{(1-y_1)^2}\right] (1-y_1)} (x-y) = \frac{(p-\tau_1)}{(1-y_1)} \frac{\left[x - y - \frac{6}{C^2} \frac{(x-y)}{(1-y)^2}\right]}{\left[1 - \frac{6}{C^2} \frac{1}{(1-y_1)^2}\right]}$	
ACCELERATION	$\ddot{w} = -\left[1 + \frac{6}{C^2} \left\{ \frac{2(x-y)}{(1-y)^3} - \frac{1}{(1-y)^2} \right\} \right]$	
TIME τ_o	$\tau_o = \tau_1 + \frac{(p-\tau_1)(1-y_1)}{\left[1 - y_1 + \frac{\sqrt{6}}{C}\right]}$	
DISPLACEMENT AT REST	$w(y_o, \tau_o)$ when $y = y_o = 1 - \frac{\sqrt{6}}{C}$ the velocity becomes zero.	(SEE NEXT PAGE FOR MORE INFORMATION.)

TABLE 19. (Cont.)

$$A_1 = x - y_1 + \left[1 - \frac{6}{C^2(1-y_1)^2}\right] \frac{(1-y_1)}{(p-\tau_1)} \quad \tau_1$$

$$B_1 = \left[1 - \frac{6}{C^2} \; \frac{1}{(1-y_1)^2}\right] \frac{(1-y_1)}{(p-\tau_1)}$$

$$C_1 = 1 - y_1 + \left[1 - \frac{6}{C^2} \; \frac{1}{(1-y_1)^2}\right] \frac{(1-y_1)}{(p-\tau_1)}$$

$$D_1 = \frac{(1-y_1) \; (p-\tau_1)}{\left[C^2(1-y_1)^2 - 6\right]}$$

TABLE 20. GEOMETRICAL AND MATERIAL CHARACTERISTICS OF MODEL

MODEL NO.	RADIUS a (IN)	THICKNESS h (=2H) (IN)	LENGTH L_T (IN)	YOUNG'S MODULUS E (psi)	POISSON'S RATIO ν	YIELD STRESS σ_y (psi)	SURFACE DENSITY (LB/IN^2)	$\Delta p_{CR} = \sigma_y (\frac{h}{a}) [1 + \frac{4ah}{L_T^2}]$ (psi)
1	17.697	0.112992	3.5430	30.0×10^6	0.300	50,183.0	0.0387618	524.569
2	17.697	0.112992	10.630	30.0×10^6	0.300	50,183.0	0.0387618	343.089
3	17.697	0.112992	10.630	30.0×10^6	0.300	50,183.0	0.0387618	343.089
4	1.503	0.0147	3.210	30.0×10^6	0.300	100,000.0	0.0041601	986.432
5	1.503	0.0147	3.210	30.0×10^6	0.300	100,000.0	0.0041601	986.432

TABLE 20. (CONTINUED)

MODEL NO.	T DURATION OF PRESSURE (SEC)	NORMALIZING AXIAL FORCE N_0 (LBF/IN)	NORMALIZING BENDING MOMENT M_0 (LBF-IN/IN)	NONDIMENSIONAL PRESSURE PARAMETER $(p = \frac{a}{2\sigma_y H} P)$	PEAK PRESSURE P (psi)	$c^2 = \frac{L_T^2}{2ah}$	TYPE OF ANALYSIS
1	0.10×10^{-3}	5670.28	160.174	1.87261	600.0	3.13881	SHORT SHELLS LOW LOADING
2	0.10×10^{-3}	5670.28	160.174	1.87261	600.0	28.2546	LONG SHELLS HIGH LOADING
3	0.10×10^{-3}	5670.28	160.174	1.63853	525.0	28.2546	LONG SHELLS HIGH LOADING
4	0.10×10^{-2}	1470.0	5.40225	1.22694	1200.0	233.187	LONG SHELLS HIGH LOADING
5	0.10×10^{-1}	1470.0	5.40225	1.00916	987.0	233.187	LONG SHELLS LOW LOADING

TABLE 21. RESULTS FOR MODEL NO. 1 (SHORT SHELLS, LOW LOADING CASE)
(P = 600 psi)

NONDIMENSIONAL HALFLENGTH	RESIDUAL DISPLACEMENT (IN.)	NONDIMENSIONAL DISPLACEMENT	DISPLACEMENT/THICKNESS
0.000000	0.000000E+00	0.000000E+00	0.000000E+00
0.100000	0.645053E-03	0.201957E-01	0.570884E-02
0.200000	0.129011E-02	0.403913E-01	0.114177E-01
0.300000	0.193516E-02	0.605870E-01	0.171265E-01
0.400000	0.258021E-02	0.807827E-01	0.228354E-01
0.500000	0.322527E-02	0.100978E+00	0.285442E-01
0.600000	0.387032E-02	0.121174E+00	0.342530E-01
0.700000	0.451537E-02	0.141370E+00	0.399619E-01
0.800000	0.516043E-02	0.161565E+00	0.456707E-01
0.900000	0.580548E-02	0.181761E+00	0.513796E-01
1.000000	0.645053E-02	0.201957E+00	0.570884E-01

NONDIMENSIONAL TIME	NORMALIZED VELOCITY (1/SEC)	TIME (SEC)	VELOCITY (IN/SEC)
0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.250000E+00	0.882834E-01	0.250000E-04	0.281979E-02
0.500000E+00	0.176567E+00	0.500000E-04	0.563958E-02
0.750000E+00	0.264850E+00	0.750000E-04	0.845937E-02
0.100000E+01	0.353134E+00	0.100000E-03	0.112792E-01
0.101797E+01	0.308992E+00	0.101797E-03	0.986926E-02
0.103595E+01	0.264850E+00	0.103595E-03	0.845937E-02
0.105392E+01	0.220709E+00	0.105392E-03	0.704947E-02
0.107190E+01	0.176567E+00	0.107190E-03	0.563958E-02
0.108987E+01	0.132425E+00	0.108987E-03	0.422968E-02
0.110785E+01	0.882834E-01	0.110785E-03	0.281979E-02
0.112582E+01	0.441417E-01	0.112582E-03	0.140989E-02
0.114380E+01	0.000000E+00	0.114380E-03	0.000000E+00

TABLE 22. RESULTS FOR MODEL NO. 2 (LONG SHELL, HIGH LOADING CASE) (P = 600 psi)

NONDIMENSIONAL DISPLACEMENT BY TRAPEZOIDAL RULE

TIME/DISTANCE	0.0000	0.1000	0.2000	0.3000	0.4000	0.4933	0.4967	0.5000
0.000000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.250000	0.0000	0.0055	0.0111	0.0166	0.0221	0.0273	0.0273	0.0273
0.500000	0.0000	0.0221	0.0442	0.0663	0.0884	0.1091	0.1091	0.1091
0.750000	0.0000	0.0497	0.0995	0.1492	0.1990	0.2454	0.2454	0.2454
1.000000	0.0000	0.0884	0.1769	0.2653	0.3538	0.4363	0.4363	0.4363
1.166028	0.0000	0.1110	0.2247	0.3385	0.4522	0.5583	0.5587	0.5591
1.332056	0.0000	0.1203	0.2522	0.3840	0.5159	0.6390	0.6399	0.6409
1.498084	0.0000	0.1217	0.2621	0.4065	0.5509	0.6856	0.6870	0.6884
1.664112	0.0000	0.1217	0.2632	0.4125	0.5631	0.7049	0.7066	0.7082
1.694489	0.0000	0.1217	0.2632	0.4125	0.5633	0.7060	0.7076	0.7093
1.724866	0.0000	0.1217	0.2632	0.4125	0.5633	0.7065	0.7082	0.7099
1.755243	0.0000	0.1217	0.2632	0.4125	0.5633	0.7066	0.7083	0.7100
1.785620	0.0000	0.1217	0.2632	0.4125	0.5633	0.7066	0.7083	0.7100

NONDIMENSIONAL DISPLACEMENT BY TRAPEZOIDAL RULE

TIME/DISTANCE	0.6000	0.7000	0.8000	0.9000	1.0000
0.000000	0.0000	0.0000	0.0000	0.0000	0.0000
0.250000	0.0273	0.0273	0.0273	0.0273	0.0273
0.500000	0.1091	0.1091	0.1091	0.1091	0.1091
0.750000	0.2454	0.2454	0.2454	0.2454	0.2454
1.000000	0.4363	0.4363	0.4363	0.4363	0.4363
1.166028	0.5674	0.5674	0.5674	0.5674	0.5674
1.332056	0.6651	0.6709	0.6709	0.6709	0.6709
1.498084	0.7251	0.7417	0.7467	0.7469	0.7469
1.664112	0.7526	0.7768	0.7895	0.7926	0.7953
1.694489	0.7546	0.7797	0.7933	0.7973	0.8009
1.724866	0.7558	0.7817	0.7960	0.8007	0.8051
1.755243	0.7565	0.7829	0.7976	0.8028	0.8076
1.785620	0.7567	0.7832	0.7981	0.8035	0.8085

TABLE 22. (CONTINUED)

NONDIMENSIONAL HALFLENGTH	RESIDUAL DISPLACEMENT (IN.)	NONDIMENSIONAL DISPLACEMENT	DISPLACEMENT/THICKNESS
0.000000	0.000000E+00	0.000000E+00	0.000000E+00
0.100000	0.388717E-02	0.121702E+00	0.344022E-01
0.200000	0.840639E-02	0.263192E+00	0.743981E-01
0.300000	0.131764E-01	0.412534E+00	0.116614E+00
0.400000	0.179908E-01	0.563266E+00	0.159222E+00
0.493312	0.225695E-01	0.706617E+00	0.199744E+00
0.496656	0.226241E-01	0.708327E+00	0.200228E+00
0.500000	0.226788E-01	0.710039E+00	0.200711E+00
0.600000	0.241686E-01	0.756683E+00	0.213897E+00
0.700000	0.250165E-01	0.783230E+00	0.221401E+00
0.800000	0.254925E-01	0.798132E+00	0.225613E+00
0.900000	0.256654E-01	0.803544E+00	0.227143E+00
1.000000	0.258238E-01	0.808506E+00	0.228546E+00

NONDIMENSIONAL TIME	NORMALIZED VELOCITY (1/SEC)	TIME (SEC)	VELOCITY (IN/SEC)
0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.250000E+00	0.218152E+00	0.250000E-04	0.696781E-02
0.500000E+00	0.436303E+00	0.500000E-04	0.139356E-01
0.750000E+00	0.654455E+00	0.750000E-04	0.209034E-01
0.100000E+01	0.872607E+00	0.100000E-03	0.278712E-01
0.116603E+01	0.706579E+00	0.116603E-03	0.225683E-01
0.133206E+01	0.540551E+00	0.133206E-03	0.172653E-01
0.149808E+01	0.374523E+00	0.149808E-03	0.119623E-01
0.166411E+01	0.208495E+00	0.166411E-03	0.665937E-02
0.169449E+01	0.161359E+00	0.169449E-03	0.515385E-02
0.172487E+01	0.111449E+00	0.172487E-03	0.355970E-02
0.175524E+01	0.580124E-01	0.175524E-03	0.185293E-02
0.178562E+01	0.254936E-16	0.178562E-03	0.814269E-18

TABLE 23. RESULTS FOR MODEL NO. 3 (LONG SHELLS, HIGH LOADING CASE)
(P = 525 psi)

NONDIMENSIONAL DISPLACEMENT BY TRAPEZOIDAL RULE

TIME/DISTANCE	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.5767	0.5883
0.000000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.250000	0.0000	0.0035	0.0069	0.0104	0.0138	0.0173	0.0200	0.0200
0.500000	0.0000	0.0138	0.0277	0.0415	0.0554	0.0692	0.0798	0.0798
0.750000	0.0000	0.0311	0.0623	0.0934	0.1246	0.1557	0.1796	0.1796
1.000000	0.0000	0.0554	0.1107	0.1661	0.2214	0.2768	0.3193	0.3193
1.113639	0.0000	0.0644	0.1310	0.1976	0.2642	0.3308	0.3819	0.3825
1.227278	0.0000	0.0671	0.1420	0.2173	0.2927	0.3680	0.4257	0.4273
1.340917	0.0000	0.0671	0.1454	0.2270	0.3088	0.3905	0.4532	0.4555
1.454556	0.0000	0.0671	0.1454	0.2296	0.3147	0.4007	0.4667	0.4695
1.481397	0.0000	0.0671	0.1454	0.2296	0.3149	0.4016	0.4681	0.4710
1.508238	0.0000	0.0671	0.1454	0.2296	0.3149	0.4020	0.4689	0.4719
1.535079	0.0000	0.0671	0.1454	0.2296	0.3149	0.4022	0.4694	0.4724
1.561920	0.0000	0.0671	0.1454	0.2296	0.3149	0.4022	0.4695	0.4725

NONDIMENSIONAL DISPLACEMENT BY TRAPEZOIDAL RULE

TIME/DISTANCE	0.6000	0.7000	0.8000	0.9000	1.0000
0.000000	0.0000	0.0000	0.0000	0.0000	0.0000
0.250000	0.0200	0.0200	0.0200	0.0200	0.0200
0.500000	0.0798	0.0798	0.0798	0.0798	0.0798
0.750000	0.1796	0.1796	0.1796	0.1796	0.1796
1.000000	0.3193	0.3193	0.3193	0.3193	0.3193
1.113639	0.3830	0.3854	0.3854	0.3854	0.3854
1.227278	0.4289	0.4374	0.4386	0.4386	0.4386
1.340917	0.4579	0.4727	0.4778	0.4788	0.4788
1.454556	0.4723	0.4915	0.5008	0.5046	0.5062
1.481397	0.4739	0.4937	0.5038	0.5083	0.5106
1.508238	0.4749	0.4953	0.5059	0.5109	0.5138
1.535079	0.4754	0.4962	0.5071	0.5126	0.5158
1.561920	0.4756	0.4965	0.5076	0.5131	0.5165

TABLE 23. (CONTINUED)

NONDIMENSIONAL HALFLENGTH	RESIDUAL DISPLACEMENT (IN.)	NONDIMENSIONAL DISPLACEMENT	DISPLACEMENT/THICKNESS
0.000000	0.000000E+00	0.000000E+00	0.000000E+00
0.100000	0.214274E-02	0.670861E-01	0.189637E-01
0.200000	0.464430E-02	0.145406E+00	0.411029E-01
0.300000	0.733194E-02	0.229552E+00	0.648890E-01
0.400000	0.100568E-01	0.314864E+00	0.890045E-01
0.500000	0.128453E-01	0.402168E+00	0.113683E+00
0.576687	0.149954E-01	0.469485E+00	0.132712E+00
0.588343	0.150927E-01	0.472531E+00	0.133573E+00
0.600000	0.151900E-01	0.475576E+00	0.134434E+00
0.700000	0.158575E-01	0.496474E+00	0.140341E+00
0.800000	0.162117E-01	0.507565E+00	0.143477E+00
0.900000	0.163893E-01	0.513124E+00	0.145048E+00
1.000000	0.164974E-01	0.516511E+00	0.146005E+00

NONDIMENSIONAL TIME	NORMALIZED VELOCITY (1/SEC)	TIME (SEC)	VELOCITY (IN/SEC)
0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.250000E+00	0.159633E+00	0.250000E-04	0.509870E-02
0.500000E+00	0.319265E+00	0.500000E-04	0.101974E-01
0.750000E+00	0.478898E+00	0.750000E-04	0.152961E-01
0.100000E+01	0.638531E+00	0.100000E-03	0.203948E-01
0.111364E+01	0.524892E+00	0.111364E-03	0.167651E-01
0.122728E+01	0.411253E+00	0.122728E-03	0.131355E-01
0.134092E+01	0.297614E+00	0.134092E-03	0.950585E-02
0.145456E+01	0.183975E+00	0.145456E-03	0.587620E-02
0.148140E+01	0.142413E+00	0.148140E-03	0.454870E-02
0.150824E+01	0.983906E-01	0.150824E-03	0.314262E-02
0.153508E+01	0.512341E-01	0.153508E-03	0.163643E-02
0.156192E+01	-.296941E-16	0.156192E-03	-.948434E-18

TABLE 24. RESULTS FOR MODEL NO. 4 (LONG SHELLS, HIGH LOADING CASE)
(P = 1200 psi)

NONDIMENSIONAL DISPLACEMENT BY TRAPEZOIDAL RULE

TIME/DISTANCE	0.0000	0.1000	0.2000	0.3000	0.3367	0.3684	0.4000	0.5000
0.000000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.250000	0.0000	0.0021	0.0042	0.0063	0.0071	0.0071	0.0071	0.0071
0.500000	0.0000	0.0084	0.0168	0.0253	0.0284	0.0284	0.0284	0.0284
0.750000	0.0000	0.0190	0.0379	0.0569	0.0638	0.0638	0.0638	0.0638
1.000000	0.0000	0.0337	0.0674	0.1011	0.1135	0.1135	0.1135	0.1135
1.056603	0.0000	0.0362	0.0734	0.1106	0.1243	0.1247	0.1247	0.1247
1.113206	0.0000	0.0367	0.0763	0.1165	0.1312	0.1324	0.1328	0.1328
1.169809	0.0000	0.0367	0.0771	0.1189	0.1344	0.1361	0.1371	0.1376
1.226411	0.0000	0.0367	0.0771	0.1194	0.1351	0.1371	0.1382	0.1392
1.226489	0.0000	0.0367	0.0771	0.1194	0.1351	0.1371	0.1382	0.1392
1.226566	0.0000	0.0367	0.0771	0.1194	0.1351	0.1371	0.1382	0.1392
1.226643	0.0000	0.0367	0.0771	0.1194	0.1351	0.1371	0.1382	0.1392
1.226720	0.0000	0.0367	0.0771	0.1194	0.1351	0.1371	0.1382	0.1392

NONDIMENSIONAL DISPLACEMENT BY TRAPEZOIDAL RULE

TIME/DISTANCE	0.6000	0.7000	0.8000	0.9000	1.0000
0.000000	0.0000	0.0000	0.0000	0.0000	0.0000
0.250000	0.0071	0.0071	0.0071	0.0071	0.0071
0.500000	0.0284	0.0284	0.0284	0.0284	0.0284
0.750000	0.0638	0.0638	0.0638	0.0638	0.0638
1.000000	0.1135	0.1135	0.1135	0.1135	0.1135
1.056603	0.1247	0.1247	0.1247	0.1247	0.1247
1.113206	0.1328	0.1328	0.1328	0.1328	0.1328
1.169809	0.1376	0.1376	0.1376	0.1376	0.1376
1.226411	0.1392	0.1392	0.1392	0.1392	0.1392
1.226489	0.1392	0.1392	0.1392	0.1392	0.1392
1.226566	0.1392	0.1392	0.1392	0.1392	0.1392
1.226643	0.1392	0.1392	0.1392	0.1392	0.1392
1.226720	0.1392	0.1392	0.1392	0.1392	0.1392

TABLE 24. (CONTINUED)

NONDIMENSIONAL HALFLENGTH	RESIDUAL DISPLACEMENT (IN.)	NONDIMENSIONAL DISPLACEMENT	DISPLACEMENT/THICKNESS
0.000000	0.000000E+00	0.000000E+00	0.000000E+00
0.100000	0.333457E+01	0.367069E-01	0.226841E+03
0.200000	0.700420E+01	0.771022E-01	0.476476E+03
0.300000	0.108434E+02	0.119365E+00	0.737649E+03
0.336720	0.122694E+02	0.135062E+00	0.834654E+03
0.368360	0.124506E+02	0.137057E+00	0.846982E+03
0.400000	0.125576E+02	0.138234E+00	0.854257E+03
0.500000	0.126458E+02	0.139205E+00	0.860259E+03
0.600000	0.126458E+02	0.139205E+00	0.860259E+03
0.700000	0.126458E+02	0.139205E+00	0.860259E+03
0.800000	0.126460E+02	0.139207E+00	0.860270E+03
0.900000	0.126466E+02	0.139213E+00	0.860310E+03
1.000000	0.126472E+02	0.139220E+00	0.860351E+03

NONDIMENSIONAL TIME	NORMALIZED VELOCITY (1/SEC)	TIME (SEC)	VELOCITY (IN/SEC)
0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.250000E+00	0.567347E-01	0.250000E-03	0.515395E+01
0.500000E+00	0.113469E+00	0.500000E-03	0.103079E+02
0.750000E+00	0.170204E+00	0.750000E-03	0.154619E+02
0.100000E+01	0.226939E+00	0.100000E-02	0.206158E+02
0.105660E+01	0.170336E+00	0.105660E-02	0.154738E+02
0.111321E+01	0.113733E+00	0.111321E-02	0.103319E+02
0.116981E+01	0.571302E-01	0.116981E-02	0.518988E+01
0.122641E+01	0.527339E-03	0.122641E-02	0.479051E-01
0.122649E+01	0.408448E-03	0.122649E-02	0.371046E-01
0.122657E+01	0.282408E-03	0.122657E-02	0.256548E-01
0.122664E+01	0.147207E-03	0.122664E-02	0.133728E-01
0.122672E+01	0.161556E-16	0.122672E-02	0.146763E-14

TABLE 25. RESULTS FOR MODEL NO. 5 (LONG SHELLS, LOW LOADING CASE)
(P = 987 psi)

NONDIMENSIONAL TIME	VELOCITY (1/SEC)	NONDIMENSIONAL POSITION OF HINGE CIRCLE	NONDIMENSIONAL HINGE VELOCITY (1/SEC)
0. 000000E+00	0. 000000E+00	0. 000000E+00	0. 000000E+00
0. 250000E+00	0. 217621E-03	0. 000000E+00	0. 000000E+00
0. 500000E+00	0. 435243E-03	0. 000000E+00	0. 000000E+00
0. 750000E+00	0. 652864E-03	0. 000000E+00	0. 000000E+00
0. 100000E+01	0. 870486E-03	0. 000000E+00	0. 000000E+00
0. 100009E+01	0. 774021E-03	0. 104949E+00	0. 111922E+04
0. 100019E+01	0. 676840E-03	0. 209898E+00	0. 111922E+04
0. 100028E+01	0. 578613E-03	0. 314847E+00	0. 111922E+04
0. 100038E+01	0. 478774E-03	0. 419796E+00	0. 111922E+04
0. 100047E+01	0. 376255E-03	0. 524746E+00	0. 111922E+04
0. 100056E+01	0. 268776E-03	0. 629695E+00	0. 111922E+04
0. 100066E+01	0. 150453E-03	0. 734644E+00	0. 111922E+04
0. 100075E+01	0. 143876E-16	0. 839593E+00	0. 111922E+04

TABLE 25. (CONTINUED)

NONDIMENSIONAL HALFLENGTH	RESIDUAL DISPLACEMENT (IN.)	NONDIMENSIONAL DISPLACEMENT	DISPLACEMENT/THICKNESS
0. 000000	0. 000000E+00	0. 000000E+00	0. 000000E+00
0. 100000	0. 395388E+00	0. 435243E-04	0. 268971E+02
0. 200000	0. 790776E+00	0. 870486E-04	0. 537943E+02
0. 300000	0. 118616E+01	0. 130573E-03	0. 806914E+02
0. 400000	0. 158155E+01	0. 174097E-03	0. 107589E+03
0. 500000	0. 197694E+01	0. 217621E-03	0. 134486E+03
0. 600000	0. 237233E+01	0. 261146E-03	0. 161383E+03
0. 700000	0. 276772E+01	0. 304670E-03	0. 188280E+03
0. 800000	0. 316310E+01	0. 348194E-03	0. 215177E+03
0. 900000	0. 356117E+01	0. 392014E-03	0. 242257E+03

NONDIMENSIONAL TIME	NORMALIZED VELOCITY (1/SEC)	TIME (SEC)	VELOCITY (IN/SEC)
0. 000000E+00	0. 000000E+00	0. 000000E+00	0. 000000E+00
0. 250000E+00	0. 217621E-03	0. 250000E-02	0. 197694E+01
0. 500000E+00	0. 435243E-03	0. 500000E-02	0. 395388E+01
0. 750000E+00	0. 652864E-03	0. 750000E-02	0. 593082E+01
0. 100000E+01	0. 870486E-03	0. 100000E-01	0. 790776E+01
0. 100009E+01	0. 774021E-03	0. 100009E-01	0. 703144E+01
0. 100019E+01	0. 676840E-03	0. 100019E-01	0. 614862E+01
0. 100028E+01	0. 578613E-03	0. 100028E-01	0. 525630E+01
0. 100038E+01	0. 478774E-03	0. 100038E-01	0. 434933E+01
0. 100047E+01	0. 376255E-03	0. 100047E-01	0. 341802E+01
0. 100056E+01	0. 268776E-03	0. 100056E-01	0. 244164E+01
0. 100066E+01	0. 150453E-03	0. 100066E-01	0. 136676E+01
0. 100075E+01	0. 143876E-16	0. 100075E-01	0. 130701E-12

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NOMENCLATURE

A	=	$\dfrac{2\,\sigma_y H\, t_o^2}{a^2 s}$
a	=	Mean Shell radius (in.)
c, C	=	Non-dimensional parameter defined as $\dfrac{\sqrt{2}L}{\sqrt{ah}}$
$\dot{D}$	=	Rate of plastic work done (plastic dissipation) $\left(\frac{d}{d\tau}D\right)$ / unit area (Lb-IN/IN^3*SEC)
$\dot{\underset{\sim}{e}}$	=	Strain rate vector with components $(\dot{\varepsilon}_\phi, \dot{\kappa}_x)$
e_x	=	Total axial strain
e_ϕ	=	Total circumferential (hoop) strain
$F(\tau)$	=	Function with respect to non-dimensional time variable τ
$\dot{F}(\tau)$	=	Derivative of $F(\tau)$ with respect to τ
H	=	Half thickness of shell (in.)
h	=	Total thickness of shell (in.)
$(\underset{\sim}{i}, \underset{\sim}{j})$	=	Unit Vectors on (n_ϕ, m_x) or (e_ϕ, κ_x) space (See Appendix.)

NOMENCLATURE (Cont.)

κ_x	=	Bending curvature in axial plane (1/in.) ($\dot{\kappa}_x^1$ = Nondimensional bending strain rate and $\dot{\kappa}_x^1 = \frac{H}{2}\dot{\kappa}_x$)
κ_ϕ	=	Bending curvature is a plane perpendicular to the shell axis (1/in.)
κ	=	Yield stress in pure shear defined in terms of σ_y yield stress as $\frac{1}{2}\sigma_y$ (psi)
L	=	Half length of shell (from stiffener to midbay) (in.)
L_T	=	Total length of shell ($L_T = 2L$) (in.)
M	=	Mass of section analyzed (Lb-sec^2/in.)
M_x	=	Bending Moment/length of section of shell perpendicular to x-axis (longitudinal axis). (Lb-in./in.) It is defined as $\int_{-H}^{H} z\, \sigma_x dz$, where z is dummy variable across thickness
m_x	=	Non-dimensional bending moment based on M_x
M_o	=	$\sigma_y H^2$ (Lb-in./in.)
$\underset{\sim}{n}_A^{(1)}, \underset{\sim}{n}_A^{(2)}$	=	Unit normals at point A of yield surface (See Appendix)
$\underset{\sim}{n}_B^{(1)}, \underset{\sim}{n}_B^{(2)}$	=	Unit normals at point B of yield surface (See Appendix)
N_x	=	Membrane force/unit length defined as $\int_{-H}^{H} \sigma_x dz$ (lb./in.)
N_o	=	$2\sigma_y H$ (lb./in.)

NOMENCLATURE (Cont.)

N_ϕ	=	Membrane force/unit length of axial section and a section perpendicular to the axis of cylindrical shell (defined as $\int_{-H}^{H} \sigma_\phi dz$) (lb./in.)
n_ϕ	=	Non-dimensional membrane force variable based on N_ϕ
P	=	External pressure, function of time, i.e., $P = P(t)$ (psi). In this report it is taken as a constant
p	=	Non-dimensional pressure parameter $\left(= \frac{a}{4\kappa H} P\right)$
p_o	=	Non-dimensional collapse load parameter defined by $p_o = 1 + \frac{2}{C^2}$
p_1	=	Non-dimensional collapse load parameter defined by $p_1 = 1 + \frac{6}{C^2}$
s	=	Surface density of material (mass/unit area) (lb.-sec.^{2}/in.^{3})
t	=	Time variable (sec.)
t_o	=	Time period over which pressure loading P(t) is acting (sec.)
X	=	Axial distance (in.)
x	=	Non-dimensional variable $\left(= \frac{X}{L}\right)$
U	=	Longitudinal displacement (in.)
u	=	Non-dimensional axial displacement $\left(\frac{U}{L}\right)$
V	=	Velocity of propagation of traveling hinge (in./sec.)
W	=	Radial displacement (in.)

NOMENCLATURE (Cont.)

w = Non-dimensional radial displacement variable defined by

$$w = \frac{sa}{4\kappa H t_o^2} W = \frac{sa}{2\sigma_y H t_o^2} = \frac{1}{aA} W$$

$(\ )_x$ = Derivative with respect to non-dimensional axial variable x

$(\dot{\ })$ = Time derivative with respect to non-dimensional time variable τ

[ ] = Square brackets indicate jump conditions of a variable (i.e., difference of its value at two points, one to the left and the other to the right of the point in question)

α, β = Arbitrary positive constants used in Appendices

ε_x = Axial strain in middle surface of skin (otherwise known as axial inplane strain)

ε_ϕ = Circumferential strain in middle surface of skin (otherwise known as hoop strain)

$\underset{\sim}{\sigma}$ = Non-dimensional stress resultant vector (n_ϕ, m_x)

σ_y = Yield stress (psi)

τ = Non-dimensional time variable (τ = t/to)

τ_0 = Non-dimensional time at which shell comes to rest

τ_1 = Non-dimensional time

APPENDIX A

PRELIMINARY ANALYSIS

This Appendix presents the derivation of the developed solutions in a detailed way. The circular cylindrical shell is ring-stiffened, fairly long, and subjected to an external dynamic pressure load P, which, in itself, is assumed axisymmetric and whose form over time is rectangular. Because the material is assumed as rigid perfectly plastic, there will be no deformation unless the external load P exceeds the critical collapse load P_o (References A-1 and A-2).

For the purpose of solution we must have at our disposal:

1. The equations of equilibrium

2. Strain-displacement relations

3. A yield condition and associated flow rule.

4. Conditions of continuity of certain variables, and initial and boundary conditions.

The analysis follows the five steps of the problem statement listed in the main text for all four cases listed below. (It will become clear after each solution in later appendices why such division takes place.)

1. Short shell and low intensity loading.

2. Long shell and low intensity loading.

3. Short shell and high intensity loading.

4. Long shell and high intensity loading.

EQUATIONS OF EQUILIBRIUM AND STRAIN - DISPLACEMENT EQUATIONS

If we consider a section of a shell element of axial length dX and circumferential length $ad\phi$ (where a stands for the shell radius) of mass M (Figures A-1 and A-2) acted on by external pressure P, we can obtain Timoshenko's equations of equilibrium[A-3] in the undeformed configuration. The external loads for this axisymmetric case are resisted by internal bending moment/unit length, M_x, and membrane force/unit length, N_ϕ. The equilibrium equation is:

$$M\frac{\partial^2}{\partial t^2}W = a\, d\phi dX\, \frac{\partial^2}{\partial X^2}M_x + a\, d\phi dX\frac{N_\phi}{a} + a\, dX\, d\phi P \tag{A-1}$$

Replacing the mass per unit area s in Equation (A-1) by

$$s = \frac{M}{a\ d\phi dX} \tag{A-2}$$

we have

$$s \frac{\partial^2 W}{\partial t^2} = \frac{\partial^2}{\partial X^2} M_x + \frac{1}{a} N_\phi + P \tag{A-3}$$

In the paragraphs that follow, equations will be written in non-dimensional form. The objective is to obtain certain relations between certain groups of parameters, on one hand, while on the other (it is easier), to work with non-dimensional groups.

We introduce the following notation:

$$\tau = t/t_o \tag{A-4}$$

$$x = \frac{X}{L} \tag{A-5}$$

$$m_x = \frac{M_x}{M_o} = \frac{M_x}{\sigma_y H^2} \tag{A-6}$$

$$n_\phi = \frac{N_\phi}{N_o} = \frac{N_\phi}{2\sigma_y H} \tag{A-7}$$

$$p = \frac{P}{\sigma_y}\,\frac{a}{2H} \tag{A-8}$$

$$w = \frac{sa}{2\sigma_y H t_o^2} W \tag{A-9}$$

$$c^2 = \frac{L^2}{aH}, \tag{A-10}$$

where L is the half length of the shell, H is the half thickness, σ_y is the yield stress, t_o is the duration (secs) of the rectangular pressure loading, and X is the axial distance.

Using these equations, Equation (A-3) is transformed to

$$\frac{1}{2c^2}\,\frac{\partial^2}{\partial x^2} m_x + n_\phi + p - \frac{\partial^2}{\partial \tau^2} w = 0 \tag{A-11}$$

This represents the equilibrium equation of an element in the undeformed configuration. It is written in terms of two generalized stress resultants only, m_x and n_ϕ. This is due to the axisymmetric nature of the problem and the absence of axial loads. These resultants appear in the plastic dissipation rate $\dot{D}$, depending on the stress-state and the yield surface (References A-1 and A-4).

DISSIPATION EQUATIONS

The purpose of the analysis that follows is to obtain the plastic dissipation rate in terms of the strain rates associated with the generalized stress resultants. The step is vital because it identifies the strain space corresponding to the space of "stress resultants" of the yield surface (References A-1 and A-4).

To further the analysis, we state the strain-displacement equations from Reference A-3. The inplane axial (ε_x) and circumferential strains (ε_ϕ) as well as curvatures κ_x and κ_ϕ are:

$$\varepsilon_x = \frac{\partial U}{\partial X} \tag{A-12}$$

$$\varepsilon_\phi = -\frac{1}{a}W \tag{A-13}$$

$$\kappa_x = \frac{\partial^2}{\partial X^2}W \tag{A-14}$$

$$\kappa_\phi = 0 \tag{A-15}$$

where U and W are the axial and radial displacements at the midsurface of the shell, independent of the thickness, variable z. They depend on X only due to axisymmetry (Reference A-3).

The total strains e_x, e_ϕ are given by

$$e_x = \varepsilon_x + z\,\kappa_x = \frac{\partial U}{\partial X} + z\frac{\partial^2}{\partial X^2}W \qquad \text{(A-16)}$$

$$e_\phi = \varepsilon_\phi + z\,\kappa_\phi = -\frac{1}{a}W \qquad \text{(A-17)}$$

We introduce a non-dimensional thickness variable ζ, as

$$\zeta = \frac{z}{H} \qquad \text{(A-18)}$$

and $$A = \frac{2\sigma_y H t_o^2}{a^2 s} \qquad \text{(A-19)}$$

Naturally, A is a positive quantity.

We can write Equations (A-16) and (A-17) as

$$e_x = \frac{\partial u}{\partial x} + \zeta H \kappa_x \qquad \text{(A-20)}$$

$$e_\phi = -Aw \qquad \text{(A-21)}$$

and Equations (A-12), (A-13), (A-14), and (A-15) as

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{A-22}$$

$$\epsilon_\phi = - Aw \tag{A-23}$$

$$\kappa_x = \frac{1}{c^2} \frac{A}{H} \frac{\partial^2 w}{\partial x^2} \tag{A-24}$$

$$\kappa_\phi = 0 \tag{A-25}$$

To identify which parts of the strain rates play a part in the analysis, we must compute the energy dissipated pl?stically (plastic dissipation rate $\dot{D}$).

From first principles (References A-1 and A-4), the internal rate of plastic work $\dot{D}$/unit area in terms of stresses and strain rates integrated over shell thickness 2H is

$$\dot{D} = \int_{-H}^{H} (\sigma_x \dot{e}_x + \sigma_\phi \dot{e}_\phi) dz \tag{A-26}$$

If we introduce the non-dimensional parameters n_x, m_x, and n_ϕ by means of

$$N_x = N_o n_x = 2\sigma_y H n_x \tag{A-27}$$

$$N_\phi = N_o n_\phi = 2\sigma_y H n_\phi \tag{A-28}$$

$$M_x = M_o m_x = \sigma_y H^2 m_x = \frac{H}{2} N_o m_x \tag{A-29}$$

Equation (A-26) can be expressed as

$$\dot{D} = \dot{\varepsilon}_x N_x + \dot{\kappa}_x^1 M_x + \dot{\varepsilon}_\phi N_\phi = \left[\dot{\varepsilon}_x n_x + \dot{\kappa}_x^1 m_x + \dot{\varepsilon}_\phi n_\phi\right] N_o \tag{A-30}$$

where

$$\dot{\kappa}_x^1 = \frac{H}{2} \dot{\kappa}_x = \frac{1}{2c^2} A \frac{\partial^2 w}{\partial x^2}$$

and this is the only time (numerical) 1 does not denote differentiation.

In the present case there is no axial load ($n_x = 0$) and Equation (A-30) becomes

$$\dot{D} = \left[\dot{\varepsilon}_\phi n_\phi + \dot{\kappa}_x^1 m_x\right] N_o \tag{A-31}$$

in terms of two generalized strain rates ($\dot{\varepsilon}_\phi$, $\dot{\kappa}_x^1$) and stress resultants (n_ϕ, m_x).

We observe that we can write the generalized strain rate in terms of a two-component vector $\dot{\underset{\sim}{e}}$, in strain space, of the form

$$\dot{\underset{\sim}{e}} = (\dot{\varepsilon}_\phi, \dot{\kappa}_x^1) \tag{A-32}$$

or in view of Equations (A-23) and (A-24)

$$\underset{\sim}{\dot{e}} = A\left(-\dot{w},\ -\frac{1}{2c^2}\dot{w}_{xx}\right) \tag{A-33}$$

where A is a constant given by Equation (A-19). Furthermore, the stress resultants, which contribute to plastic dissipation, form a two-component stress-resultant vector, $\underset{\sim}{\sigma}$ of the form

$$\underset{\sim}{\sigma} = (n_\phi,\ m_x) \tag{A-34}$$

In the analysis, a yield surface and associated flow rule must be used. The yield surface, which satisfies the Tresca yield criterion, has been developed by Drucker, Hodge, and Onat. (See References A-4 and A-5 for more details.) For the present case, the exact condition will be replaced by the simplified Tresca rectangle ABCD (Figure A-3). This locus envelopes the exact curve from the outside. According to plasticity theory, the strain rate vector $\underset{\sim}{\dot{e}}$ must always be normal to the yield surface and directed outwards during plastic flow, except at the corners, where $\underset{\sim}{\dot{e}}$ must lie within the space described by the (two or three) normals to the yield surface there. Furthermore, the yield surface must be convex. (See References A-4 through A-10.)

The flow rule (outward normality to yield surface), therefore, determines the direction of growth of plastic flow, which takes place along the gradient of the yield surface. For perfectly plastic materials, however, the magnitude of the strain rate cannot be determined.

Furthermore, notice that on a straight side, while the strain rate vector does not uniquely determine the state of stress [i.e., components (n_ϕ and m_x)] the plastic dissipation is unique. [See Hodge (Reference A-5), Chapter 8, pp. 195-201.] Accordingly, the strain rate vector $\dot{\underset{\sim}{e}}$ must be of the form indicated in Table A-1.

Here we explain how Table A-1 is constructed.

1. <u>Region AD</u>

We consider the region AD (except the two end points A and D which must be treated separately as they constitute "corners" on the yield surface). The equation of the outward normal to AD is:

$$n_\phi = c_1 \qquad -1 \leq c_1 \leq 1 \tag{A-35}$$

Its slope is $-\infty$ on the (n_ϕ, m_x) plane. The strain components of $\dot{\underset{\sim}{e}}$ are given by Equation (A-33). Since $m_x = -1$, this means that

$$-\frac{A}{2c^2}\dot{w}_{xx} \leq 0 \tag{A-36}$$

and since $A > 0$, this leads to $\dot{w}_{xx} \geq 0$.

Noting that n_ϕ is arbitrary [to the extent that it lies between (-1, 1)] and the slope of the $\dot{\underset{\sim}{e}}$ vector is

$$\frac{-\frac{1}{2c^2}\dot{w}_{xx}}{-\dot{w}} = -\infty \qquad \text{(A-37)}$$

this can only be satisfied by $\dot{w} = 0$.

2. Point A

At point A there are two outward normals.

a. Normal to AD = $\underset{\sim}{n}_A^{(1)} = (0, -1)$

or in $(\underset{\sim}{i}, \underset{\sim}{j})$ notation

$$\underset{\sim}{n}_A^{(1)} = -\underset{\sim}{j} \qquad \text{(A-38)}$$

b. Normal to AB = $\underset{\sim}{n}_A^{(2)} = (-1, 0) = -\underset{\sim}{i}$

As already mentioned (and discussed in References A-1, A-4, and A-5), at a corner of the yield surface the strain rate vector $\dot{\underset{\sim}{e}}$ must be a positive linear combination (α, $\beta > 0$) of the unit exterior normals, i.e.,

$$\alpha\, \underset{\sim}{n}_A^{(1)} + \beta \underset{\sim}{n}_A^{(2)} = (-\beta, -\alpha) \qquad \text{(A-39)}$$

By Equation (A-33), however, we note that

$$\dot{\underset{\sim}{e}} = A\left(-\dot{w},\ -\frac{1}{2c^2}\dot{w}_{xx}\right) = (-\beta,\ -\alpha) \tag{A-40}$$

i.e., $-\beta$, $-\alpha \leq 0$ and, hence,

$$\begin{aligned} -\dot{w} &\leq 0 \\ -\frac{1}{2c^2}\dot{w}_{xx} &\leq 0 \end{aligned} \tag{A-41}$$

or

$$\begin{aligned} \dot{w} &\geq 0 \\ \dot{w}_{xx} &\geq 0 \end{aligned} \tag{A-42}$$

3. Point B

Similarly, at point B the outward normals are given by

a. Normal to AB at B: $\underset{\sim}{n}_B^{(1)} = (-1,\ 0) = -\underset{\sim}{i}$

b. Normal to BC at B: $\underset{\sim}{n}_B^{(2)} = (0,\ 1) = \underset{\sim}{j}$

Since $\dot{\underset{\sim}{e}}$ must be a positive linear combination of $\underset{\sim}{n}_B^{(1)}$ and $\underset{\sim}{n}_B^{(2)}$, we have

$$\alpha\ \underset{\sim}{n}_B^{(1)} + \beta\ \underset{\sim}{n}_B^{(2)} = (-\alpha,\ \beta) \tag{A-43}$$

Comparing Equation (A-43) with (A-33) we deduce that

$$-\dot{w} \leq 0 \qquad \text{or} \qquad \dot{w} \geq 0 \tag{A-44}$$

$$\frac{-1}{2c^2}\dot{w}_{xx} \geq 0 \qquad \text{or} \qquad \dot{w}_{xx} \leq 0 \tag{A-45}$$

Table A-1 can be completed by similar arguments.

CONTINUITY CONDITIONS

The shell in question is made of material without voids or other sources of discontinuity. Consequently, the radial displacement w and velocity $\frac{\partial w}{\partial \tau}$ are continuous. If we denote "jump" conditions through brackets, then

$$[w] = 0 \tag{A-46}$$

$$\left[\frac{\partial w}{\partial \tau}\right] = 0 \tag{A-47}$$

In addition, the bending moment, m_x, membrane force, n_ϕ, and shear force distributions must be continuous across the length of the shell.

Finally, we need to quote some results from the travelling hinge theory of Lee and Symonds.[A-11,A-12] Recall that a hinge forms at some point on the structure, when the limit moment has been reached. In structural dynamics, this hinge, hinge line, or hinge circle may either be stationary or travelling with a velocity or propagation V. In the present case of our cylindrical shell, note that

a. If the hinge is stationary ($V = 0$), then the slope may be discontinuous, i.e.,

$$\left[\frac{\partial w}{\partial x}\right] \neq 0 \quad \text{or } 0 \tag{A-48}$$

b. If the hinge is travelling ($V \neq 0$), then the slope is continuous

$$\left[\frac{\partial w}{\partial x}\right] = 0 \tag{A-49}$$

INITIAL AND LOADING CONDITIONS

The shell is initially at rest. Up to time $t = 0$ ($\tau = 0$) there is no applied load. Suddenly, at $t = 0$ ($\tau = 0$) a rectangular pressure pulse of magnitude P (p) is applied on the cylindrical shell. The pressure pulse acts over a time t_o ($\tau = 1$) and then becomes zero. Consequently, the initial velocity and displacement conditions are

$$\dot{w}(x, 0) = 0 \tag{A-50}$$

$$w(x, 0) = 0 \tag{A-51}$$

At some time $t_1 > t_0$ ($\tau_1 > 1$) the body is brought to rest, i.e.,

$$\dot{w}(x, \tau_1) = 0 \tag{A-52}$$

The loading conditions are

$$p(x, 0) = 0$$

$$p(x, \tau) = p \quad \text{for} \quad 0 \leq \tau \leq 1 \tag{A-53}$$

$$p(x, \tau) = 0 \quad \text{for} \quad \tau > 1$$

where p is a constant.

BOUNDARY CONDITIONS

With reference to Figure A-1, the first ring stiffener is located at $x = 0$ ($X = 0$), and the second one (end of bay) at $x = 2$ ($X = 2L$). At the first stiffener ($x = 0$) the radial displacement and velocities are zero for all times $\tau \geq 0$, i.e.,

$$w(0, \tau) = 0 \tag{A-54}$$

$$\dot{w}(0, \tau) = 0 \tag{A-55}$$

From the discussion in "conditions of continuity," we conclude that whether the slope of a hinge is continuous or not will depend on whether the initially formed hinge is "travelling" or not, respectively. However, at the center of the bay ($x = 1$) the slope must be zero

$$w_x (1, \tau) = 0 \tag{A-56}$$

Due to symmetry, the shear at the center must be zero, i.e.,

$$\frac{\partial}{\partial x} m_x (1, \tau) = 0 \tag{A-57}$$

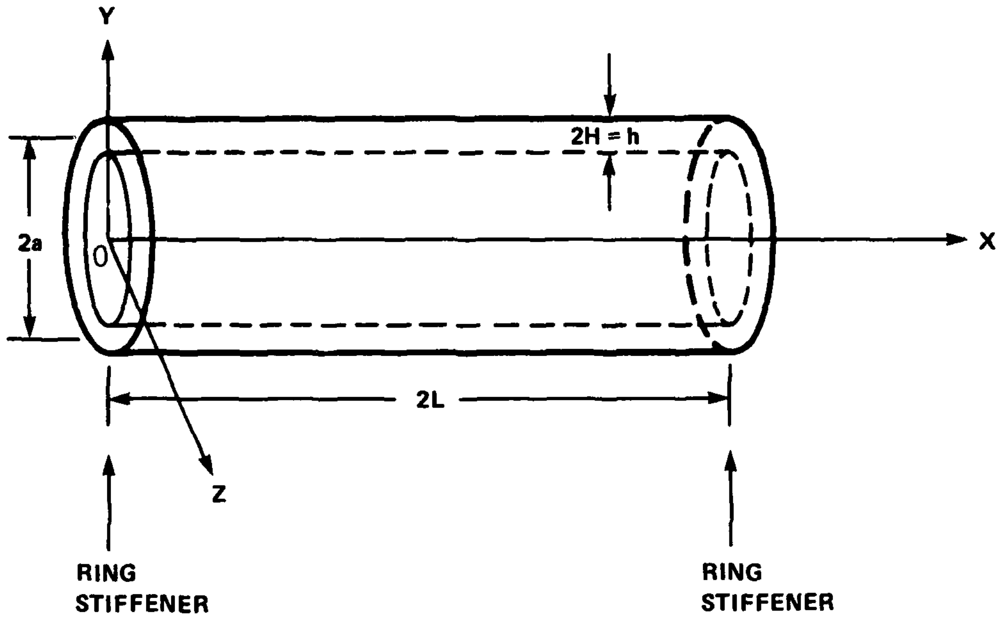
Also observe that the non-dimensional bending moment m_x and membrane force n_ϕ must lie on the yield locus when the structure is deforming.

From beam theory, recall that the bending moment distribution of a clamped beam has a maximum negative value at the supports ($x = 0$ and $x = 2$), while it attains a maximum positive value at the center ($x = 1$). Interpreting this in terms of the square yield surface, the bending moment distribution could lie anywhere along the DA, AB, BC (including corners) portions of the Tresca locus. However, we also know that due to the compressive nature of the external pressure, n_ϕ must be negative. This limits us along BA and the portion of AB (including corners) where n_ϕ is negative.

GENERAL METHOD

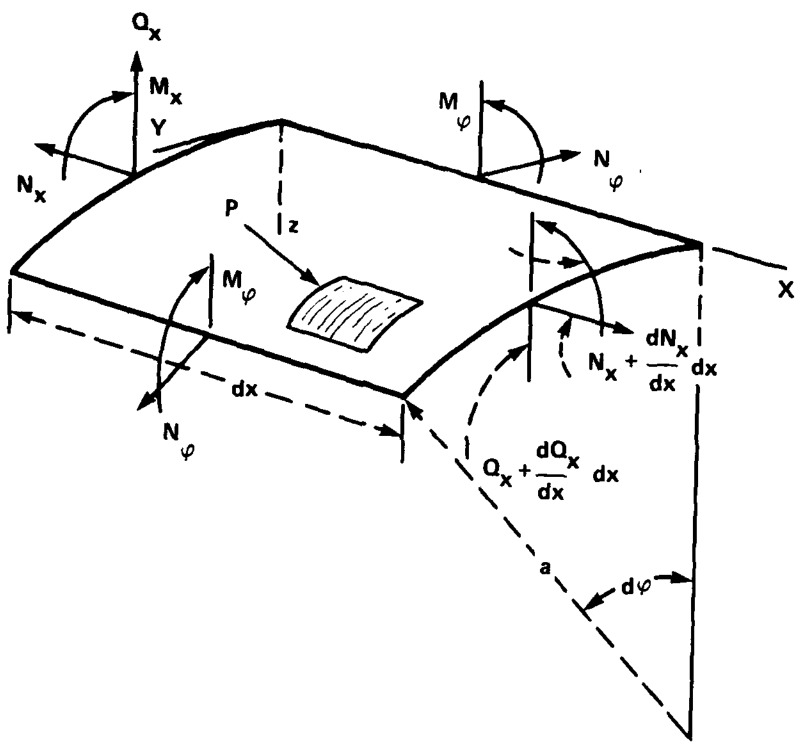
As already outlined in the "PROBLEM STATEMENT AND SOLUTION METHOD" section of the body of the report, the approach consists of solving the equilibrium equations involving two stress resultants, m_x and n_ϕ, subject to initial, boundary, and jump conditions. The load consists of a non-dimensional pressure load, exceeding the collapse load p_o (References A-13, A-14, and A-1) which was calculated by a limit analysis and a rigid, perfectly plastic material. In the process, we must assume (1) a kinematically admissible velocity profile, and (2) the portion of the applicable yield locus.

Based on these assumptions, we obtain a solution. We must verify its validity, i.e., none of the stress resultants must violate the yield surface. It is these conditions that impose certain constraints on our parameters, and allow certain cases to emerge.



NOTE: GLOBAL FRAME OF REFERENCE (X, Y, Z) IS THROUGH LEFT END, WHERE FIRST RING STIFFENER IS LOCATED.

FIGURE A-1. CIRCULAR CYLINDRICAL SHELL OF LENGTH 2L, DIAMETER 2a, SKIN THICKNESS h = 2H



NOTE: MASS OF SHELL IS M.

FIGURE A-2. CYLINDRICAL SHELL SUBJECTED TO RADIALLY INWARD PRESSURE P

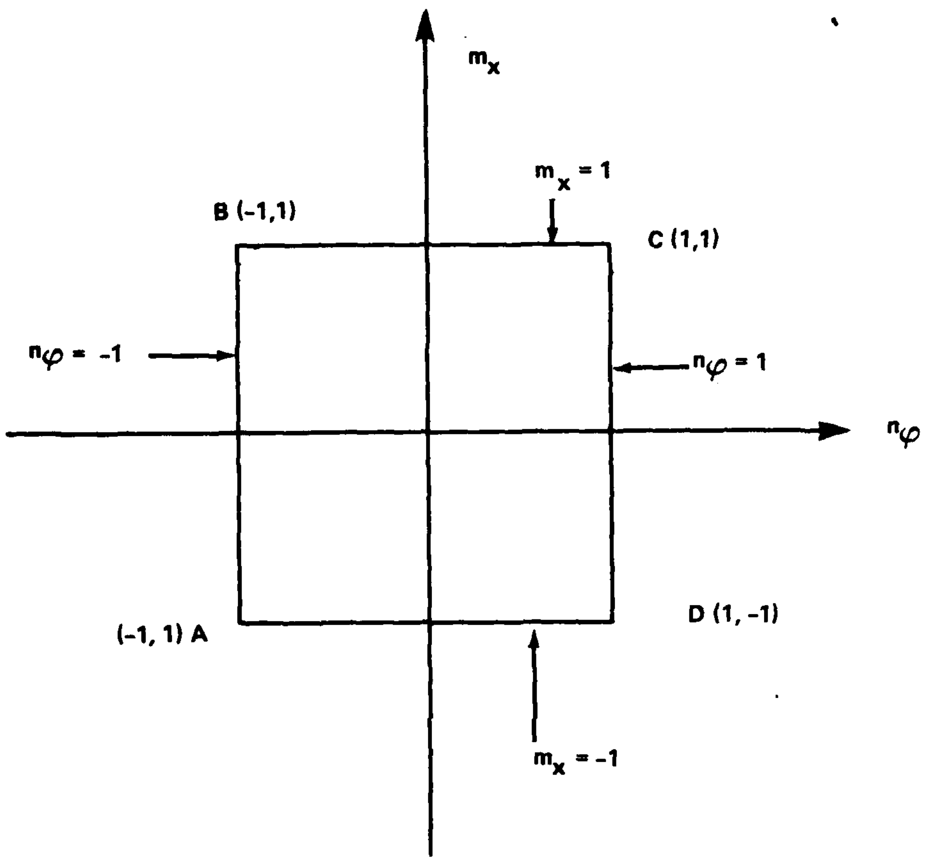


FIGURE A-3. SIMPLIFIED TRESCA YIELD SQUARE ABCD

TABLE A-1. REGIONS ON SIMPLIFIED TRESCA SQUARE (YIELD LOCUS)

Plastic Region	Stress Resultants: Components		Stress Resultants: Inequalities	Strain-Rate Vector: Components		Strain-Rate Vector: Equations	Strain-Rate Vector: Inequalities
	n_ϕ	m_x		$\dot{w}$	$\dot{w}''/2c^2$		
AD	*	-1	$-1 \leq n_\phi \leq 1$	0	λ	$\dot{w} = 0$	$\dot{w}'' \geq 0$
A	-1	-1	*	μ	λ	*	$\dot{w} \geq 0$, $\dot{w}'' \geq 0$
AB	-1	*	$-1 \leq m_x \leq 1$	μ	0	$\dot{w}'' = 0$	$\dot{w} \geq 0$
B	-1	+1	*	μ	$-\nu$	*	$\dot{w} \geq 0$, $\dot{w}'' \leq 0$

REFERENCES--APPENDIX A

A-1. Hodge, P. G. Jr., Plastic Analysis of Structures, McGraw-Hill, 1959, Chapters 11 and 13.

A-2. Hodge, P. G. Jr., The Rigid-Plastic Analysis of Symmetrically Loaded Closed Cylindrical Shells, Polytechnic Institute of Brooklyn, PIBAL Report No. 246, Mar 1954.

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A-7. Kachanov, L. M., Foundations of the Theory of Plasticity, North-Holland Publishing Co., Amsterdam, 1971.

A-8. Mendelson, A., Plasticity: Theory and Application, McMillan Co., New York, 1970. (For rigid-plastic model, see pp. 16-20; for limit theorems, see pp. 312-317.)

A-9. Hill, R., The Mathematical Theory of Plasticity, Oxford University Press, 1979, pp. 38-45.

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APPENDIX B

CASE A - SHORT SHELLS, LOW LOADING

$$\left(1 + \frac{2}{c^2} \leq p \leq 1 + \frac{6}{c^2}\right)$$

$$\left(0 < c^2 \leq 6\right)$$

SHORT SHELLS, LOW LOADING, $1 + \frac{2}{c^2} \leq p \leq 1 + \frac{6}{c^2}$ WITH $c^2 > 0$

In this section, we show the relevant stress resultants m_x and n_ϕ, displacement, velocity, and acceleration $(w, \dot{w}, \ddot{w})$, as well as displacement at rest $w(x, \tau_o)$, and time to come to rest τ_o in Tables B-1 and B-2. Figure B-1 displays the kinematic assumption with regard to the velocity field. The analysis proceeds as follows.

The non-dimensional collapse load p_o was derived by Hodge in Reference B-1. It is defined as

$$p_o = 1 + \frac{2}{c^2} \tag{B-1}$$

We must solve (Equation A-11 of Appendix A)

$$\frac{1}{2c^2} m_x'' + n_\phi + p - \ddot{w} = 0 \tag{B-2}$$

subject to initial, boundary, and jump conditions for

$$p > p_o > 0 \tag{B-3}$$

and $p = 0$.

Equation (B-2) must be solved in two time intervals.

1. For $0 \leq \tau \leq 1$, when $p \neq 0$ and

2. For $1 \leq \tau \leq \tau_o$, when $p = 0$,

where τ_o represents the time at which the shell comes to rest.

SOLUTION FOR TIME INTERVAL $0 \leq \tau \leq 1$, $(p \neq 0)$

Referring to Figure A-3, assume we are on the AB side of the Tresca square. Using Table A-1, this translates to the following requirements for the two resultants and velocity

$$n_\phi = -1 \tag{B-4}$$

$$-1 \leq m_x \leq 1 \tag{B-5}$$

$$\dot{w}'' = 0 \tag{B-6}$$

$$\dot{w} \geq 0 \tag{B-7}$$

Since $\dot{w}'' = 0$, the velocity profile can only be linear in x. Since we have two initial conditions (zero displacement and velocity at time $\tau = 0$) we assume a displacement profile of the form (linear in distance from ring stiffener)

$$w(x, \tau) = x\,[A_o + A_1\tau + A_2\tau^2] \qquad \text{(B-8)}$$

with $\dot{w} = x[A_1 + 2A_2\tau]$ (B-9)

$$\ddot{w} = 2A_2 x \qquad \text{(B-10)}$$

But

$$w(x,o) = 0 \qquad \text{(B-11)}$$

implies

$$A_o = 0 \qquad \text{(B-12)}$$

and $\dot{w}(x,o) = 0$ (B-13)

implies

$$A_1 = 0 \qquad \text{(B-14)}$$

Therefore,

$$w(x, \tau) = xA_2\tau^2 \qquad \text{(B-15)}$$

where A_2 is not a function of time or distance but a function of the load level, i.e.,

$$A_2 = A_2(p) \qquad \text{(B-16)}$$

Also, in view of $\dot{w} \geq 0$

$$A_2 \geq 0 \qquad \text{(B-17)}$$

We now replace $n_\phi = -1$ and $w = xA_2\tau^2$ in the equilibrium equation, and solve for m_x

$$m_x'' + 2c^2(p-1) - 4c^2xA_2 = 0 \qquad \text{(B-18)}$$

$$m_x'' = 4c^2A_2x - 2c^2(p-1) \qquad \text{(B-19)}$$

Integrating once

$$m_x' = 2c^2A_2x^2 - 2c^2(p-1)x + B_1 \qquad \text{(B-20)}$$

But at $x = 1$, $m_x'(1,\tau) = 0$, i.e.,

$$B_1 = - 2c^2A_2 + 2c^2(p-1) \qquad \text{(B-21)}$$

$$m_x' = 2c^2A_2(x^2-1) + 2c^2(p-1)(1-x) \qquad \text{(B-22)}$$

Integrating again,

$$m_x = 2c^2A_2(\frac{1}{3}x^3-x) + 2c^2(p-1)(x - \frac{1}{2}x^2) + D_1 =$$

$$\frac{2}{3}c^2A_2x^3 - c^2(p-1)x^2 + 2c^2(p-1-A_2)x + D_1 \tag{B-23}$$

At the end support ($x = 0$), $m_x = -1$

This defines

$$D_1 = -1 \tag{B-24}$$

Thus,

$$m_x = \frac{2}{3}c^2A_2x^3 - c^2(p-1)x^2 + 2c^2(p-1-A_2)x - 1 \tag{B-25}$$

At the middle ($x = 1$), $m_x = 1$.

This determines that

$$A_2 = \frac{3}{4}(p-1) - \frac{3}{2c^2} \tag{B-26}$$

and, finally, the bending moment profile assumes the form

$$m_x = \left[\frac{c^2}{2}(p-1) - 1\right]x^3 - c^2\,(p-1)x^2 + \left[\frac{c^2}{2}(p-1) + 3\right]x - 1 \tag{B-27}$$

and the displacement w becomes

$$w = \frac{3}{2c^2}\left[\frac{c^2}{2}(p-1) - 1\right]x\tau^2 \tag{B-28}$$

while the velocity $\dot{w}$ is

$$\dot{w} = \frac{3}{2c^2}\left[c^2(p-1) - 2\right]x\tau \tag{B-29}$$

Now we must require that $|m_x| \leq 1$ everywhere. We obtain the first and second derivatives of m_x. They are

$$\frac{\partial m_x}{\partial x} = 3\left[\frac{c^2}{2}(p-1) - 1\right]x^2 - 2c^2(p-1)x + \left\{\frac{c^2}{2}(p-1) + 3\right\} \tag{B-30}$$

$$\frac{\partial^2 m_x}{\partial x^2} = 6\left[\frac{c^2}{2}(p-1) - 1\right]x - 2c^2(p-1) \tag{B-31}$$

We observe that one of the roots of Equation (B-30) is $x_1 = 1$ (midpoint). The other root is

$$x_2 = \frac{\left[3 + \frac{c^2}{2}(p-1)\right]}{3\left[\frac{c^2}{2}(p-1) - 1\right]} = \frac{1}{3}\left[1 + \frac{4}{\left\{\frac{c^2}{2}(p-1) - 1\right\}}\right] \tag{B-32}$$

which is a positive quantity since

$p > p_0 = 1 + \frac{2}{c^2}$. As p approaches p_0 (from the right), $x_2 \rightarrow \infty$.

This means that if we choose this root (x_2) to exceed or equal 1, in the interval

$$0 \leq x \leq 1$$

Equation (B-30) will always be non-negative, i.e.,

$$\frac{1}{3}\left[1 + \frac{4}{\left[\frac{c^2}{2}(p-1) - 1\right]}\right] \geq 1$$

$$p \leq 1 + \frac{6}{c^2} \tag{B-33}$$

We need to examine later the sign of the second derivative, Equation (B-31), to determine whether the yield condition will be violated. We started the analysis based on the fact that for plastic deformations to take place

$$p > p_0 = 1 + \frac{2}{c^2} \tag{B-34}$$

This means that $\frac{c^2}{2}(p-1) - 1 \geq 0$ (B-35)

This also means that $p - 1 > 0$

At the left end (ring support), $x = 0$

$$\left.\frac{\partial^2 m_x}{\partial x^2}\right]_{x=0} = -\ 2c^2(p-1) < 0 \tag{B-36}$$

At the center ($x = 1$)

$$\left.\frac{\partial^2 m_x}{\partial x^2}\right]_{x=1} = 6\left[\frac{c^2}{2}(p-1)-1\right] - 2c^2(p-1) = c^2(p-1) - 6 \tag{B-37}$$

Since $\frac{c^2}{2}\ (p-1) - 1 > 0$ (B-38)

$$c^2(p-1) > 2 \tag{B-39}$$

Set $$c^2(p-1) = 2 + \varepsilon \tag{B-40}$$

where $\varepsilon > 0$

Then

$$\left.\frac{\partial^2 m_x}{\partial x^2}\right]_{x=1} = -4 + \varepsilon \tag{B-41}$$

At the two end points, the second derivative varies from $-4 - 2\varepsilon$ (which is always negative), to $-4 + \varepsilon$. The function m_x will have a maximum if its first derivative vanishes and its second derivative is negative or zero.

We must have

$$\frac{\partial^2 m_x}{\partial x^2} = 6\left[\frac{c^2}{2}(p-1)-1\right]x - 2c^2(p-1) \leq 0 \tag{B-42}$$

or $$x \leq \frac{c^2(p-1)}{3\left[\frac{c^2}{2}(p-1) - 1\right]} \tag{B-43}$$

for all $0 \leq x \leq 1$.

At $x = 1$ $$-4 + \varepsilon \leq 0 \tag{B-44}$$

or $$\varepsilon \leq 4 \tag{B-45}$$

or $$c^2 (p-1) = 2 + \varepsilon \leq 6 \tag{B-46}$$

or $$p \leq 1 + \frac{6}{c^2} \tag{B-47}$$

At the lowest possible load $p = p_0 = 1 + \frac{2}{c^2}$ the moment resultant m_x assumes the form

$$m_x^{lower} (x,\tau) = - 2x^2 + 4x - 1 \tag{B-48}$$

while at the upper possible load (without violating the yield surface)

$$p = 1 + \frac{6}{c^2} \tag{B-49}$$

$$m_x^{upper} (x,\tau) = 2x^3 - 6x^2 + 6x - 1 \tag{B-50}$$

Table B-1 summarizes the obtained results so far.

SOLUTION FOR TIME INTERVAL $1 \leq \tau \leq \tau_o$, $(p = 0)$

At time $\tau = 1$ the pressure ceases acting $(p = 0)$. There is an acting acceleration, velocity, and displacement. They must match with the solution in this range. Therefore, at $\tau = 1$

$$\ddot{w}\,(x,1) = \left\{\frac{3}{2}\,(p-1) - \frac{3}{c^2}\right\}_{p=0} x =$$

$$-\left[\frac{3}{2} + \frac{3}{c^2}\right]x = -\frac{3}{c^2}\left[\frac{c^2}{2} + 1\right]\,x \tag{B-51}$$

Furthermore, the moment resultant with p = 0 becomes

$$m_x(x,1) = -\left[\frac{c^2}{2} + 1\right]x^3 + c^2x^2 + \left[3 - \frac{c^2}{2}\right]x - 1 \tag{B-52}$$

Assuming the range AB on the yield surface,

$$\dot{w}'' = 0 \tag{B-53}$$

$$\dot{w} \geq 0 \tag{B-54}$$

to follow normality requirements. This means that the velocity profile must be linear in x ($\dot{w}'' = 0$).

Assume

$$w(x,\tau) = x[B_0 + B_1\tau + B_2\tau^2] \tag{B-55}$$

Then

$$\dot{w} = x[B_1 + 2B_2\tau] \tag{B-56}$$

$$\ddot{w} = 2\,B_2\,x \tag{B-57}$$

But from Equation (B-51)

$$B_2 = -\frac{3}{2c^2}\left[\frac{c^2}{2} + 1\right] \tag{B-58}$$

and $$\ddot{w} = -\frac{3}{c^2}\left[\frac{c^2}{2} + 1\right]x \tag{B-59}$$

Therefore,

$$\dot{w} = x\left[B_1 - \frac{3}{c^2}\left(\frac{c^2}{2} + 1\right)\tau\right] \tag{B-60}$$

Again, using normality ($\dot{w} \geq 0$)

$$B_1 \geq \frac{3}{c^2}\left[\frac{c^2}{2} + 1\right] \tag{B-61}$$

Furthermore,

$$p \geq 1 + \frac{c^2}{2} \tag{B-62}$$

and, hence,

$$B_1 \geq \frac{3}{2}\,p \tag{B-63}$$

Taking the lowest value of B_1 for which normality is satisfied,

$$\dot{w} = x\left[\frac{3}{2}p - \frac{3}{c^2}\left(\frac{c^2}{2} + 1\right)\tau\right] \qquad (B-64)$$

Therefore,

$$\dot{w} \geq 0 \qquad (B-65)$$

gives

$$\frac{3}{2}p - \frac{3}{2}\left(\frac{c^2}{2} + 1\right)\tau \geq 0 \qquad (B-66)$$

or

$$\tau \geq \frac{c^2}{c^2+2}p = \frac{p}{p_o} \qquad (B-67)$$

When time τ becomes τ_o, velocity is zero if the shell comes to rest.

Consequently, $\tau = \tau_o = \frac{p}{p_o}$

Also,

$$w = x\left[B_o + \frac{3}{2}p\tau - \frac{3}{2c^2}\left(\frac{c^2}{2} + 1\right)\tau^2\right] \qquad (B-68)$$

Since displacements must match for time $\tau = 1$

$$\frac{3}{2c^2}\left[\frac{c^2}{2}(p-1) - 1\right]x = x\left[B_o + \frac{3}{2}p - \frac{3}{2c^2}\left(\frac{c^2}{2} + 1\right)\right] \tag{B-69}$$

yielding

$$B_o = -\frac{3}{4}p \tag{B-70}$$

and

$$w = x\left[-\frac{3}{2c^2}\left(\frac{c^2}{2} + 1\right)\tau^2 + \frac{3}{2}p\tau - \frac{3}{4}p\right] \tag{B-71}$$

The displacement at time $\tau = \tau_o$ can be calculated by substituting the value of τ_o. Hence,

$$w\,(x,\tau_o) = \frac{px}{\left(1 + \frac{2}{c^2}\right)^2}\left[\frac{3}{2}\left(\frac{1}{2} + \frac{1}{c^2}\right)p - \frac{3}{4}\left(1 + \frac{2}{c^2}\right)^2\right] \tag{B-72}$$

We must solve again

$$\frac{1}{2c^2}m_x'' + n_\phi + p - \ddot{w} = 0 \tag{B-73}$$

with $p = 0$, $n_\phi = -1$ and $\ddot{w} = -\frac{3}{c^2}\left[\frac{c^2}{2} + 1\right]x$ (B-74)

$$\frac{1}{2c^2}m_x'' = -\frac{3}{c^2}\left[\frac{c^2}{2} + 1\right]x + 1 \tag{B-75}$$

$$m_x'' = -6\left[\frac{c^2}{2} + 1\right]x + 2c^2 \tag{B-76}$$

Integrating

$$m_x' = -3\left[\frac{c^2}{2} + 1\right]x^2 + 2c^2x + E_1 \tag{B-77}$$

However, this must match the solution of time $\tau = 1$, i.e.,

$$m_x'(x,1) = -3\left[\frac{c^2}{2} + 1\right]x^2 + 2c^2x + E_1 =$$

$$-3\left[\frac{c^2}{2} + 1\right]x^2 + 2c^2x + \left[3 - \frac{c^2}{2}\right] \tag{B-78}$$

Hence,

$$E_1 = 3 - \frac{c^2}{2} \tag{B-79}$$

Integrating again

$$m_x = -\left[\frac{c^2}{2} + 1\right]x^3 + c^2x^2 + \left[3 - \frac{c^2}{2}\right]x + E_2 \tag{B-80}$$

This must match the solution at time $\tau = 1$, i.e., $E_2 = -1$

Therefore,

$$m_x = -\left[\frac{c^2}{2} + 1\right]x^3 + c^2x^2 + \left[3 - \frac{c^2}{2}\right]x - 1 \qquad \text{(B-81)}$$

We observe again that

$$m_x\ (0,\tau) = -1 \qquad \text{(B-82)}$$

$$m_x\ (1,\tau) = 1 \qquad \text{(B-83)}$$

Again the yield condition must not be violated along any portion, which means that $-1 \leq m_x \leq 1$. The moment resultant is an increasing function from $x = 0$ to $x = 1$ and, for it to stay that way without exceeding the maximum absolute value of 1, the first derivative must be non-negative in the x range (0,1). Furthermore, near $x = 1$, the second derivative must be non-positive for an increasing function.

The first condition is

$$\frac{\partial m_x}{\partial x} = -3\left[\frac{c^2}{2} + 1\right]\left\{x^3 - \frac{2c^2}{3\left(\frac{c^2}{2} + 1\right)}x - \frac{\left(3 - \frac{c^2}{2}\right)}{3\left(1 + \frac{c^2}{2}\right)}\right\} \geq 0 \qquad \text{(B-84)}$$

Since

$$1 + \frac{c^2}{2} > 0 \qquad \text{(B-85)}$$

the quantity in curly brackets must be non-positive. For this to be so, the value of x must lie within the range of the two roots of the quadratic. Since by inspection one of them is $x_1 = 1$, the other must necessarily be

$$x_2 = -\frac{\left[3 - \frac{c^2}{2}\right]}{3\left[1 + \frac{c^2}{2}\right]} \tag{B-86}$$

Now x will always be in the non-negative range $0 \leq x \leq 1$, which means that if we make x_2 negative or zero, x will always satisfy

$$\frac{\partial m_x}{\partial x} \geq 0.$$

Therefore,

$$x_2 = -\frac{\left[3 - \frac{c^2}{2}\right]}{3\left[1 + \frac{c^2}{2}\right]} \leq 0 \tag{B-87}$$

gives $\quad c^2 \leq 6$ (B-88)

The second condition is always satisfied for points $x = 1 - \delta$ (when $1 > \delta \geq 0$, and δ is a small quantity)

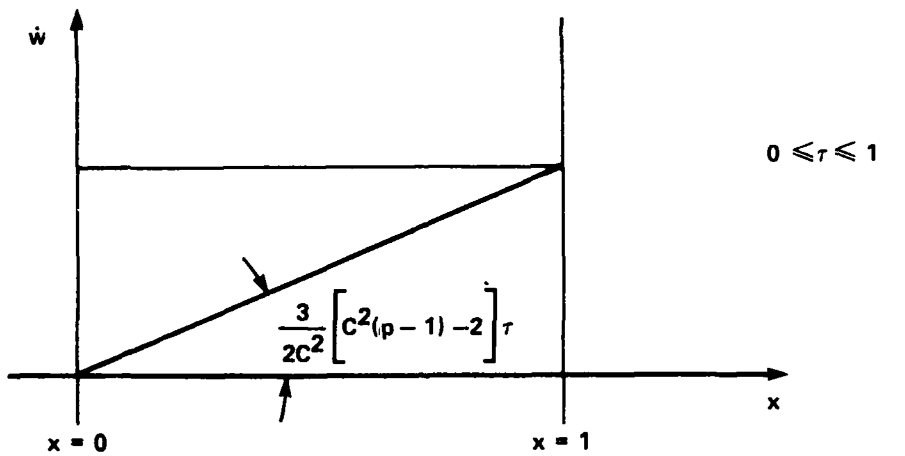
$$\left.\frac{\partial^2 m_x}{\partial x^2}\right|_{x=1-\delta} = -6\left[\frac{c^2}{2} + 1\right](1-\delta) + 2c^2 \leq 0 \tag{B-89}$$

or when

$$\delta \leq \frac{3 + \frac{1}{2} c^2}{3\left[1 + \frac{1}{2} c^2\right]} \qquad \text{(B-90)}$$

For small c^2 (which is the current case), δ must be less than 1 (as assumed).

Finally, Figure B-1 summarizes the velocity profiles for this case, and Table B-2 summarizes the used quantities.



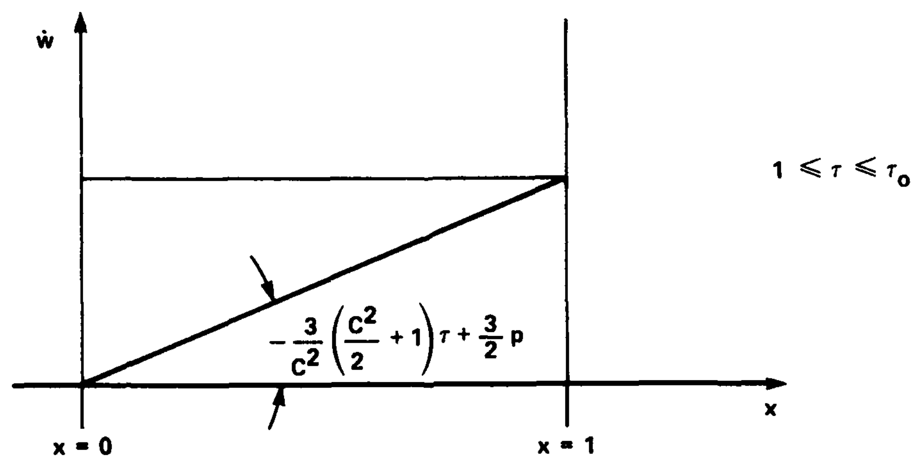


FIGURE B-1. VELOCITY PROFILES FOR SHORT SHELLS ($C^2 < 6$), LOW LOADING $\left(1+\frac{2}{C^2} \leqslant p \leqslant 1+\frac{1}{C^2}\right)$

TABLE B-1. SUMMARY, SHORT SHELLS, LOW LOADING, 1

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(0 < C^2 < 6\right)$	LOW $\left(1 + \frac{2}{C^2} \leqslant p \leqslant 1 + \frac{6}{C^2}\right)$
CONDITIONS	$0 \leqslant \tau \leqslant 1$	
MOMENT RESULTANT	$m_x(x,\tau) = \left[\frac{C^2}{2}(p-1) - 1\right]x^3 - C^2(p-1)x^2 + \left[\frac{C^2}{2}(p-1) + 3\right]x - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \frac{3}{2C^2}\left[\frac{C^2}{2}(p-1) - 1\right]x\tau^2$	
VELOCITY	$\dot{w}(x,\tau) = \frac{3}{2C^2}\left[C^2(p-1) - 2\right]x\tau$	
ACCELERATION	$\ddot{w}(x,\tau) = \frac{3}{2C^2}\left[C^2(p-1) - 2\right]x$	
TIME τ_o	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE B-2. SUMMARY, SHORT SHELLS, LOW LOADING, 2

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(0 < c^2 \leq 6\right)$	LOW $\left(1+\frac{2}{c^2} \leq p \leq 1+\frac{6}{c^2}\right)$
CONDITIONS	$1 \leq \tau \leq \frac{p}{p_o}$	
MOMENT RESULTANT	$m_x(x,\tau) = -\left[\frac{c^2}{2}+1\right]x^3 + c^2x^2 + \left[3-\frac{c^2}{2}\right]x - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \left[-\frac{3}{2c^2}\left(\frac{c^2}{2}+1\right)\tau^2 + \frac{3}{2}p\tau - \frac{3}{4}p\right]x$	
VELOCITY	$\dot{w}(x,\tau) = \left[-\frac{3}{c^2}\left(\frac{c^2}{2}+1\right)\tau + \frac{3}{2}p\right]x$	
ACCELERATION	$\ddot{w}(x,\tau) = -\frac{3}{c^2}\left(\frac{c^2}{2}+1\right)x$	
TIME τ_o	$\tau_o = \frac{p}{p_o} = \frac{c^2}{\left(c^2+2\right)}p$	
DISPLACEMENT AT REST	$w(x,\tau_o) = \frac{px}{\left(1+\frac{2}{c^2}\right)^2}\left[\frac{3}{2}\left(\frac{1}{2}+\frac{1}{c^2}\right)p - \frac{3}{4}\left(1+\frac{2}{c^2}\right)^2\right]$	

REFERENCE--APPENDIX B

B-1. Hodge, P. G. Jr., The Rigid-Plastic Analysis of Symmetrically Loaded Closed Cylindrical Shells, Polytechnic Institute of Brooklyn, PIBAL Report No. 246, Mar 1954.

APPENDIX C

CASE B - LONG SHELLS, LOW LOADING

$$\left(1 + \frac{2}{c^2} \leq p \leq 1 + \frac{6}{c^2}\right)$$

$$\left(c^2 > 6\right)$$

LONG SHELLS, LOW LOADING, $\left(1 + \frac{2}{c^2} \le p \le 1 + \frac{6}{c^2}\right)$ WITH $c^2 > 0$

Figure C-1 displays the assumed kinematics for the velocity distribution for times $0 \le \tau \le 1$ and $1 \le \tau \le \tau_o$. Tables C-1 through C-5 summarize the results we are about to obtain. Observe that there are two intervals of interest, depending on whether $o \le \tau \le 1$ and $1 \le \tau \le \tau_o$. Furthermore, the last interval is subdivided depending on whether the point of interest x is to the left $(o \le x \le u)$ or to the right $(u < x \le 1)$ of the travelling hinge.

When the pressure loading is restricted in the range

$$1+ \frac{2}{c^2} \le p \le 1 + \frac{6}{c^2} \tag{C-1}$$

the moment resultant m_x of the previous analysis [Equation (B-27)] for $o \le \tau \le 1$ does not violate the yield surface requirements $-1 \le m_x \le 1$. In the second range, however $(1 \le \tau \le \tau_o)$ [Equation (B-81)], when $c^2 > 6$, we obtain, setting $c^2 = 6 + \varepsilon$ and $\varepsilon > 0$ and ε being small

$$m_x(x,\tau) = \left[4 + \frac{\varepsilon}{2}\right]x^3 + (6+\varepsilon)x^2 - \frac{\varepsilon}{2}\,x - 1 \tag{C-2}$$

It is quite obvious that if $x = \frac{1}{3}\,\frac{\varepsilon}{(8+\varepsilon)}$ (i.e., near a point where $x \to + 0$), the yield condition is violated since

$$m_x\left(\frac{1}{3}\,\frac{\varepsilon}{(8+\varepsilon)},\tau\right) = -\,\frac{[4\varepsilon^3+90\varepsilon^2+864\varepsilon+3456]}{54[8+\varepsilon]^2} \leq -1 \tag{C-3}$$

Table C-1 summarizes the result for $0 \leq \tau \leq 1$. We just observed that when $c^2 > 6$, the yield condition is violated near $x = +0$.

Consider the possibility of two regions on the yield surface:

1. Region AD for point x such that $0 \leq x \leq u$

2. Region AB for point x such that $u < x \leq 1$

where u is the point along the shell's length where the regions change. It actually represents the position where a hinge circle develops.

REGION AD

The hinge in question is travelling because it changes position with time. Its initial location must be at the ring support at time $\tau = 1$. Along AD, the strain rate vector must satisfy

1. $\dot{w} = 0$ (no velocity) (C-4)

2. $\dot{w}'' = 0$ (C-5)

3. $m_x = -1$ (C-6)

and equilibrium

$$\frac{1}{2c^2} m_x'' + n_\phi + p - \ddot{w} = 0 \tag{C-7}$$

(See Figure A-3.)

Since $\dot{w} = \ddot{w} = 0$ and at time $\tau = 1$ the pressure has ceased acting ($p = 0$)

$$n_\phi = 0 \tag{C-8}$$

and

$$w = c_1 \tag{C-9}$$

The deformation is rigid plastic for $0 < x < u$. This means that all points $0 < x < u$ have no motion for $1 \leq \tau \leq \tau_0$, i.e., they do not deform further than the deformation they acquired at time $\tau = 1$, which is

$$w(x,1) = \frac{3}{2c^2}\left[\frac{c^2}{2}(p-1)-1\right]x \tag{C-10}$$

REGION AB FOR $u < x \leq 1$

The conditions the strain rate must satisfy are

$$\dot{w}'' = 0 \tag{C-11}$$

$$\ddot{w} \geq 0 \tag{C-12}$$

$$n_\phi = -1 \tag{C-13}$$

where

$$-1 \leq m_x \leq 1$$

while equilibrium gives:

$$\frac{1}{2c^2} m_x'' + n_\phi + p - \ddot{w} = 0 \qquad \text{(C-14)}$$

Since the second derivative of the velocity with respect to position must vanish it must be, at the most, linear in distance. Assume, therefore,

$$\dot{w} = \dot{A}\, x + \dot{B} \qquad \text{(C-15)}$$

and

$$\ddot{w} = \ddot{A}\, x + \ddot{B} \qquad \text{(C-16)}$$

Replacing Equation (C-16) in the equilibrium Equation (C-14) and setting p = 0, we have

$$\frac{1}{2c^2} m_x'' - 1 - (\ddot{A}\, x + \ddot{B}) = 0 \qquad \text{(C-17)}$$

Integrating once

$$m_x' = 2c^2 x + 2c^2 \left(\frac{1}{3}\ddot{A}\, x^2 + \ddot{B}x\right) + c_1 \qquad \text{(C-18)}$$

Integrating again

$$m_x = c^2x^2 + c^2\left[\frac{1}{3}\ddot{A}\ x^3 + \ddot{B}\ x^2\right] + c_1x + c_2 \tag{C-19}$$

At the two ends x = u and x = 1 the moment resultant becomes -1 and 1, respectively.

$$m_x = c^2x^2 + c^2\left[\frac{1}{3}\ddot{A}\ x^3 + \ddot{B}\ x^2\right] + c_1x + c_2 \tag{C-20}$$

At $x = u$ $\quad m_x = -1$. Therefore,

$$c^2\left[\frac{1}{3}\ddot{A}u^3 + (1+\ddot{B})u^2\right] + c_1u + c_2 = -1 \tag{C-21}$$

Continuity of shearing force $\left(m_x' = 0\right)$ at $x = u$ yields

$$m_x'(u) = c^2\left[\ddot{A}\ u^2 + 2(1 + \ddot{B})\ u\right] + c_1 = 0 \tag{C-22}$$

At the middle ($x = 1$) we must have

$$m_x(1) = 1 \tag{C-23}$$

$$m_x'(1) = 0 \tag{C-24}$$

i.e.

$$c^2[\ddot{A} + 2(1 + \ddot{B})] + c_1 = 0 \tag{C-25}$$

$$c^2\left[\frac{1}{3}\ddot{A} + (1 + \ddot{B})\right] + c_1 + c_2 = 1 \tag{C-26}$$

or

$$c_1 = - c^2[\ddot{A} + 2(1 + \ddot{B})] \tag{C-27}$$

and

$$c_2 = 1 + c^2 \left[\frac{2}{3}\ddot{A} + (1 + \ddot{B})\right] \tag{C-28}$$

Substituting Equation (C-27) for c_1 in Equation (C-22) we get

$$\ddot{A} = -2(1 + \ddot{B}) \frac{(u-1)}{(u^2-1)} = -2(1 + \ddot{B})\frac{1}{(1+u)} \quad \text{for } u \neq 1 \tag{C-29}$$

Substituting c_1 and c_2 in Equations (C-21) and (C-22) we get

$$\frac{c^2(u-1)(u^2 + u-2)}{3}\ddot{A} + c^2(1 + \ddot{B})(u^2 - 2u + 1) = -2 \tag{C-30}$$

However, from Equation (C-29)

$$1 + \ddot{B} = - \frac{1}{2}(u + 1)\ddot{A} \tag{C-31}$$

and after algebra we obtain

$$\ddot{A} = - \frac{12}{c^2} \frac{1}{(1-u)^3} \tag{C-32}$$

$$1 + \ddot{B} = \frac{6}{c^2} \frac{(u+1)}{(1-u)^3} \tag{C-33}$$

This, in turn, defines c_1 and c_2 from Equations (C-27) and (C-28). They are

$$c_1 = - \frac{12}{(1-u)^3} u \tag{C-34}$$

$$c_2 = 1 - \frac{2(1-3u)}{(1-u)^3} \tag{C-35}$$

The two solutions [Equation (C-15)] in the intervals $0 < x < u$ and $u < x < 1$ must have continuous velocities at $x = u$, i.e.,

$$\dot{w} = \dot{A}(\tau)\, u + \dot{B}(\tau) = 0 \tag{C-36}$$

Differentiating Equation (C-36) again with respect to time we obtain

$$\ddot{A}\, u + \dot{A}\, \dot{u} + \ddot{B} = 0 \tag{C-37}$$

or

$$\dot{A}\, \dot{u} = -(\ddot{A}\, u + \ddot{B}) = \frac{6}{c^2} \left[\frac{c^2}{6} - \frac{1}{(1-u)^2} \right] \tag{C-38}$$

or

$$\dot{A} = \frac{1}{\dot{u}} - \frac{6}{c^2} \frac{1}{\dot{u}(1-u)^2} \tag{C-39}$$

Differentiating Equation (C-39) with respect to time we get

$$\ddot{A} = - \frac{\ddot{u}}{\dot{u}^2} + \frac{6}{c^2} \frac{-\left(\dot{u}(1-u)^2\right)'}{[\dot{u}(1-u)^2]^2} = - \frac{\ddot{u}}{\dot{u}^2} + \frac{6}{c^2} \frac{\ddot{u}}{\dot{u}^2(1-u)^2} - \frac{12}{c^2} \frac{1}{(1-u)^3} \tag{C-40}$$

We already obtained [Equation (C-32)] that

$$\ddot{A} = -\frac{12}{c^2}\frac{1}{(1-u)^3} \tag{C-41}$$

Equating the two, we obtain the following differential equation

$$\frac{\ddot{u}}{\dot{u}^2}\left[\frac{6}{c^2}\frac{1}{(1-u)^2} - 1\right] = 0 \tag{C-42}$$

Since

$$\dot{A} = \frac{6}{u^2}\left[\frac{c^2}{6} - \frac{1}{(1-u)^2}\right]\frac{1}{\dot{u}} \neq 0 \tag{C-43}$$

we must necessarily have

$$\frac{\ddot{u}}{\dot{u}} = 0 \tag{C-44}$$

or

$$\ddot{u} = 0 \tag{C-45}$$

and

$$\dot{u} \neq 0 \tag{C-46}$$

This means that the location of the hinge circle is linear in time, τ

$$u = E_1\,\tau + E_2 \tag{C-47}$$

$$\dot{u} = E_1 \tag{C-48}$$

$$\ddot{u} = 0 \tag{C-49}$$

Velocities must match at time $\tau = 1$. [See Equations (C-36) and (B-29).]

Therefore,

$$\dot{w}(1) = \left\{ \frac{3}{c^2} \left[\frac{c^2}{2}(p - 1) - 1 \right] \tau x \right\}_{\tau=1} = \dot{A}(1)x + \dot{B}(1) \tag{C-50}$$

i.e., we get two initial conditions for $\dot{A}$, $\dot{B}$

$$\frac{3}{c^2} \left[\frac{c^2}{2}(p-1) - 1 \right] = \dot{A}(1) \tag{C-51}$$

$$0 = \dot{B}(1) \tag{C-52}$$

From Equation (C-36)

$$\dot{B} = - \dot{A}u \tag{C-53}$$

and from Equation (C-43)

$$\dot{A} = \frac{6}{c^2} \left[\frac{c^2}{6} - \frac{1}{(1-u)^2} \right] \frac{1}{\dot{u}} \tag{C-54}$$

Therefore, by Equation (C-50)

$$\left\{ \frac{6}{c^2} \left[\frac{c^2}{6} - \frac{1}{(1-u)^2} \right] \frac{1}{\dot{u}} \right\}_{\tau=1} = \frac{3}{c^2} \left[\frac{c^2}{2}(p-1) - 1 \right] \tag{C-55}$$

with

$$u(\tau) = E_1\tau + E_2 \tag{C-56}$$

$$\dot{u}(\tau) = E_1 \tag{C-57}$$

and $$u(1) = 0 \tag{C-58}$$

Equations (C-58) and (C-56) imply $E_2 = - E_1$, i.e.,

$$u(\tau) = E_1(\tau-1) \tag{C-59}$$

$$\dot{u} = E_1 \tag{C-60}$$

Replacing Equations (C-59) and (C-60) at time $\tau = 1$ in Equation (C-55) we obtain

$$\frac{3}{c^2}\left[\frac{c^2}{2}(p-1) - 1\right] = \frac{6}{c^2\dot{u}(1)}\left[\frac{c^2}{6} - 1\right] \tag{C-61}$$

or

$$\dot{u}(1) = E_1 = \frac{\left(c^2 - 6\right)}{3\left[\frac{c^2}{2}(p-1) - 1\right]} \tag{C-62}$$

$$u = \frac{\left(c^2 - 6\right)}{3\left[\frac{c^2}{2}(p-1) - 1\right]}(\tau-1) \tag{C-63}$$

Therefore, by Equation (C-53) the velocity field for $u < x < 1$ is

$$\dot{w} = \dot{A}(\tau)x + \dot{B}(\tau) = \dot{A}(\tau)(x - u) \tag{C-64}$$

However, using Equations (C-54), (C-62), and (C-63)

$$\dot{w}(\tau) = \frac{6}{c^2}\frac{1}{\dot{u}}\left[\frac{c^2}{6} - \frac{1}{(1-u)^2}\right](x - u) \tag{C-65}$$

where $\dot{u}$ and u are given by Equations (C-62) and (C-63), respectively.

Replacing the values of c_1, c_2, $\ddot{A}$, and $\ddot{B}$ in Equation (C-20) we obtain

$$m_x(x,\tau) = c^2\left[-\frac{4}{c^2}\frac{1}{(1-u)^3}x^3 + \frac{6}{c^2}\frac{(1+u)}{(1-u)^3}x^2\right] -$$

$$\frac{12}{(1-u)^3}u\,x + 1 - \frac{2(1-3u)}{(1-u)^3} =$$

$$1 + \frac{1}{(1-u)^3}\left[-4x^3 + 6(1+u)x^2 - 12ux - 2(1-3u)\right] \tag{C-66}$$

Also

$$n_\phi = -1 \tag{C-67}$$

$$1 \le \tau \le \tau_o \tag{C-68}$$

Equation (C-66) yields

$$m_x(1,\tau) = 1 \tag{C-69}$$

$$m_x(u,\tau) = 1 + 2\frac{(u-1)^3}{(1-u)^3} = -1 \tag{C-70}$$

and

$$\dot{w} \geq 0 \tag{C-71}$$

$$\dot{w}'' = 0 \tag{C-72}$$

Equation (C-71) implies

$$\frac{6}{c^2}\frac{1}{\dot{u}}\left[\frac{c^2}{6} - \frac{1}{(1-u)^2}\right](x-u) \geq 0 \tag{C-73}$$

with u given by Equation (C-63) and, hence, $\dot{u}$ given by Equation (C-62)

$$\dot{u} = \frac{c^2-6}{3\left[\frac{c^2}{2}(p-1)-1\right]} \tag{C-74}$$

Since $\dot{u} \geq 0$ $\left(c^2 \geq 6\right)$, Equation (C-73) can be rewritten as

$$\frac{6}{c^2}\frac{\left(c^2-6\right)}{3\left[\frac{c^2}{2}(p-1)-1\right]}\left[\frac{c^2(1-u)^2-6}{6(1-u)^2}\right](x-u) \geq 0 \tag{C-75}$$

This implies that

$$c^2(1-u)^2 - 6 \geq 0 \tag{C-76}$$

or

$$|1-u| \geq \frac{\sqrt{6}}{c} \tag{C-77}$$

Since $1 \geq u \geq 0$, Equation (B-77) reduces to

$$u \geq 1 - \frac{\sqrt{6}}{c} \tag{C-78}$$

At time $\tau = \tau_o$ (≥ 1) the motion stops

$$\dot{w}(x, \tau_o) = 0 \tag{C-79}$$

i.e.,

$$\dot{A}\left(\tau_o\right)\left[x-u\left(\tau_o\right)\right] = 0 \tag{C-80}$$

or since in general $x \neq u\left(\tau_o\right)$

$$\dot{A}\left(\tau_o\right) = 0 \tag{C-81}$$

or by Equation (C-54)

$$\left(\frac{6}{c^2}\right) \frac{1}{\dot{u}\left(\tau_o\right)} \left[\frac{c^2}{6} - \frac{1}{\left[1-u\left(\tau_o\right)\right]^2}\right] = 0 \tag{C-82}$$

Since in general, $u\left(\tau_o\right) = \dfrac{\left(c^2-6\right)}{3\left[\dfrac{c^2}{2}(p-1)-1\right]} > 0$

by Equation (C-82)

$$\frac{c^2}{6} - \frac{1}{\left[1-u\left(\tau_o\right)\right]^2} = 0 \tag{C-83}$$

yielding

$$u\left(\tau_o\right) = 1 - \frac{\sqrt{6}}{c} \tag{C-84}$$

Replacing $\tau = \tau_o$ in Equation (C-63) and equating it to Equation (C-84), we solve for τ_o from

$$1 - \frac{\sqrt{6}}{c} = \frac{1}{3} \frac{\left(c^2-6\right)}{\left[\dfrac{c^2}{2}(p-1)-1\right]} \left(\tau_o-1\right) \tag{C-85}$$

$$\tau_o = 1 + \frac{3\left[\dfrac{c^2}{2}(p-1)-1\right]}{c\ (c+\sqrt{6})} \tag{C-86}$$

or if we set

$$\alpha = \frac{\left(c^2-6\right)}{3\left[\dfrac{c^2}{2}(p-1)-1\right]} \tag{C-87}$$

$$\tau_o = 1 + \frac{1}{\alpha}\left(1 - \frac{\sqrt{6}}{c}\right) \tag{C-88}$$

and Equation (C-65) for the velocity field can be rewritten as

$$\dot{w} = \frac{6}{c^2}\left(\frac{1}{\alpha}\right)\left[\frac{c^2}{6} - \frac{1}{(1-u)^2}\right](x-u) = \frac{6}{c^2}\left(\frac{1}{\alpha}\right)\left[\frac{c^2}{6}\left\{x - \alpha(\tau-1)\right\} - \right.$$

$$\left.\left\{\frac{1}{[(1+\alpha)-\alpha\tau]} + \frac{(x-1)}{[(1+\alpha)-\alpha\tau]^2}\right\}\right] \tag{C-89}$$

Also observe that

$$1 - \alpha(\tau-1) = \frac{\sqrt{6}}{c} \tag{C-90}$$

and

$$\frac{(\tau-1)}{[1-\alpha(\tau-1)]} = \frac{1}{\alpha}\left(\frac{c}{\sqrt{6}} - 1\right) \tag{C-91}$$

and

$$\frac{\alpha}{2}\left(\tau^2-1\right) = \frac{1}{2\alpha}\left[\left\{\left(1-\frac{\sqrt{6}}{c}\right) + \alpha\right\}^2 - \alpha^2\right] \tag{C-92}$$

Integrating Equation (C-89) and applying the boundary condition that [Equation (B-28)]

$$w(x,1) = \frac{3}{2c^2}\left[\frac{c^2}{2}(p-1)-1\right]x \tag{C-93}$$

we have

$$w(x,\tau) = \frac{3}{2c^2}\left[\frac{c^2}{2}(p-1)-1\right]x + \int_{\tau=1}^{\tau} \dot{w}\, d\tau \tag{C-94}$$

Observing that (see Tables C-2 and C-3)

$$\frac{d}{d\tau}\left\{-\frac{1}{\alpha}\log_e|(1+\alpha)-\alpha\tau| + \frac{(x-1)}{\alpha}\,\frac{1}{[1+\alpha-\alpha\tau]}\right\} = \frac{1}{[1+\alpha-\alpha\tau]} + \frac{(x-1)}{[1+\alpha-\alpha\tau]^2} \tag{C-95}$$

Equation (C-94) becomes

$$w(x,\tau) = \frac{3}{2c^2}\left[\frac{c^2}{2}(p-1)-1\right]x + \frac{6}{c^2}\left(\frac{1}{\alpha}\right)\int_{\tau=1}^{\tau}\left\{\frac{c^2}{6}[x-\alpha(\tau-1)] -\right.$$

$$\left.\left[\frac{1}{[(1+\alpha)-\alpha\tau]} + \frac{(x-1)}{[(1+\alpha)-\alpha\tau]^2}\right]\right\} d\tau = \frac{3}{2c^2}\left[\frac{c^2}{2}(p-1)-1\right]x +$$

$$\frac{6}{c^2}\left(\frac{1}{\alpha}\right)\left\{\frac{c^2}{6}\left[(x+\alpha)(\tau-1) - \frac{\alpha}{2}\left(\tau^2-1\right)\right] +\right.$$

$$\left.\left[\frac{1}{\alpha}\log_e|1-\alpha(\tau-1)| + \frac{(1-x)}{\alpha}\cdot\frac{\alpha(\tau-1)}{(1+\alpha-\alpha\tau)}\right]\right\} \tag{C-96}$$

or using Equations (C-90), (C-91), and (C-92) the displacement at time τ_o [i.e., when $w(x,\tau_o) = 0$] becomes:

$$w(x,\tau_o) = \frac{3}{2c^2}\left[\frac{c^2}{2}(p-1)-1\right]x +$$

$$\frac{1}{\alpha^2}\left\{(x+\alpha)\left(1 - \frac{\sqrt{6}}{c}\right) - \frac{1}{2}\left[\left(\left(1 - \frac{\sqrt{6}}{c}\right) + \alpha\right)^2 - \alpha^2\right]\right\}$$

$$\frac{1}{\alpha^2}\left\{\frac{6}{c^2}\left[\frac{1}{2}\log_e 6 - \log_e|c|\right] + (1-x)\left(\frac{\sqrt{6}}{c} - \frac{6}{c^2}\right)\right\} \qquad \text{(C-97)}$$

Observe that $x = u$ only at time $\tau = 1$. At that time, $x = u = 0$ and Equations (C-96) and (C-10) give the same answer.

Equation (C-96) can also be written in terms of u as follows:

$$w(x,\tau) = \frac{3}{2c^2}\left[\frac{c^2}{2}(p-1)-1\right]x + \frac{6}{c^2}\left(\frac{1}{\alpha}\right)\left[\frac{c^2}{6}\left\{\frac{(x+\alpha)u}{\alpha} - \frac{(\tau+1)}{2}u\right\} +\right.$$

$$\left.\frac{1}{\alpha}\left\{\log_e|1-u| + \frac{(1-x)u}{(1-u)}\right\}\right] \qquad \text{(C-98)}$$

The acceleration $\ddot{w}$ is given by replacing Equations (C-32) and (C-33) in (C-16)

$$\ddot{w} = \ddot{A}x + \ddot{B} = \frac{6}{c^2}\frac{1}{(1-u)^3}[(1+u)-2x] - 1 \qquad \text{(C-99)}$$

If we further substitute u from Equation (C-63) we get

$$\Psi(x,\tau) = \frac{6}{c^2} \frac{1}{[1+\alpha-\alpha\tau]^3}[1+\alpha(\tau-1)-2x] - 1 \tag{C-100}$$

Figure C-1 summarizes the velocity profiles employed for long shells ($c^2 > 6$) and low loading $\left(1 + \frac{2}{c^2} \leq p \leq 1 + \frac{6}{c^2}\right)$

Tables C-4 and C-5 are a summary of all the derived quantities for the same case.

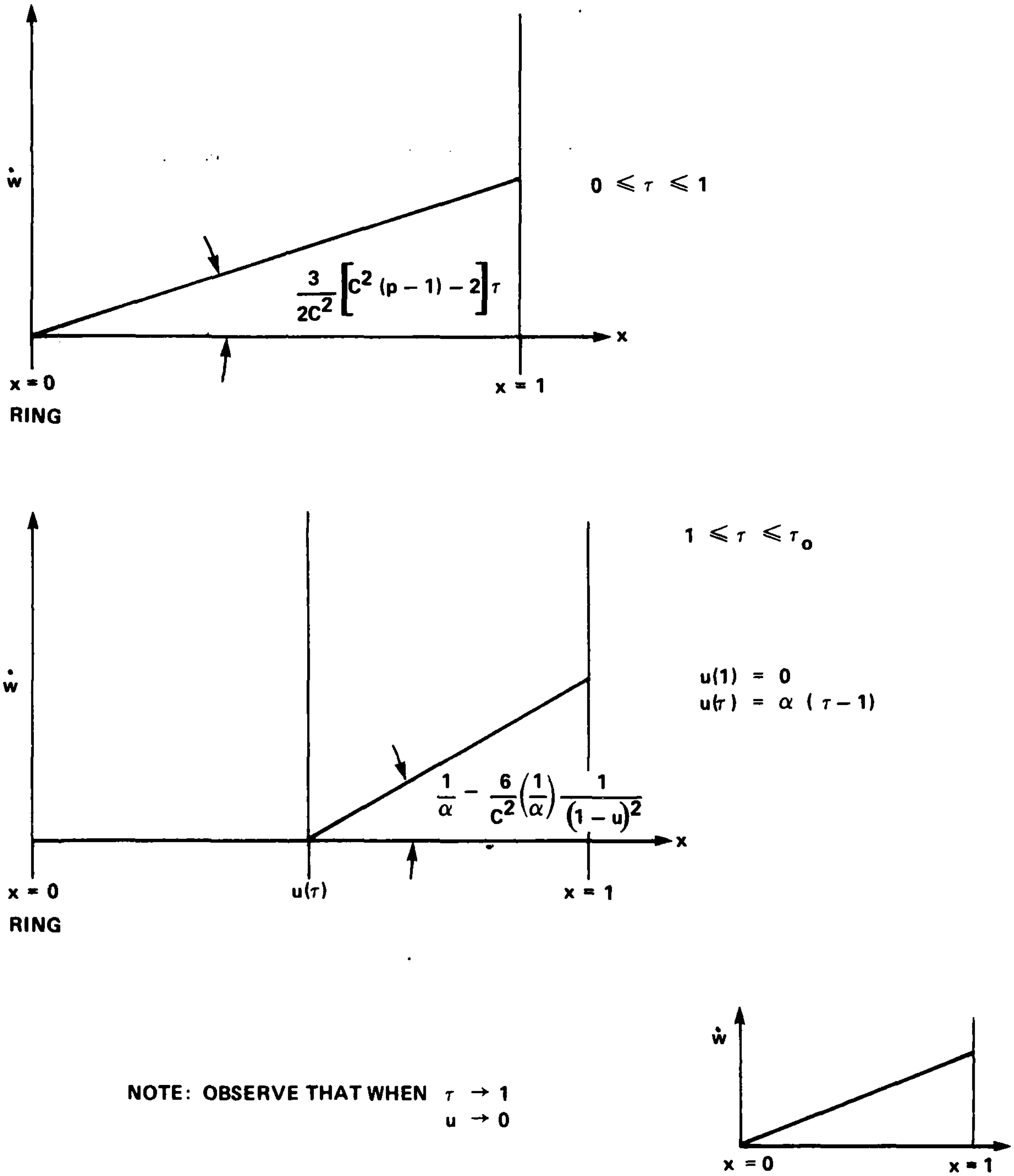


FIGURE C-1. VELOCITY PROFILES FOR LONG SHELLS ($C^2 > 6$) AND LOW LOADING $\left(1 + \frac{2}{C^2} \leqslant p \leqslant 1 + \frac{6}{C^2}\right)$

TABLE C-1. SUMMARY, LONG SHELLS, LOW LOADING, 1

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $\left(C^2 > 6\right)$	LOW $\left(1+\frac{2}{C^2} \leq p \leq 1+\frac{6}{C^2}\right)$
CONDITIONS	$0 \leq \tau \leq 1$	
MOMENT RESULTANT	$m_x(x,\tau) = \left[\frac{C^2}{2}(p-1)-1\right]x^3 - C^2(p-1)x^2 + \left[\frac{C^2}{2}(p-1)+3\right]x - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \frac{3}{2C^2}\left[\frac{C^2}{2}(p-1)-1\right]x\tau^2$	
VELOCITY	$\dot{w}(x,\tau) = \frac{3}{2C^2}\left[C^2(p-1)-2\right]x\tau$	
ACCELERATION	$\ddot{w}(x,\tau) = \frac{3}{2C^2}\left[C^2(p-1)-2\right]x$	
TIME τ_0	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE C-2. SUMMARY OF THE OBTAINED CONSTANTS

PARAMETER	EXPRESSION
C_1	$-\frac{12}{(1-u)^3}u$
C_2	$1-\frac{2(1-3u)}{(1-u)^3}$
$\ddot{A}(\tau)$	$-\frac{12}{C^2}\frac{1}{(1-u)^3}$
$\dot{A}(\tau)$	$\frac{6}{C^2}\frac{1}{\dot{u}}\left[\frac{C^2}{6}-\frac{1}{(1-u)^2}\right]$
$\ddot{B}(\tau)$	$\frac{6}{C^2}\frac{(1+u)}{(1-u)^3}-1$
$\dot{B}(\tau)$	$-\frac{6}{C^2}\frac{u}{\dot{u}}\left[\frac{C^2}{6}-\frac{1}{(1-u)^2}\right]$
$u(\tau)$	$\frac{(C^2-6)}{3\left[\frac{C^2}{2}(p-1)-1\right]}(\tau-1)$
$\dot{u}(\tau)$	$\frac{(C^2-6)}{3\left[\frac{C^2}{2}(p-1)-1\right]}$

TABLE C-3. FUNCTIONS AND THEIR DERIVATIVES

(τ IS THE ONLY TIME VARIABLE. x IS TAKEN AS A CONSTANT)

FUNCTION $F(\tau)$	DERIVATIVE WITH RESPECT TO TIME VARIABLE τ $\dot{F}(\tau)$
$-\frac{1}{\alpha}\log_e \lvert(1+\alpha)-\alpha\tau\rvert+\frac{(x-1)}{\alpha}\frac{1}{(1+\alpha-\alpha\tau)}$	$\frac{1}{(1+\alpha-\alpha\tau)}+\frac{(x-1)}{(1+\alpha-\alpha\tau)^2}$
$\frac{(2\tau-p)}{4}\sqrt{\tau(p-\tau)}+\frac{p^2}{8}\cos^{-1}\left\{1-\frac{2}{p}\tau\right\}$ for $\lvert 1-\frac{2}{p}\tau\rvert \leqslant 1$	$\sqrt{\tau(p-\tau)}$

TABLE C-4. SUMMARY, LONG SHELLS, LOW LOADING, 2

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $\left(C^2 > 6\right)$	LOW $\left(1 + \frac{2}{C^2} \leqslant p \leqslant 1 + \frac{6}{C^2}\right)$
CONDITIONS	$0 \leqslant x < u$ $u = \alpha(\tau - 1)$, $\dot{u} = \alpha$	$\alpha = \dfrac{(C^2 - 6)}{3\left[\frac{C^2}{2}(p-1) - 1\right]}, 1 \leqslant \tau \leqslant \tau_o$ $\left(u \leqslant 1 - \frac{\sqrt{6}}{C}\right)$
MOMENT RESULTANT	$m_x = -1$	
MEMBRANE RESULTANT	$n_\varphi = 0$	
DISPLACEMENT	$w = \dfrac{3}{2C^2}\left[\frac{C^2}{2}(p-1) - 1\right] x$	
VELOCITY	$\dot{w} = 0$	
ACCELERATION	$\ddot{w} = 0$	
TIME τ_o	$\tau_o = 1 + \dfrac{1}{\alpha}\left(1 - \frac{\sqrt{6}}{C}\right)$	
DISPLACEMENT AT REST	$w(x, \tau_o) = \dfrac{3}{2C^2}\left[\frac{C^2}{2}(p-1) - 1\right] x$ FOR $0 \leqslant x < \alpha(\tau_o - 1)$	

TABLE C-5. SUMMARY, LONG SHELLS, LOW LOADING, 3

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $\left(C^2 > 6\right)$	LOW $\left(1 + \frac{2}{C^2} \leqslant p \leqslant 1 + \frac{6}{C^2}\right)$
CONDITIONS	$\alpha = \dfrac{\left(C^2 - 6\right)}{3\left[\frac{C^2}{2}(p-1) - 1\right]}$, $1 \leqslant \tau \leqslant \tau_o$ $u \leqslant \tau \leqslant 1$ $u \leqslant 1 - \frac{\sqrt{6}}{C}$	
MOMENT RESULTANT	$m_x(x,\tau) = 1 + \dfrac{1}{(1-u)^3}\left[-4x^3 + 6(1+u)x^2 - 12ux - 2(1-3u)\right]$ where $u = \alpha(\tau - 1)$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \dfrac{3}{2C^2}\left[\dfrac{C^2}{2}(p-1) - 1\right]x + \dfrac{6}{C^2}\left(\dfrac{1}{\alpha}\right)\left\{\dfrac{C^2}{6}\left[(x+\alpha)(\tau-1) - \dfrac{\alpha}{2}(\tau^2 - 1)\right] + \left[\dfrac{1}{\alpha}\log_e\lvert 1 - \alpha(\tau-1)\rvert + \dfrac{(1-x)}{\alpha}\cdot\dfrac{\alpha(\tau-1)}{(1+\alpha-\alpha\tau)}\right]\right\}$	
VELOCITY	$\dot{w}(x,\tau) = \dfrac{6}{C^2}\left(\dfrac{1}{\alpha}\right)\left[\dfrac{C^2}{6} - \dfrac{1}{(1-u)^2}\right](x-u)$	
ACCELERATION	$\ddot{w}(x,\tau) = \left(\dfrac{6}{C^2}\right)\dfrac{[1 + \alpha(\tau-1) - 2x]}{[1+\alpha-\alpha\tau]^3} - 1$	
TIME τ_o	$\tau_o = 1 + \dfrac{3\left[\frac{C^2}{2}(p-1) - 1\right]}{C\left(C + \sqrt{6}\right)} = 1 + \dfrac{1}{\alpha}\left(1 - \dfrac{\sqrt{6}}{C}\right)$	
DISPLACEMENT AT REST	$w(x,\tau_o) = \dfrac{3}{2C^2}\left[\dfrac{C^2}{2}(p-1) - 1\right]x + \dfrac{1}{\alpha^2}\left\{(x+\alpha)\left(1 - \dfrac{\sqrt{6}}{C}\right) - \dfrac{1}{2}\left[\left\{1 - \dfrac{\sqrt{6}}{C} + \alpha\right\}^2 - \alpha^2\right]\right\} + \dfrac{1}{\alpha^2}\left\{\dfrac{6}{C^2}\left[\dfrac{1}{2}\log_e\lvert C\rvert\right] + (1-x)\left(\dfrac{\sqrt{6}}{C} - \dfrac{6}{C^2}\right)\right\}$	

APPENDIX D

CASE C - SHORT SHELLS, HIGH LOADING

$$\left(p > 1 + \frac{6}{c^2}\right)$$

$$\left(0 < c^2 < 6\right)$$

SHORT SHELLS, HIGH LOADING, $\left(p > 1 + \frac{6}{c^2}\right)$, $0 < c^2 \leq 6$

Figure D-1 gives the assumed velocity profile, which will be justified in this section.

Tables D-1, D-2, D-4, D-5, D-6, D-7, and D-8 summarize the solution in all intervals. Tables D-4 and D-5 summarize the solution in the interval $1 \leq \tau \leq \tau'$, while Tables D-6, D-7, and D-8 for $\tau' \leq \tau \leq \tau_o$.

First, we observe that in view of Equation (B-42) or (B-47), the solution for short shells (portion AB on the yield surface) in the range $0 \leq \tau \leq 1$ cannot be used, as it predicts a moment resultant m_x outside the (-1,1) range of the Tresca square.

From the point of view of time intervals, it turns out that we must consider three intervals:

1. STAGE 1 ($0 \leq \tau \leq 1$) for times during which the excess pressure p is acting.

2. STAGE 2 ($1 \leq \tau \leq \tau'$) for times during which the pressure load has been removed ($p = 0$), but motion continues ($\dot{w}(\tau') \neq 0$), τ' will be defined there.

3. STAGE 3 ($\tau' \leq \tau \leq \tau_o$) for times during which the pressure p has been removed ($p = 0$) and the shell comes to rest ($\dot{w}(\tau_o) = 0$).

STAGE 1

We consider two regions as a possible assumed profile on the yield surface: (1) for points such that $0 \leq x < u_o$ on AB (Figure A-3) and (2) for points such that $u_o < x \leq 1$ at corner A (Figure A-3). Other combinations were considered and eliminated by Hodge.[D-1,D-2] They violate the $-1 \leq m_x \leq 1$ requirement.

1. First interval $0 \leq x \leq u_o$ (AB). The requirements on the strain rate vector are such that

$$\dot{w}'' = 0 \tag{D-1}$$

$$\dot{w} \geq 0 \tag{D-2}$$

$$n_\phi = -1 \tag{D-3}$$

Equations (D-1) and (D-2) together yield that $\dot{w}$ is linear in distance x, such that

$$\dot{w} = \dot{A}x \tag{D-4}$$

with $$\dot{A}(\tau) \geq 0 \tag{D-5}$$

Therefore,

$$\ddot{w} = \ddot{A}(\tau)x \tag{D-6}$$

Equilibrium requires

$$\frac{1}{2c^2} m_x'' + n_\phi + p - \ddot{w} = 0 \tag{D-7}$$

or using Equations (D-3) and (D-6)

$$m_x'' + 2c^2 (-1+p) - 2c^2 \ddot{A}x = 0 \tag{D-8}$$

Integrating twice Equation (D-8)

$$m_x' = 2c^2 \left[\frac{1}{2} \ddot{A} x^2 - (p-1)x\right] + C_1 \tag{D-9}$$

$$m_x = \frac{1}{3} c^2 \ddot{A} x^3 - c^2 (p-1)x^2 + C_1 x + D_1 \tag{D-10}$$

At $x = 0$, $m_x(o) = -1$, yields

$$D_1 = -1 \tag{D-11}$$

Hence,

$$m_x = \frac{1}{3} c^2 \ddot{A} x^3 - c^2(p-1)x^2 + C_1 x - 1 \tag{D-12}$$

To proceed further we need to consider the second region $u_o < x \leq 1$ and then examine both solutions at $x = u_o$.

2. Second interval $u_o < x \leq 1$ (point B on yield surface). The strain rate requirements are

$$\dot{w} \geq 0 \tag{D-13}$$

$$\dot{w}'' \leq 0 \tag{D-14}$$

and

$$n_\phi = -1 \tag{D-15}$$

$$m_x = 1 \tag{D-16}$$

together with

$$\frac{1}{2c^2} m_x'' + n_\phi + p - \ddot{w} = 0 \tag{C-17}$$

Substituting Equations (D-15) and (D-16) in Equation (D-17)

$$\ddot{w} = n_\phi + p = p - 1 \tag{D-18}$$

and integrating once

$$\dot{w} = (p-1)\tau + D_1 \tag{D-19}$$

Since, however, $\dot{w}(x,o) = 0$, $D_1 = 0$.

Therefore,

$$\dot{w} = (p-1)\tau \tag{D-20}$$

Integrating again

$$w = \frac{1}{2}(p-1)\tau^2 + E_1 \tag{D-21}$$

where E_1 will be determined by matching displacements of $x = u_o$ from the first interval $(0 \leq x < u_o)$.

Since in the second interval $(u_o < x \leq 1)$ $m_x = 1$, it must match m_x of the first interval $(0 \leq x < u_o)$, i.e, by Equation (D-12)

$$\frac{1}{3} c^2 \ddot{A} u_o^3 - c^2(p-1) u_o^2 + C_1 u_o - 1 = 1 \tag{D-22}$$

Furthermore, the shearing forces must match, i.e,

$$m_x'(u_o,\tau) = \left[c^2 \ddot{A}x^2 - 2c^2(p-1)x + C_1\right]_{x = u_o} =$$

$$c^2 \ddot{A} u_o^2 - 2c^2(p-1)u_o + C_1 = 0 \tag{D-23}$$

Since the hinge may be stationary, there may be a discontinuity in slope. However, displacements, velocities, and accelerations must be equal at $x = u_o$, i.e., from Equations (D-6) and (D-18)

$$\ddot{A}(\tau)u_o = p-1 \qquad \text{(D-24)}$$

Therefore,

$$\ddot{A}(\tau) = \frac{p-1}{u_o} \qquad \text{(D-25)}$$

and by Equations (D-25) and (D-6)

$$\ddot{w}(\tau) = \frac{(p-1)}{u_o}\, x \qquad \text{(D-26)}$$

represents the acceleration in the first interval ($0 \leq x < u_o$). By Equations (D-4) and (D-19)

$$\dot{A}\, u_o = (p-1)\tau \qquad \text{(D-27)}$$

i.e.,

$$\dot{A}(\tau) = \frac{(p-1)}{u_o}\, \tau \qquad \text{(D-28)}$$

This means that u_o is a constant and in $0 \leq x \leq u_o$

$$\dot{w}(x,\tau) = \dot{A}x = \frac{(p-1)}{u_o} \tau x \tag{D-29}$$

and

$$w(x,\tau) = \frac{1}{2} \frac{(p-1)}{u_o} \tau^2 x \tag{D-30}$$

because $w(x,o) = 0$.

At this point we consider that at $x = u_o$ the displacements must agree. By Equations (D-30) and (D-21), $E_1 = 0$. Hence, in the second interval $(u_o < x \leq 1)$

$$w = \frac{1}{2}(p-1)\tau^2 \tag{D-31}$$

Furthermore, replacing Equation (D-25) in Equation (D-12) we get

$$m_x = 2 \left[\left(\frac{x}{u_o}\right)^3 - 3\left(\frac{x}{u_o}\right)^2 + 3\left(\frac{x}{u_o}\right) - \frac{1}{2}\right] \tag{D-32}$$

for $(0 \leq x \leq u_o)$

Substitute $\ddot{A}$ from Equation (D-25) in Equation (D-23) to obtain

$$C_1 = c^2 (p-1) u_o \tag{D-33}$$

Substitute the values of C_1 from Equation (D-33) and $\ddot{A}$ from Equation (D-25) in Equation (D-22) to determine u_o

$$u_o^2 = \frac{6}{c^2(p-1)} \tag{D-34}$$

In Equation (D-32) the moment resultant can be written in an alternate form

$$m_x\,(x,\tau) = 2\left(\frac{x}{u_o} - 1\right)^3 + 1 \tag{D-35}$$

Therefore, Tables D-1 and D-2 display the solutions for times $0 \leq \tau \leq 1$ in the two intervals, $o \leq x \leq u_o$ and $u_o < x \leq 1$. Figure D-1 gives information on the velocity profiles for times $o \leq \tau \leq 1$ as well as $1 \leq \tau \leq \tau'$, which will be analyzed next.

Note once more that since

$$\frac{\partial m_x}{\partial x} = \frac{6}{u_o}\left(\frac{x}{u_o} - 1\right)^2 \geq 0 \tag{D-36}$$

$$\frac{\partial^2 m_x}{\partial x^2} = \frac{12}{u_o^2}\left(\frac{x}{u_o} - 1\right) \leq 0 \tag{D-37}$$

The function m_x is uniformly increasing in the interval $0 \leq x \leq u_o$. This is to be considered later (long shells, high loading).

STAGE 2

Pressure load p is removed, $p = 0$, where τ' is a time that will be defined later. We must solve the equilibrium equation

$$\frac{1}{2c^2} m_x' + p + n_\phi - \ddot{w} = 0 \tag{D-38}$$

with $p = 0$, with initial conditions as to velocity and displacement given by Stage 1. We must also satisfy the conditions imposed by the normality requirements of the strain rate vector to the yield surface. We assume that the same two ranges on the yield surface apply, i.e.,

1. First interval. For $0 \leq x \leq u$, we are along AB.

$$n_\phi = -1 \tag{D-39}$$

$$-1 \leq m_x \leq 1 \tag{D-40}$$

$$\dot{w} \geq 0 \tag{D-41}$$

$$\dot{w}'' = 0 \tag{D-42}$$

and

$$\dot{w}(1) = \frac{(p-1)}{u_o} x \tag{D-43}$$

$$u_o^2 = \frac{6}{c^2(p-1)} \tag{D-44}$$

$$\ddot{w}(1) = \frac{(p-1)}{u_o} \tag{D-45}$$

$$w(x,1) = \frac{1}{2}\,\frac{(p-1)}{u_o}\,x \tag{D-46}$$

$$m_x(x,1) = 2\left(\frac{x}{u_o} - 1\right)^3 + 1 \tag{D-47}$$

2. Second interval. For $u < x \leq 1$ (at corner B of yield surface; see Figure A-3).

$$n_\phi = -1 \tag{D-48}$$

$$m_x = -1 \tag{D-49}$$

$$\dot{w} \geq 0 \tag{D-50}$$

$$\dot{w}'' \leq 0 \tag{D-51}$$

and

$$\dot{w}(1) = p - 1 \tag{D-52}$$

$$u_o^2 = \frac{(p-1)}{u_o} \tag{D-53}$$

$$\ddot{w}(1) = p - 1 \tag{D-54}$$

$$w(x,1) = \frac{1}{2}(p-1) \tag{D-55}$$

1. <u>First Interval $0 \leq x \leq u$</u>

$(1 \leq \tau \leq \tau')$

Take

$$\dot{w}(\tau) = (p-\tau)\frac{x}{u(\tau)} \tag{D-56}$$

Then by differentiating with time τ

$$\ddot{w} = \frac{d}{d\tau}\left[(p-\tau)\frac{x}{u}\right] = -\frac{x}{u}\left[1 + \frac{(p-\tau)\dot{u}}{u}\right] \tag{D-57}$$

Solving

$$\frac{1}{2c^2}\, m''_x - 1 - \ddot{w} = 0 \tag{D-58}$$

$$\frac{1}{2c^2}\, m''_x = 1 - \frac{x}{u}\left[1 + \frac{(p-\tau)}{u}\, u\right] \tag{D-58}$$

$$m''_x = 2c^2\left[1 - \left(\frac{u}{u^2} + \frac{(p-\tau)\dot{u}}{u^2}\right)x\right] \tag{D-59}$$

$$m_x' = 2c^2\left[x - \frac{1}{2}\left(\frac{u}{u^2} + \frac{(p-\tau)\dot{u}}{u^2}\right)x^2\right] + C_1 \tag{D-60}$$

At $x = u$, $m_x' = 0$, i.e.,

$$C_1 = -2c^2\left[\frac{u}{2} - \frac{1}{2}(p-\tau)\dot{u}\right] \tag{D-61}$$

$$m_x' = 2c^2\left[\left(x - \frac{u}{2}\right) + \frac{1}{2}\left\{(p-\tau)\dot{u} - \left(\frac{1}{u} + \frac{(p-\tau)\dot{u}}{u^2}\right)x^2\right\}\right] \tag{D-62}$$

$$m_x = 2c^2\left[\left(\frac{x^2}{2} - \frac{u}{2}x\right) + \frac{1}{2}\left\{(p-\tau)\dot{u}x - \frac{1}{3}\left(\frac{1}{u} + \frac{(p-\tau)\dot{u}}{u^2}\right)x^3\right\}\right] + D_1 \tag{D-63}$$

But

$$m_x(o) = -1 \quad \text{gives} \quad D_1 = -1$$

Hence,

$$m_x = c^2x^2 - c^2ux + c^2(p-\tau)\dot{u}x - \frac{c^2}{3}\left(\frac{1}{u} + \frac{(p-\tau)\dot{u}}{u^2}\right)x^3 - 1 \tag{D-64}$$

Also at $x = u$, $m_x = 1$ yields the following ordinary differential equation

$$\frac{2}{3}c^2(p-\tau)u\dot{u} = 2 + \frac{1}{3}c^2u^2 \tag{D-65}$$

or

$$\frac{2}{3}\,\frac{(p-\tau)udu}{d\tau} = \frac{2}{c^2} + \frac{1}{3}\,u^2 \tag{D-66}$$

$$\frac{d\tau}{p-\tau} = \frac{2udu}{\left(u^2 + \frac{6}{c^2}\right)} \tag{D-67}$$

Hence integrating we get

$$-\log_e|p-\tau| = \log_e\left|u^2 + \frac{6}{c^2}\right| + C_1 \tag{D-68}$$

But at $\tau = 1$

$$u^2 = u_o^2 = \frac{6}{c^2(p-1)} \tag{D-69}$$

Therefore,

$$C_1 = \log_e\left|\frac{c^2}{6p}\right| \tag{D-70}$$

and, finally,

$$u^2 = \frac{6\tau}{c^2(p-\tau)} \tag{D-71}$$

Using the differential Equation (D-65) in Equation (D-57), the acceleration can be rewritten as

$$\ddot{w} = -\frac{x}{u}\left[1 + \frac{(p-\tau)\dot{u}}{u}\right] = -\left(\frac{3}{2}\right)\left(\frac{x}{u}\right)\left[1 + \frac{2}{u^2c^2}\right] \tag{D-72}$$

However, using Equation (D-71) we get

$$\ddot{w} = -\frac{3}{2}\left(\frac{x}{u}\right)\left[1 + \frac{2}{\left\{\frac{6\tau}{(p-\tau)}\right\}}\right] = -\frac{1}{2}\left(\frac{x}{u}\right)\left[2 + \frac{p}{\tau}\right] \tag{D-73}$$

$$\ddot{w} = -\frac{1}{2}\left(\frac{x}{u}\right)\left[2 + \frac{p}{\tau}\right] \tag{D-74}$$

which makes acceleration linear with distance for points $0 \leq x \leq u$. We also note that it is a "deceleration" (negative sign) since the load was removed while in Stage 1 ($0 \leq \tau \leq 1$) [Equations (D-6) and (D-25)].

$$\ddot{w}(\tau) = \frac{(p-1)}{u_o}x \tag{D-75}$$

represents a positive acceleration.

Using Equation (D-65) in the moment resultant Equation (D-64), we obtain

$$m_x(x,\tau) = -\frac{1}{2}(2 + c^2u^2)\left(\frac{x}{u}\right)^3 + c^2u^2\left(\frac{x}{u}\right)^2 + \left(3 - \frac{c^2u^2}{2}\right)\left(\frac{x}{u}\right) - 1 \tag{D-76}$$

which becomes 1 at $x = u$.

Therefore, for $0 \leq x \leq u_o$

$$m'_x = -\frac{3}{2}\left(\frac{2}{u^3} + \frac{c^2}{u}\right)x^2 + 2c^2x + \left(\frac{3}{u} - \frac{c^2u}{2}\right) \tag{D-77}$$

and

$$m''_x = -3\left[\frac{2}{u^3} + \frac{c^2}{u}\right]x + 2c^2 \tag{D-78}$$

One of the roots ρ_1 of $m'_x = o$ is $\rho_1 = u$. Thus the other one, ρ_2, is

$$\rho_2 = \frac{-2\left[\frac{3}{u^2} - \frac{c^2}{2}\right]}{3\left[\frac{2}{u^3} + \frac{c^2}{u}\right]} = \frac{\left(c^2 - \frac{6}{u^2}\right)}{3\left[\frac{2}{u^3} + \frac{c^2}{u}\right]} \tag{D-79}$$

We can rewrite m'_x as

$$m'_x = -\frac{3}{2}\left[\frac{2}{u^3} + \frac{c^2}{u}\right]\left\{x^2 - \frac{4c^2}{3\left[\frac{2}{u^3} + \frac{c^2}{u}\right]}x^2 - \frac{\left(\frac{3}{u} - \frac{c^2u}{2}\right)}{3\left[\frac{2}{u} - \frac{c^2u}{2}\right]}\right\} =$$

$$-\frac{3}{2}\left[\frac{2}{u^3} + \frac{c^2}{u}\right][x-\rho_1][x-\rho_2] \tag{D-80}$$

Since $m_x'(x) \geq 0$ and $u,c > 0$ this means that

$$(x-\rho_1)(x-\rho_2) \leq 0 \tag{D-81}$$

or x must lie within the range of the roots ρ_1 and ρ_2. This means that since $0 \leq x \leq u$, the second root ρ_2 cannot be positive, because then there will be an interval $(0,\rho_2)$ for which a value of x will yield $(x-\rho_1)(x-\rho_2)$ non-negative.

Thus,

$$\rho_2 = \frac{\left(c^2 - \frac{6}{u^2}\right)}{3\left[\frac{2}{u^3} + \frac{c^2}{u}\right]} \leq 0 \tag{D-82}$$

means

$$u^2 \leq \frac{6}{c^2} \tag{D-83}$$

At $x = 0$, where the supports are, m_x' becomes

$$m_x'(o) = \frac{3}{u} - \frac{c^2 u}{2} \tag{D-84}$$

For the function not to become less than -1, $m_x'(o)$ must be non-negative, i.e.,

$$m_x'(o) = \frac{3}{u} - \frac{c^2 u}{2} \geq 0 \tag{D-85}$$

or

$$u^2 \leq \frac{6}{c^2} \tag{D-86}$$

Since the shell is short, $c^2 < 6$ means that

$$u^2 \leq \frac{6}{c^2} > 1 \tag{D-87}$$

However, at the beginning and time $\tau = 1$ the initial value that u assumes is u_o given by Equation (D-71), i.e.,

$$u_o^2 = \frac{6}{c^2(p-1)} \tag{D-88}$$

which cannot exceed 1, i.e.,

$$u_o^2 = \frac{6}{c^2(p-1)} \leq 1 \quad \text{or} \quad p \geq 1 + \frac{6}{c^2}$$

As time goes on, we know that u increases, as per

$$u^2 = u_o^2\left[1 + \frac{(\tau-1)}{(p-\tau)}\right]\tau \tag{D-89}$$

until at $\tau \leq \frac{c^2}{c^2+6}p$, u becomes 1.

In summary, therefore, $c^2 < 6$ and $p > 1 + \frac{6}{c^2}$ guarantees that $m_x'(o) \geq o$. At the other end, $x = u, {}_x m'(u) = o$, and it is the second derivative that dictates that the moment resultant is less than or equal to 1, since

$$m_x''(u) = -\frac{6}{u^2} - c^2 = -\left(\frac{6}{u^2} + c^2\right) < 0 \qquad \text{(D-90)}$$

Returning to the velocity equation we have, noting that

$$u^2 = \frac{6\tau}{c^2(p-\tau)} \qquad \text{(D-91)}$$

$$\tau = \frac{c^2u^2}{\left(6+c^2u^2\right)}p \qquad \text{(D-92)}$$

$$(p-\tau)^2 = \frac{36\tau^2}{c^4u^4} = \frac{36}{\left(6+c^2u^2\right)^2}\, p^2 \qquad \text{(D-93)}$$

and

$$6 + c^2u^2 = \frac{6p}{p-\tau} \qquad \text{(D-94)}$$

we proceed as follows

$$\dot{w} = x\,\frac{(p-\tau)}{u} \tag{D-95}$$

or

$$dw = x\frac{(p-\tau)}{u}\,d\tau \tag{D-96}$$

However, by the differential equation that u satisfies Equation (D-66)

$$\frac{d\tau}{u} = \frac{2}{3}\,\frac{(p-\tau)}{\left(\frac{2}{c^2} + \frac{1}{3}u^2\right)} \tag{D-97}$$

or

$$dw = \frac{2}{3}x(p-\tau)^2\,\frac{du}{\left(\frac{2}{c^2} + \frac{1}{3}u^2\right)} \tag{D-98}$$

Writing $(p-\tau)^2$ in terms of u only, we get

$$dw = \frac{72x}{c^4}\,p^2\,\frac{du}{\left(u^2 + \frac{6}{c^2}\right)^3} \tag{D-99}$$

By Table D-3, however,

$$w = \int_{u=u_o}^{u} \frac{72x}{c^4}\, p^2 \frac{du}{\left(u^2 + \frac{6}{c^2}\right)^3} =$$

$$xp^2 \left[\frac{6u}{c^2} \left\{ \frac{1}{2\left(u^2 + \frac{6}{c^2}\right)^2} + \frac{c^2}{8\left(u^2 + \frac{6}{c^2}\right)} \right\} + \right.$$

$$\left. \frac{3c}{4\sqrt{6}} \tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) \right]_{u=u_o}^{u=u} + f(x) \qquad \text{(D-100)}$$

or

$$w(x,u) = f(x) + xp^2 \left[\frac{6u}{c^2} \left\{ \frac{1}{2\left(u^2 + \frac{6}{c^2}\right)^2} + \frac{c^2}{8\left(u^2 + \frac{6}{c^2}\right)} \right\} - \right.$$

$$\frac{6}{c^2} \frac{\sqrt{6}}{c\sqrt{p-1}} \left\{ \frac{c^4 (p-1)^2}{72p^2} + \frac{c^4 (p-1)}{48p} \right\} +$$

$$\left. \frac{3c}{4\sqrt{6}} \left(\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right) \right) \right] \qquad \text{(D-101)}$$

and we have replaced $u_o^2 + \frac{6}{c^2}$ by

$$u_o^2 + \frac{6}{c^2} = \frac{6}{c^2}\left(\frac{p}{p-1}\right) \tag{D-102}$$

or

$$w(x,u) = f(x) + xp^2 \left[\frac{6}{c^2} \left\{ \frac{uc^4}{2\left(6+c^2u^2\right)^2} - \frac{\sqrt{6}}{c\sqrt{p-1}} \frac{c^4(p-1)^2}{72p^2} \right\} + \right.$$

$$\frac{6}{c^2} \quad \frac{uc^4}{8\left(6+c^2u^2\right)} - \frac{\sqrt{6}}{c\sqrt{p-1}} \frac{c^4(p-1)}{48p} +$$

$$\left. \frac{3c}{4\sqrt{6}}\left(\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right)\right] \tag{D-103}$$

or

$$w(x,u) = f(x) + xp^2 \left[\left\{ \frac{3c^2u}{\left(6+c^2u^2\right)^2} - \frac{c}{2\sqrt{6}} \quad \frac{(p-1)^{3/2}}{p^2} \right\} + \right.$$

$$\left\{ \frac{3c^2u}{4\left(6+c^2u^2\right)} - \frac{\sqrt{6}}{8}\, c\, \frac{\sqrt{p-1}}{p} \right\} +$$

$$\left. \frac{\sqrt{6}}{8}\, c \left(\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right)\right] \tag{D-104}$$

This can also be rewritten in terms of time τ as

$$w(x,\tau) = f(x) + x\left[\frac{c}{2\sqrt{6}}\sqrt{\tau}(p-\tau)^{3/2} + \frac{3c}{4\sqrt{6}}p\sqrt{\tau(p-\tau)} + \frac{3c}{4\sqrt{6}}p^2\tan^{-1}\left(\sqrt{\frac{\tau}{p-\tau}}\right)\right]_{\tau=1}^{\tau=\tau} =$$

$$f(x) + x\left[\frac{c}{2\sqrt{6}}\left\{\sqrt{\tau}(p-\tau)^{3/2} - (p-1)^{3/2}\right\} + \frac{3c}{4\sqrt{6}}p\left\{\sqrt{\tau(p-\tau)} - \sqrt{(p-1)}\right\} + \frac{3c}{4\sqrt{6}}p^2\left\{\tan^{-1}\sqrt{\frac{\tau}{p-\tau}} - \tan^{-1}\sqrt{\frac{1}{p-1}}\right\}\right] \quad \text{(D-105)}$$

Using the expression for w in terms of (x,τ), we observe that at one end at time $\tau = 1$ for all $x \leq u_o$, the expression in brackets vanishes, while f(x) assumes the form

$$f(x) = \frac{1}{2}(p-1)\left(\frac{x}{u_o}\right) \quad \text{for} \quad x \leq u_o \quad \text{(D-106)}$$

By Equation (D-34)

$$f(x) = \frac{3}{c^2u_o^2}\left(\frac{x}{u_o}\right) \quad \text{for} \quad x \leq u_o \quad \text{(D-107)}$$

To obtain the solution for $u_o \leq x \leq u$ we proceed as follows. At this point we need to use the value of w of x = u, which has not been obtained. Using this quantity [(see second interval for $u < x \leq 1$) (Equation (D-130)] we must have

$$w(x,\tau)\Big]_{u=x} = \frac{1}{2}\left[2p\tau - \tau^2 - p\right] = -\frac{1}{2}\left[p - 2p\tau + \tau^2\right] \tag{D-108}$$

Rewriting all τ's in terms of u [Equation (D-92)].

$$w(x,\tau)_{u=x} = -\frac{p}{2}\left[1 - 2p\frac{c^2u^2}{\left(6+c^2u^2\right)} + p\frac{c^4u^4}{\left(6+c^2u^2\right)^2}\right] \tag{D-109}$$

and using Equations (D-103), (D-108), and (D-109) we obtain Equation (D-110).

$$f(u) + up^2\left[\frac{6}{c^2}\left\{\frac{uc^4}{2\left(6+c^2u^2\right)^2} - \frac{\sqrt{6}}{c\sqrt{p-1}}\frac{c^4(p-1)^2}{72p^2}\right\} + \right.$$

$$\frac{6}{c^2}\left\{\frac{uc^4}{8\left(6+c^2u^2\right)} - \frac{\sqrt{6}}{c\sqrt{p-1}}\frac{c^4(p-1)}{48p}\right\} +$$

$$\left.\frac{\sqrt{6c}}{8}\left(\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{p-1}}\right)\right)\right] =$$

$$-\frac{p}{2}\left[1 - 2p\frac{c^2u^2}{\left(6+c^2u^2\right)} + p\frac{c^4u^4}{\left(6+c^2u^2\right)^2}\right] \tag{D-110}$$

Hence,

$$f(u) = -\frac{p}{2}\left[1 - 2p\frac{c^2u^2}{(6+c^2u^2)} + p\frac{c^4u^4}{(6+c^2u^2)^2}\right] -$$

$$\frac{\sqrt{6}c}{8}\,p^2u\left\{\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\} -$$

$$up^2\left[\frac{6}{c^2}\left\{\frac{uc^4}{2(6+c^2u^2)^2} - \frac{\sqrt{6}}{c\sqrt{p-1}}\,\frac{c^4(p-1)^2}{72p^2}\right\} +\right.$$

$$\left.\frac{6}{c^2}\left\{\frac{uc^4}{8(6+c^2u^2)} - \frac{\sqrt{6}}{c\sqrt{p-1}}\,\frac{c^4(p-1)}{48p}\right\}\right] \tag{D-111}$$

Consequently ($1 \le \tau \le \tau'$)

$$w(x,\tau) = \frac{1}{2}(p-1)\left(\frac{x}{u_o}\right) + x\left[\frac{c}{2\sqrt{6}}\left\{\sqrt{\tau}(p-\tau)^{3/2} - (p-1)^{3/2}\right\} +\right.$$

$$\frac{3c}{4\sqrt{6}}p\left\{\sqrt{\tau(p-\tau)} - \sqrt{p-1}\right\} +$$

$$\left.\frac{\sqrt{6}}{8}cp^2\left\{\tan^{-1}\left(\sqrt{\frac{\tau}{p-\tau}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\}\right] \tag{D-112}$$

in the interval

$$0 \le x \le u_o$$

and

$$w(x,\tau) = -\frac{p}{2}\left[1 - 2p\,\frac{c^2x^2}{(6+c^2x^2)} + p\,\frac{c^4x^4}{(6+c^2x^2)^2}\right] +$$

$$\frac{\sqrt{6}}{8}\,cp^2x\left[\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{cx}{\sqrt{6}}\right)\right] +$$

$$xp^2\left[\frac{3c^2u}{(6+c^2u^2)^2} - \frac{3c^2x}{(6+c^2x^2)^2} + \frac{3c^2u}{4(6+c^2u^2)} - \frac{3c^2x}{(6+c^2x^2)}\right]$$

for $\quad u_o \le x \le u.$ (D-113)

This agrees with the two end values. Furthermore, $\dot{w} > 0$ for all $p > \tau$ and $\dot{w}'' = 0$ as required.

2. Second interval ($u < x \le 1$) and $1 \le \tau \le \tau'$, where τ' is defined later.

Solve

$$\frac{1}{2c^2}\,m_x'' + p + n_\phi - \ddot{w} = 0 \qquad \text{(D-114)}$$

with

$$p = 0 \qquad \text{(D-115)}$$

$$m_x = -1 \qquad \text{(D-116)}$$

$$n_\phi = -1 \qquad \text{(D-117)}$$

$$\dot{w} \geq 0 \qquad \text{(D-118)}$$

$$\dot{w}'' \leq 0 \qquad \text{(D-119)}$$

The equilibrium equation reduces to

$$\ddot{w} = -1 \qquad \text{(D-120)}$$

By integrating with respect to time

$$\dot{w} = -\tau + C_1 \qquad \text{(D-121)}$$

At $\tau = 1 \quad , \dot{w}(1) = p - 1$ (D-122)

hence,

$$p - 1 = -1 + C_1 \qquad \text{(D-123)}$$

$$C_1 = p \qquad \text{(D-124)}$$

i.e.,

$$\dot{w}(\tau) = p - \tau \qquad \text{(D-125)}$$

for all $p - \tau \geq 0$. (D-126)

Integrating again

$$w(\tau) = p\tau - \frac{1}{2}\tau^2 + C_2 \tag{D-127}$$

However, at $\tau = 1$ by Equation (D-55)

$$w(1) = \frac{1}{2}(p-1) = p - \frac{1}{2} + C_2 \tag{D-128}$$

$$C_2 = -\frac{p}{2} \tag{D-129}$$

Hence,

$$w(\tau) = \frac{1}{2}\left[2p\tau - \tau^2 - p\right] \tag{D-130}$$

STAGE 3 ($\tau' \leq \tau \leq \tau_o$)

We observe that when u reaches the midpoint ($u = 1$), the velocity $\dot{w}$ is still not zero, i.e., for time τ' when

$$u^2 = \frac{6\tau'}{c^2(p-\tau')} = 1 \tag{D-131}$$

i.e.,

$$\tau' = \frac{c^2}{c^2+6}p = \frac{p}{\left(1 + \frac{6}{c^2}\right)} = \frac{p}{p_{high}} \tag{D-132}$$

$$\dot{w}(\tau') = (p-\tau')x \neq o \tag{D-133}$$

the hinge has reached the center and motion continues.

Again, we must consider the possible portion on the yield surface. Hodge concludes that portion AB, for which

$$n_\phi = -1 \tag{D-134}$$

$$-1 \leq m_x \leq 1 \tag{D-135}$$

$$\dot{w}'' = 0 \tag{D-136}$$

$$\dot{w} \geq 0 \tag{D-137}$$

is the correct location. In addition to the above we must have continuity of velocities, accelerations, displacements, and stress resultants, at time $\tau = \tau'$.

The equilibrium equation to be solved then is

$$\frac{1}{2c^2} m_x'' + p + n_\phi - \ddot{w} = 0 \tag{D-138}$$

with

$$n_\phi = -1 \quad , \quad p = 0 \quad , \text{ i.e.,}$$

$$\frac{1}{2c^2} m_x'' = 1 + \ddot{w} \qquad \text{(D-139)}$$

However, by Equation (D-66) or Table D-3 we know that at $\tau = \tau'$

$$\ddot{w}(\tau') = -\frac{1}{2}\left\{\left(\frac{x}{u}\right)\left[2 + \frac{p}{\tau}\right]\right\}_{\substack{\tau=\tau'\\ u=1}} =$$

$$-\frac{1}{2}x\left(\frac{3c^2+6}{c^2}\right) = -\frac{3}{2}x\left(\frac{c^2+2}{c^2}\right) \qquad \text{(D-140)}$$

and

$$\dot{w}(\tau') = \left\{(p-\tau)\left(\frac{x}{u}\right)\right\}_{\substack{\tau=\tau'\\ u=1}} = (p-\tau')x =$$

$$\left(p - \frac{c^2}{c^2+6}p\right)x = x\frac{6}{\left(c^2+6\right)}p = \frac{3}{2}x\frac{4}{\left(c^2+6\right)}p \qquad \text{(D-141)}$$

Since the velocity must be non-negative for the strain vector to satisfy the flow rule on the yield surface and since its second derivative with respect to the space variable x is zero, the velocity can only be linear in x. Assume

$$\dot{w}(\tau) = \left(A_1 + A_2\tau\right)x \qquad \text{(D-142)}$$

then $\ddot{w}(\tau) = A_2x$ (D-143)

From the boundary condition

$$A_2 = -\frac{3}{2}\left(\frac{c^2+2}{c^2}\right) \tag{D-144}$$

$$\ddot{w}(\tau) = -\frac{3}{2}\left(\frac{c^2+2}{c^2}\right)x \tag{D-145}$$

Hence,

$$\dot{w}(\tau) = \left[A_1 - \frac{3}{2}\left(\frac{c^2+2}{c^2}\right)\tau\right] x \tag{D-146}$$

From the boundary condition on velocity, however,

$$\frac{3}{2}\ x\ \frac{4}{\left(c^2+6\right)}p = \left[A_1 - \frac{3}{2}\left(\frac{c^2+2}{c^2}\right) \cdot \frac{c^2}{c^2+6}p\right]x \tag{D-147}$$

or

$$A_1 = \frac{3}{2}\ p \tag{D-148}$$

Hence,

$$\dot{w}(\tau) = \frac{3}{2}\ x\left[p - \left(\frac{c^2+ 2}{c^2}\right)\tau\right] \tag{D-149}$$

This equation is valid for all times for which $\dot{w}(\tau) \geq o$, i.e.,

$$p - \left(\frac{c^2+2}{c^2}\right)\tau \geq 0 \tag{D-150}$$

i.e., for times τ in excess of τ' $\left(\tau' = \frac{c^2}{c^2+6} p\right)$, but less than or equal to τ_o

$$\tau_o \leq \frac{c^2}{c^2+2} p \tag{D-151}$$

τ_o represents the time for which velocity $\dot{w}(\tau)$ vanishes and motion stops.

Before we integrate the velocity to obtain the displacement distribution we use, integrate the equation of equilibrium

$$\frac{1}{2c^2} m_x'' = 1 - \frac{3}{2}\left(\frac{c^2+2}{c^2}\right)x \tag{D-152}$$

$$\frac{1}{2c^2} m_x' = x - \frac{3}{4}\left(\frac{c^2+2}{c^2}\right)x^2 + C_1 \tag{D-153}$$

$$\frac{1}{2c^2} m_x = \frac{x^2}{2} - \frac{1}{4} \frac{\left(c^2+2\right)}{c^2} x^3 + C_1 x + C_2 \tag{D-154}$$

At $x = 1$, $m_x' = o$, i.e.,

$$C_1 = \frac{6-c^2}{4c^2} \tag{D-155}$$

$$\frac{1}{2c^2}\, m_x' = x - \frac{3}{4}\left(\frac{c^2+2}{c^2}\right)x^2 + \frac{6-c^2}{4c^2} \tag{D-156}$$

$$m_x' = 2c^2x - \frac{3}{2}\left(c^2+2\right)x^2 + \frac{6 - c^2}{2} =$$

$$- 3\left(\frac{c^2}{2} + 1\right)x^2 + 2c^2x + \left(3 - \frac{c^2}{2}\right) \tag{D-157}$$

$$m_x = -\left(\frac{c^2}{2} + 1\right)x^3 + c^2x^2 + \left(3 - \frac{c^2}{2}\right)x + c_2 \tag{D-158}$$

At $x = 1$,

$$m_x(x=1) = 1 \tag{D-159}$$

giving $C_2 = -1$. Therefore,

$$m_x = -\left(\frac{c^2}{2} + 1\right)x^3 + c^2x^2 + \left(3 - \frac{c^2}{2}\right)x - 1 \tag{D-160}$$

We also note that $m_x(o) = -1$, as it should.

Furthermore, as Equation (B-88) indicates, the yield criterion is not violated. [See Appendix B, Equation (B-81), short shells, low loading.]

Integrating the velocity $\dot{w}(\tau)$ and applying the boundary conditions that, in the two regimes (and for $\tau' \le \tau \le \tau_o$)

(a) $0 \le x \le u_o$, and

(b) $u_o < x \leq 1$

displacements at $\tau = \tau'$ must match, we have

$$w(\tau) = \frac{3}{2}x\left[p\tau' - \frac{1}{2}\frac{(c^2+2)}{c^2}\tau'^2\right] + E_1(x) = \frac{3}{4}xp^2\frac{(c^2+10)c^2}{(c^2+6)^2} + E_1(x) \qquad \text{(D-161)}$$

In the interval $0 \leq x \leq u_o$

$$\text{at } \tau = \tau' = \frac{c^2}{c^2+6}p \qquad \text{(D-162)}$$

we have

$$w(x,\tau') = \frac{1}{2}(p-1)\frac{x}{u_o} + x\left[\frac{c}{2\sqrt{6}}\left\{\sqrt{\tau'}\,(p-\tau')^{3/2} - (p-1)^{3/2}\right\} + \frac{3c}{4\sqrt{6}}p\left\{\sqrt{\tau'(p-\tau')} - \sqrt{p-1}\right\} + \frac{\sqrt{6}}{8}cp^2\left\{\tan^{-1}\left(\sqrt{\frac{\tau'}{p-\tau'}}\right) - \tan^{-1}\sqrt{\frac{1}{p-1}}\right\}\right] \qquad \text{(D-163)}$$

Observe that at that time

$$\frac{\tau'}{p-\tau'} = \frac{c^2}{6} \qquad \text{(D-164)}$$

$$\tau'(p-\tau') = \frac{6c^2}{(c^2+6)^2}\, p^2 \tag{D-165}$$

$$\sqrt{\tau'}(p-\tau')^{3/2} = 6\sqrt{6c}\, \frac{p^2}{(c^2+6)^2} \tag{D-166}$$

Therefore, for $0 \leq x \leq u_o$

$$w(x,\tau') = \frac{1}{2}(p-1)\frac{x}{u_o} + x\left[\left\{\frac{3c^2}{(c^2+6)^2}\, p^2 - \frac{c}{2\sqrt{6}}(p-1)^{3/2}\right\} + \right.$$

$$\left\{\frac{3c^2p^2}{4(c^2+6)} - \frac{3c}{4\sqrt{6}}\, p\, \sqrt{p-1}\right\} +$$

$$\left.\frac{\sqrt{6}}{8}\, cp^2 \left\{\tan^{-1}\left(\frac{c}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\}\right] \tag{D-167}$$

Using Equations (D-167) and (D-161) to determine $E_1(x)$, we have:

$$\frac{3}{4}\, xp^2\, \frac{(c^2+10)}{(c^2+6)^2}\, c^2 + E_1(x) = \frac{1}{2}(p-1)\frac{x}{u_o} +$$

$$x\left[\frac{3c^2}{(c^2+6)^2}\, p^2 - \frac{c}{2\sqrt{6}}\, (p-1)^{3/2} + \frac{3c^2p^2}{4(c^2+6)} - \frac{3c}{4\sqrt{6}}\, p\, \sqrt{p-1} + \right.$$

$$\left.\frac{\sqrt{6}}{8}\, cp^2 \left\{\tan^{-1}\left(\frac{c}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\}\right] \tag{D-168}$$

Hence for $o \leq x \leq u_o$

$$E_1(x) = \frac{1}{2}(p-1)\frac{x}{u_o} - \frac{3}{4}\,xp^2\,\frac{\left(c^2+10\right)c^2}{\left(c^2+6\right)^2} +$$

$$x\left[\frac{3c^2}{\left(c^2+6\right)^2}\,p^2 - \frac{c}{2\sqrt{6}}\,(p-1)^{3/2} + \frac{3c^2p^2}{4\left(c^2+6\right)} - \frac{3c}{4\sqrt{6}}\,p\,\sqrt{p-1} + \right.$$

$$\left.\frac{\sqrt{6}}{8}\,cp^2\left\{\tan^{-1}\left(\frac{c}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\}\right] \qquad \text{(D-169)}$$

For $u_o < x \leq u$ and $\tau = \tau'$

$$w(x,\tau') = -\frac{p}{2}\left[1 - 2p\,\frac{c^2x^2}{\left(6+c^2x^2\right)} + p\,\frac{c^4x^4}{\left(6+c^2x^2\right)^2}\right] -$$

$$\frac{\sqrt{6}}{8}\,cp^2\left\{\tan^{-1}\left(\frac{cx}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\} -$$

$$xp^2\left[\left(\frac{3c^2x}{\left(6+c^2x^2\right)^2} - \frac{c}{2\sqrt{6}}\,\frac{(p-1)^{3/2}}{p^2}\right) + \left(\frac{3c^2x}{4\left(6+c^2x^2\right)} - \frac{\sqrt{6}}{8}\,c\,\frac{\sqrt{p-1}}{p}\right)\right] +$$

$$xp^2\left[\left(\frac{3c^2u}{\left(6+c^2u^2\right)^2} - \frac{c}{2\sqrt{6}}\,\frac{(p-1)^{3/2}}{p^2}\right) + \left(\frac{3c^2u}{4\left(6+c^2u^2\right)} - \frac{\sqrt{6}}{8}\,c\,\frac{\sqrt{p-1}}{p}\right)\right] +$$

$$\frac{\sqrt{6}}{8}\,cp^2x\left\{\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\} \qquad \text{(D-170)}$$

When u' = 1

$$w(x,\tau') = -\frac{p}{2}\left[1 - 2p\,\frac{c^2x^2}{(6+c^2x^2)} + p\,\frac{c^4x^4}{(6+c^2x^2)^2}\right]$$

$$\frac{\sqrt{6}}{8}\,cp^2x\left\{\tan^{-1}\left(\frac{cu'}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{cx}{\sqrt{6}}\right)\right\} +$$

$$xp^2\left[\frac{3c^2u'}{(6+c^2u'^2)^2} - \frac{3c^2x}{(6+c^2x^2)^2} + \frac{3c^2u'}{4(6+c^2u'^2)^2} - \frac{3c^2x}{4(6+c^2x^2)^2}\right] \qquad \text{(D-171)}$$

For $u_o < x < u$

$$w(x,\tau) = -\frac{p}{2}\left[1 - 2p\,\frac{c^2x^2}{(6+c^2x^2)} + p\,\frac{c^4x^4}{(6+c^2x^2)^2}\right] +$$

$$\frac{\sqrt{6}}{8}\,cp^2x\left\{\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{cx}{\sqrt{6}}\right)\right\} +$$

$$xp^2\left[\frac{3c^2u}{(6+c^2u^2)^2} - \frac{3c^2x}{(6+c^2x^2)^2} + \frac{3c^2u}{4(6+c^2u^2)} - \frac{3c^2x}{4(6+c^2x^2)}\right] \qquad \text{(D-172)}$$

or

$$w(x,\tau') = -\frac{p}{2}\left[1 - 2p\,\frac{c^2\,x^2}{\left(6+c^2x^2\right)} + p\,\frac{c^4\,x^4}{\left(6+c^2x^2\right)^2}\right] +$$

$$\frac{\sqrt{6}}{8}\,cp^2x\left\{\tan^{-1}\left(\frac{c}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{cx}{\sqrt{6}}\right)\right\} +$$

$$xp^2\left\{\frac{3c^2}{\left(6+c^2\right)^2} - \frac{3c^2x}{\left(6+c^2x^2\right)^2} + \frac{3c^2}{4\left(6+c^2\right)} - \frac{3c^2x}{4\left(6+c^2x^2\right)}\right\} \qquad \text{(D-173)}$$

For $u_o \leq x \leq 1$

$$\frac{3}{4}\,xp^2\,\frac{\left(c^2+10\right)c^2}{\left(c^2+6\right)^2} + E_1(x) =$$

$$-\frac{p}{2}\left[1 - 2p\,\frac{c^2x^2}{\left(6+c^2x^2\right)} + p\,\frac{c^4x^4}{\left(6+c^2x^2\right)^2}\right] +$$

$$-\frac{\sqrt{6}}{8}\,cp^2x\left\{\tan^{-1}\left(\frac{c}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{cx}{\sqrt{6}}\right)\right\} +$$

$$xp^2\left\{\frac{3c^2}{\left(6+c^2\right)^2} - \frac{3c^2x}{\left(6+c^2x^2\right)^2} + \frac{3c^2}{4\left(6+c^2\right)} - \frac{3c^2x}{4\left(6+c^2x^2\right)}\right\} \qquad \text{(D-174)}$$

Hence, in the interval $u_o \le x \le 1$

$$E_1(x) = -\frac{3}{4}\, xp^2 \,\frac{\left(c^2+10\right)c^2}{\left(c^2+6\right)^2} -$$

$$\frac{p}{2}\left[1 - 2p\,\frac{c^2x^2}{\left(6+c^2x^2\right)} + p\,\frac{c^4x^4}{\left(6+c^2x^2\right)^2}\right] +$$

$$\frac{\sqrt{6}}{8}\, cp^2x \left\{\tan^{-1}\left(\frac{c}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{cx}{\sqrt{6}}\right)\right\} +$$

$$xp^2\left\{\frac{3c^2}{\left(6+c^2\right)^2} - \frac{3c^2x}{\left(6+c^2x^2\right)^2} + \frac{3c^2}{4\left(6+c^2\right)} - \frac{3c^2x}{4\left(6+c^2x^2\right)}\right\} \tag{D-175}$$

We can calculate the displacement at rest ($\tau = \tau_o$) by replacing τ by τ_o. We obtain

$$w\left(\tau_o\right) = \left[\frac{3}{4}\,\frac{c^2}{\left(c^2+2\right)}\, p^2x + E_1(x)\right]_{\tau=\tau_o} \tag{D-176}$$

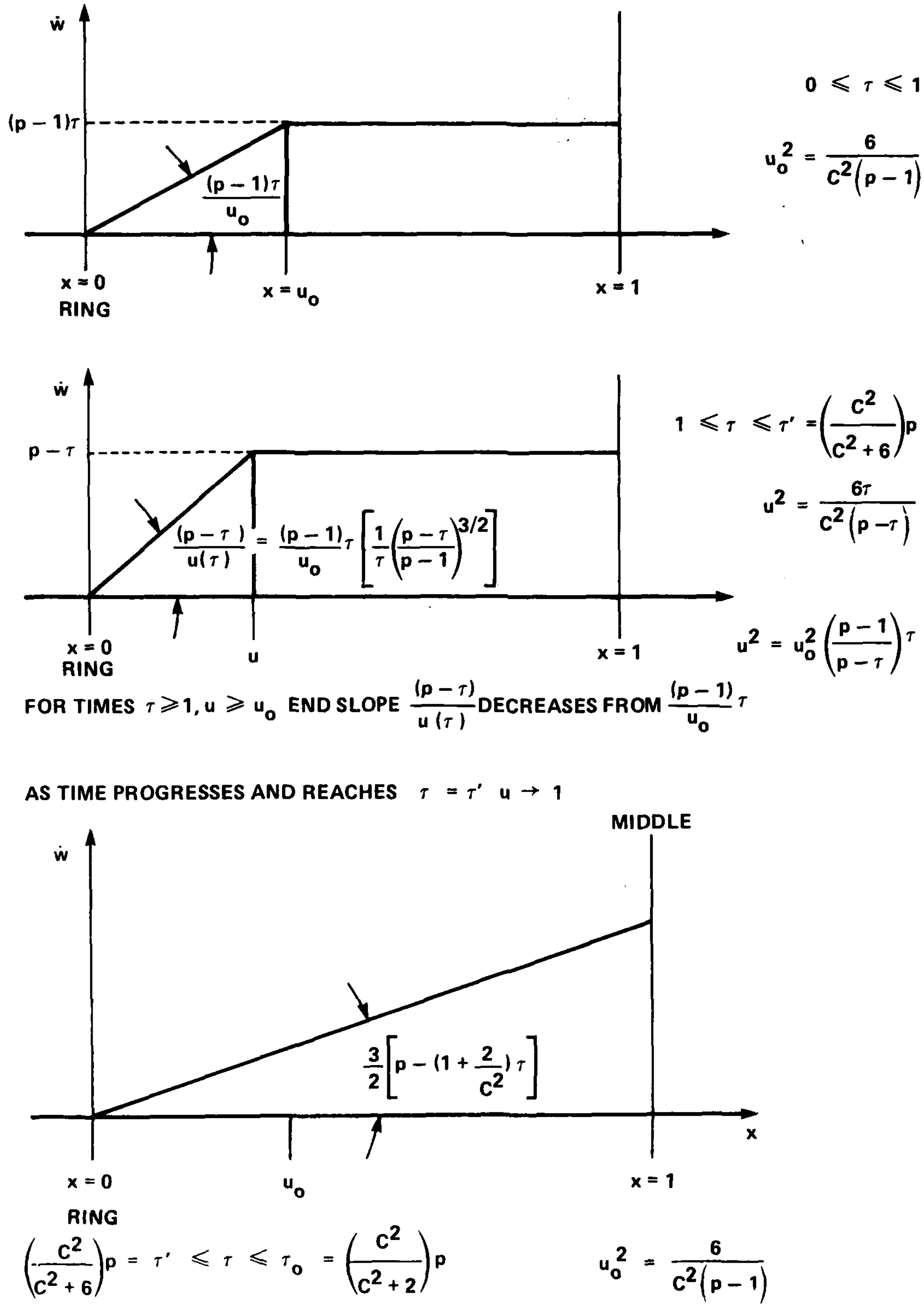


FIGURE D-1. VELOCITY PROFILES FOR TIME τ SHORT SHELLS, HIGH LOADING

TABLE D-1. SUMMARY, SHORT SHELLS, HIGH LOADING, 1

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(c^2 < 6\right)$	HIGH LOAD $\left(p > 1 + \frac{6}{c^2}\right)$
CONDITIONS	$u_o^2 = \frac{6}{c^2 (p-1)}$ $\quad 0 \leqslant \tau \leqslant 1 \quad 0 \leqslant x < u_o$	
MOMENT RESULTANT	$m_x (x, \tau) = 2\left[\frac{x}{u_o} - 1\right]^3 + 1$ OR $= 2\left[\frac{x}{u_o}\right]^3 - 6\left[\frac{x}{u_o}\right]^2 + 6\left[\frac{x}{u_o}\right] - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x, \tau) = \frac{1}{2} \frac{(p-1)}{u_o} \tau^2 x$	
VELOCITY	$\dot{w} = \frac{(p-1)}{u_o} \tau x$	
ACCELERATION	$\ddot{w} = \frac{(p-1)}{u_o} x$	
TIME τ_o	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE D-2. SUMMARY, SHORT SHELLS, HIGH LOADING, 2

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(c^2 < 6\right)$	HIGH LOAD $\left(p > 1 + \frac{6}{c^2}\right)$
CONDITIONS	$u_o^2 = \frac{6}{c^2\left(p-1\right)}$	$0 \leqslant \tau \leqslant 1$ $u_o < x \leqslant 1$
MOMENT RESULTANT	$m_x = 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w = \frac{1}{2}\left(p-1\right)\tau^2$	
VELOCITY	$\dot{w} = (p-1)\tau$	
ACCELERATION	$\ddot{w} = p-1$	
TIME τ_o	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE D-3. USEFUL FUNCTIONS AND THEIR DERIVATIVES

FUNCTION F (u)	DERIVATIVE $\dot{F}$ (u) WITH RESPECT TO u
$-\dfrac{u}{2\left(u^2+\dfrac{6}{C^2}\right)^2}$	$\dfrac{\left(\dfrac{6}{C^2}-3u^2\right)}{2\left(u^2+\dfrac{6}{C^2}\right)^3}$
$\dfrac{u}{\left(u^2+\dfrac{6}{C^2}\right)}$	$\dfrac{\left(\dfrac{6}{C^2}-u^2\right)}{\left(u^2+\dfrac{6}{C^2}\right)^2}$
$\tan^{-1}\left(\dfrac{Cu}{\sqrt{6}}\right)$	$\dfrac{\sqrt{6}}{C\left(u^2+\dfrac{6}{C^2}\right)}$
$\dfrac{C^2}{12}\left[\dfrac{u}{2\left(u^2+\dfrac{6}{C^2}\right)^2}+\dfrac{C^2}{8}\,\dfrac{u}{\left(u^2+\dfrac{6}{C^2}\right)}\right]+\dfrac{C^5}{96\sqrt{6}}\tan^{-1}\left(\dfrac{Cu}{\sqrt{6}}\right)$	$\dfrac{1}{\left(u^2+\dfrac{6}{C^2}\right)^3}$

TABLE D-4. SUMMARY, SHORT SHELLS, HIGH LOADING, 3

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $\left(c^2 < 6\right)$	HIGH LOAD $\left(p > 1 + \frac{6}{c^2}\right)$ $\quad p > \tau$
CONDITIONS	$u^2 = \dfrac{6\tau}{c^2(p-\tau)}$ $\quad 0 \leqslant x < u$ $\quad 1 \leqslant \tau \leqslant \tau'$	$\tau' = \dfrac{p}{1+\frac{6}{c^2}} = \dfrac{p}{p_1}$
MOMENT RESULTANT	$m_x(x,\tau) = -\frac{1}{2}\left(2+c^2u^2\right)\left(\frac{x}{u}\right)^3 + c^2u^2\left(\frac{x}{u}\right)^2 + \left(3-\frac{c^2u^2}{2}\right)\left(\frac{x}{u}\right) - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$u_o^2 = \dfrac{6}{c^2(p-1)}$ $w(x,\tau) = \frac{1}{2}(p-1)\left(\frac{x}{u_o}\right) + x\left[\frac{c}{2\sqrt{6}}\left\{\sqrt{\tau}\,(p-\tau)^{3/2} - (p-1)^{3/2}\right\} + \frac{3c}{4\sqrt{6}}\,p\left\{\sqrt{\tau(p-\tau)} - \sqrt{(p-1)}\right\} + \frac{\sqrt{6}}{8}c\,p^2\left\{\tan^{-1}\left(\sqrt{\frac{\tau}{p-\tau}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\}\right]$ FOR $0 \leqslant x \leqslant u_o$ $w(x,\tau) = -\frac{p}{2}\left[1 - 2p\frac{c^2x^2}{\left(6+c^2x^2\right)} + p\frac{c^4x^4}{\left(6+c^2x^2\right)^2}\right] + \frac{\sqrt{6}}{8}c\,p^2x\left[\tan^{-1}\left(\frac{cu}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{cx}{\sqrt{6}}\right)\right]$ $+ x\,p^2\left[\frac{3c^2u}{\left(6+c^2u^2\right)^2} - \frac{3c^2x}{\left(6+c^2x^2\right)^2} + \frac{3c^2u}{4\left(6+c^2u^2\right)} - \frac{3c^2x}{4\left(6+c^2x^2\right)}\right]$ FOR $u_o \leqslant x \leqslant u$	
VELOCITY	$\dot{w}(\tau) = (p-\tau)\left(\frac{x}{u}\right) = \dfrac{c}{\sqrt{6}}\,\dfrac{(p-\tau)^{3/2}}{\sqrt{\tau}}\,x$	
ACCELERATION	$\ddot{w}(\tau) = -\frac{x}{u}\left[1 + \frac{(p-\tau)\dot{u}}{u}\right] = \frac{1}{2}\left(\frac{x}{u}\right)\left[2 + \frac{p}{\tau}\right]$	
TIME τ_o	AT TIME $\tau' = \dfrac{p}{p_1}$, TRAVELING HINGE u MOVES TO MIDLENGTH $x = u = 1$	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE D-5. SUMMARY; SHORT SHELLS, HIGH LOADING, 4

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $C^2 < 6$	HIGH LOAD $p > 1 + \frac{6}{C^2}$
CONDITIONS	$u^2 = \frac{6\tau}{C^2(p-\tau)}$ $u < x \leqslant 1$, $p > \tau$, $1 \leqslant \tau \leqslant \tau'$ $\tau' = \frac{p}{\left(1+\frac{6}{C^2}\right)} = \frac{p}{p_1}$	
MOMENT RESULTANT	$m_x = -1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(\tau) = \frac{1}{2}\left[2p\tau - \tau^2 - p\right]$	
VELOCITY	$\dot{w}(\tau) = p - \tau$	
ACCELERATION	$\ddot{w}(\tau) = -1$	
TIME τ_o	AT TIME $\tau' = \frac{p}{p_1}$, TRAVELING HINGE u MOVES TO MIDLENGTH $x = u = 1$ AND THIS REGIME SHRINKS TO ZERO	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE D-6. SUMMARY, SHORT SHELLS, HIGH LOADING, 5

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	SHORT $C^2 < 6$	HIGH LOAD $p > 1 + \frac{6}{C^2}$
CONDITIONS	$p > \tau \qquad \tau' \leqslant \tau \leqslant \tau_o \qquad \tau' = \frac{C^2}{C^2+6}\,p$	
MOMENT RESULTANT	$m_x = -\left[1 + \frac{C^2}{2}\right]x^3 + C^2x^2 + \left[3 - \frac{C^2}{2}\right]x - 1$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(\tau) = \frac{3}{2}x\left[p\tau - \frac{1}{2}\frac{(C^2+2)}{C^2}\tau^2\right] + E_1(x)$ WHERE FOR $0 \leqslant x \leqslant u_o$ E_1 IS GIVEN IN TABLE D-7 $u_o < x \leqslant 1$ E_1 IS GIVEN IN TABLE D-8	
VELOCITY	$\dot{w} = \left(\frac{3}{2}\right)x\left[p - \frac{(C^2+2)}{C^2}\tau\right]$	
ACCELERATION	$\ddot{w} = -\frac{3}{2}x\left[\frac{C^2+2}{C^2}\right]$	
TIME τ_o	$\tau_o = \left[\frac{C^2}{C^2+2}\right]p$	
DISPLACEMENT AT REST	$w(\tau_o) = \frac{3}{4}\left(\frac{C^2}{C^2+2}\right)p^2x + E_1(x)\Big]_{\tau=\tau_o}$	WHERE E_1 WILL BE CALCULATED EITHER BY TABLE D-7 OR TABLE D-8

TABLE D-7. $E_1(x)$ FOR $0 \leqslant x \leqslant u_o$

$$E_1(x) = \frac{1}{2}(p-1)\frac{x}{u_o} - \frac{3}{4}\,x\,p^2\,\frac{(C^2+10)C^2}{(C^2+6)^2} + x\left[\frac{3C^2}{(C^2+6)^2}\,p^2 - \frac{C}{2\sqrt{6}}(p-1)^{3/2} + \right.$$

$$\left.\frac{3C^2p^2}{4(C^2+6)} - \frac{3C}{4\sqrt{6}}\,p\sqrt{p-1} + \frac{\sqrt{6}}{8}\,C\,p^2\left\{\tan^{-1}\left(\frac{C}{\sqrt{6}}\right) - \tan^{-1}\left(\sqrt{\frac{1}{p-1}}\right)\right\}\right]$$

TABLE D-8. $E_1(x)$ FOR $u_o \leqslant x \leqslant 1$

$$
\begin{aligned}
E_1(x) = & -\frac{3}{4} x\, p^2 \frac{\left(C^2+p\right) C^2}{\left(C^2+6\right)^2} - \frac{p}{2}\left[1 - 2p \frac{C^2 x^2}{\left(6+C^2 x^2\right)} + p \frac{C^4 x^4}{\left(6+C^2 x^2\right)^2}\right] \\
& + \frac{\sqrt{6}}{8} C\, p^2 x \left\{ \tan^{-1}\left(\frac{C}{\sqrt{6}}\right) - \tan^{-1}\left(\frac{C x}{\sqrt{6}}\right)\right\} + x\, p^2 \left\{\frac{3C^2}{\left(6+C^2\right)^2} - \frac{3C^2 x}{\left(6+C^2 x^2\right)^2}\right. \\
& \left. + \frac{3C^2}{4\left(6+C^2\right)} - \frac{3C^2 x}{4\left(6+C^2 x^2\right)}\right\}
\end{aligned}
$$

REFERENCES--APPENDIX D

D-1. Hodge, P. G. Jr., Impact Pressure Loading of Rigid-Plastic Cylindrical Shells, Polytechnic Institute of Brooklyn, PIBAL Report No. 255, May 1954.

D-2. Hodge, P. G. Jr., "Impact Pressure Loading of Rigid-Plastic Cylindrical Shells," Journal of the Mechanics and Physics of Solids, Vol. 3, 1955, pp. 176-188.

APPENDIX E

CASE D - LONG SHELLS, HIGH LOADING

$$\left(c^2 > 6 \quad , \quad p > 1 + \frac{6}{c^2}\right)$$

LONG SHELLS, HIGH LOADING, $\left(c^2 > 6 \quad , \quad p > 1 + \frac{6}{c^2}\right)$

This case is summarized in Tables E-1 through E-7.

Let us examine first the time interval $0 \leq \tau \leq 1$. Table E-1 (it applies for $0 \leq x \leq u_o$) and Table E-2 (it applies for $u_o \leq x \leq 1$) summarize the results. The pertinent intervals on the yield surface lie along AB and at point B (corner), respectively. The analysis for "short shells under high loading," therefore, applies for times $0 \leq \tau \leq 1$. Equations (D-36) and (D-37) will be satisfied and hence there will be no violation of the yield locus. Furthermore, the velocity distribution from Equations (D-29) and (D-20) is such that the flow rule is satisfied. Therefore,

$$\dot{w}(x,\tau) = \frac{(p-1)}{u_o}\tau\, x \qquad \text{for} \qquad 0 \leq x \leq u_o \tag{E-1}$$

$$\dot{w}(x,\tau) = (p-1)\tau \qquad \text{for} \qquad u_o \leq x \leq 1 \tag{E-2}$$

At time $\tau = 1$ the load is removed. We observe that the analysis of the previous section for the next time interval ($\tau \geq 1$) is only valid for $c^2 < 6$. For long shells, however, $c^2 > 6$.

Hence a new assumption is required as to the ranges on the yield surface. This must be done in association with the fact that in the interval $o \leq \tau \leq 1$ the velocity profile is given on Figure E-1 (this figure comes from Figure D-1 applicable to short shells and high loads). Figure E-1 suggests that for $1 \leq \tau \leq \tau_o$ the logical compatible velocity profile is given by Figure E-2 in such a way that

1. at time $\tau = 1, y = 0$

2. at time $\tau = 1, u = u_o$ and the range AD has disappeared.

Based on the previous assumptions we must solve the equilibrium equation in three intervals for $1 \leq \tau \leq \tau_o$, and account for initial and boundary conditions.

1. For times $1 \leq \tau \leq \tau_1$

a. For $0 \leq x \leq y$ (Range AD, Figure A-3, Table A-1)

$$\dot{w} = 0 \tag{E-3}$$

$$\dot{w}'' \geq 0 \tag{E-4}$$

$$m_x = -1 \tag{E-5}$$

$$\frac{1}{2c^2} m_x'' + n_\phi + p - \ddot{w} = 0 \tag{E-6}$$

$$p = 0 \tag{E-7}$$

b. For $y \leq x \leq u$ (Range AB, Figure A-3, Table A-1)

$$n_\phi = -1 \tag{E-8}$$

$$\dot{w} \geq 0 \tag{E-9}$$

$$\dot{w}'' = 0 \tag{E-10}$$

$$\frac{1}{2c^2} m_x'' + n_\phi + p - \ddot{w} = 0 \tag{E-11}$$

$$p = 0 \tag{E-12}$$

c. For $u \leq x \leq 1$ (Point B, Figure A-3, Table A-1)

$$n_\phi = -1 \tag{E-13}$$

$$m_x = 1 \tag{E-14}$$

$$\dot{w} \geq 0 \tag{E-15}$$

$$\dot{w}'' \leq 0 \tag{E-16}$$

$$\frac{1}{2c^2} m_x'' + n_\phi + p - \ddot{w} = 0 \tag{E-17}$$

$$p = 0 \tag{E-18}$$

We know that at time $\tau = 1$, when the pressure load ceases to act (see Tables D-5 and D-6), the initial conditions are

1. For $0 \leq x \leq u_o = \sqrt{\dfrac{6}{c^2(p-1)}}$

$$\dot{w}(x,1) = \frac{(p-1)}{u_o}x \tag{E-18a}$$

$$w(x,1) = \frac{1}{2}\frac{(p-1)}{u_o}x \tag{E-18b}$$

2. For $u_o \leq x \leq 1$

$$\dot{w}(x,1) = p-1 \tag{E-19a}$$

$$w(x,1) = \frac{1}{2}(p-1) \tag{E-19b}$$

We now examine the time interval $\tau \geq 1$. The space interval is subdivided in three segments

1. $0 \leq x \leq y$,

2. $y \leq x \leq u$, and

3. $u \leq x \leq 1$

Both y and u will be defined later.

FIRST INTERVAL $0 \leq x \leq y$

The equilibrium equation yields

$$n_\phi = \ddot{w} \tag{E-20}$$

Since $\dot{w} = 0$ (E-21)

for all times $\tau \geq 1$, however, this means that

$$\ddot{w} = 0 \tag{E-22}$$

and, hence,

$$n_\phi = 0 \tag{E-23}$$

and integrating Equation (E-21) once with time

$$w = C_1(x) \tag{E-24}$$

Hence,

$$w = \frac{1}{2}\frac{(p-1)}{u_o}x \tag{E-25}$$

In summary, for $0 \leq x \leq y$

$$m_x = -1 \tag{E-26}$$

$$n_\phi = 0 \tag{E-27}$$

$$\ddot{w} = 0 \tag{E-28}$$

$$\dot{w} = 0 \tag{E-29}$$

$$w = \frac{1}{2}\frac{(p-1)}{u_o}x \tag{E-30}$$

and y has not been determined.

SECOND INTERVAL $y \leq x \leq u$

The equilibrium equation yields

$$\frac{1}{2c^2}m''_x = 1 + \ddot{w} \tag{E-31}$$

Since by Equation (E-10) $\dot{w}'' = 0$, the velocity profile must be linear in x.

Assume

$$\dot{w} = \dot{A}(\tau)x + \dot{B}(\tau) \tag{E-32}$$

Hence,

$$\ddot{w} = \ddot{A}(\tau)x + \ddot{B}(\tau) \tag{E-33}$$

Hence,

$$\frac{1}{2c^2}m''_x = 1 + \ddot{A}(\tau)x + \ddot{B}(\tau) \tag{E-34}$$

Integrating once with respect to x

$$\frac{1}{2c^2}m'_x = \frac{\ddot{A}(\tau)}{2}x \;+ (\ddot{B}(\tau)+1)x + D(\tau) \tag{E-35}$$

and once more

$$\frac{1}{2c^2}m_x = \frac{1}{6}\ddot{A}(\tau)x^3 + \frac{1}{2}(1+\ddot{B}(\tau))x^2 + D(\tau)x + E(\tau) \tag{E-36}$$

At x = y the moment resultant and shearing forces must agree, i.e.,

$$\frac{1}{2c^2}m'(y,\tau) = \frac{\ddot{A}(\tau)}{2}y^2 + (1+\ddot{B}(\tau))y + D(\tau) = 0 \tag{E-37}$$

$$\frac{1}{2c^2}m_x(y,\tau) = \frac{1}{6}\ddot{A}(\tau)y^3 + \frac{1}{2}(1+\ddot{B}(\tau))y^2 + D(\tau)y + E(\tau) = -\frac{1}{2c^2} \tag{E-38}$$

Also at x = u

$$\frac{1}{2c^2}m'(u,\tau) = \frac{1}{2}\ddot{A}(\tau)u^2 + (1+\ddot{B}(\tau))u + D(\tau) = 0 \tag{E-39}$$

$$\frac{1}{2c^2}m_x(u,\tau) = \frac{1}{6}\ddot{A}(\tau)u^3 + \frac{1}{2}(1+\ddot{B}(\tau))u^2 + D(\tau)u + E(\tau) = -\frac{1}{2c^2} \tag{E-40}$$

Solving Equation (E-37) for $2c^2D$, we get

$$2c^2D = -c^2\ddot{A}\,y^2 - 2c^2(1+\ddot{B})y \tag{E-41}$$

Solving Equation (E-38) for $2c^2E$ and replacing $2c^2D$ from Equation (E-41), we obtain

$$2c^2E = -1 + \frac{2}{3}c^2\ddot{A}y^3 + c^2(1+\ddot{B})y^2 \tag{E-42}$$

Replacing $2c^2D$ in Equation (E-39) from Equation (E-41) we obtain

$$c^2\ddot{A}\left(u^2-y^2\right) + 2c^2(1+\ddot{B})(u-y) = 0 \tag{E-43}$$

Replacing $2c^2D$ and $2c^2E$ in Equation (E-40) from Equations (E-41) and (E-42), we obtain a second relation

$$\frac{1}{3}c^2\ddot{A}\left[u^3-3uy^2+2y^3\right] + c^2(1+\ddot{B})(u-y)^2 = 2 \tag{E-44}$$

or

$$\frac{1}{3}c^2\ddot{A}(u-y)\left(u^2+yu-2y^2\right) + c^2(1+\ddot{B})(u-y)^2 = 2 \tag{E-45}$$

Substituting $c^2(1+\ddot{B})(u-y)$ from Equation (E-43) in Equation (E-45) we finally obtain the value of $\ddot{A}$, i.e.,

$$\ddot{A} = -\frac{12}{c^2}\frac{1}{(u-y)^3} \tag{E-46}$$

Hence (for $u \neq y$)

$$2(1+\ddot{B}) = -\ddot{A}(u+y)$$

which, in view of Equation (E-46), becomes

$$2(1+\ddot{B}) = -(u+y)x\left(\frac{-12}{c^2}\right)\frac{1}{(u-y)^3}$$

or

$$1+\ddot{B} = \frac{6}{c^2}\frac{(u+y)}{(u-y)^3} \tag{E-47}$$

Therefore,

$$\ddot{A} = -\frac{12}{c^2}\frac{1}{(u-y)^3} \tag{E-48}$$

$$1+\ddot{B} = \frac{6}{c^2}\frac{(u+y)}{(u-y)^3} \tag{E-49}$$

$$\ddot{B} = \frac{6}{c^2}\frac{(u+y)}{(u-y)^3} - 1 \tag{E-50}$$

$$2c^2D = \frac{-12uy}{(u-y)^3} \tag{E-51}$$

$$2c^2E = -\frac{(y+u)\left(y^2-4yu+u^2\right)}{(u-y)^3} \tag{E-52}$$

Also by Equation (E-41)

$$2c^2D = -\frac{12uy}{(u-y)^3} \tag{E-53}$$

By *Equation* (E-42)

$$2c^2E = -1 + \frac{2y^2[3u-y]}{(u-y)^3} \tag{E-54}$$

$$m_x(x,\tau) = \frac{1}{(u-y)^3}\left[-4x^3+6(u+y)x^2-12uyx-(y+u)\left(y^2-4yu+u^2\right)\right] \tag{E-55}$$

$$m_x'(x,\tau) = \frac{1}{(u-y)^3}\left[-12x^2+12(u+y)x-12uy\right] \tag{E-56}$$

and

$$m_x''(x,\tau) = \frac{12}{(u-y)^3}[-2x+(u+y)] \tag{E-57}$$

We observe, that, in fact

$$m_x(y,\tau) \to -1$$

$$m_x'(y,\tau) \to 0$$

$$m_x(u,\tau) \to 1$$

$$m_x'(u,\tau) \to 0$$

$$m_x''(y,\tau) \to \frac{12}{(u-y)^2}$$

$$m_x''(u,\tau) \to -\frac{12}{(u-y)^2}$$

Before we can combine all three ranges, we examine the third interval ($u \leq x \leq 1$).

THIRD INTERVAL ($u \leq x \leq 1$)

The equilibrium equation yields

$$\ddot{w} = -1 \qquad \text{(E-58)}$$

Integrating once we get

$$\dot{w}(\tau) = -\tau + C_1 \qquad \text{(E-59)}$$

However, by the initial condition in velocity [see Equation (E-18)] at $\tau = 1$, we have

$$[-\tau + C_1]_{\tau=1} = p - 1 \qquad \text{(E-60)}$$

Hence,

$$C_1 = p \qquad \text{(E-61)}$$

and $$\dot{w}(x,\tau) = p - \tau \qquad \text{(E-62)}$$

Integrating again

$$w(x,\tau) = p\tau - \frac{1}{2}\tau^2 + D_1 \qquad \text{(E-63)}$$

By Equation (E-19) at $\tau = 1$

$$D_1 = -\frac{1}{2}p \qquad \text{(E-64)}$$

Leaving Equation (E-63) with the known constant D_1 we observe that the displacement must agree with that one obtained from the second interval for all times $\tau > 1$. We must also retrieve the value of the displacement at time $\tau = 1$ as given by Equation (E-19), i.e.,

$$w(x,1) = \frac{1}{2}(p-1) \qquad \text{(E-65)}$$

and, therefore, D_1 takes on the value already given by Equation (E-64) and

$$w(x,\tau) = p\left(\tau - \frac{1}{2}\right) - \frac{1}{2}\tau^2 \qquad \text{(E-66)}$$

It can easily be seen that for the time range of interest, $w(x,\tau)$ is non-negative.

We proceed to match solutions in the three intervals for $\tau \geq 1$, since

1. $0 \leq x \leq y \qquad \dot{w} = 0 \qquad$ (E-67)

2. $y \leq x \leq u \qquad \dot{w} = \dot{A}(\tau)x + \dot{B}(\tau) \qquad$ (E-68)

3. $u \leq x \leq 1 \qquad \dot{w} = p - \tau \qquad$ (E-69)

For $\quad x = y \qquad \dot{A}(\tau)y + \dot{B}(\tau) = 0 \qquad$ (E-70)

For $\quad x = u \qquad \dot{A}(\tau)u + \dot{B}(\tau) = p - \tau \qquad$ (E-71)

Therefore,

$$\dot{B}(\tau) = -\dot{A}(\tau)y \qquad \text{(E-72)}$$

$$\dot{A}(\tau)(u-y) = p - \tau \qquad \text{(E-73)}$$

$$\dot{A}(\tau) = \frac{(p-\tau)}{(u-y)} \qquad \text{(E-74)}$$

Differentiate Equation (E-74) with time

$$\dot{A}(\tau) = -\frac{1}{(u-y)} - \frac{(p-\tau)}{(u-y)^2}\frac{d}{d\tau}(u-y) \tag{E-75}$$

By Equation (E-48) however and setting $\theta = u-y$

$$\ddot{A} = -\frac{12}{c^2}\frac{1}{(u-y)^3}\text{, i.e.,} \tag{E-76}$$

we get

$$-\frac{1}{\theta} - \frac{(p-\tau)}{\theta^2}\frac{d}{d\tau}\theta = -\frac{12}{c^2}\frac{1}{\theta^3} \tag{E-77}$$

or

$$-\frac{(p-\tau)}{\theta^2}\frac{d}{d\tau}\theta = \frac{1}{\theta} - \frac{12}{c^2}\frac{1}{\theta^3} \tag{E-78}$$

$$\frac{d}{d\tau}\left[\frac{1}{2}(u-y)^2\right] = \frac{1}{(p-\tau)}\left[\frac{12}{c^2} - (u-y)^2\right] \tag{E-79}$$

$$\frac{d}{d\tau}(u-y) = -\frac{\theta^2}{(p-\tau)}\left[\frac{1}{\theta} - \frac{12}{c^2}\frac{1}{\theta^3}\right] = -\frac{\theta}{(p-\tau)} + \frac{12}{c^2}\frac{1}{(p-\tau)}\frac{1}{\theta} \tag{E-80}$$

$$\frac{d}{d\tau}(u-y) = \frac{1}{(p-\tau)}\left[\frac{12}{c^2}\frac{1}{(u-y)} - (u-y)\right] \tag{E-81}$$

$$-(p-\tau)\theta\frac{d}{d\tau}\theta = \theta^2 - \frac{12}{c^2} \tag{E-82}$$

$$\frac{d}{d\tau}\left(\frac{1}{2}\theta^2\right) = \left(\theta^2 - \frac{12}{c^2}\right)\left[\frac{-1}{(p-\tau)}\right] = \frac{\left(\frac{12}{c^2} - \theta^2\right)}{(p-\tau)} \tag{E-83}$$

$$\frac{d\left(\frac{1}{2}\theta^2\right)}{\left(\frac{12}{c^2} - \theta^2\right)} = \frac{d\tau}{(p-\tau)} = -\frac{1}{2}\frac{d(\theta^2)}{\left(\theta^2 - \frac{12}{c^2}\right)} \tag{E-84}$$

Set $\phi = \theta^2$

$$-\frac{1}{2}\frac{d\phi}{\left(\phi - \frac{12}{c^2}\right)} = \frac{d\tau}{p-\tau} \tag{E-85}$$

$$-\frac{1}{2}\log_e\left|\phi - \frac{12}{c^2}\right| = -\log_e\left|p-\tau\right| + \log_e D \tag{E-86}$$

When $\tau = 1$, $\quad \phi = \theta^2 = (u-y)^2 = u_o^{\,2} = \dfrac{6}{c^2(p-1)}$ (E-87)

$$-\frac{1}{2}\log_e\left|\frac{6}{c^2(p-1)} - \frac{12}{c^2}\right| = -\log_e\left|p-1\right| + \log_e D \tag{E-88}$$

$$\log_e D = \log_e\left|p-1\right| - \frac{1}{2}\log_e\left|\frac{6}{c^2}\left(\frac{1}{(p-1)} - 2\right)\right| =$$

$$\log_e\left|p-1\right| - \frac{1}{2}\log_e\left|\frac{6}{c^2}\,\frac{(3-2p)}{(p-1)}\right| \tag{E-89}$$

$$-\frac{1}{2}\log_e\left|\phi - \frac{12}{c^2}\right| = \log_e\left|\frac{(p-1)}{(p-\tau)}\right| - \frac{1}{2}\log_e\left|\frac{6}{c^2}\,\frac{(3-2p)}{(p-1)}\right| \qquad \text{(E-90)}$$

$$\frac{1}{2}\log_e\left|\frac{\frac{6}{c^2}\left(\frac{3-2p}{p-1}\right)}{\left(\phi - \frac{12}{c^2}\right)}\right| = \log_e\left|\frac{(p-1)}{(p-\tau)}\right| \qquad \text{(E-91)}$$

$$\sqrt{\frac{\frac{6}{c^2}\,\frac{(3-2p)}{(p-1)}}{\left(\phi - \frac{12}{c^2}\right)}} = \frac{(p-1)}{(p-\tau)} \qquad \text{(E-92)}$$

or

$$(u-y)^2 = \phi = \theta^2 = \frac{6}{c^2}\left[2 - (2p-3)\frac{(p-\tau)^2}{(p-1)^3}\right] \qquad \text{(E-93)}$$

Therefore,

$$(u-y)^2 = \frac{6}{c^2}\left[2 - \frac{(p-\tau)^2}{(p-1)^3}(2p-3)\right]$$

Taking the positive root only since $u > y$ we obtain

$$\theta(\tau) = u-y = \frac{\sqrt{6}}{c}\left[2 - \frac{(p-\tau)^2}{(p-1)^3}(2p-3)\right]^{\frac{1}{2}} \qquad \text{(E-94)}$$

Equation (E-50) gives $\ddot{B}$

$$\ddot{B} = \frac{6}{c^2} \frac{(u+y)}{(u-y)^3} - 1$$

Also Equation (E-71) relates $\dot{A}(\tau)$ to $\dot{B}(\tau)$

$$\dot{A}(\tau)u + \dot{B}(\tau) = p - \tau \qquad \text{(E-95)}$$

Differentiating (E-71) with respect to time τ we obtain

$$\ddot{A}(\tau)u + \dot{A}(\tau)\frac{du}{d\tau} + \ddot{B} = -1 \qquad \text{(E-96)}$$

By Equation (E-74)

$$\dot{A}(\tau) = \frac{(p-\tau)}{(u-y)} \qquad \text{(E-96a)}$$

By Equation (E-48)

$$\ddot{A}(\tau) = -\frac{12}{c^2} \frac{1}{(u-y)^3} \qquad \text{(E-96b)}$$

Replace $\ddot{A}(\tau)$ from Equation (E-48), $\dot{A}(\tau)$ from Equation (E-74), and $\ddot{B}$ from Equation (E-50) in the differential Equation (E-74)

$$-\frac{12}{c^2} \frac{u}{(u-y)^3} + \frac{(p-\tau)}{(u-y)} \frac{du}{d\tau} + \frac{6}{c^2} \frac{(u+y)}{(u-y)^3} - 1 = -1 \qquad \text{(E-97)}$$

or

$$\frac{(p-\tau)}{(u-y)}\frac{du}{d\tau} = \frac{6}{c^2}\frac{1}{(u-y)^2} \tag{E-98}$$

or

$$du = \frac{\sqrt{6}}{c}\frac{d\tau}{(p-\tau)\left[2 - \frac{(2p-3)}{(p-1)^3}(p-\tau)^2\right]^{\frac{1}{2}}} =$$

$$\frac{6}{c^2}\frac{d\tau}{(p-\tau)\frac{\sqrt{6}}{c}\left[2 - \frac{(2p-3)}{(p-1)^3}(p-\tau)^2\right]^{\frac{1}{2}}}$$

and by Equation (E-93)

$$du = \frac{6}{c^2}\frac{d\tau}{(p-\tau)\theta} \tag{E-99}$$

By Equation (E-93) θ is given in terms of τ

$$\theta = \frac{\sqrt{6}}{c}\left[2 - \frac{(2p-3)}{(p-1)^3}(p-\tau)^2\right]^{\frac{1}{2}}$$

$$\frac{d\theta}{d\tau} = \frac{(2p-3)}{(p-1)^3}\frac{\left(\frac{\sqrt{6}}{c}\right)(p-\tau)}{\left[2 - \frac{(2p-3)}{(p-1)^3}(p-\tau)^2\right]^{\frac{1}{2}}} = \left(\frac{6}{c^2}\right)\left(\frac{(2p-3)}{(p-1)^3}\right)\frac{(p-\tau)}{\theta} \tag{E-100}$$

$$\frac{d\theta}{d\tau} = \left(\frac{6}{c^2}\right)\frac{(2p-3)}{(p-1)^3}\,\frac{(p-\tau)}{\theta} \tag{E-101}$$

and since

$$d\theta = \frac{(2p-3)}{(p-1)^3}\,\frac{6}{c^2}(p-\tau)\,\frac{d\tau}{\theta} \tag{E-102}$$

$$d\tau = \frac{\theta d\theta}{(p-\tau)}\quad\frac{(p-1)^3}{(2p-3)}\quad\frac{c^2}{6} \tag{E-103}$$

and

$$du = \frac{6}{c^2}\,\frac{1}{(p-\tau)}\,\frac{1}{\theta}\quad\frac{\theta d\theta}{(p-\tau)}\quad\frac{(p-1)^3}{(2p-3)}\quad\frac{c^2}{6} = \frac{1}{(p-\tau)^2}\,\frac{(p-1)^3}{(2p-3)}d\theta \tag{E-104}$$

and solving Equation (E-94) for $(p-\tau)^2$ and replacing it in Equation (E-104), we get

$$(p-\tau)^2 = \frac{(p-1)^3}{(2p-3)}\left[2 - \frac{c^2}{6}\theta^2\right] \tag{E-105}$$

$$\frac{1}{(p-\tau)^2} = \frac{(2p-3)}{(p-1)^3}\,\frac{1}{\left[2 - \frac{c^2}{6}\theta^2\right]} \tag{E-106}$$

$$du = \frac{(2p-3)}{(p-1)^3}\,\frac{1}{\left[2 - \frac{c^2}{6}\theta^2\right]}\quad\frac{(p-1)^3}{(2p-3)}\,d\theta = \frac{d\theta}{\left[2 - \frac{c^2}{6}\theta^2\right]} = \frac{6}{c^2}\;\frac{d\theta}{\left[\frac{12}{c^2} - \theta^2\right]} \tag{E-107}$$

$$du = \frac{6}{c^2}\frac{d\theta}{\left(\frac{12}{c^2} - \theta^2\right)} = \left(\frac{6}{c^2}\right)\frac{c}{4\sqrt{3}}\left[\frac{1}{\left(\frac{2\sqrt{3}}{c} + \theta\right)} + \frac{1}{\left(\frac{2\sqrt{3}}{c} - \theta\right)}\right] d\theta =$$

$$\frac{\sqrt{3}}{2c}\left[\frac{1}{\left(\theta + \frac{2\sqrt{3}}{c}\right)} - \frac{1}{\left(\theta - \frac{2\sqrt{3}}{c}\right)}\right] d\theta \tag{E-107a}$$

$$u = \frac{\sqrt{3}}{2c}\log_e\left|\frac{\left(\theta + \frac{2\sqrt{3}}{c}\right)}{\left(\theta - \frac{2\sqrt{3}}{c}\right)}\right| + C_1 \tag{E-107b}$$

at $\tau = 1$, $\theta = u_o$, $C_1 = u_o - \frac{\sqrt{3}}{2c}\log_e\left|\frac{\left(u_o + \frac{2\sqrt{3}}{c}\right)}{\left(u_o - \frac{2\sqrt{3}}{c}\right)}\right|$ (E-107c)

Hence,

$$u = u_o + \frac{\sqrt{3}}{2c}\log_e\left|\frac{\left(\theta + \frac{2\sqrt{3}}{c}\right)\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(\theta - \frac{2\sqrt{3}}{c}\right)\left(u_o + \frac{2\sqrt{3}}{c}\right)}\right| \tag{E-107d}$$

Integrating once with respect to θ we get

$$u = \frac{6}{c^2}\int\frac{d\theta}{\left[\frac{12}{c^2} - \theta^2\right]} + C_1 = \frac{6}{c^2} \times \frac{c}{4\sqrt{3}}\log_e\left\{\frac{\left(\theta + \frac{2\sqrt{3}}{c}\right)}{\left(\theta - \frac{2\sqrt{3}}{c}\right)}\right\} + C_1 =$$

$$\frac{\sqrt{3}}{2c}\log_e\left\{\left|\frac{\left(\theta + \frac{2\sqrt{3}}{c}\right)}{\left(\theta - \frac{2\sqrt{3}}{c}\right)}\right|\right\} + C_1 \tag{E-108}$$

However, at $\tau = 1$, $\theta = u_o = \dfrac{\sqrt{6}}{c\sqrt{p-1}}$ and $u = u_o$ (since $y = o$ and $\theta = u-y$)

$$u_o - \frac{\sqrt{3}}{2c}\log_e\left\{\left|\frac{\left(u_o + \frac{2\sqrt{3}}{c}\right)}{\left(u_o - \frac{2\sqrt{3}}{c}\right)}\right|\right\} = C_1 \qquad \text{(E-109)}$$

Hence,

$$u = u_o + \frac{\sqrt{3}}{2c}\log_e\left\{\left|\frac{\left(\theta + \frac{2\sqrt{3}}{c}\right)\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(\theta - \frac{2\sqrt{3}}{c}\right)\left(u_o + \frac{2\sqrt{3}}{c}\right)}\right|\right\} \qquad \text{(E-110)}$$

and

$$y = u - \theta = u_o - \theta + \frac{\sqrt{3}}{2c}\log_e\left\{\left|\frac{\left(\theta + \frac{2\sqrt{3}}{c}\right)\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(\theta - \frac{2\sqrt{3}}{c}\right)\left(u_o + \frac{2\sqrt{3}}{c}\right)}\right|\right\} \qquad \text{(E-111)}$$

When $u = 1$, $\tau = \tau_1$, and $\theta = \theta_1$ and hence

$$u - u_o = 1 - u_o = \frac{\sqrt{3}}{2c}\log_e\left\{\left|\frac{\left(\theta_1 + \frac{2\sqrt{3}}{c}\right)\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(\theta_1 - \frac{2\sqrt{3}}{c}\right)\left(u_o + \frac{2\sqrt{3}}{c}\right)}\right|\right\} \qquad \text{(E-112)}$$

or

$$e^{\frac{2c}{\sqrt{3}}\left(1-u_o\right)} = \left|\frac{\left(\theta_1 + \frac{2\sqrt{3}}{c}\right)}{\left(\theta_1 - \frac{2\sqrt{3}}{c}\right)}\right| \left|\frac{\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(u_o + \frac{2\sqrt{3}}{c}\right)}\right| \qquad \text{(E-113)}$$

and by Equation (E-100)

$$\theta_1 = \frac{\sqrt{6}}{c}\left[2 - \frac{(2p-3)}{(p-1)^3}\left(p-\tau_1\right)^2\right]^{\frac{1}{2}} \tag{E-114}$$

$$\left|\frac{\left(u_o + \frac{2\sqrt{3}}{c}\right)}{\left(u_o - \frac{2\sqrt{3}}{c}\right)}\right| e^{\frac{2c}{\sqrt{3}}\left(1-u_o\right)} = \left|\frac{\left(\theta_1 + \frac{2\sqrt{3}}{c}\right)}{\left(\theta_1 - \frac{2\sqrt{3}}{c}\right)}\right| \tag{E-115}$$

Also

$$e^{\frac{2c}{\sqrt{3}}\left(1-u_o\right)} = \frac{\left(\theta_1 + \frac{2\sqrt{3}}{c}\right)\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(\theta_1 - \frac{2\sqrt{3}}{c}\right)\left(u_o + \frac{2\sqrt{3}}{c}\right)} \tag{E-116}$$

for $\theta_1 > \frac{2\sqrt{3}}{c}$, $u_o > \frac{2\sqrt{3}}{c}$

$$\left(\theta_1 + \frac{2\sqrt{3}}{c}\right)\left[\frac{\left(u_o + \frac{2\sqrt{3}}{c}\right)}{\left(u_o - \frac{2\sqrt{3}}{c}\right)}\right] e^{\frac{2c}{\sqrt{3}}\left(1-u_o\right)} = \theta_1 - \frac{2\sqrt{3}}{c} \tag{E-117}$$

$$\theta_1\left[\frac{\left(u_o + \frac{2\sqrt{3}}{c}\right)}{\left(u_o - \frac{2\sqrt{3}}{c}\right)} e^{\frac{2c\left(1-u_o\right)}{\sqrt{3}}} - 1\right] =$$

$$\frac{2\sqrt{3}}{c}\left[1 + \frac{\left(u_o + \frac{2\sqrt{3}}{c}\right)}{\left(u_o - \frac{2\sqrt{3}}{c}\right)} e^{\frac{2c\left(1-u_o\right)}{\sqrt{3}}}\right] \tag{E-118}$$

$$\theta_1 = \frac{2\sqrt{3}}{c}\left\{\frac{\left\{\frac{\left(u_o + \frac{2\sqrt{3}}{c}\right)}{\left(u_o - \frac{2\sqrt{3}}{c}\right)} e^{\frac{2c\left(1-u_o\right)}{\sqrt{3}}} + 1\right\}}{\left\{\frac{\left(u_o + \frac{2\sqrt{3}}{c}\right)}{\left(u_o - \frac{2\sqrt{3}}{c}\right)} e^{\frac{2c\left(1-u_o\right)}{\sqrt{3}}} - 1\right\}}\right\} \tag{E-119}$$

Multiply top and bottom with $e^{-c(1-u_o)/\sqrt{3}}$ to obtain

$$\theta_1 = \frac{2\sqrt{3}}{c}\frac{\left[u_o \cosh\left(\frac{c\left(1-u_o\right)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{c}\sinh\left(\frac{c\left(1-u_o\right)}{\sqrt{3}}\right)\right]}{\left[u_o \sinh\left(\frac{c\left(1-u_o\right)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{c}\cosh\left(\frac{c\left(1-u_o\right)}{\sqrt{3}}\right)\right]} \tag{E-120}$$

For the general case of any θ, using Equation (E-110) we obtain

$$\theta = \frac{2\sqrt{3}}{c}\frac{\left[u_o \cosh\left(\frac{c\left(u-u_o\right)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{c}\sinh\left(\frac{c\left(u-u_o\right)}{\sqrt{3}}\right)\right]}{\left[u_o \sinh\left(\frac{c\left(u-u_o\right)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{c}\cosh\left(\frac{c\left(u-u_o\right)}{\sqrt{3}}\right)\right]} \tag{E-121}$$

Equation (E-120) can be written in the alternative form

$$\theta_1 = \frac{2\sqrt{3}}{c}\frac{\left[\coth\left(\frac{c\left(1-u_o\right)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{cu_o}\coth\left(\frac{c\left(1-u_o\right)}{\sqrt{3}}\right)\right]} \tag{E-122}$$

Solving Equation (E-100) for τ_1 we obtain

$$\tau_1 = p - \sqrt{\frac{(p-1)^3}{(2p-3)} \left\{2 - \frac{c}{6}\theta_1^2\right\}} \tag{E-123}$$

We distinguish three cases

1. $2 - \frac{c^2}{6}\theta_1^2 = o$ (it will turn out that $p = \frac{3}{2}$ and $c^2 > 12$) (E-124)

2. $2 - \frac{c^2}{6}\theta_1^2 > o$ for $p > \frac{3}{2}$ (E-125)

3. $2 - \frac{c^2}{6}\theta_1^2 < o$ for $p < \frac{3}{2}$ (E-126)

CASE 1

We now examine case $\theta = \frac{2\sqrt{3}}{c}$

The differential Equation (E-82) reduces to

$$-(p-\tau)\theta\frac{d\theta}{d\tau} = \theta^2 - \frac{12}{c^2} = 0 \qquad \text{(E-127)}$$

or

$$\frac{d\theta}{d\tau} = 0 \qquad \text{(E-128)}$$

Since $p \neq \tau$ and $\theta \neq 0$

Equation (E-98) reduces to

$$du = \frac{6}{c^2}\,\frac{1}{(p-\tau)}\,\frac{d\tau}{\theta} \qquad \text{(E-129)}$$

when θ is a constant, i.e.,

$$du = \frac{6}{c^2}\,\frac{1}{\frac{2\sqrt{3}}{c}}\,\frac{d\tau}{(p-\tau)} = \frac{\sqrt{3}}{c}\,\frac{d\tau}{(p-\tau)} \qquad \text{(E-130)}$$

and integrating

$$u = \frac{\sqrt{3}}{c} \left\{ \log_e \left| p-\tau \right| + \log_e C \right\} \tag{E-131}$$

At time $\tau = 1$, $u = u_o$, i.e.,

$$u_o = \frac{\sqrt{3}}{c} \left\{ \log_e \left| p-1 \right| + \log_e C \right\} \tag{E-132}$$

i.e.,

$$u = \frac{\sqrt{3}}{c} \left\{ \log_e \left| \frac{(p-\tau)}{(p-1)} \right| \right\} + u_o$$

And

$$y = u - \theta = u_o - \theta + \frac{\sqrt{3}}{c} \log_e \left| \frac{(p-\tau)}{(p-1)} \right| =$$

$$\frac{\sqrt{6}}{c\sqrt{p-1}} - \frac{2\sqrt{3}}{c} + \frac{\sqrt{3}}{c} \log_e \left| \frac{(p-\tau)}{(p-1)} \right| \tag{E-133}$$

However, at $\tau = 1$, $y = 0$. This leads to $p = \frac{3}{2}$ and since

$$p > 1 + \frac{6}{c^2} \tag{E-134}$$

$$c^2 > 12 \tag{E-135}$$

By Equation (E-74)

$$\dot{A}(\tau) = \frac{(p-\tau)}{u-y} = \frac{(p-\tau)}{\theta} = \frac{c(p-\tau)}{2\sqrt{3}} \qquad \text{(E-136)}$$

By Equation (E-48)

$$\ddot{A}(\tau) = -\frac{12}{c^2}\frac{1}{(u-y)^3} = -\frac{12}{c^2}\frac{1}{\theta^3} = -\frac{c}{2\sqrt{3}} \qquad \text{(E-137)}$$

By Equation (D-72)

$$\dot{B}(\tau) = -\dot{A}(\tau)y = -\frac{c(p-\tau)}{2\sqrt{3}}y \qquad \text{(E-138)}$$

By Equation (E-50)

$$\ddot{B} = \frac{6}{c^2}\frac{(u+y)}{(u-y)^3} - 1 = \frac{c}{4\sqrt{3}}\left[2u_o - \theta + \frac{2\sqrt{3}}{c}\log_e\left|\frac{p-\tau}{p-1}\right|\right] - 1 \qquad \text{(E-139)}$$

Therefore,

$$p = \frac{3}{2},\ p > \tau,\ c^2 > 12 \qquad \text{(E-140)}$$

$$u_o = \frac{\sqrt{6}}{c\sqrt{p-1}} \qquad \text{(E-141)}$$

$$\theta = \frac{2\sqrt{3}}{c} \qquad \text{(E-142)}$$

$$u = u_o + \frac{\sqrt{3}}{c}\log_e\left|\frac{(p-\tau)}{(p-1)}\right| \qquad \text{(E-143)}$$

$$y = u_o - \theta + \frac{\sqrt{3}}{c}\log_e\left|\frac{(p-\tau)}{(p-1)}\right| \qquad \text{(E-144)}$$

and adding and subtracting θ inside the square brackets $\ddot{B}$ can be written as

$$\ddot{B} = \frac{c}{2\sqrt{3}}y - \frac{1}{2} \qquad \text{(E-145)}$$

Since

$$\dot{B}(\tau) = -\frac{c}{2\sqrt{3}}(p-\tau)y \qquad \text{(E-146)}$$

$$\ddot{B} = -\frac{c}{2\sqrt{3}}\left[-y + (p-\tau)\frac{d}{d\tau}y\right] = \frac{c}{2\sqrt{3}}y - \frac{1}{2} \qquad \text{(E-147)}$$

as above.

Also by Equation (E-68) the velocity profile in the second interval $y \leq x \leq u$ becomes

$$\dot{w} = \dot{A}(\tau)x + \dot{B}(\tau) = \frac{c}{2\sqrt{3}}(p-\tau)(x-y) \qquad \text{(E-148)}$$

and the acceleration

$$\ddot{w} = \ddot{A}(\tau)x + \ddot{B}(\tau) = -\frac{c}{2\sqrt{3}}x + \frac{c}{2\sqrt{3}}y - \frac{1}{2} = \frac{c}{2\sqrt{3}}(-x + y) - \frac{1}{2} \qquad \text{(E-149)}$$

When u reaches the midpoint, let the time taken be indicated by τ_1. Then we will have

$$u_1 - u_o = \frac{\sqrt{3}}{c}\log_e\left|\frac{(p-\tau_1)}{(p-1)}\right| \tag{E-150}$$

i.e.,

$$\left|\frac{(p-\tau_1)}{(p-1)}\right| = e^{\frac{c}{\sqrt{3}}(1-u_o)} \tag{E-151}$$

i.e.,

$$\left|(p-\tau_1)\right| = [p-1]\ e^{\frac{c}{\sqrt{3}}(1-u_o)} \tag{E-152}$$

If

$$p > \tau_1 \tag{E-153}$$

$$p - (p-1)\ e^{\frac{c}{\sqrt{3}}(1-u_o)} = \tau_1 \tag{E-154}$$

$$p\left[1 - e^{\frac{c}{\sqrt{3}}(1-u_o)}\right] + e^{\frac{c}{\sqrt{3}}(1-u_o)} = \tau_1 \tag{E-155}$$

If, on the other hand, $p < \tau_1$

$$\tau_1 - p = (p-1)\, e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} \qquad \text{(E-156)}$$

$$\tau_1 = p + (p-1)\, e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} = p\left(1 + e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)}\right) - e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} \qquad \text{(E-157)}$$

Hence,

$$\text{if} \quad p > \tau_1,\ \tau_1 = p - (p-1)\, e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} \qquad \text{(E-158)}$$

$$\text{if} \quad p < \tau_1,\ \tau_1 = p + (p-1)\, e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} \qquad \text{(E-159)}$$

and always $\tau_1 \geq 1$

It turns out that the conditions

$$p > \tau_1 > 1 \qquad \text{(E-160)}$$

reduce to

$$p > p - (p-1)\, e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} > 1 \qquad \text{(E-161)}$$

or

$$0 < e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} \quad \text{which is satisfied} \tag{E-162}$$

and

$$e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} < 1 \tag{E-163}$$

which cannot be satisfied for $0 < u_o \leq 1$

Hence for $p = \frac{3}{2}$, $p > \tau_1$, and $\tau_1 \geq 1$, the postulated profile violates the flow rule. While for $p = \frac{3}{2}$, $p < \tau_1$, and $\tau_1 \geq 1$

$$e^{\frac{c}{\sqrt{3}}\left(1-u_o\right)} > 0 \tag{E-164}$$

is satisfied for

$$0 \leq u_o \leq 1.$$

Equation (E-148) for the velocity profile and Equation (E-9) ($\dot{w} \geq o$) imply that the inequality is violated and, therefore, a different velocity profile must be assumed for this case. This particular case will not be further studied here, since other considerations, such as inclusion of nonlinearities, have priority.

CASE 2

Corresponding to a value of θ, call it θ_1 $\left(\text{for time } \tau = \tau_1\right)$ and $u = 1$, there exists a value of y, call it y_1, such that

$$y_1 = 1 - \theta_1 \qquad \text{(E-165)}$$

i.e.,

$$0 < 1 - \theta_1 = y_1 < 1 \qquad \text{(E-166)}$$

with

$$1 > \theta_1 > 0 \qquad \text{(E-167)}$$

By Equations (E-125) and (E-167), therefore,

$$\theta_1 < \min \left\{1, \frac{2\sqrt{3}}{c}\right\} \qquad \text{(E-168)}$$

with $\quad p > \frac{3}{2}$

CASE 3

Corresponding to Equations (E-126) and (E-167) we must have

$$\frac{2\sqrt{3}}{c} < \theta_1 < 1 \tag{E-169}$$

for $p < \frac{3}{2}$

Consequently, we combine cases 2 and 3, and treat them as a single case, i.e., for

$$p \neq \frac{3}{2} \text{ and } p > \tau.$$

We proceed now to calculate the displacement distribution for times τ such that $1 \leq \tau \leq \tau_1$ and τ_1 are given by Equation (E-123). For this purpose, we observe that at time $\tau = 1$ the displacement has an initial value, depending on the position of the point x at which it is considered. This initial value is given by the following table:

INTERVAL	INITIAL CONDITION ON DISPLACEMENT W
$0 \leq x \leq y$	$\frac{1}{2}\frac{(p-1)}{u_o}x$
$y \leq x \leq u_o$	$\frac{1}{2}\frac{(p-1)}{u_o}x$
$u_o \leq x \leq u$	$\frac{1}{2}(p-1)$
$u \leq x \leq 1$	$\frac{1}{2}(p-1)$

Therefore

1. For $0 \leq x \leq y$ and $\tau \geq 1$, and by Equation (E-21)

$$w(x,\tau) = \frac{1}{2}\frac{(p-1)}{u_o}x \qquad \text{(E-170)}$$

2. For $y \leq x \leq u_o$ and $\tau \geq 1$

$$w(x,\tau) = \frac{1}{2}\frac{(p-1)}{u_o}x + \int_{\tau=1}^{\tau} \dot{w}\,d\tau \qquad \text{(E-171)}$$

However, by Equations (E-32), (E-74), and (E-72)

$$\dot{w}(x,\tau) = (p-\tau)\,\frac{(x-y)}{(u-y)} \qquad \text{(E-172)}$$

and, therefore, Equation (E-171) becomes

$$w(x,\tau) = \frac{1}{2}\frac{(p-1)}{u_o}x + \frac{c^2}{6}\frac{(p-1)^3}{(2p-3)}\Bigg[\left(x-u_o\right)\left(\theta-u_o\right) + \frac{1}{2}\left(\theta^2-u_o^{\,2}\right) - \frac{\sqrt{3}}{2c}\left(\theta-u_o\right)\log_e\left|\frac{\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(u_o + \frac{2\sqrt{3}}{c}\right)}\right| - \frac{\sqrt{3}}{2c}\left\{\left(\theta + \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(\theta + \frac{2\sqrt{3}}{c}\right)\right| - \left(u_o + \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(u_o + \frac{2\sqrt{3}}{c}\right)\right| - \left(\theta - \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(\theta - \frac{2\sqrt{3}}{c}\right)\right| + \left(u_o - \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(u_o - \frac{2\sqrt{3}}{c}\right)\right|\right\}\Bigg] \qquad \text{(E-173)}$$

3. For $u_o \leq x \leq u$ and $\tau \geq 1$, the expression for $w(x,\tau)$ is like the one given by Equation (E-173) except that instead of $\frac{1}{2}\frac{(p-1)}{u_o}x$, the first term is $\frac{1}{2}(p-1)$.

4. For $u \leq x \leq 1$ and $\tau \geq 1$ we must proceed as follows:

At time $\tau = \tau^*$, when $x = u = u^*$ and $\theta = \theta(u) = \theta(u^*) = \theta^*$ the displacements in the third and fourth intervals must agree. This means that

$$w(u^*,\tau^*) = \int_{\tau=1}^{\tau=\tau^*} \dot{w}\,d\tau + C_1 \tag{E-174}$$

However, by Equation (E-62)

$$w(u^*,\tau^*) = \int_{\tau=1}^{\tau=\tau^*} (p-\tau)\,d\tau + C_1 \tag{E-175}$$

and also

$$w(u^*,\tau^*) = \frac{1}{2}(p-1) + \frac{c^2}{6}\,\frac{(p-1)^3}{(2p-3)}\left[\left(u^*-u_o\right)\left(\theta^*-u_o\right) + \frac{1}{2}\left(\theta^{*2}-u_o^2\right) - \right.$$

$$\frac{\sqrt{3}}{2c}\left(\theta^*-u_o\right)\log_e\left|\frac{\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(u_o + \frac{2\sqrt{3}}{c}\right)}\right| - \frac{\sqrt{3}}{2c}\left\{\left(\theta^* + \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(\theta^* + \frac{2\sqrt{3}}{c}\right)\right| - \right.$$

$$\left(u_o + \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(u_o + \frac{2\sqrt{3}}{c}\right)\right| - \left(\theta^* - \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(\theta^* - \frac{2\sqrt{3}}{c}\right)\right| +$$

$$\left.\left.\left(u_o - \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(u_o - \frac{2\sqrt{3}}{c}\right)\right|\right\}\right] = p\left(\tau^*-1\right) - \frac{1}{2}\left(\tau^{*2}-1\right) + C_1 \qquad \text{(E-176)}$$

Equation (E-176) determines C_1 in terms of θ^*, u^*, and τ^*.

Therefore,

$$w(x,\tau) = \frac{1}{2}(p-1) + p(\tau-\tau^*) - \frac{1}{2}\left(\tau^2-\tau^{*2}\right) +$$

$$\frac{c}{6}\,\frac{(p-1)^3}{(2p-3)}\left[\left(u^*-u_o\right)\left(\theta^*-u_o\right) + \frac{1}{2}\left(\theta^{*2}-u_o^2\right) - \right.$$

$$\frac{\sqrt{3}}{2c}\left(\theta^*-u_o\right)\log_e\left|\frac{\left(u_o - \frac{2\sqrt{3}}{c}\right)}{\left(u_o + \frac{2\sqrt{3}}{c}\right)}\right| - \frac{\sqrt{3}}{2c}\left\{\left(\theta^* + \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(\theta^* + \frac{2\sqrt{3}}{c}\right)\right| - \right.$$

$$\left(u_o + \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(u_o + \frac{2\sqrt{3}}{c}\right)\right| - \left(\theta^* - \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(\theta^* - \frac{2\sqrt{3}}{c}\right)\right| +$$

$$\left.\left.\left(u_o - \frac{2\sqrt{3}}{c}\right)\log_e\left|\left(u_o - \frac{2\sqrt{3}}{c}\right)\right|\right\}\right] \qquad \text{(E-177)}$$

The previous analysis is valid until the first hinge circle, located by u (see Figure E-2) reaches the midpoint, i.e., until $u(\tau_1) = 1$. Let the corresponding time be denoted by τ_1 and the corresponding values of y, θ, and u be denoted by y_1, θ_1, and u_1 (=1), respectively, i.e.,

$$y_1 = y(\tau_1) \tag{E-178}$$

$$u_1 = u(\tau_1) = 1 \tag{E-179}$$

$$\theta_1 = \theta(\tau_1) \tag{E-180}$$

and θ_1 will be given by Equation (E-113), (E-120), or (E-122), while τ_1 will be obtained by Equation (E-114) or (E-123). Equation (E-111) defines y_1. At that instant in time, the velocity profile is given by Figure E-3. A different assumption must be made for the motion to continue. This must be so, since $y(\tau)$, where the other hinge is located, varies with time and has not reached the midpoint yet. We must also satisfy all geometrical inequalities, such as $y \geq 0$ and $0 \leq \theta \leq 1$ for the solution to be valid.

2. $\tau_1 \leq \tau \leq \tau_o$

The analysis of long shells under low loading (Appendix C) applies in this case. We observe that the initial conditions are different then the ones in Appendix C. Both velocity and displacement profiles must agree for times $\tau = \tau_1$, which represent our starting time for this interval.

We consider two intervals.

a. For $0 \leq x \leq y$ (along AD on the yield surface)

we must have

$$\dot{w} = 0 \tag{E-181}$$

$$\dot{w}'' = 0 \tag{E-182}$$

$$m_x = -1 \tag{E-183}$$

$$p = 0 \tag{E-184}$$

and by Equation (E-3)

$$\dot{w}(x,\tau_1) = 0 \tag{E-185}$$

and by Equation (E-25)

$$w(x,\tau_1) = \frac{1}{2}\frac{(p-1)}{u_o}x \tag{E-186}$$

Also the equilibrium equation is

$$\frac{1}{2c^2}m_x'' + n_\phi + p - \ddot{w} = 0 \tag{E-187}$$

Therefore,

$$n_\phi = o \tag{E-188}$$

$$\ddot{w} = \dot{w} = o \tag{E-189}$$

and, hence. displacement is independent of time τ, but dependent on location x, i.e.,

$$w(x,\tau) = C(x) \tag{E-190}$$

This constant is the value of the displacement from the previous time range, i.e.,

$$C(x) = \frac{1}{2}\frac{(p-1)}{u_o}x \tag{E-191}$$

Thus,

$$w(x,\tau) = \frac{1}{2}\,\frac{(p-1)}{u_o}x \tag{E-192}$$

b. For $y \leq x \leq 1$ (along AB on yield surface)

we must have

$$\dot{w} \geq 0 \tag{E-193}$$

$$\dot{w}'' = 0 \tag{E-194}$$

$$n_\phi = -1 \tag{E-195}$$

$$-1 \leq m_x \leq 1 \tag{E-196}$$

$$p = 0 \tag{E-197}$$

with initial condition

$$\dot{w}\left(x,\tau_1\right) = \left(p-\tau_1\right)\left[\frac{x-y\left(\tau_1\right)}{1-y\left(\tau_1\right)}\right] \tag{E-198}$$

We must satisfy the equilibrium equation

$$\frac{1}{2c^2}m_x'' + n_\phi + p - \ddot{w} = 0 \tag{E-199}$$

and assuming a distribution of velocity of the form

$$\dot{w}(x,\tau) = \dot{A}x + \dot{B} \tag{E-200}$$

$$\ddot{w}(x,\tau) = \ddot{A}x + \ddot{B} \tag{E-201}$$

and, hence, replacing $\ddot{w}$ in Equation (E-199) we get

$$\frac{1}{2c^2}m_x'' - 1 = \ddot{A}x + \ddot{B} \tag{E-202}$$

As before, integrating we obtain

$$m_x' = 2c^2x + 2c^2\left[\frac{1}{2}\ddot{A}x^2 + \ddot{B}x\right] + C_1 \tag{E-203}$$

$$m_x = c^2x^2 + c^2\left[\frac{1}{3}\ddot{A}x^3 + \ddot{B}x^2\right] + C_1\ x + C_2 \tag{E-204}$$

We must satisfy the following boundary conditions

$$m_x(y,\tau) = -1 \tag{E-205}$$

$$m_x'(y,\tau) = 0 \tag{E-206}$$

$$m_x(1,\tau) = 1 \tag{E-207}$$

$$m_x'(1,\tau) = 0 \tag{E-208}$$

Therefore,

$$c^2\left[\frac{1}{3}\ddot{A}\ y^3 + (1+\ddot{B})y^2\right] + C_1 y + C_2 = -1 \tag{E-209}$$

$$c^2\left[\ddot{A}\ y^2 + 2(1+\ddot{B})y\right] + C_1 = 0 \tag{E-210}$$

$$c^2[\ddot{A} + 2(1+\ddot{B})] + C_1 = 0 \tag{E-211}$$

$$c^2\left[\frac{1}{3}\ddot{A} + (1+\ddot{B})\right] + C_1 + C_2 = 1 \tag{E-212}$$

After similar operations, we obtain

$$\ddot{A} = -\ 2(1+\ddot{B})\frac{(y-1)}{(y^2-1)} = -\ 2(1+\ddot{B})\frac{1}{(1+y)} \tag{E-213}$$

and for $y \neq 1$

$$1 + \ddot{B} = -\ \frac{1}{2}(1+y)\ddot{A} \tag{E-214}$$

and

$$\ddot{A} = -\ \frac{12}{c^2}\ \frac{1}{(1-y)^3} \tag{E-215}$$

$$1 + \ddot{B} = \frac{6}{c^2}\ \frac{(1+y)}{(1-y)^3} \tag{E-216}$$

and

$$C_1 = -\ \frac{12}{(1-y)^3}y \tag{E-217}$$

$$C_2 = 1 - \frac{2(1-3y)}{(1-y)^3} \tag{E-218}$$

The bending moment distribution, Equation (E-204), becomes

$$m_x = 1 + \frac{1}{(1-y)^3}\left[-4x^3 + 6(1+y)x^2 - 12y\,x - 2(1-3y)\right] \tag{E-219}$$

When $x \rightarrow y$ (for $\tau > \tau_1$) the velocity vanishes, i.e.,

$$\dot{w}(y,\tau) = 0 \tag{E-220}$$

or

$$\dot{A}(\tau)y + \dot{B}(\tau) = 0 \tag{E-221}$$

However, y is a function of time τ. Similar differentiations, as with Equation (C-36) gives

$$\ddot{A}y + \dot{A}\dot{y} + \ddot{B} = 0 \tag{E-222}$$

or

$$\dot{A} = \frac{1}{\dot{y}} - \frac{6}{c^2}\,\frac{1}{\dot{y}(1-y)^2} \tag{E-223}$$

and the differential equation

$$\frac{\ddot{y}}{\dot{y}} = 0 \tag{E-224}$$

or $\ddot{y} = 0$ (E-225)

since $\dot{y} \neq 0$

Therefore, $y(\tau)$ is a linear function of time τ and such that at time $\tau = \tau_1$ it equals y_1 $(y(\tau_1))$

$$y(\tau) = E_1\left(\tau - \tau_1\right) + E_2 \tag{E-226}$$

$$E_2 = y\left(\tau_1\right) = y_1 \tag{E-227}$$

$$\dot{y}(\tau) = E_1 \tag{E-228}$$

Comparing the x coefficient of Equation (E-198), which in fact is $\dot{A}$, with Equation (E-223), for times $\tau = \tau_1$, we have

$$\frac{6}{c^2}\left[\frac{c^2}{6} - \frac{1}{\left(1-y_1\right)^2}\right]\frac{1}{\dot{y}_1} = \frac{\left(p-\tau_1\right)}{\left(1-y_1\right)} \tag{E-229}$$

or

$$\dot{y}_1 = \frac{6}{c^2}\left[\frac{c^2}{6} - \frac{1}{\left(1-y_1\right)^2}\right]\frac{\left(1-y_1\right)}{\left(p-\tau_1\right)} \tag{E-230}$$

But

$$E_1 = \dot{y} = \dot{y}_1 \tag{E-231}$$

We also observe that the velocity of the hinge circle is given by

$$\dot{y}(\tau) = \dot{y}_1 = E_1 = \left[1 - \frac{6}{c^2}\frac{1}{\left(1-y_1\right)^2}\right]\frac{\left(1-y_1\right)}{\left(p-\tau_1\right)} \tag{E-232}$$

Also at time $\tau = \tau_1$ the velocities for all $y \leq x \leq 1$ must agree, i.e.,

$$\dot{w}\left(x,\tau_1\right) = \dot{A}\left(\tau_1\right)x + \dot{B}\left(\tau_1\right) = \left(p-\tau_1\right)\frac{\left(x-y_1\right)}{\left(1-y_1\right)} \tag{E-233}$$

i.e.,

$$\dot{A}\left(\tau_1\right) = \frac{\left(p-\tau_1\right)}{\left(1-y_1\right)} \tag{E-234}$$

$$\dot{B}\left(\tau_1\right) = -\frac{\left(p-\tau_1\right)}{\left(1-y_1\right)}y_1 \tag{E-235}$$

and, therefore,

$$y(\tau) = \left[1 - \frac{6}{c^2}\frac{1}{\left(1-y_1\right)^2}\right]\frac{\left(1-y_1\right)}{\left(p-\tau_1\right)}\left(\tau-\tau_1\right) + y_1 \tag{E-236}$$

and the velocity distribution becomes

$$\dot{w} = \dot{A}(\tau)x + \dot{B}(\tau) = \dot{A}(\tau)(x-y) =$$

$$\frac{\left(p-\tau_1\right)}{\left(1-y_1\right)\left[1 - \frac{6}{c^2}\frac{1}{\left(1-y_1\right)^2}\right]}\left[(x-y) - \frac{6}{c^2}\frac{(x-y)}{(1-y)^2}\right] \tag{E-237}$$

Except the case when either $x = y$ or $p = \tau_1$, the velocity becomes zero when

$$1 \mp \frac{\sqrt{6}}{c} = y$$

Thus

$$\dot{w}(x,\tau) = \frac{(1-y_1)}{(1-y)^2} \frac{\left[c^2(1-y)^2-6\right]}{\left[c^2\left(1-y_1\right)^2-6\right]} \left(p-\tau_1\right)(x-y) \qquad \text{(E-238)}$$

The acceleration is given by

$$\ddot{w} = \frac{\left(p-\tau_1\right)}{\left(1-y_1\right)\left[1 - \frac{6}{c^2} \frac{1}{\left(1-y_1\right)^2}\right]} \left[-\dot{y} - \frac{6}{c^2} \frac{d}{d\tau} \left\{\frac{(x-y)}{(1-y)^2}\right\}\right] =$$

$$- \left[1 + \frac{6}{c^2}\left\{\frac{2(x-y)}{(1-y)^3} - \frac{1}{(1-y)^2}\right\}\right] \qquad \text{(E-239)}$$

For consistency $\dot{w} \geq 0$. Since $1 > y_1$, $1 > y$, and $x > y$, we must also have

$$p \geq \tau_1 \qquad \text{(E-240)}$$

$$1 - \frac{6}{c^2} \frac{1}{(1-y)^2} > 0 \qquad \text{(E-241)}$$

and $$1 - \frac{6}{c^2} \frac{1}{\left(1-y_1\right)^2} > 0 \qquad \text{(E-242)}$$

i.e., $$y < 1 - \frac{\sqrt{6}}{c} \qquad \text{(E-243)}$$

$$y_1 < 1 - \frac{\sqrt{6}}{c} \qquad \text{(E-244)}$$

At

$$y = 1 - \frac{\sqrt{6}}{c} \tag{E-245}$$

the velocity becomes zero. The time taken, indicated by τ_o, is given by

$$\tau_o = \tau_1 + \frac{\left\{1 - \frac{\sqrt{6}}{c} - y_1\right\}\left(p-\tau_1\right)}{\left\{1 - \frac{6}{c^2}\frac{1}{\left(1-y_1\right)^2}\right\}\left(1-y_1\right)} = \tau_1 + \frac{\left(p-\tau_1\right)\left(1-y_1\right)}{\left[1-y_1 + \frac{\sqrt{6}}{c}\right]} \tag{E-246}$$

Table E-6 summarizes the results for the interval $\tau_1 \leq \tau \leq \tau_o$ and points lying in $0 \leq x \leq y$.

The objective here is to obtain the displacement distribution in the interval $y \leq x \leq 1$ for times $\tau \geq \tau_1$. To do this we observe that at $x = y$ the displacements must be equal. Therefore

$$w(y,\tau) = \frac{1}{2}\frac{(p-1)}{u_o}y\left(\tau_1\right) + \int_{\tau=\tau_1}^{\tau} \dot{w}d\tau \tag{E-247}$$

Using Equation (E-237) written in a different format

$$\int_{\tau_1}^{\tau} \dot{w}d\tau = \frac{\left(1-y_1\right)\left(p-\tau_1\right)}{\left[c^2\left(1-y_1\right)^2-6\right]}\int_{\tau_1}^{\tau}\left[c^2 - \frac{6}{(1-y)^2}\right](x-y)d\tau \tag{E-248}$$

we get

$$A_1 = x-y_1 + \left[1 - \frac{6}{c^2\left(1-y_1\right)^2}\right]\frac{\left(1-y_1\right)}{\left(p-\tau_1\right)}\ \tau_1 \qquad \text{(E-249)}$$

$$B_1 = \left[1 - \frac{6}{c^2}\ \frac{1}{\left(1-y_1\right)^2}\right]\frac{\left(1-y_1\right)}{\left(p-\tau_1\right)} \qquad \text{(E-250)}$$

$$C_1 = 1 - y_1 + \left[1 - \frac{6}{c^2}\ \frac{1}{\left(1-y_1\right)^2}\right]\frac{\left(1-y_1\right)}{\left(p-\tau_1\right)} \qquad \text{(E-251)}$$

$$D_1 = \frac{\left(1-y_1\right)\left(p-\tau_1\right)}{\left[c^2\left(1-y_1\right)^2-6\right]} \qquad \text{(E-252)}$$

where $\quad C_1 - A_1 = 1 - x \qquad$ (E-253)

Finally, Equation (E-247) assumes the form

$$w(y,\tau) = \frac{1}{2}\ \frac{(p-1)}{u_o} y\left(\tau_1\right) + D_1\left[c^2\left\{A_1\left(\tau-\tau_1\right) - \frac{1}{2}B_1\left(\tau^2-\tau_1^2\right)\right\} + \right.$$

$$\left. 6\left\{\frac{1}{B_1}\log_e\left|\frac{\left(C_1-B_1\tau\right)}{\left(C_1-B_1\tau_1\right)}\right| + \frac{(1-x)}{B_1}\left\{\frac{1}{\left(C_1-B_1\tau\right)} - \frac{1}{\left(C_1-B_1\tau_1\right)}\right\}\right\}\right] \qquad \text{(E-254)}$$

and the displacement at which the velocity vanishes is given by

$$w\left(y_o,\tau_o\right), \text{ where } \quad y_o = 1 - \frac{\sqrt{6}}{c}$$

and τ_o is given by Equation (E-246).

Table E-7 summarizes these results.

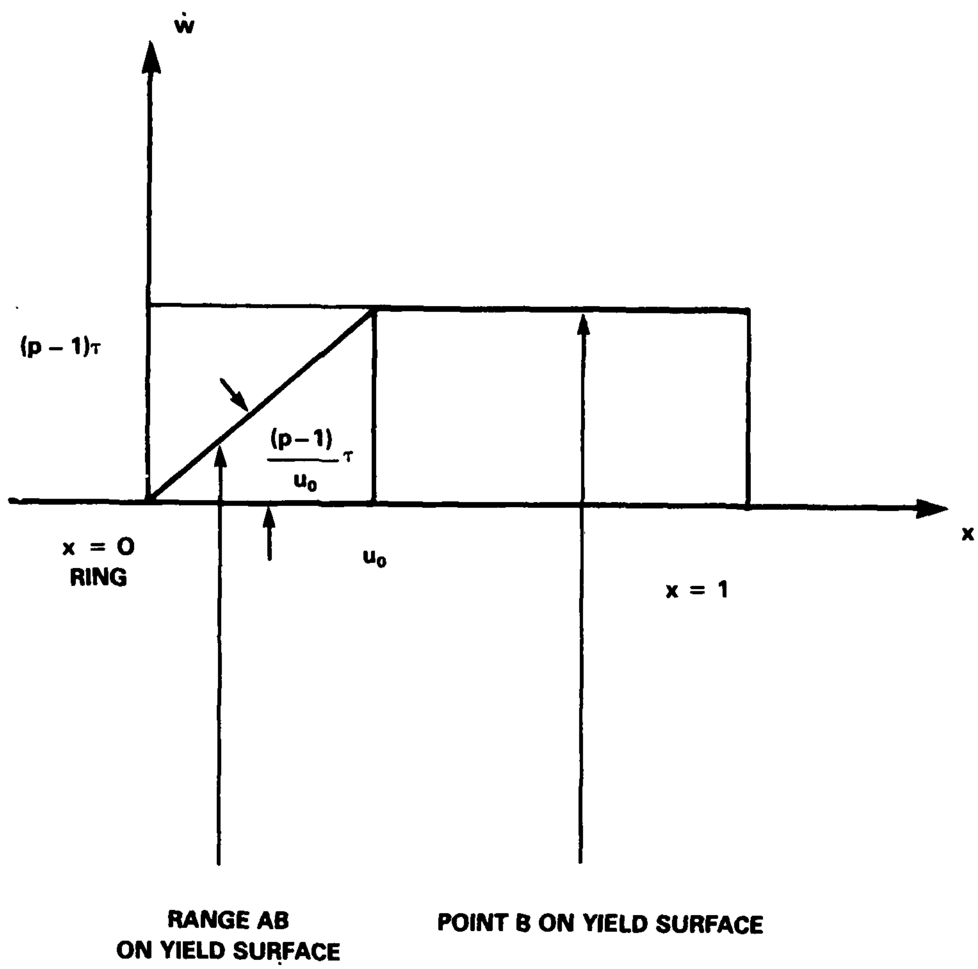


FIGURE E-1. VELOCITY PROFILE FOR LONG SHELLS AND HIGH PRESSURES FOR IN $0 < \tau \leq 1$

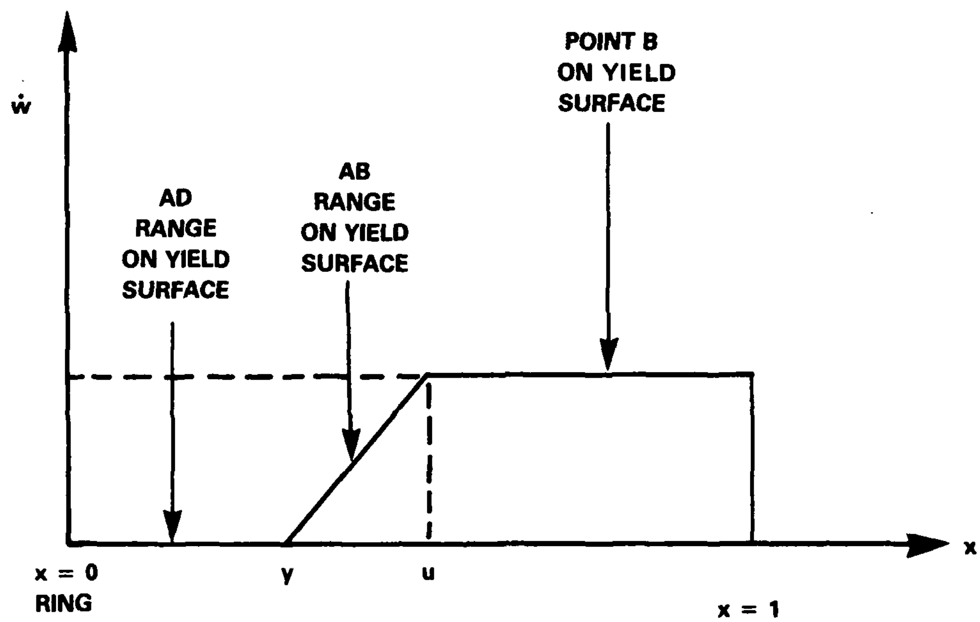


FIGURE E-2. ASSUMED VELOCITY PROFILE FOR LONG SHELLS AND HIGH LOADS FOR $1 \leqslant \tau \leqslant \tau_o$

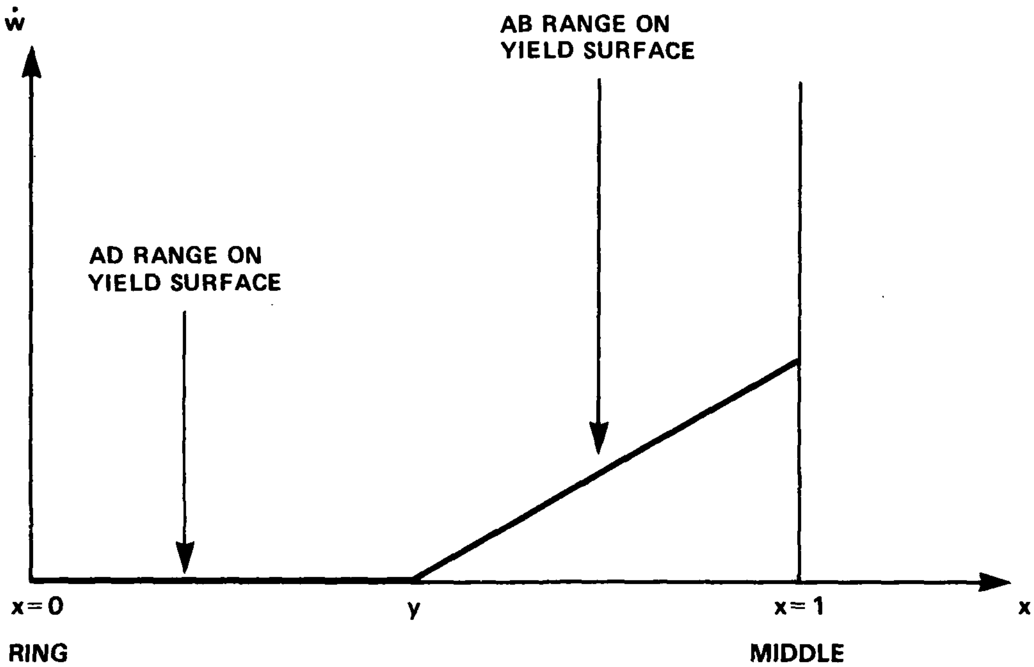


FIGURE E-3. VELOCITY PROFILE FOR LONG SHELLS AND HIGH PRESSURES $\left(p \neq \frac{3}{2}\right)$ FOR $1 \leqslant \tau \leqslant \tau_1$

TABLE E-1. SUMMARY, LONG SHELLS, HIGH LOADING, 1

TYPE	SHELL TYPE	AB PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH LOAD $p > 1 + \frac{6}{C^2}$
CONDITIONS	$u_o^2 = \frac{6}{C^2(p-1)}$	$0 \leqslant \tau \leqslant 1$ $0 \leqslant x \leqslant u_0$
MOMENT RESULTANT	$m_x(x,\tau) = 2\left(\frac{x}{u_o} - 1\right)^3 + 1$ or $= 2\left(\frac{x}{u_o}\right)^3 - 6\left(\frac{x}{u_0}\right)^2 + 6\left(\frac{x}{u_0}\right) - 1$	POINTS ALONG AB ON TRESCA SQUARE
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w(x,\tau) = \frac{1}{2}\frac{(p-1)}{u_o}\tau^2 x$	
VELOCITY	$\dot{w} = \frac{(p-1)}{u_o}\tau x$	
ACCELERATION	$\ddot{w} = \frac{(p-1)}{u_o}x$	
TIME τ_O	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE E-2. SUMMARY, LONG SHELLS, HIGH LOADING, 2

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH LOAD $p > 1 + \frac{6}{C^2}$
CONDITIONS	$u_o^2 = \frac{6}{C^2(p-1)}$	$0 \leq \tau \leq 1$ $u_0 < x \leq 1$
MOMENT RESULTANT	$m_x = 1$	POINT B ON TRESCA SQUARE
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	$w = \frac{1}{2}(p-1)\,\tau^2$	
VELOCITY	$\dot{w} = (p-1)\,\tau$	
ACCELERATION	$\ddot{w} = p-1$	
TIME τ_O	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE E-3. SUMMARY, LONG SHELLS, HIGH LOADING, 3

TYPE	SHELL TYPE	AB PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH $p > 1 + \frac{6}{C^2}$ $(p \neq \frac{3}{2})$
CONDITIONS	$o \leqslant x \leqslant y \qquad 1 \leqslant \tau \leqslant \tau_1$ $u_o = \sqrt{\frac{6}{C^2(p-1)}} \qquad \tau_1 = p - \sqrt{\frac{(p-1)^3}{(2p-3)}} \left\{2 - \frac{C^2}{6}\Theta_1^2\right\}$ Θ_1 defined in TABLE E-4, ATTACHMENT 1	
MOMENT RESULTANT	$m_x = -1$	
MEMBRANE RESULTANT	$n_\varphi = 0$	
DISPLACEMENT	$w = \frac{1}{2} \frac{(p-1)}{u_o} x$	
VELOCITY	$\dot{w} = 0$	
ACCELERATION	$\ddot{w} = 0$	
TIME τ_O	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE E-4. SUMMARY, LONG SHELLS, HIGH LOADING, 4

TYPE	SHELL TYPE	AB PRESSURE LOADING TYPE
	LONG $C^2 > 6$ SEE	HIGH LOAD $p > 1 + \frac{6}{C^2} \quad (p \neq \frac{3}{2})$
CONDITIONS	ATTACHMENT 1. $y \leqslant x \leqslant u \qquad 1 \leqslant \tau \leqslant \tau_1$ $\Theta^2 = \frac{6}{C^2}\left[2 - \frac{(2p-3)}{(p-1)^3}(p-\tau)^2\right]$, $\tau_1 = p - \sqrt{\frac{(p-1)^3}{(2p-3)}\left\{2 - \frac{C^2}{6}\Theta_1^2\right\}}$	$\Theta = \left(\frac{2\sqrt{3}}{C}\right)\frac{\left[\coth\left(\frac{C(u-u_o)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{Cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{Cu_o}\coth\left(\frac{C(u-u_o)}{\sqrt{3}}\right)\right]}$ $\theta_1 < \min\left(1, \frac{2\sqrt{3}}{C}\right)$, if $p > \frac{3}{2}$ $\frac{2\sqrt{3}}{C} < \theta_1 < 1$, if $p < \frac{3}{2}$
MOMENT RESULTANT	$m_x(x,\tau) = \frac{1}{(u-y)^3}\left[-4x^3 + 6(u+y)x^2 - 12uyx - (y+u)(y^2 - 4yu + u^2)\right]$	
MEMBRANE RESULTANT	$n_\varphi = -1$	
DISPLACEMENT	SEE ATTACHMENT 2.	
VELOCITY	$\dot{w} = (p-\tau)\frac{(x-y)}{\Theta}$	
ACCELERATION	$\ddot{w} = \left(\frac{6}{C^2}\right)\frac{1}{(u-y^3)}\left[u + y - 2x\right] - 1$	
TIME τ_0	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE E-4. ATTACHMENT 1

$$\Theta_1 = \frac{2\sqrt{3}}{C}\,\frac{\left[\coth\left(\frac{C(1-u_o)}{\sqrt{3}}\right) + \frac{2\sqrt{3}}{Cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{Cu_o}\coth\left(\frac{C(1-u_o)}{\sqrt{3}}\right)\right]}$$

$$u = u_o + \frac{2\sqrt{3}}{C}\,\log_e\left[\left|\frac{\left(\Theta + \frac{2\sqrt{3}}{C}\right)\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(\Theta - \frac{2\sqrt{3}}{C}\right)\left(u_o + \frac{2\sqrt{3}}{C}\right)}\right|\right]$$

$$y = u_o - \Theta + \frac{\sqrt{3}}{2C}\,\log_e\left[\left|\frac{\left(\Theta + \frac{2\sqrt{3}}{C}\right)\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(\Theta - \frac{2\sqrt{3}}{C}\right)\left(u_o + \frac{2\sqrt{3}}{C}\right)}\right|\right]$$

TABLE E-4. ATTACHMENT 2

$y < x \leqslant u_o$

$$w(x,\tau) = \frac{1}{2}\,\frac{(p-1)}{u_o}\,x +$$

$$+\frac{C^2}{6}\,\frac{(p-1)^3}{(2p-3)}\left[(x-u_o)(\Theta-u_o) + \frac{1}{2}(\Theta^2-u_o^2) - \frac{\sqrt{3}}{2C}(\Theta-u_o)\log_e\left|\frac{\left(u_o-\frac{2\sqrt{3}}{C}\right)}{\left(u_o+\frac{2\sqrt{3}}{C}\right)}\right| -\right.$$

$$\frac{\sqrt{3}}{2C}\left\{\left(\Theta+\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta+\frac{2\sqrt{3}}{C}\right)\right| - \left(u_o+\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o+\frac{2\sqrt{3}}{C}\right)\right| -\right.$$

$$\left.\left.\left(\Theta-\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta-\frac{2\sqrt{3}}{C}\right)\right| + \left(u_o-\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o-\frac{2\sqrt{3}}{C}\right)\right|\right\}\right]$$

$u_o < x \leqslant u$

$$w(x,\tau) = \frac{1}{2}(p-1) +$$

$$+\frac{C^2}{6}\,\frac{(p-1)^3}{(2p-3)}\left[(x-u_o)(\Theta-u_o) + \frac{1}{2}(\Theta^2-u_o^2) - \frac{\sqrt{3}}{2C}(\Theta-u_o)\log_e\left|\frac{\left(u_o-\frac{2\sqrt{3}}{C}\right)}{\left(u_o+\frac{2\sqrt{3}}{C}\right)}\right| -\right.$$

$$\frac{\sqrt{3}}{2C}\left\{\left(\Theta+\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta+\frac{2\sqrt{3}}{C}\right)\right| - \left(u_o+\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o+\frac{2\sqrt{3}}{C}\right)\right| -\right.$$

$$\left.\left.\left(\Theta-\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta-\frac{2\sqrt{3}}{C}\right)\right| + \left(u_o-\frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o-\frac{2\sqrt{3}}{C}\right)\right|\right\}\right]$$

TABLE E-5. SUMMARY, LONG SHELLS, HIGH LOADING, 5

TYPE	SHELL TYPE	AB PRESSURE LOADING TYPE
	LONG $C^2>6$	HIGH $p>1+\frac{6}{C^2}$ $\left(p\neq\frac{3}{2}\right)$
CONDITIONS	$u\leqslant x\leqslant 1$	
MOMENT RESULTANT	$m_x=1$	
MEMBRANE RESULTANT	$n_\varphi=-1$	
DISPLACEMENT	SEE ATTACHMENT 1.	
VELOCITY	$\dot{w}=p-\tau$	
ACCELERATION	$\ddot{w}=-1$	
TIME τ_0	N/A	
DISPLACEMENT AT REST	HAS NOT COME TO REST YET	

TABLE E-5. ATTACHMENT 1

$$u < x \leqslant 1 \quad u = u^* \text{ at } \tau = \tau^*, \quad \Theta = \Theta^*$$

$$w(x,\tau) = \frac{1}{2}(p-1) + p\,[\tau - \tau^*] - \frac{1}{2}\,[\tau^2 - \tau^{*2}] +$$

$$\frac{C^2}{6}\,\frac{(p-1)^3}{(2p-3)}\Bigg[(u^* - u_o)\,(\Theta^* - u_o) + \frac{1}{2}\,(\Theta^{*2} - u_o^2) -$$

$$-\frac{\sqrt{3}}{2C}\,(\Theta^* - u_o)\,\log_e\left|\frac{\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(u_o + \frac{2\sqrt{3}}{C}\right)}\right| -$$

$$\frac{\sqrt{3}}{2C}\Bigg\{\left(\Theta^* + \frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta^* + \frac{2\sqrt{3}}{C}\right)\right| - \left(u_o + \frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o + \frac{2\sqrt{3}}{C}\right)\right| -$$

$$\left(\Theta^* - \frac{2\sqrt{3}}{C}\right)\log_e\left|\left(\Theta^* - \frac{2\sqrt{3}}{C}\right)\right| + \left(u_o - \frac{2\sqrt{3}}{C}\right)\log_e\left|\left(u_o - \frac{2\sqrt{3}}{C}\right)\right|\Bigg\}\Bigg]$$

TABLE E-6. SUMMARY, LONG SHELLS, HIGH LOADING, 6

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH $p > 1 + \frac{6}{C^2}$ $\quad p \neq \frac{3}{2}$
CONDITIONS	$y_1 = u_o - \theta_1 + \frac{\sqrt{3}}{2C} \log_e \left[\left\| \frac{\left(\theta_1 + \frac{2\sqrt{3}}{C}\right)\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(\theta_1 - \frac{2\sqrt{3}}{C}\right)\left(u_o + \frac{2\sqrt{3}}{C}\right)} \right\| \right]$ $\quad u_o = \frac{\sqrt{6}}{C\sqrt{p-1}}$ $\tau_1 \leqslant \tau \leqslant \tau_o$ $\quad \theta_1 = \frac{2\sqrt{3}}{C} \frac{\left[\coth(A) + \frac{2\sqrt{3}}{Cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{Cu_o}\coth(A)\right]}, A = \frac{C\left(1-u_o\right)}{\sqrt{3}}$ $\tau_1 = p - \left\{\frac{(p-1)^3}{(2p-3)}\left[2 - \frac{C^2}{6}\theta^2\right]\right\}^{1/2},$ $0 \leqslant x \leqslant y \quad y \leqslant 1 - \frac{\sqrt{6}}{C}$	
MOMENT RESULTANT	$m_x = -1$	
MEMBRANE RESULTANT	$n_\varphi = 0$	
DISPLACEMENT	$w(x,\tau) = \frac{1}{2}\frac{(p-1)}{u_o} x$	
VELOCITY	$\dot{w} = 0$	
ACCELERATION	$\ddot{w} = 0$	
TIME τ_o	$\tau_o = \tau_1 + \frac{\left(p-\tau_1\right)\left(1-y_1\right)}{\left[1 - y_1 + \frac{\sqrt{6}}{C}\right]}$	
DISPLACEMENT AT REST	$w(x,\tau_o)$	

TABLE E-7. SUMMARY, LONG SHELLS, HIGH LOADING, 7

TYPE	SHELL TYPE	PRESSURE LOADING TYPE
	LONG $C^2 > 6$	HIGH $p > 1 + \frac{6}{C^2}$ $\quad p \neq \frac{3}{2}$
CONDITIONS	$y_1 = u_o - \theta_1 + \frac{\sqrt{3}}{2C}\log_e\left[\left\lvert\frac{\left(\theta_1 + \frac{2\sqrt{3}}{C}\right)\left(u_o - \frac{2\sqrt{3}}{C}\right)}{\left(\theta_1 - \frac{2\sqrt{3}}{C}\right)\left(u_o + \frac{2\sqrt{3}}{C}\right)}\right\rvert\right]$ $\quad u_o = \frac{\sqrt{6}}{C\sqrt{p-1}}$ $\tau_1 \leqslant \tau \leqslant \tau_o$ $\quad \theta_1 = \frac{2\sqrt{3}}{C}\frac{\left[\coth(A) + \frac{2\sqrt{3}}{Cu_o}\right]}{\left[1 + \frac{2\sqrt{3}}{Cu_o}\coth(A)\right]}$, $\quad A = \frac{C\left(1 - u_o\right)}{\sqrt{3}}$ $\tau_1 = p - \left\{\frac{(p-1)^3}{(2p-3)}\left[2 - \frac{C^2}{6}\theta^2\right]\right\}^{1/2}$, $y \leqslant x \leqslant 1 \quad y \leqslant 1 - \frac{\sqrt{6}}{C}$ $\quad y = \left[1 - \frac{6}{C^2}\frac{1}{\left(1-y_1\right)^2}\right]\frac{\left(1-y_1\right)}{\left(p-\tau_1\right)}\left(\tau - \tau_1\right) + y_1$ $\dot{y}(\tau) = \left[1 - \frac{6}{C^2}\frac{1}{\left(1-y_1\right)^2}\right]\cdot\frac{\left(1-y_1\right)}{\left(p-\tau_1\right)}$	
MOMENT AND MEMBRANE RESULTANT	$m_x = 1 + \frac{1}{(1-y)^3}\left[-4x^3 + 6(1+y)x^2 - 12yx - 2(1-3y)\right]$ $n_\varphi = -1$	
DISPLACEMENT	$w(y,\tau) = \frac{1}{2}\frac{(p-1)}{u_o}y\left(\tau_1\right) + D_1\left[C^2\left\{A_1\left(\tau-\tau_1\right) - \frac{1}{2}B_1\left(\tau^2 - \tau_1^2\right)\right\} + 6\left\{\frac{1}{B_1}\log_e\left\lvert\frac{\left(C_1 - B_1\tau\right)}{\left(C_1 - B_1\tau_1\right)}\right\rvert + \frac{(1-x)}{B_1}\left\{\frac{1}{\left(C_1 - B_1\tau\right)} - \frac{1}{\left(C_1 - B_1\tau_1\right)}\right\}\right\}\right]$ (SEE NEXT PAGE FOR MORE INFORMATION.)	
VELOCITY	$\dot{w} = \frac{\left[1 - \frac{6}{C^2}\frac{1}{(1-y)^2}\right]}{\left[1 - \frac{6}{C^2}\frac{1}{\left(1-y_1\right)^2}\right]}\frac{\left(p-\tau_1\right)}{\left(1-y_1\right)}(x-y) = \frac{\left(p-\tau_1\right)}{\left(1-y_1\right)}\frac{\left[x - y - \frac{6}{C^2}\frac{(x-y)}{(1-y)^2}\right]}{\left[1 - \frac{6}{C^2}\frac{1}{\left(1-y_1\right)^2}\right]}$	
ACCELERATION	$\ddot{w} = -\left[1 + \frac{6}{C^2}\left\{\frac{2(x-y)}{(1-y)^3} - \frac{1}{(1-y)^2}\right\}\right]$	
TIME τ_o	$\tau_o = \tau_1 + \frac{\left(p-\tau_1\right)\left(1-y_1\right)}{\left[1 - y_1 + \frac{\sqrt{6}}{C}\right]}$	
DISPLACEMENT AT REST	$w\left(y_o, \tau_o\right)$ when $y = y_o = 1 - \frac{\sqrt{6}}{C}$ the velocity becomes zero. (SEE CONTINUATION PAGE.)	

TABLE E-7. (Cont.)

$$A_1 = x - y_1 + \left[1 - \frac{6}{C^2(1-y_1)^2}\right] \frac{(1-y_1)}{(p-\tau_1)} \quad \tau_1$$

$$B_1 = \left[1 - \frac{6}{C^2} \; \frac{1}{(1-y_1)^2}\right] \frac{(1-y_1)}{(p-\tau_1)}$$

$$C_1 = 1 - y_1 + \left[1 - \frac{6}{C^2} \; \frac{1}{(1-y_1)^2}\right] \frac{(1-y_1)}{(p-\tau_1)}$$

$$D_1 = \frac{(1-y_1)\ (p-\tau_1)}{\left[C^2(1-y_1)^2 - 6\right]}$$

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