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ON THE SOLUTION OF LAMBERT'S ORBITAL BOUNDARY-VALUE PROBLEM

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SUMMARY

During the past 30 years there has been a resurgence of interest in the classical orbital boundary-value problem of Lambert, largely because of its relevance to space rendezvous and interception. The most notable contribution to the subject was by Lancaster, Blanchard and Devaney, in 1966, but more recent researchers have failed to build on that work; the present Report is aimed at remedying this neglect by providing details of a universal solution of Lambert's problem based on the approach of Lancaster *et al*. In particular, the Report presents starting formulae for Halley's cubic iteration process, used for evaluation of the unknown parameter, x, at the heart of the approach; this process always gives highly accurate values of x after three iterations.

A Fortran-77 computing procedure for a general solution of Lambert's problem has been developed, and its three main subroutines are listed. Details are given of the testing of this procedure.

Much of the Report is devoted to a classification of the set of all Lambert problems, and to a discussion of various geometric and physical aspects.

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LIST OF CONTENTS

1 INTRODUCTION

The 'orbital boundary-value problem', constrained by two points and an elapsed time, is usually associated with the name of Lambert[1], though Euler had studied the problem some 20 years before Lambert (but only for parabolic orbits); other celebrated mathematicians whose names are associated with the problem and its solution include Gauss and Lagrange. Thus it is a problem of classical celestial mechanics, and one that (like the solution of Kepler's equation) continues to attract the attention of mathematicians searching for solution procedures of ever-greater generality, accuracy and efficiency. Good text-book introductions are to be found in Refs 2 to 4, whilst Refs 5 to 26 are studies, chronologically listed, from the last 30 years; the outstanding paper on this list, though from as far back as 1966, is the one by Lancaster, Blanchard and Devaney[8]. Classically, Lambert's problem arose as a core component in the determination of an orbit from three observations of direction alone, the central observation being used (on a trial-and-error basis) as a source of the missing distance data for the other two observations. In the Space Age, with direction measurement a commonplace, the solving of Lambert's problem is directly applicable to the important subject of orbital rendezvous.

Lambert's problem may be stated as follows: an (unperturbed) orbit, about a given inverse-square-law centre of force, C say, is to be found connecting two given points, P_1 and P_2, with a flight time $\Delta t (= t_2 - t_1)$ that has been specified*. The problem must always have at least one solution and the actual number, which we denote by N, depends on the geometry of the triangle CP_1P_2 and the value of Δt - it is assumed, for convenience and with no loss of generality, that $\Delta t > 0$.

To get an immediate feel for the problem, let us suppose first that the triangle CP_1P_2 is not degenerate, so that θ, the angle subtended by P_1P_2 at C, lies between 0 and π. Then it would appear there must be at least two solutions, since an orbital path (in the plane CP_1P_2) can be found that subtends an angle $2\pi - \theta$ (*ie* going the 'long way around') as well as one that subtends θ. We can avoid this duality, however, by supposing the direction of motion to be specified in advance, so that the two angles can be deemed to define different problems. There is a further complication, since if Δt is large enough, other paths (necessarily elliptical) will be possible, each of which includes a number of complete revolutions. It turns out (and will be apparent when Fig 2 is

* A list of symbols is provided at the end of the Report.

introduced) that paths incorporating a specific number of complete revolutions, m say, normally occur in pairs; thus as Δt increases, N (for a given triangle and specified direction of motion) is an increasing odd integer, apart from being even (instantaneously) at each (critical) value of Δt at which two new solutions emerge, coincident for that precise value of Δt. We simplify the approach to multiple revolutions by extending the distinction between individual Lambert problems so that '0' is regarded as an angle of unrestricted positive magnitude defined by the geometry of the path and not just by that of the triangle; we write θ_r for the reduced angle (such that $0 < \theta_r < 2\pi$) when it is necessary to discriminate. (We do not have to consider negative θ, because to do so would imply negative Δt which we have already excluded for convenience.) We have effectively redefined N such that $N = 1$ if $\theta < 2\pi$; if $\theta > 2\pi$, on the other hand, $N = 0$, 1 or 2, depending on the relation of Δt to the appropriate critical value.

Turning to degenerate triangles, we consider these on the basis of the unlimited values of θ just introduced, so that θ is now $k\pi$ for some integer k (≥ 0). Then if orbital paths exist that are not rectilinear, their number must be infinite, since any plane through the degenerate triangle contains valid paths. If we choose an orbital plane (as well as the direction of motion) arbitrarily, however, we have $N = 0$, 1 or 2, if k is odd, exactly as in the last paragraph; this is actually the simplest of all cases to deal with in practice, though the literature contains a number of solution procedures that fail here quite unnecessarily. But there are real difficulties when k is even ($= 2m$), associated with a type of discontinuity that is described in section 3. The effect of this discontinuity is that we would like to be able to distinguish the angle $(k\pi)_-$, which symbolizes the representation of θ as $2(m - 1)\pi$ plus a θ_r of 2π, from $(k\pi)_+$, which symbolizes its representation as $2m\pi$ plus a θ_r of zero; if this distinction (or an equivalent one) is not made, then (for an arbitrarily chosen orbital plane) $N = 1$ or 3 if $m = 1$, and $N = 0$, 1, 2 or 4 if $m > 1$. The orbital path has to be rectilinear (when k is even) unless P_1 and P_2 coincide.

We can now summarize the data involved in the solution procedure to be developed in the present Report. The input quantities are the constant μ (strength of the given force centre at C), r_1 and r_2 (equal to CP_1 and CP_2), θ (the unrestricted angle P_1CP_2) and Δt. (We assume $\mu > 0$, but there is also a Lambert problem when $\mu < 0$; if $\theta < \pi$, there is then a unique hyperbolic solution, wholly internal to the triangle. The transitional case,

with $\mu = 0$, is of course trivial, but even this can be treated as a Lambert problem.) The output quantities, $4N + 1$ in number, consist of N itself and N sets of four quantities, *viz* V_R (radial velocity) and V_T (transverse velocity) at both P_1 and P_2. It is assumed here that values of θ equal to $(2m\pi)_-$ and $(2m\pi)_+$ can be distinguished, so that N does not exceed 2. In reality, of course, the real-number system does not permit this distinction, though this is a somewhat academic point in a computing procedure that can only operate for the finite set of computable numbers; more importantly, with 'multiple revolutions' of a rectilinear orbit the problem has become completely academic anyway, since it involves at least one infinite-velocity 'bounce' off the force centre. Nevertheless, we shall find it advantageous to compute with a pair of quantities, q (introduced in section 3) and m, in place of just θ; this avoids the academic difficulties, and is also more efficient.

As with Kepler's equation, Lambert's problem has no satisfactory direct solution - we have to proceed by an iterative technique (trial and error) and this inevitably dominates the solution procedure being developed. The following issues then arise, and will be discussed in successive sections of the Report: first (in section 2) the choice of a suitable parameter of the motion to use as the iteration variable x (it is sometimes claimed, for example in Ref 9, that the problem is inherently a two-parameter problem, with simultaneous iteration needed on both parameters, but this claim is unwarranted); second (in section 3) the 'direct' algorithm that generates a quantity equivalent to Δt, together with such of its derivatives as are required, from x, r_1, r_2 and θ; third (in section 4) the iteration process, by means of which successive x_i (estimates of x) are computed; fourth (in section 5) the starting formula (or formulae) for provision of x_0; fifth (in section 6) the basis for the cessation of iteration (when 'convergence is complete'), and the accuracy obtained as a result; sixth (in section 7) the formulae for computing the $4N$ velocity components; and lastly (in section 8) the rationale behind, and results of, the testing of the solution procedure.

2 CHOICE OF PARAMETER FOR THE ITERATION VARIABLE

2.1 Lambert's theorem and the relations of L-congruence and L-similarity

For the iteration variable, x, it is desirable to use a quantity that is a 'Lambert invariant' of the problem, if possible. To explain this (in section 2.2), we require a preliminary digression on Lambert's theorem; as a result of this theorem, and the equivalence classification of triangles that it makes possible, individual Lambert problems can be divided into equivalence classes.

Lambert's theorem is usually stated, with an extension of the notation already introduced, as follows: if c is the chord (side P_1P_2) of the triangle CP_1P_2, then Δt (for a connecting orbital path) can be expressed as a (multivalued) function of just three quantities (not counting μ), *viz.* $r_1 + r_2$, c and a, this last being the semi-major axis of the path (taken as negative for hyperbolic orbits[3,27]); many text-books[2-4] prove the theorem for elliptic orbits, and a general 'minimalist' proof has recently been given by Sarnecki[28]. Defining s as the semi-perimeter of the triangle, so that $r_1 + r_2 = 2s - c$, and noting that a is equivalent to the total energy of the motion per unit mass, we can also express Δt as a function of s, c and energy.

It follows from the theorem that triangles with the same values of s and c are equivalent, from the viewpoint of the relation between Δt and energy, and the set of all triangles CP_1P_2 can thus be divided into equivalence classes, as illustrated in Fig 1. Each class contains a unique (apart from orientation) isosceles triangle with $r_1 = r_2$, and the general class (with $0 < c < s$, illustrated in Fig 1a) contains a pair of degenerate triangles such that one of the points P_1 and P_2 lies between the other point and C; a connecting orbit for either of these degenerate triangles is necessarily rectilinear. Classes with $c = 0$ (illustrated in Fig 1b) contain only a single member each, which is simultaneously isosceles and doubly degenerate. The other extreme (illustrated in Fig 1c), occurring when $c = s$, is such that the classes have their widest membership, in regard to the $r_2{:}r_1$ ratios possible, though *all* members are now degenerate; each class contains a pair of doubly degenerate triangles such that either P_1 or P_2 coincides with C, whilst the remaining (singly degenerate) triangles (infinite in number, as in the general case) all have P_1 and P_2 on *opposite* sides of C. Connecting orbits for the singly degenerate triangles of this extreme case cannot be rectilinear, on the usual assumption that C is a point of reflexion (at infinite velocity) for rectilinear orbits; however, a connecting orbit for either of the doubly degenerate triangles is *bound* to be rectilinear (as with singly degenerate triangles in the general case). For the extreme classes with $c = s$, therefore, it is convenient to regard the term 'degenerate' as referring only to the doubly degenerate triangles. If, further, we cease to distinguish between the pair of degenerate triangles with $r_1 < r_2$ and $r_1 > r_2$, then we can say, in all cases, that an equivalence class contains exactly one degenerate triangle.

Triangles in one of the foregoing equivalence classes may be described as 'Lambert congruent', or L-congruent for brevity, and introduction of the concept of congruence suggests the allied one of similarity, just as in elementary geometry. Thus two triangles may be described as L-similar whenever they have the same value of c/s, this being a dimensionless quantity. Though we continue, in section 2.2, to introduce Lambert invariance on the basis of L-congruence, in section 3 we shall find that thinking in terms of L-similarity, with its wider equivalence classes, has the effect of reducing (by one) the number of arguments in the algorithm for the flight time.

2.2 Lambert-invariant parameters, Lambert-equivalent problems, and the parameter x

Since Δt is a function only of s, c and energy, it follows that the equivalence of L-congruent triangles provides the basis for a classification of individual Lambert problems into their own equivalence classes, each such class being defined by the underlying class of triangles and the given value of Δt. Then a Lambert-invariant parameter may be defined as one that has the same value for all members of an equivalence class of problems. It is unfortunate that, though s and c are Lambert-invariant, θ is not, which at first sight negates the virtues of the unrestricted angle that were noted in section 1. We can get the best of both worlds, however, by taking Θ (instead of θ) as a parameter of the general triangle, where Θ is defined as being the θ for the equivalent isosceles triangle: then Θ can be regarded (like θ) as an angle of unrestricted magnitude. (The quantity $\frac{1}{2}\Theta$, denoted by f, was recognized as an important parameter in the paper[19] by Battin, Fill and Shepperd.)

The energy-equivalent orbital parameter a is certainly Lambert-invariant, but this is not true of e (eccentricity) or p (semi-latus rectum). The use of p as iteration variable is intuitively appealing, because of its direct relation to true anomaly and hence to θ, and it is recommended in a paper as recent as Ref 24. But p fails, in a somewhat paradoxical fashion, when $\theta = \pi$. The paradox is that p is given, as $2r_1r_2/(r_1 + r_2)$, without the need to iterate at all in these circumstances: because Δt is not involved in this formula, however, no further progress can then be made without iterating on some other variable.

The advantage in using a Lambert-invariant parameter as the iteration variable is that its determination is a numerically identical procedure for all the individual problems of an equivalence class. The resulting 'reduction in cases' is a very practical consideration for the solution procedure to be

developed, not least when it comes to the number of tests that have to be run. Thus a is an immediate candidate for iteration variable, but its direct use would be unsatisfactory because (as shown in Ref 2, for example) orbital paths with a particular value of a occur in pairs or not at all. There is, in fact, an upper limit (corresponding to minimum possible energy) to the possible values of $1/a$; it is given by $2/s$, for which value the pair of paths coincide. It follows from this that if we write

$$x^2 = 1 - \frac{s}{2a} \quad , \tag{1}$$

then x is a satisfactory substitute for a , such that the choice of sign in the implied square root distinguishes between the two paths of each pair. The parameter x is universal (defined independently of the type of orbital path), unlike, for example, the parameter used by Sun, Vinh and Chern in their recent paper[26]; it also has the advantage of being non-dimensional, which facilitates the switch from L-congruence to L-similarity.

The parameter x , as just introduced, is the iteration variable used in the milestone paper by Lancaster, Blanchard and Devaney[8] (of which Ref 10 is a somewhat expanded version) underlying much of the work reported here. It has been shown by Sarnecki[28] that x has a dynamical interpretation, being a non-dimensionalized value of the velocity in the (rectilinear) solution of the Lambert problem for the degenerate triangle that is L-congruent to the given one and is such that $r_1 \geqslant r_2$; x is positive or negative according to whether the direction of V (now pure radial) is inward or outward, and (in principle - see also section 3) this resolves the ambiguity in the sign of x outstanding from the last paragraph. (Battin[12] finds an interpretation for x in terms of the actual problem, whilst a geometrical interpretation of the parameters of the classical Lambert-Euler equation, also involving the rectilinearly equivalent problem, is given in Ref 20.) Sarnecki's interpretation, which requires P_1 to be the more distant point, involves an intrinsic (and perhaps surprising) lack of symmetry; thus $s = \max(r_1, r_2)$ when P_1 and P_2 are connected by a rectilinear orbit. It is clear from equation (1), now that there is no longer an ambiguity in the sign of x , that $|x| < 1$ for elliptic orbits, $x = 1$ for parabolas and $x > 1$ for hyperbolas. Values of $x \leqslant -1$ do not arise; they would be associated with negative values of Δt . There is an apparent distinction (asymmetry) between the elliptic and hyperbolic paths corresponding to a given pair of points P_1 and P_2 : if a positive value of $1/a$, less than $2/s$,

is given, equation (1) gives two values of x, numerically equal but opposite in sign, and to each value there corresponds an elliptic orbital path; if $1/a$ is negative, on the other hand, only the positive root of (1) is legitimate, implying only one hyperbolic path. But suppose that, for a given value of θ less than π, we consider the path for $2\pi - \theta$ as well as the path for θ; then for negative $1/a$ there are two hyperbolic paths that are quite distinct (though having the same value of x), whereas for positive $1/a$ the two 'additional' elliptic paths, one associated with positive x and one with negative x, are merely the 'orbital complements' of the two original paths (with reversed signs of x). Thus the apparent asymmetry between the two types of orbit (and parabolic orbits behave in the same way as hyperbolic orbits) has no real significance. The nature of the various orbital paths is well illustrated by Figs 3.7 and 3.8 in chapter 3 of Ref 2.

3 THE ALGORITHM FOR COMPUTING Δt

An algorithm for Δt, in terms of x, requires a pair of Lambert-invariant parameters to specify the relevant triangle. By thinking in terms of L-similarity, rather than L-congruence, however, we can reduce the number of triangle parameters from two to one, so long as the output is made non-dimensional; we follow Lancaster *et al*[8,10] in replacing t by T, where

$$T = \left(\frac{8\mu}{s^3}\right)^{\frac{1}{2}} \Delta t \quad . \tag{2}$$

The choice of s, as 'length scale' in equation (2), pays dividends (and was also made in some of the papers by Battin and his colleagues, *eg* in Ref 18). In the classical procedure of Gauss, on the other hand, the length scale is effectively $\sqrt{s(s-c)}$, whilst in the modification of Gauss's method due to Battin and Vaughan[23] it is somewhat more complicated - Ref 23 is perhaps the most interesting of the papers since the pair by Lancaster *et al*, and Appendix A is devoted to a brief discussion of this Report and the underlying method of Gauss. Finally, in Ref 26 the scale length is effectively $2s - c$, *ie* $r_1 + r_2$. The advantage of s over any of these alternatives is that T (as opposed to the quantities corresponding to it) is monotonic with respect to θ (for fixed x), which makes for the essential simplicity of Fig 2, introduced in the next paragraph; further, we have already seen that s is the quantity that is directly associated with the minimum-energy path. The importance of using s as the length scale must not be exaggerated, however, as the use of one of the alternatives does not affect the progress of the iteration process, except of course if

the scale becomes zero, in particular with Gauss's scale when $s = c$. (The presence of μ in (2) debars the solution of Lambert's problem when μ is zero; x is then infinite, since $a = 0$ for the hyperbola into which the solution degenerates, and we can include this case in a general algorithm if we sacrifice full Lambert-invariance and solve for $x\sqrt{\mu}$ rather than x .)

The quantity T is a function of the two parameters x and θ (defined in section 2.2), but Lancaster *et al* use q , rather than θ , where

$$q = \cos \tfrac{1}{2}\theta_r/(1 + \sin \tfrac{1}{2}\theta_r) = \tan \tfrac{1}{4}(\pi - \theta_r) \ . \qquad (3)$$

Here θ_r is θ reduced to the range $(0, 2\pi)$, so that

$$\sin \tfrac{1}{2}\theta_r = \frac{c}{2s - c} \ . \qquad (4)$$

Clearly, (3) and (4) lead to the simple result that

$$q^2 = (1 - \sin \tfrac{1}{2}\theta_r)/(1 + \sin \tfrac{1}{2}\theta_r) = 1 - c/s \ , \qquad (5)$$

but (5) does not define the sign of q , whereas (3) does. We also have, from (3) and (4),

$$q = (1 - c/2s) \cos \tfrac{1}{2}\theta_r \ , \qquad (6a)$$

which is just a particular case of the result that, for the general triangle, follows from the standard 'cosine formula' and may be expressed as

$$q = \frac{\sqrt{r_1 r_2}}{s} \cos \tfrac{1}{2}\theta_r \ ; \qquad (6b)$$

we refer to (6a) and (6b) in Appendix A.

When q is the second parameter of the T-function, m is required in addition (as a third parameter) to specify the number of completed circuits. This brings us to Fig 2, which plots (as in Refs 8 and 10) T against x for particular values of q and m (corresponding to selected values of θ). The relation (almost linear) between q and θ_r is indicated by Fig 3, which also plots $\theta_{T,0}$ and $\theta_{T,\infty}$ against θ_r . Here $\theta_{T,0}$ is a quantity that may be regarded as a 'time-linearized' version of θ_r , *ie* it is defined over the range $(0 \ , \ 2\pi)$ in such a way as to make T a linear function of $\theta_{T,0}$ when $x = 0$;

similarly, $\Theta_{T,\infty}$ becomes time-linearized as x tends to ∞. These two curves are much less linear, the non-linearity for $\Theta_{T,0}$ being pronounced. (As $x \to \infty$, T behaves like $2x^{-1}(1 - q|q|)$, so $\Theta_{T,\infty} = \pi(1 - q|q|)$.)

The most striking feature of Fig 2 consists in the gaps (unrealizable regions) that occur in the part of the figure associated with elliptic orbits. Thus, as Θ increases through a multiple of 2π, for a fixed value of x such that $0 < |x| < 1$, q jumps from -1 back to $+1$ and there is a jump in the value of T. This brings us to the difficulty, referred to in section 1 in relation to the use of θ, in the functional representation of T as $T(x,\theta)$ rather than $T(x,q,m)$; when $\Theta = 2m\pi$ and $x \neq 0$, in fact, $T(x,\Theta)$ is not unique, the values of $T(x, -1, m - 1)$ and $T(x,1,m)$ not being the same. Though this is not a major defect in the $T(x,\theta)$ representation, as we saw, it combines with the greater efficiency of the $T(x,q,m)$ representation to make us regard the latter as 'standard'.

We can account for the unrealizable regions of Fig 2 by studying (for a fixed x with $|x| < 1$) the variation of T with Θ for the isosceles representatives of the equivalence classes of triangles. If $x = 0$, the connecting orbital paths are all minimum-energy trajectories, by equation (1), and there is no jump at $\theta = 2m\pi$ (we now write θ, rather than Θ, since they are identical for isosceles triangles). When $x \neq 0$, for which a number of cases (with a fixed value of $|x|$, *viz* 0.5) are illustrated in Fig 4, consider first the situation for $\theta = 0$, regarded as the limit case as θ reduces to zero. Then our triangle is degenerate as well as isosceles, but the orbital path does not itself have to be degenerate (rectilinear). In fact, any orbit through P_1 trivially passes through P_2 after zero time (since the points coincide), and this is reflected in the zero value of T when $x > 0$ (left-hand illustration of Fig 4a); uniqueness here derives from the isosceles-triangle assumption, the limit case of this being such that the velocity is pure transverse and hence the orbit non-degenerate - for $x = 1/\sqrt{2}$, indeed, the orbit is circular. (The general formula for eccentricity, under these conditions, is $|1 - 2x^2|$.) An entirely different path is available, however, that can be seen to be intrinsically unique. It is given by a degenerate (rectilinear) orbit with V directed outward from P_1, such that P_2 (still supposed coincident with P_1) is reached after non-zero T, even though $\theta = 0$; this applies when $x < 0$ (right-hand illustration of Fig 4b), the velocity now being pure radial.

Now consider what happens as θ increases from zero to 2π, with a fixed x that is either positive or negative. The two orbital paths are always such

that $T(x < 0)$ exceeds $T(x > 0)$; and when $\theta = \pi$ (Fig 4c), the path for $T(x < 0)$ is the reflexion in the line P_1CP_2 of the orbital complement of the path for $T(x > 0)$, such that the point half way along the path is pericentre if $x > 0$ but apocentre if $x < 0$. For $\theta > \pi$, it becomes clear, from symmetry considerations, that the pairs of orbital paths are always shaped as the orbital complements of the paths for $\theta < \pi$, but with the sign of x reversed in the correspondence (compare the pair of illustrations of Fig 4d with the pair of Fig 4b). As θ comes up to 2π (Fig 4e), the position is as follows: for $x < 0$, the orbital path approaches the complete circuit of the (non-degenerate) orbit from which originated the infinitesimal arc that applied to $\theta = 0$ and $x > 0$; for $x > 0$, on the other hand, the path approaches the incomplete rectilinear orbit that complements the path applying to $\theta = 0$ and $x < 0$. In the latter case, the (initial) radial velocity now has to be inwards and (in the limit) the starting point is reached after reflexion (infinite-V 'bounce') from the force centre.

In studying $T(x,\theta)$ beyond $\theta = 2\pi$, still assuming isosceles triangles, we have a choice in the way we ascribe the sign of x . To follow Lancaster *et al* (and also Sarnecki[28]), we effectively start again with a complete revolution behind us, so that the position represented by Fig 4e is followed by the position represented by Fig 4a with an orbit grafted on. This implies a double discontinuity, so that for both $x > 0$ and $x < 0$ there is a jump in the value of T at $\theta = 2\pi$, by ΔT say, equal to the value that T has when $x < 0$ and $\theta = 0$ (and hence equal to twice the journey time from P_1 to apocentre). With the double discontinuity associated with this specification of the sign of x , we have accounted for the unrealizable regions of Fig 2, as promised; at $\theta = 2\pi$ itself, there are three (not four) possible values of T (*cf* the original discourse on Lambert solutions in section 1) corresponding to a given $|x|$, since the limiting post-jump value for $x > 0$ is equal to the limiting pre-jump value for $x < 0$ (same non-degenerate orbital path), both being equal to the orbital period. But this very fact leads to the alternative way of defining x when θ lies between $2m\pi$ and $2(m + 1)\pi$ for any odd value of m ; this simply reverses the sign assumed by Lancaster *et al*. Thus, instead of just 'starting again' at $\theta = 2\pi$, we 'switch over' (between left-hand and right-hand illustrations) in returning from Fig 4e to Fig 4a. So there is now no jump in T when $x < 0$; for $x > 0$, on the other hand, there is a jump of $2\Delta T$. When we get to $\theta = 4\pi$, the position is the exact opposite: there is a jump for $x < 0$, with x as redefined, but no jump for $x > 0$; the sign of x now reverts to being

the same as in the definition by Lancaster *et al*. This alternation continues, *ad infinitum*, for successive values of m.

Thus, if the alternative definition of (the sign of) x is adopted, it is possible to plot a figure with only half the number of unrealizable regions, and this has been done in Fig 5; when the sign of x is reversed, the sign of q has to be reversed as well, which makes q always continuous with respect of θ (though $dq/d\theta$ will have discontinuities - see again Fig 3). It is emphasized that both methods of definition of x are legitimate; the original definition, due to Lancaster *et al*, leads to the simpler algorithm, however, and will be adopted for the rest of this Report. In this context, it is worth remarking that Sarnecki[28] effectively presents an understanding of $T(x,\theta)$, obtained here via the isosceles-triangle representatives of the equivalence classes, by considering the degenerate-triangle representatives. Then increasing values of θ, for a fixed value of x, can be interpreted in terms of the 'time-line' of a single orbital path, so long as r_2 (now evolving) remains less than r_1 (fixed initial value), so that x (and hence s) remains constant. From Sarnecki's viewpoint, the unrealized values of T correspond to a breakdown of the constraint (that $r_1 \geqslant r_2$), but, regardless of this breakdown, the original definition of x clearly fits this interpretation more naturally than the alternative one. In Ref 23 (the paper that is discussed in Appendix A), on the other hand, Battin and Vaughan make explicit use of the isosceles-triangle equivalent of a given problem, in the solution that they offer as an improvement on Gauss's method. (It is noted, for completeness, that Figs 2 and 5 can be extended to cover negative values of T, by reflecting the existing contours in the origin; for $T < 0$ it would be values of $x \geqslant 1$, not $x \leqslant -1$, that would be impossible, however, and this complication is one reason for restricting consideration to positive Δt.)

Further light is thrown on the evolution of the orbital paths for isosceles triangles, if the variation of the eccentricity is considered, and Fig 6 gives contours of e in the (x,θ)-plane, with x as 'alternatively defined'. (For $\theta = 0$ and $x > 0$, we have $e = |1 - 2x^2|$, as already noted.)

The basic formulae for computing T from x, q and m were given in the papers by Lancaster *et al* and are repeated here (without proof) in Appendix B, which gives some details of the Fortran-77 subroutine, TLAMB, that has been written as the core of a universal procedure for solving Lambert's problem; further details of TLAMB may be elicited from the listing in Appendix C. For the present, a few remarks will suffice. The first is that, in addition to T,

the subroutine can generate, if needed, (partial) derivatives (up to the third) with respect to x - we denote these derivatives by T', T'' and T'''. Secondly, the basic formulae of Lancaster *et al* do not give full accuracy in all circumstances; it should be clear from Appendices B and C, however, that rounding error is minimized in TLAMB. Thirdly, one of the quantities used by the subroutine is conveniently introduced here, as it is needed in section 7. The quantity is z, defined by

$$z = +\sqrt{(1 - q^2 + q^2x^2)} \quad . \tag{7}$$

It has a dynamical interpretation, similar to that given for x in section 2, following Ref 28; thus z/q is a non-dimensionalized value of the velocity at the *closer* point in the degenerate Lambert problem (z is never negative because the direction of this velocity is looked after by the sign of q). The computation of z by equation (7) is itself an example of the potential loss of accuracy, since in many situations it is desirable that $1 - q^2$ be regarded as a quantity available independently of q; this is true, in particular, when $m = 0$ and q is close to 1, *ie* when Θ is small. Because of this danger, TLAMB has $1 - q^2$ as an extra argument, the assumption being that it may be computed directly from Θ via the formula

$$1 - q^2 = c/s = 2 \sin \tfrac{1}{2}\Theta_r/(1 + \sin \tfrac{1}{2}\Theta_r) \quad , \tag{8}$$

which is immediate from (5).

4 ITERATION PROCESS

Though the starting value, x_0, has to be available before numerical iteration can commence, the iteration process is considered now, before we look at starting formulae, because the iteration process drives the starting formulae rather than *vice versa*. Thus, if just the basic Newton-Raphson method of iteration is used, the devising of starting formulae to cover all cases becomes an almost impossible task. When the Halley process, found to work extremely well in the solution of Kepler's equation (for hyperbolic[29] as well as elliptic[30] orbits), is used instead, however, the task is greatly eased. This is well illustrated by the solution for x when its true value is 0.5 and Θ is small, say $10^{-5}\pi$. Then (with the starting formula given by equation (10) in the next section) x_0 is roughly double the true x, after which the Newton-Raphson process leads to an x_1 very close to zero and the iterative process effectively

stagnates; the Halley process, on the other hand, gives an x_1 very close to the true value, after which convergence is rapid. (The reason for the accurate x_1 computation, in circumstances such as these, is that for very small θ and $x \gg \sqrt{\theta}$, T behaves like $1/x$, which is a bilinear function, and the Halley method gives an immediately-correct solution for bilinear functions[31].) In defence of the Newton-Raphson process, on the other hand, it is remarked that it works very well when the value of θ approaches 2π and $|x|$ is small, so long as the appropriate starting formula from section 5.2 is used; this is notwithstanding the explicit warning (against using the process in these circumstances) issued at the end of Ref 10, an unwarranted warning to which there will be further reference in the present paper.

The Halley method is essentially the Newton-Raphson method extended to give third-order convergence, so it requires T'' (the second derivative of T with respect to x) as well as T' (the first). Since Halley's method was also adopted for the iteration involved in a subsidiary problem that arises when $m = 0$ (see section 5.3), to satisfy an equation expressed in terms of T', we sometimes also need T'''; this is why the subroutine TLAMB generates derivatives up to the third.

In the present study, the iteration process is incorporated in the Fortran-77 subroutine, XLAMB, that generates solutions for x and is listed in Appendix D. The input of XLAMB consists of m, q, $1 - q^2$ (supplied separately for the same reason as in TLAMB) and T, and its output is as follows: the integer N (defined in section 1) that specifies the actual number of solutions (this should get set to 0, 1 or 2, though a value of -1 is also theoretically possible, constituting the flag to be defined in section 6); x, a solution when at least one exists; and x^+, the second solution when there are two (x^+ is actually the *first* of the two solutions to be described in section 5.3, so it is always positive).

5 STARTING FORMULAE

5.1 Introductory remarks

The requirement in regard to starting formulae for the iteration process is for an approximation to the inverse of the function that (for given m and q) generates T from x. When $m = 0$, there is formally no difficulty, since there is a unique x to which the 'starter' is to approximate (in view of the assumption $T \neq 0$). When $m > 0$, on the other hand, this uniqueness does not

normally apply: the normal cases are of 'no solution' and 'two solutions', but the case of just one solution, arising as two solutions merge, is also covered.

All the starters have been incorporated in the subroutine XLAMB.

5.2 Single-revolution starters

For $m = 0$, the curves in Fig 2 are strictly monotonic, except when $q = 1$ and $x > 0$ (excluded case of $T = 0$). As an obvious first move, we can determine the sign of x by evaluating (by TLAMB) the value of T, T_o say*, corresponding to $x = 0$. Then the sign of x is the sign of $T_o - T$, and it is used to distinguish two cases.

Suppose first that $x > 0$. Then we can approximate to the contour $T(x,q,0)$, for the given q (regarded as fixed), by the bilinear curve

$$T = T_o^2/(T_o + 4x) \quad . \tag{9}$$

The rationale for this, as a particularization of the general bilinear expression $(a + bx)/(c + dx)$, is that it satisfies the following three constraints: first that T tends to 0 as x tends to infinity; secondly, that $T = T_o$ for $x = 0$; and finally, that $T' = -4$ for $x = 0$. (T_o' has a fixed value, though this is not obvious from Fig 2; when $|q| = 1$, it is not defined, but the one-sided derivatives both exist, being -8 on one side and 0 on the other so that T_o' still has a conventional value of -4.) The merit of the bilinear approximation is that it is immediately invertible, the inverse function being also bilinear; thus from (9) we get, as our starter for $x > 0$,

$$x_0 = T_o(T_o - T)/4T \quad . \tag{10}$$

If $x < 0$, we proceed in the same way, this time approximating the $T(x,q,0)$ contour by the bilinear curve that is most naturally expressed as

$$T = T_o - \frac{4x}{x + 1} \quad . \tag{11}$$

The first of the three constraints that lead to (11) is that T tends to infinity as x tends to -1; the other two constraints are the same as for the case $x > 0$. On inverting, to get x as a bilinear function of T, we have

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* The zero suffix in T_o is notated differently from the zero suffix in x_0, to reduce possible confusion between the two different meanings.

$$x_{01} = -\frac{T - T_o}{T - T_o + 4} \quad . \tag{12}$$

This formula, (12), is as elementary as the complementary one, (10), but it was found to work much less well and it was necessary to patch it; this is why the left-hand side of (12) has been written x_{01} rather than x_0. The patching was of a complicated nature, with some arbitrary features, but it will be summarized (in the next three paragraphs) for completeness.

Equation (12) actually requires two patches. The necessity for the first became apparent for values of Θ greater than about 1.999π, and it was ascribed to the proximity of the (left-hand half of the) curve for $\Theta = 2\pi$ $(q = -1)$, for which the value of T_o' should actually be zero (as opposed to -4, for $0 < \Theta < 2\pi$). Now T_o' is automatically zero for a curve that is bilinear in x^2 (rather than x). Hence we are led to consider the alternative approximation

$$T = T_o(1 + \tfrac{1}{2}x^2)/(1 - x^2) \quad , \tag{13}$$

in which only the term $\frac{1}{2}x^2$ requires explanation - it gives T_o'' its correct value (6π) for $\Theta = 2\pi$, where $T_o = 2\pi$. The behaviour of T'' follows from equation (B-16) of Appendix B. Thus, if $|q| < 1$, T'' is defined and finite for all x, but it is unbounded as $|q|$ approaches unity and x approaches zero. If $|q|$ is actually constrained to be unity, however, T'' is bounded in the neighbourhood of $x = 0$; it is strictly undefined at $T = 0$, but may be regarded as 'effectively defined' since it has the same limiting value, equal to $3T_o$, for an approach to the limit from either side. The inversion of (13) may be written

$$x_{02} = -\sqrt{\{(T - T_o)/(T + \tfrac{1}{2}T_o)\}} \quad . \tag{14}$$

At this stage we have two possible formulae for our desired starter, the first (and simpler) being normally to be preferred; thus the possibility of a weighted combination arises. We start by computing what has been found to be a suitable empirical criterion, given by

$$W = x_{01} + 1.7\sqrt{(2 - \Theta/\pi)} \quad , \tag{15}$$

in which, since $x_{01} < 0$, the two terms are of opposite sign. If $W \geqslant 0$, we

use x_{01} without adulteration, but if $W < 0$ we use a weighted combination of x_{01} with x_{02}, so that the starter is potentially of the form

$$\left.\begin{aligned} x_{03} &= x_{01} \ , && \text{if } W \geqslant 0 \\ x_{03} &= x_{01} + w(x_{02} - x_{01}) && \text{if } W < 0 \end{aligned}\right\} \ , \qquad (16)$$

where (still empirically) the weight w is $(-W)^{1/16}$, the 16th root being computed as $\sqrt{\sqrt{\sqrt{\sqrt{}}}}$; the transition from the pure x_{01} to the weighted x_{03} is clearly continuous.

The second limitation of the simple starter, (12), was associated with (true) values of x close to -1, the potential starter, x_{03}, giving a value much *too* close to -1. This flaw has been dealt with by applying an empirical factor λ to x_{03}, so that the final starter (for $x < 0$) is given by

$$x_0 = \lambda x_{03} \ . \qquad (17)$$

The formula used for λ is

$$\lambda = 1 + c_1 x_{03}(1 + x_{01}) - c_2 x_{03}^2 \sqrt{(1 + x_{01})} \ , \qquad (18)$$

with values of c_1 and c_2 equal (empirically) to 0.5 and 0.03 respectively; to minimize rounding error, $1 + x_{01}$ in (18) is computed as $4/(4 + T - T_o)$, in conformity with (12).

A point concerning the patching of (12) in the vicinity of $(x,q) = (0,-1)$ is worth discussing. We see from (15) that W is zero at this point, so the patching associated with (16) can have little effect in its vicinity; moreover, the effect of the patching associated with (17) is very slight for $x \approx 0$, so (12) is essentially unpatched. But the x_{02}/x_{03} patching was only introduced because x_{01} has the 'wrong' left-hand derivative at $(0,-1)$, so there is the appearance of a contradiction here. The paradox is resolved if we bear two points in mind. First, the basing of (12) on the value of T_o means that there can never be a problem when $x = 0$. Secondly, so long as q is not *exactly* -1, the derivative actually has the 'right' value (-4) at $(0,q)$, no matter how close Θ is to 2π. Thus, for a value of Θ such as 1.99999π, the unpatched starter fails in the vicinity of an x-value around -0.05; much closer to zero, however, all is well again. In this context, the inflexion-point curve of Lancaster and Blanchard (Fig 4 of Ref 10) is relevant, though their associated

statement, about the need to abandon the Newton-Raphson process in the region in question, is unjustified (as already noted in section 4).

5.3 Multirevolution starters

For $m > 0$, we again make use of the value of T_o (corresponding to $x = 0$, for the given q), but this is not now the primary source of information in devising starting formulae. The primary source is T_M, the minimum value of T for the given q, together with the associated value of x, x_M. As seen from Fig 2 (and from consideration of the behaviour of T' and T''), T_M and x_M are uniquely defined; further, $x_M = 0$ when $q = 1$, and otherwise $x_M > 0$. Once T_M is known, we immediately know whether the Lambert problem will have two solutions ($T > T_M$), one ($T = T_M$), or none at all ($T < T_M$), but the evaluation of T_M (and x_M) is a non-trivial matter, itself requiring an iterative process - a solution is required to the equation $T' = 0$. For this subsidiary problem, the same iteration process has been adopted as for the Lambert problem (with the resulting need for T''', as remarked in section 4), so there are two topics to be covered here: first, the starter for this subsidiary problem; and secondly, the formulae for the two Lambert-problem starters that will be needed as soon as values for T_M and x_M are available, assuming that $T_M < T$.

In regard to the starter for x_M, we need the value of Θ_r, which can be recovered from q, since, from (3) and (8),

$$\tfrac{1}{2}\Theta_r = \arg(2q\ ,\ 1 - q^2) \quad , \tag{19}$$

where $\arg(x,y)$ is $\tan^{-1}(y/x)$ computed unambiguously and implemented in Fortran by the ATAN2 function. It can be shown that for $\Theta_r = \pi$ (*ie* for $q = 0$) a good approximation to x_M is given by $4/\{3\pi(2m + 1)\}$, which we denote by $x_{M,\pi}$. From this it has been found, empirically, that for $\Theta_r < \pi$ a good starting formula is (omitting the formal zero-suffix from x_M)

$$x_M = x_{M,\pi}(\Theta_r/\pi)^{\frac{1}{8}} \quad , \tag{20}$$

whilst for $\Theta_r > \pi$ it is (symmetrically)

$$x_M = x_{M,\pi}\{2 - (2 - \Theta_r/\pi)^{\frac{1}{8}}\} \quad . \tag{21}$$

Having determined x_M and the corresponding T_M, we now suppose that $T_M < T$, so that the Lambert problem has a pair of solutions with a starting formula required for both. For one solution we have $x_M < x$, so x is certainly positive, and this leads to the simpler of the two starters, since T_o is not used. In principle we get x_0 (the required starter) by inverting the particular bilinear approximation to T, as a function of $(x - x_M)^2$, that (i) gives $T = T_M$ when $x = x_M$, (ii) makes T tend to infinity as x tends to unity, and (iii) has the correct second derivative (T''_M) at $x = x_M$. This bilinear is

$$T = T_M + \tfrac{1}{2}T''_M(x - x_M)^2 / \left\{1 - (x - x_M)^2/(1 - x_M)^2\right\} \quad , \tag{22}$$

and its inversion leads to the starter

$$x_0 = x_M + \left[(T - T_M)/\left\{\tfrac{1}{2}T''_M + (T - T_M)/(1 - x_M)^2\right\}\right]^{\frac{1}{2}} \quad . \tag{23}$$

We should really not write x_0 in equation (23), as it requires a patch that is based on the one defined by (17) and (18) but is a little more complicated; the details of this patch are omitted, however, but they are available from the listing of Appendix D. (A new constant, $c_{4,1}$, is involved, as well as the constants, c_1 and c_2, from (18); also, m and Θ are arguments of the patch.)

For the starter for the other solution we have $x < x_M$, so we use the sign of $T - T_o$ to distinguish between two possibilities (ignoring the third possibility, $T = T_o$, which is as trivial as the one-solution case, $T = T_M$). If $T > T_o$, then $x < 0$ and we proceed exactly as when $m = 0$ and $x < 0$; thus we just use T_o, not requiring T_M, and we patch the elementary formula for x_0 in two different ways, as before (with the second involving a new constant, $c_{4,2}$, as well as c_1 and c_2 from (18)). Finally, if $T < T_o$, we have $0 < x < x_M$, and now we use both T_o and T_M. We base x_0 on the particular bilinear approximation to T, as a function of $(x - x_M)^2$, that (i) passes through the points with (x, T) equal to (x_M, T_M) and $(0, T_o)$, and (ii) has the correct second derivative (T''_M) at the former point. (We could match T'_o, instead of T''_M, but the resulting formula would be as complicated as equation (25) following.) This bilinear approximation is

$$T = T_M + \frac{\tfrac{1}{2}T''_M(x - x_M)^2}{1 + (x - x_M)^2\left\{\tfrac{1}{2}T''_M/(T_o - T_M) - 1/x_M^2\right\}} \quad , \tag{24}$$

and its inversion leads to the starter

$$x_0 = x_M - \left\{ \frac{T - T_M}{\tfrac{1}{2}T''_M - (T - T_M)\left\{\tfrac{1}{6}T''_M/T_o - T_M) - 1/x_M^2\right\}} \right\}^{\frac{1}{2}} \quad ; \qquad (25)$$

equation (25) is a complicated formula, but no patching of this starter has been found necessary.

6 COMPLETION OF CONVERGENCE

It was decided (again following the work[29-31] on the two Kepler equations) to aim at a fixed number of iterations for x, rather than employ a convergence test. In program testing, carried out on a PRIME computer providing 14 decimal digits in double precision, it was found, for $m = 0$, that three iterations always sufficed for the determination of x to 13 digits, suggesting that (for levels of precision up to this) no truncation error remained; it was legitimate, therefore, to fix the number of iterations at three. The foregoing is an over-simplified statement of the accuracy achieved, however, and an amplified version now follows.

It was arranged, in testing, that the relative error in x (true value assumed known - see section 8), after a pre-set number of iterations, should be computed, as well as the relative error (residual) in the value of T finally computed by TLAMB (as compared with the given T); the *smaller* of these two relative errors, ε say, was registered for each test case, following the rationale given in section 1 of Ref 29, and after three iterations it was found that ε never exceeded 10^{-13}. (This rationale, to be invoked again in section 8, is based on the proposition that the accuracy in a numerical solution of the general equation $f(x) = Y$ should always be assessed in terms of the numerically smaller of the relative error in x and the relative residual in Y; assessment in terms of just the former could amount to a demand for the impossible.) For completeness, it is worth remarking that when the process was reduced to two iterations the maximum value of ε was found to be about 2×10^{-6}, this being the relative error in x that arises when $\theta \approx 1.71\pi$ and $x \approx -0.9944$. For just a single iteration, on the other hand, the maximum value of ε is 4.3×10^{-3}, occurring for values of θ approaching 2π and for $x \approx 2.28$. Finally, the maximum value of ε prior to any iteration, *ie* due to the starter itself, is about 0.5, being associated with the 100% over-estimate by x_o, when θ is small, noted in section 4. Consideration of the convergence when $m > 0$ is held over to section 8.

For the convergence of x_M in the 'subsidiary problem' (as defined in section 5.3), it was soon clear that a fixed number of iterations would not be right. There was something of a dilemma here, in regard to the philosophy to adopt, since, on the one hand, great accuracy in x_M should not really be required, as x_M is only a step to the starter for the main problem; in critical cases, on the other hand, an incorrect x_M could lead to the erroneous conclusion that the particular Lambert problem possesses no solution. It was eventually decided to op . e on the (conservative) basis that iteration would end as soon as the value of x_M changed by less than three parts in 10^7 during the current iteration. With the cubic convergence of Halley's method, this meant, in principle, that, in a hypothetical further iteration, x_M would not change by more than three parts in 10^{20} - this follows from the discussion of convergence, in the context of Kepler's equation, in Ref 31. This expectation was confirmed when it was found that there was usually no change in the value of T itself if the 'further iteration' was actually performed. Tests for values of θ covering the full range of q were carried out; the maximum number of iterations needed to satisfy the x_M criterion was found to be nine, but the number was only three in the vast majority of cases. This was true even for large values of θ, the convergence being essentially dependent on q rather than m. To provide a guaranteed exit, however, it was arranged that if the criterion was not satisfied within twelve iterations, then the process would be abandoned and a flag set; reference to this has already been made (section 4).

It is worth noting how convergence for x_M would be affected by substituting the Newton-Raphson process for the Halley process. For most of the q-range, the effect would be to add only a single iteration, taking the total from three to four. For values of q approaching 1, however, *ie* for correspondingly small values of θ_r, there is a steady rise in the number of iterations required by the Newton-Raphson process - the maximum number experienced in the PRIME testing was 16, corresponding to nine for the Halley process. To avoid this behaviour, it would be necessary to improve the starting formula, (20).

7 COMPUTATION OF VELOCITY

We come at last to the solution of an individual Lambert problem. It is assumed that we have obtained a value of x (and also, where appropriate, x^+) for the Lambert-equivalent class of problems, and it remains to compute, for both of the points P_1 and P_2, the velocity components (V_R and V_T) corresponding to the specific quantities (r_1, r_2, θ and Δt) of the original problem. This computation is performed by the overall Lambert-solving subroutine, VLAMB, which

calls XLAMB (which itself calls TLAMB). VLAMB is listed in Appendix E; it will be seen that its arguments are precisely the quantities summarized (for the full Lambert solution procedure) in section 1. (For 'minimalist' derivation of the formulae for V_R and V_T, see Ref 28.)

A formula for $V_{R,1}$ was given by Lancaster and Blanchard[10]. This can be improved upon, however, to make it more accurate and efficient. The resulting formula is

$$V_{R,1} = \gamma\left\{(qz - x) - \rho(qz + x)\right\}/r_1 \quad , \tag{26}$$

where $\gamma = \sqrt{(\mu s/2)}$ and $\rho = (r_1 - r_2)/c$. The formula for $V_{R,2}$, similarly, is

$$V_{R,2} = -\gamma\left\{(qz - x) + \rho(qz + x)\right\}/r_2 \quad . \tag{27}$$

Sarnecki's interpretations[28] of x and z are immediately apparent, on putting $\rho = 1$.

We must have $|\rho| < 1$, of course, except that ρ is indeterminate when $c = 0$ (P_1 and P_2 coincide), in which case VLAMB arbitrarily sets it to zero. But if $c = 0$, $q = \pm 1$, so that $z = |x|$ by (7), and then for $x \neq 0$ there are two possibilities, depending on whether q and x are of the same or opposite sign. Only when the signs are the same do $V_{R,1}$ and $V_{R,2}$ themselves become indeterminate (see also the explanatory material in sections 1, 2.1 and 3), and then the arbitrary value of c yields conventional values for $V_{R,1}$ and $V_{R,2}$. There is no indeterminacy when q and x have opposite signs, on the other hand, since $V_{R,1}$ and $V_{R,2}$ (also $V_{T,1}$ and $V_{T,2}$) are then independent of ρ; we now have the rectilinear orbits illustrated by (the limiting form of) Fig 4a (on the right) and Fig 4e (on the left).

A formula for $V_{T,1}$ was not given directly in Ref 10, but only via the orbital elements a and e, with a serious threat to accuracy in awkward cases. The direct formula is

$$V_{T,1} = \gamma\sigma(z + qx)/r_1 \quad , \tag{28}$$

where in principle σ is defined as $(1 - \rho^2)^{\frac{1}{2}}$; to minimize rounding error, however, in practice we compute

$$\sigma = 2(r_1 r_2/c^2)^{\frac{1}{2}} \sin\tfrac{1}{2}\theta_r \quad , \tag{29}$$

setting $\sigma = 1$ (compatible with ρ) when $c = 0$. Similarly,

$$V_{T,2} = \gamma\sigma(z + qx)/r_2 \quad . \tag{30}$$

Naturally, $V_{T,1}$ and $V_{T,2}$ are indeterminate in the same circumstances as $V_{R,1}$ and $V_{R,2}$, with compatible resolution of this indeterminacy.

For use in equations (26) to (28) and (30), the values of $qz - x$, $qz + x$ and $z + qx$ are given, to optimum accuracy, as special-case output from the subroutine TLAMB, in a post-XLAMB direct call by VLAMB. The special case is signalled by the setting of (input) N to -1, which causes the quantities in question to be computed in place of T', T'' and T''' (see Appendices B and C).

8 TESTING RATIONALE AND RESULTS

The subroutines TLAMB, XLAMB and VLAMB were tested for a wide range of data, starting with TLAMB which is a self-contained procedure implementing a direct algorithm (for computing T from x). Problems of accuracy with this subroutine seemed most likely to arise with input for which q was close to ±1, or else x was close to ±1 or zero, so testing was particularly thorough for data in these categories. Further, there are (for $m = 0$) two transitions (from ellipse through 'series' to hyperbola), one each side of $x = 1$ (see Appendix B); the regions in the vicinity of these transitions, defined (in TLAMB) in practice by $x = \sqrt{0.6}$ and $x = \sqrt{1.4}$, were tested the most carefully of all, with consistency carefully monitored. The tests were entirely satisfactory, and Fig 2 was based on the output - its consistency with the Figure given by Lancaster *et al* was an additional confirmation that all was in order.

The testing of XLAMB followed a natural procedure for validating the iterative solution of an equation. Thus, the parameters of this subroutine are effectively θ and T, but instead of ranging over T the test data actually ranged over x. Then each 'true' x (with a given θ) was the source of a nominal test value of T (via TLAMB), after which the testing of XLAMB could proceed, with the true x used (at the end of the test) merely as an accuracy evaluator. Only one solution (in two-solution cases) could be relevant to this evaluation; further, the specific one-solution cases and the no-solution cases then required separate testing. The general testing was very thorough, with values of θ_r taken very close (and even equal) to 0 and 2π, and (as already remarked in section 6) it has indicated that, for $m = 0$, 13-digit accuracy is always achieved within three iterations. (The testing extended to a value of

10^{25} for x ; this confirmed that full accuracy was maintained as the attractive force was effectively reduced to zero.)

A particular case was examined in detail, namely, that for $\theta \approx 1.9936\pi$ and $x \approx 0.159$, chosen because this implied an input of $q = 0.99$ and $T = 5.0$, for which the paper by Battin and Vaughan[23] indicated the worst convergence behaviour of the cases the authors considered. In fact Ref 23 registers 14 iterations as necessary in the modified version of Gauss's method of alternating substitution (see Appendix A), just to get a solution to 8-digit accuracy. (Boltz[24] claims to achieve faster convergence than Battin and Vaughan, but he does no better in the worst cases compared.) Ref 23 suggests that methods of the Newton-Raphson type fail in these circumstances, but this is far from so (in spite of the remark of Lancaster and Blanchard[10] that originated this suggestion, already commented upon here in sections 4 and 5.2). Thus the values of ε (from section 6) after zero, one and two iterations are, respectively, 0.043, 1.3×10^{-4} and 5.2×10^{-12}; after three iterations, ε is too small to detect, but it would evidently be around 4×10^{-34} in the absence of rounding error. The figures quoted are for the Halley iteration process incorporated in XLAMB, but (as it happens) the results are no worse if the Halley process is replaced by the Newton-Raphson process. However, this conspicuously good behaviour is largely due to the bilinear starter, so it was worth seeing what would happen if, for compatibility with Ref 23, we took x_0 from its value for the circular orbit through P_1 and P_2 . The formula for this is $x_0 = q(1 + q^2)^{-\frac{1}{2}}$, giving about 0.704 for the case considered. Then the values of ε , with the built-in Halley process, increased to 0.36, 0.012 and 3.4×10^{-6}, for iterations up to the second, the value after three iterations still being 'submerged in rounding noise'. When the Halley process was replaced by the Newton-Raphson, however, it was another story: values of ε , taken now as far as the third iteration, were 0.36, 0.23, 0.034 and 6.2×10^{-4}. For this example, the conclusion is clear: only if the starting formula and the iteration process are *both* degraded will convergence deteriorate seriously; even then our normal criterion would be met after at most another two iterations.

For $m > 0$, the value of ε is no longer less than 10^{-13} in all cases, though for $m = 1$ the maximum value obtained was still only 1.1×10^{-13}. For $m > 1$, $\varepsilon(\max)$ grows steadily - the initial growth rate of its common logarithm, with respect to m , is about 0.3, but the rate falls off gradually, $\varepsilon(\max)$ being about 3×12^{-12} for $m = 5$, 8×10^{-11} for $m = 10$, 6×10^{-8} for $m = 30$ and 3×10^{-5} for $m = 100$. (Up to about this point, full accuracy would

be restored by going to a fourth iteration, since this reduces $\varepsilon(\max)$ to about 1.7×10^{-13} when $m = 100$; it rises to about 2.4×10^{-10} when $m = 200$, however.) The growth of $\varepsilon(\max)$ with m is entirely due to a defect in the patching, for $x < 0$, such that the condition for invocation of the patch involving $c_{4,2}$ is sometimes not satisfied when one would like it to be. As the patching is empirical, it could in principle be improved without much difficulty, but the need to do this is so obviously academic that the matter has not been pursued. It is worth remarking, however, that for large values of m , the accuracy of XLAMB is surprisingly sensitive to the particular values assigned to c_3 and $c_{4,2}$ - small changes can easily lead to a worsening in accuracy by several orders of magnitude. In view of the empirical nature of the patching, it is perhaps to be wondered at that the multirevolution starters have performed as well as they have.

The testing of VLAMB was based on the idea that, for each Lambert solution, the output $V_{R,1}$ and $V_{T,1}$ of the subroutine can be combined with r_1 (together with a polar reference angle, taken as zero) to constitute the four necessary components of data, associated with the point P_1 , for input to the subroutine PV2ELS. This subroutine[27] then generates the corresponding set of four universal two-dimensional orbital elements, one of which is τ , the time from pericentre (a conventional point if the orbit is circular). When τ is updated to $\tau + \Delta t$, with the other three elements unchanged, a nominal position (plus velocity) for the point P_2 can be obtained from ELS2PV, the subroutine that is inverse to PV2ELS. This position is specified as a radius vector and a polar angle, so the performance of VLAMB, and hence the success of the overall solution procedure, can be assessed by a direct comparison of these quantities with the input (test) values r_2 and θ ; we denote the (absolute values of) the differences, in r_2 and θ , by δr and $\delta\theta$. Ideally, looking at relative errors, we should like $\delta r/r_2$ and $\delta\theta/\theta$, which we can denote by δ_r and δ_θ , to be no greater than, say, 5×10^{-13} (on the PRIME computer).

For various reasons, this 'ideal' requirement (in VLAMB testing) is too stringent. First, if the velocity is (in relative terms) very great at P_2 , then a large magnification of the relative rounding error may occur that is completely unavoidable. For δ_r , this magnification will be allowed for, in principle, if we do not automatically divide δr by r_2 , but instead use r_R , where

$$r_R = \max(r_2 , |V_{R,2}| \Delta t) \quad . \tag{31}$$

(We define r_T similarly, and apply the factor r_2/r_T to $\delta\theta$.) This does not fully compensate for the velocity effect, however, since Δt may have a large derivative with respect to the value of x determined by XLAMB; this secondary effect is allowed for if we replace Δt in (31) by $\max(\Delta t , |x\, d(\Delta t)/dx|)$. (The rationale here is the same as the one involved in the definition of ϵ in section 6; the value of the derivative comes at once from T' , using the relation between Δt and T in equation (1).)

With the foregoing allowance for the velocity at P_2 , the modified value of δ_r satisfied the 'ideal requirement' for all tests with $r_1 \geqslant r_2$, but a further complication with the test criteria was observed in tests with $r_2 > r_1$. (We have hitherto thought of r_1 as being at least as great as r_2 , because of the association with the definition of x , but this restriction did not apply in the testing of VLAMB.) The difficulty may be understood in relation to situations with $r_2 >> r_1$, and with a value of x corresponding to a semi-major axis with $a >> r_2$, since then the determination of the element $\alpha(=\mu/a)$ by PV2ELS is inevitably inaccurate. In principle, it would be better to replace it by the value given (from x and s) by equation (1), but as the object of the exercise was to test the new subroutine (VLAMB) with the existing subroutines PV2ELS and ELS2PV, this has not been done. Instead, the question "what empirical further relaxation in the definition of δ_r (and similarly δ_θ) would cause the difficulty to disappear?" was addressed. It was found that an additional division, in computing the relative errors, by the ratio r_2/r_1 , or rather by $\max(1, r_2/r_1)$ to cover the other case $(r_1 \geqslant r_2)$ as well, would for the most part resolve the problem; complete resolution became possible on replacing the empirical ratio r_2/r_1 by the even more empirical $(r_2/r_1)^{1.3}$.

With δ_r and δ_θ adjusted on the rationale of the last two paragraphs, it was found that δ_r always satisfied the 'ideal requirement', but that δ_θ could fail to do so when θ was less than about 1 radian. Under these circumstances, replacement of the relative error (δ_0) by the absolute error $(\delta\theta)$ permitted the requirement to be satisfied, but this is hardly surprising, and, in view of the desirability of maintaining relative accuracy for Lambert problems with small values of θ and Δt , it was important to know whether the loss of accuracy arose in the Lambert procedure itself or in the test procedure via PV2ELS and ELS2PV. Tests for very small values of both the input parameters have indicated that the errors are entirely in the test procedure. The explanation is that PV2ELS and ELS2PV were not designed for extreme accuracy in moving between close points on a given orbital path; in particular, this follows from

the subroutines' built-in reference to a pericentre, which is likely to be a point that is remote from the given points that are neighbours. (It would be possible to formulate the subroutines in a different way, but this would involve considerable complication in the solution of the two forms of Kepler's equation; thus, for an elliptic orbit it would be necessary to solve an equation for the eccentric-anomaly difference $E_1 - E_2$, rather than solving the classical equation for E_2. Lancaster has written[13] on the Lambert problem for short arcs, whilst Battin[12] has considered the effective inversion of Lambert's boundary-value problem to provide a solution for Kepler's initial-value problem!)

With the original definitions of δ_r and δ_0 modified as indicated, including the replacement of the latter by $\delta\theta$ when θ is less than 1, their values remained below 5×10^{-13} in all the tests carried out. In this testing, the same range of values for θ was used as in testing VLAMB, with the same range of input-x used as source for the nominal values of Δt. Separate values of r_1 and r_2 were now provided, though only their ratio was actually significant; this meant, of course, that the input θ was no longer identical with θ. Various values for r_2/r_1 were selected, covering the range from 10^{-6} to 10^{6}, so the fact that (modified) δ_r and δ_θ could be held to 5×10^{-13}, over such a wide range of x, θ and r_1/r_2, must be regarded as an entirely successful outcome to the VLAMB testing; it incidentally increased the confidence placed in the robustness of PV2ELS and ELS2PV, except in regard to the point of the preceding paragraph.

9 CONCLUSIONS

A study of the literature on Lambert's orbital boundary-value problem shows that the crucial contribution to the subject was made by Lancaster, Blanchard and Devaney[8] in 1966. Some of the recent papers, unfortunately, do not refer to Ref 8 at all, and their treatment of the problem is, to this extent, retrogressive.

The original paper of Lancaster *et al*[8] was very short (less than 1½ pages) and, though their approach was amplified in a later paper[10], there has been a need for the approach to be extended to a general computing procedure for solving the Lambert problem. The present Report fills this gap, and has addressed, in particular, the following topics not covered by Lancaster *et al*: the provision of universal starting formulae for the procedure's iteration process; the method of iteration; the minimization of rounding error in all circumstances (not just around the parabolic-transition region); the accurate computation of velocity; and the test-validation of the procedure (with emphasis on extreme cases).

The overall conclusion is that the procedure provides excellent accuracy, in an efficient manner, for any problem that might arise in practice. Three iterations of the iteration process suffice for the effective elimination of rounding error, when working to an accuracy of not more than 13 significant figures. When only 5- or 6-figure accuracy is necessary, two iterations will be adequate - in most cases they will be a great deal more than merely 'adequate'.

The Report has attempted to facilitate a deeper understanding of Lambert's problem by introducing the concepts of L-similarity and L-congruence (for the basic triangle involved in the problem) and of Lambert invariance (for physical parameters).

Appendix A

GAUSS'S METHOD AND THE PAPER BY BATTIN AND VAUGHAN

Ref 23 represents the culmination of a long study of Lambert's problem by Battin and various co-workers. Ref 2 (his text-book) and Refs 12, 18, 19 and 23 do not exhaust the list of Battin's contributions to the subject, but Ref 23 represents a change of approach in which the authors return to the classical method of Gauss. The essence of Gauss's method, of which Moulton[32] and Plummer[33] give traditional text-book accounts, lies in its iterative determination, by 'alternating substitution' (not a standard term), of a pair of Lambert-invariant parameters, traditionally denoted by x and y. For an elliptic orbit, Gauss's x is $\sin^2 \frac{1}{4}(E_2 - E_1)$, where E_1 and E_2 are the eccentric anomalies of the points P_1 and P_2; x also identifies with $\frac{1}{2}(1 - g)$, where g is defined by equation (B-9) of Appendix B. Again, y is a quantity defined by the Lambert triangle CP_1P_2 and the required orbital path, being the ratio of the area of the 'curvilinear' triangle to that of the ordinary 'linear' triangle, where the curvilinear triangle has, for its 'side' P_1P_2, the orbital arc rather than the chord (so y is infinite when $\theta = \pi$); this identifies Gauss's y with $T/4q\alpha$ in the notation of Appendix B, where α is given by equation (B-4). Two equations are available for connecting x and y, *viz*

$$y^3 - y^2 = mx \qquad \text{(A-1)}$$

and

$$x = m/y^2 - \ell \quad . \qquad \text{(A-2)}$$

In (A-1) and (A-2), ℓ is equivalent to q (in the notation of the main text) since

$$\ell = (1 - q)^2/4q \qquad \text{(A-3a)}$$

$$= \sin^2 \tfrac{1}{4}\theta_r / \cos \tfrac{1}{2}\theta_r \quad . \qquad \text{(A-3b)}$$

Computation of ℓ from θ, via (A-3b), is more accurate than computation from q, via (A-3a), when θ_r is small. For computation from θ_r, rather than θ_r, Gauss recommended a formula that may be derived from the basic relation

$$\sqrt{(r_1 r_2)} \cos \tfrac{1}{2}\theta = qs = \tfrac{1}{2}(r_1 + r_2) \cos \tfrac{1}{2}\theta \quad ,$$

which is just a combination of equations (6a) and (6b) of the main text. (This, incidentally, identifies Θ with $2f$ in Ref 17.) Gauss's formula is

$$\ell = (\sin^2 \tfrac{1}{4}\theta_r + \tan^2 2\omega)/\cos \tfrac{1}{2}\theta_r \qquad \text{(A-3c)}$$

where ω is given by

$$\tan(\omega + \tfrac{1}{4}\pi) = (r_2/r_1)^{\frac{1}{4}} ,$$

and it is evident that (A-3c) is just a generalization of (A-3b). However, we can get a still more accurate value of ℓ if we replace $\tan^2 2\omega$ in (A-3c) by $(\sqrt{r}_1 - \sqrt{r}_2)^2/4\sqrt{(r_1 r_2)}$, and we do best of all, when an accurate value of $|r_1 - r_2|$ is available, if the replacement is by

$$(r_1 - r_2)^2 \Big/ \left\{4\sqrt{(r_1 r_2)}(\sqrt{r}_1 + \sqrt{r}_2)^2\right\} .$$

To return to the quantities introduced in (A-1) and (A-2), m (actually written m^2 by Moulton, following Gauss's Theoria Motus[34]) is closely related to T (present main text) since

$$m = \mu(\Delta t)^2/(8s^3q^3) , \qquad \text{(A-4a)}$$

from which, using equation (2) of the main text, we get

$$m = T^2/64q^3 . \qquad \text{(A-4b)}$$

Finally, X is a function of x alone, most concisely expressed, in terms of the usual hypergeometric function, as $\frac{4}{3}F(3,1,2\frac{1}{2};x)$, but most effectively computed using continued fractions. The determination of y from x in (A-1) then involves the solution of a cubic equation.

Gauss based the iterative determination of x and y, to satisfy (A-1) and (A-2) simultaneously, on a method of successive substitution (a form of relaxation) that can more descriptively be referred to as 'alternating substitution' when only two equations are involved. To see how the method operates, consider (A-1) and (A-2) generalized to $y = f(x)$ and $x = g(y)$ respectively, with an initial estimate available for x; then we compute $y_1 = f(x_0)$, $x_2 = g(y_1)$, $y_3 = f(x_2)$ *etc*, and need to assess how the alternating sequences converge, if at all. So let f' and g' denote the derivatives of f and g,

and suppose that f' and g' are reasonably constant over the square defined by (x_0 , y_1) and the true solution (x , y). Then if $|f'g'| < 1$, an error, ϵ say, in x_0 leads to an error of order $f'g'\epsilon$ in x_2. On this assumption, the process is linearly convergent; the rate of convergence is rapid if $|fg| \ll 1$, but still only linear. Thus, though the process is likely to be very robust, it is inherently inferior to the standard iteration processes for solving a single equation in a single variable - in particular[31] the Newton-Raphson process which gives quadratic convergence and the Halley process which gives cubic convergence. The main weakness in Gauss's method comes from the infinities in ℓ and m when $q = 0$ (which should be the simplest of all cases), and a subsidiary defect lies with the slow convergence in other cases. Battin and Vaughan have improved the method significantly in both respects, dealing with the infinities by a redefinition of all the quantities ℓ, m, x, y and X, in such a way that (A-1) and (A-2) are (formally) still the equations to be solved. In particular, the redefinition of m is as

$$m = T^2/(1 + q)^6 \quad ;$$

this still leaves an infinity, but now it is for $q = -1$ (θ a multiple of 2π, in the notation of the main text), which is a lesser fault but a fault nonetheless. Battin and Vaughan deal with the other weakness by an adjustment of the equations (A-1) and (A-2) themselves, in a way that leads to a dramatic speeding up of the convergence (via a reduction in the value of $|f'g'|$, though they do not express the alternating substitution process in this way).

It may well be supposed, however, that a much more drastic modification of Gauss's method could be devised. If we remove the denominator in (A-5), and then formulate (A-1) and (A-2) as an equation in a single unknown (as is hinted at in Ref 14), we remove both the deficiencies completely. In doing this in the most efficient way, however, we effectively just recover the method of Lancaster *et al*, and it is ironic that the only reference to this method made by Battin and Vaughan is a remark to the effect that "the difficulties arising when θ is a multiple of 2π were recognized by Lancaster and Blanchard[10] and necessitate the abandonment of the Newton-Raphson process". It is perfectly true that the possibility that this process might have to be abandoned was suggested in Ref 10. In reality, however, as should be clear from section 8 of the present Report, this is the last circumstance in which it is actually profitable to change the iteration process: the Gauss process is inevitably still at its worst, in spite of the improvements of Battin and Vaughan, whereas the Newton-Raphson process (or better

the Halley process) continues to do extremely well in three iterations; this success must be attributed to the use of a good starting formula for the x of Lancaster *et al.*

Appendix B

DETAILS OF THE TLAMB ALGORITHM

The normal function of the subroutine TLAMB, which is listed in Appendix C, is to compute the value of the Lambert-invariant parameter T , defined in section 3 of the main text, together with as many of its first three derivatives (with respect to the Lambert-invariant parameter x) as are required. The input arguments of the subroutine are m , q , $1 - q^2$, x and n , where n , when $0 \leq n \leq 3$ (normal operation), specifies the number of derivatives to be output, the output arguments being T , T' , T" and T"' . When $n = -1$, the subroutine has a special function, namely, to compute the quantities β , B and Λ that will be defined in due course; these quantities then replace the arguments T' , T" and T"' , there being no output T .

The basic algorithm for T is unchanged from the definitive work of Lancaster *et al*[8,10], and we use the same notation, except for the introduction of α , Λ , β and B , and for the use of u as a quantity that is the negative of E in Refs 8 and 10. We start by summarizing the basic algorithm for T , T' , T" and T"' , including such comments as seem necessary, in particular in regard to the computation of quantities with minimal rounding error. The rounding-error problems referred to here do not extend to the most serious one of all, that arises for orbital paths that are sufficiently close to parabolic: the entire 'basic algorithm' then has to be replaced by a series-based algorithm, which is also rooted in the formulae of Lancaster *et al* and is summarized after the basic algorithm.

The basic algorithm operates whenever any one (or more) of the following three conditions applies: (i) $m > 0$ (multi-revolution elliptic path); (ii) $x < 0$ (remote from parabolic path, for which $x = 1$); (iii) $|u| > 0.4$, where

$$u = 1 - x^2 \quad , \tag{B-1}$$

the criterion (0.4) being an essentially arbitrary one. The appropriate part of the basic algorithm also always operates when the special function of the subroutine is required, since there is never a need for series expansion when the computation does not proceed as far as T . The following formulae apply to both ellipses and hyperbolas:

$$y = \sqrt{|u|} \quad , \tag{B-2}$$

$$z = \sqrt{(1 - q^2 + q^2x^2)} \quad , \tag{B-3}$$

$$\alpha = z - qx \quad , \tag{B-4}$$

$$A = z + qx \quad , \tag{B-5}$$

$$\beta = qz - x \quad , \tag{B-6}$$

$$B = qz + x \quad , \tag{B-7}$$

$$f = \alpha y \tag{B-8}$$

and

$$g = xz + qu \quad . \tag{B-9}$$

In equation (B-3), z^2 is computed using the input argument $1 - q^2$, provided independently of q, as part of the rounding-error minimization philosophy. In respect of the next four quantities, it is actually α and β, rather than A and B, that are normally required, *viz* α in (B-8) above and β in (B-13) below, so the flow will often bypass A and B (required, in their own right, only in the special operation of the subroutine). However, about 50% of the time it will be necessary to compute α and β from A and B, to avoid rounding error in subtractions, and sometimes (in the special operation) the opposite computations will be necessary; the connecting formulae are

$$\alpha A = 1 - q^2 \tag{B-10}$$

and

$$\beta B = (1 - q^2)(q^2u - x^2) \quad . \tag{B-11}$$

It will also be necessary, 50% of the time, to compute g from

$$g = (x^2 - q^2u)/(xz - qu)$$

instead of from (B-9).

We now require a quantity, d, for which there are formulae that differentiate between ellipses and hyperbolas. For an elliptic path,

$$d = m\pi + \arg(g , f) \quad , \tag{B-12a}$$

where the arg function is as for equation (19) of the main text. The corresponding formula for a hyperbolic path is

$$d = \tanh^{-1}(f/g) \quad , \qquad \text{(B-12b)}$$

which may in principle be evaluated as $\ln(f + g)$, as assumed by Lancaster *et al* (since f and g are equal to, not just proportional to, sinh d and cosh d , respectively). This would lead to serious error when f is small, however (not noted by Lancaster *et al*), so in this case we evaluate the inverse tanh function by series; as in Ref 29, it is more efficient to evaluate d as $2 \tanh^{-1}\{f/(g + 1)\}$, rather than as $\tanh^{-1}(f/g)$. It is perhaps worth remarking on what may appear to be curious in the preceding formulae, namely, that the elliptic and hyperbolic régimes can be dealt with by just two versions of a single formula, equations (B-12); the explanation is that the distinction really starts with the absolute-value sign in (B-2), as a result of which (B-8) and (B-9) are reached with values of f and g such that (apart from rounding error) $g^2 + f^2 = 1$ for ellipses and $g^2 - f^2 = 1$ for hyperbolas.

The formula for T can now be given, to cover all orbits that are not 'too parabolic'. It is

$$T = 2(d/y + \beta)/u \quad . \qquad \text{(B-13)}$$

This is taken directly from Refs 8 and 10, where the formula for T' is given as well, *viz*

$$T' = (3xT + 4q^3x/z - 4)/u \quad . \qquad \text{(B-14)}$$

In (B-14), as in (B-13), we are not concerned with non-zero values of the u-denominator, since they only arise with the near-parabolic régime that we are not yet considering, but the z-denominator requires attention. From (B-3) we see that z can only vanish when x is zero and $|q| = 1$, so we do not get an infinity in (B-14) but an indeterminate term of the form 0/0 (herein lies a further explanation of the discontinuities in Fig 2 that were remarked upon in section 5.2). The effect of this indeterminacy is entirely localized, however, with no spread of rounding error in the vicinity of the zero in z ; thus all that is necessary is that the computation of T' be bypassed when z is zero.

To complete the description of the basic algorithm, it only remains to present the formulae for T'' and T''' . These were not given by Lancaster *et al*, but are easy to derive, on proceeding from (B-14) and bearing in mind that

$$z' = 2q^2x/z \quad . \tag{B-15}$$

The formulae may be expressed in the same way as (B-14), but it will be seen that real infinities now arise when z is zero; thus the possibility of overflow exists, even when $z \neq 0$, but (as with T') there should be no question of the spread of rounding error. The formulae are

$$T'' = \left\{3T + 5xT' + 4(q/z)^3(1 - q^2)\right\}/u \tag{B-16}$$

and

$$T''' = \left\{8T' + 7xT'' - 12x(q/z)^5(1 - q^2)\right\}/u \quad . \tag{B-17}$$

We now come to the series-based formulae that are required for T, T', T'' and T''' when m is zero, with x positive and such that $|u| \leq 0.4$. As Lancaster *et al* indicate, following Gedeon[6,7], the basic expression for T can be written

$$T = \phi(u) - q^3\,\phi(q^2u) \quad , \tag{B-18}$$

where

$$\phi(u) = \sum_{n=0}^{\infty} A_n u^n \quad , \tag{B-19}$$

with

$$A_n = (2n)!/\left\{2^{2n-2}(n!)^2(2n + 3)\right\} \quad . \tag{B-20}$$

But (B-18) is inaccurate for computing when q has a value close to 1, so we replace it by a single series, *viz*

$$T = \sum_{n=0}^{\infty} B_n u^n \quad , \tag{B-21}$$

where

$$B_n = A_n b_n \tag{B-22}$$

with

$$b_n = 1 - q^{2n+3} \quad . \tag{B-23}$$

Accuracy demands that b_n be computed with minimal rounding error. The critical computation is of b_0, which we simply set to $1 - q^3$ if $q < 0.5$. If $q \geq 0.5$, however, we set

$$b_0 = \left\{q + 1/(1 + q)\right\}(1 - q^2) \quad . \tag{B-24}$$

The two expressions for b_0 are formally identical, but computing accuracy (when needed) is given by (B-24), because of the assumed accuracy of the independent argument $(1 - q^2)$ of TLAMB. For $n > 0$, b_n is given by the recurrence formula

$$b_n = b_{n-1} + q^{2n+1}(1 - q^2) \quad . \tag{B-25}$$

The computation of A_n is straightforward, since we may express it as

$$A_n = a_n/(2n + 3) \quad , \tag{B-26}$$

where a_n is given by the recurrence formula

$$a_n = \left\{(2n - 1)/2n\right\} a_{n-1} \quad , \tag{B-27}$$

with $a_0 = 4$.

In principle, then, T is computed from (B-21), with terms continually added until there is no further change of value. But for values of $|u|$ close to the maximum (0.4) for which the series would be used, the convergence was found to be rather slow. At the expense of a little extra computation, however, it was possible to accelerate it by computing the series for Tx^2 instead of T, and then dividing by x^2 $(= 1 - u^2$, of minimum value 0.84). Writing

$$Tx^2 = \sum_{n=0}^{\infty} C_n u^n \quad , \tag{B-28}$$

therefore, we have (for $n > 0$, C_0 being just B_0)

$$C_n = B_n - B_{n-1} = (A_n - A_{n-1})b_n + A_{n-1}(b_n - b_{n-1}) \quad , \tag{B-29}$$

where, from (B-26) and (B-27), we get

$$A_n - A_{n-1} = -\left\{(6n + 1)/(4n^2 - 1)\right\} A_n \quad . \tag{B-30}$$

Thus the computation of T is based on (B-28) rather than (B-21).

Two additional complications arise with the computation of T', T'' and T'''. First, it is derivatives with respect to x that we want, whereas the series for T is in terms of u. Writing $\bar{T}$ for dT/du etc, we have

$$T' = -2x\bar{T} \quad , \tag{B-31}$$

$$T'' = -2\bar{T} + 4x^2\bar{\bar{T}} \tag{B-32}$$

and

$$T''' = 12x\bar{\bar{T}} - 8x^3\bar{\bar{\bar{T}}} \quad . \tag{B-33}$$

The other complication arose from the desire to compute x^2T, $\bar{T}$, $\bar{\bar{T}}$, and $\bar{\bar{\bar{T}}}$ efficiently within the same loop of code. It was decided that negligible accuracy would be lost if the criterion for an unchanging value of Tx^2 was used to terminate the computation of all four quantities, but there was another difficulty created by the fact that the initial value of n in the series for $\bar{T}$, $\bar{\bar{T}}$ and $\bar{\bar{\bar{T}}}$ should be 1, 2 and 3 respectively; to overcome this, it was decided that the simplest procedure was to allow the initial value to be zero in each case, even though initially-zero contributions to $\bar{T}$, *etc* would be computed in consequence.

As a footnote to the function $\phi(u)$, it is remarked that it can be expressed in terms of the same hypergeometric function as was involved in the quantity X used in Appendix A. Thus we have $\phi(u) = \frac{4}{3}F\left(3,1,2\frac{1}{2};\frac{1}{2}(1 - x)\right)$, where $\frac{1}{2}(1 - x) = \frac{1}{2}u/\left\{1 + \sqrt{1 - u}\right\}$ when $x > 0$; this identifies (B-18) with equation (28) of Ref 18. Also, Ref 18 indicates a way of expressing (B-18) with only one occurrence of the function ϕ; in the notation of the present Appendix, the expression for T is then

$$T = \alpha\left\{\alpha^2\phi(f^2) + 4q\right\} \quad .$$

Appendix C

FORTRAN-77 SUBROUTINE TLAMB

```
      SUBROUTINE TLAMB (M, Q, QSQFM1, X, N, T, DT, D2T, D3T)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      LOGICAL LM1, L1, L2, L3
      DATA PI, SW /3.14159265358979300, 0.400/
      LM1 = N.EQ.-1
      L1 = N.GE.1
      L2 = N.GE.2
      L3 = N.EQ.3
      QSQ = Q*Q
      XSQ = X*X
      U = (1D0 - X)*(1D0 + X)
      IF (.NOT.LM1) THEN
C           (NEEDED IF SERIES, AND OTHERWISE USEFUL WHEN Z = 0)
        DT = 0D0
        D2T = 0D0
        D3T = 0D0
      END IF
      IF (LM1 .OR. M.GT.0 .OR. X.LT.0D0 .OR. DABS(U).GT.SW) THEN
C         DIRECT COMPUTATION (NOT SERIES)
        Y = DSQRT(DABS(U))
        Z = DSQRT(QSQFM1 + QSQ*XSQ)
        QX = Q*X
        IF (QX.LE.0D0) THEN
          A = Z - QX
          B = Q*Z - X
        END IF
        IF (QX.LT.0D0 .AND. LM1) THEN
          AA = QSQFM1/A
          BB = QSQFM1*(QSQ*U - XSQ)/B
        END IF
        IF (QX.EQ.0D0.AND.LM1 .OR. QX.GT.0D0) THEN
          AA = Z + QX
          BB = Q*Z - X
        END IF
        IF (QX.GT.0D0) THEN
          A = QSQFM1/AA
          B = QSQFM1*(QSQ*U - XSQ)/BB
        END IF
        IF (.NOT.LM1) THEN
          IF (QX*U.GE.0D0) THEN
            G = X*Z + Q*U
          ELSE
            G = (XSQ - QSQ*U)/(X*Z - Q*U)
          END IF
          F = A*Y
          IF (X.LE.1D0) THEN
            T = M*PI + DATAN2(F, G)
          ELSE
            IF (F.GT.SW) THEN
              T = DLOG(F + G)
            ELSE
              FG1 = F/(G + 1D0)
              TERM = 3D0*FG1
              FG1SQ = FG1*FG1
              T = TERM
              TWOI1 = 1D0
    1         TWOI1 = TWOI1 + 2D0
              TERM = TERM*FG1SQ
              TOLD = T
              T = T + TERM/TWOI1
              IF (T.NE.TOLD) GO TO 1
C           (CONTINUE LOOPING FOR INVERSE TANH)
```

```
            END IF
          END IF
          T = 2D0*(T/Y + B)/U
          IF (L1 .AND. Z.NE.0D0) THEN
            QZ = Q/Z
            QZ2 = QZ*QZ
            QZ = QZ*QZ2
            DT = (3D0*X*T - 4D0*(A + QX*QSQFM1)/Z)/U
            IF (L2) D2T = (3D0*T + 5D0*X*DT + 4D0*QZ*QSQFM1)/U
            IF (L3) D3T = (8D0*DT + 7D0*X*D2T - 12D0*QZ*QZ2*X*QSQFM1)/U
          END IF
        ELSE
          DT = B
          D2T = BB
          D3T = AA
        END IF
      ELSE
C           COMPUTE BY SERIES
        U0I = 1D0
        IF (L1) U1I = 1D0
        IF (L2) U2I = 1D0
        IF (L3) U3I = 1D0
        TERM = 4D0
        TQ = Q*QSQFM1
        I = 0
        IF (Q.LT.5D-1) TQSUM = 1D0 - Q*QSQ
        IF (Q.GE.5D-1) TQSUM = (1D0/(1D0 + Q) + Q)*QSQFM1
        TTMOLD = TERM/3D0
        T = TTMOLD*TQSUM
C           (START OF LOOP)
    2   I = I + 1
        P = I
        U0I = U0I*U
        IF (L1 .AND. I.GT.1) U1I = U1I*U
        IF (L2 .AND. I.GT.2) U2I = U2I*U
        IF (L3 .AND. I.GT.3) U3I = U3I*U
        TERM = TERM*(P - 0.5D0)/P
        TQ = TQ*QSQ
        TQSUM = TQSUM + TQ
        TOLD = T
        TTERM = TERM/(2D0*P + 3D0)
        TQTERM = TTERM*TQSUM
        T = T - U0I*((1.5D0*P - 0.25D0)*TQTERM/(P*P - 0.25D0)
     A                                                - TTMOLD*TQ)
        TTMOLD = TTERM
        TQTERM = TQTERM*P
        IF (L1) DT = DT + TQTERM*U1I
        IF (L2) D2T = D2T + TQTERM*U2I*(P - 1D0)
        IF (L3) D3T = D3T + TQTERM*U3I*(P - 1D0)*(P - 2D0)
        IF (I.LT.N .OR. T.NE.TOLD) GO TO 2
C           (END OF LOOP)
        IF (L3) D3T = 8D0*X*(1.5D0*D2T - XSQ*D3T)
        IF (L2) D2T = 2D0*(2D0*XSQ*D2T - DT)
        IF (L1) DT = -2D0*X*DT
        T = T/XSQ
      END IF
      RETURN
      END
```

Appendix D

FORTRAN-77 SUBROUTINE XLAMB

```
      SUBROUTINE XLAMB (M, Q, QSQFM1, TIN, N, X, XPL)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (PI=3.14159265358979300, TOL=3D-7, C0=1.7D0,
      PARAMETER (C1=0.5D0, C2=0.03D0, C3=0.15D0, C41=1D0, C42=0.24D0,
      D8RT(X) = DSQRT(DSQRT(DSQRT(X)))
      THR2 = DATAN2(QSQFM1, 2D0*Q)/PI
      IF (M.EQ.0) THEN
C          SINGLE-REV STARTER FROM  T (AT X = 0) & BILINEAR (USUALLY,
        N = 1
        CALL TLAMB (M, Q, QSQFM1, 0D0, 0, T0, DT, D2T, D3T)
        TDIFF = TIN - T0
        IF (TDIFF.LE.0D0) THEN
          X = T0*TDIFF/(-4D0*TIN)
C           (-4 IS THE VALUE OF DT, FOR X = 0)
         ELSE
          X = -TDIFF/(TDIFF + 4D0)
          W = X + C0*DSQRT(2D0*(1D0 - THR2))
          IF (W.LT.0D0)
     A     X = X - DSQRT(D8RT(-W))*(X + DSQRT(TDIFF/(TDIFF + 1.5D0*PI)),
          W = 4D0/(4D0 + TDIFF)
          X = X*(1D0 + X*(C1*W - C2*X*DSQRT(W)))
        END IF
       ELSE
C          WITH MULTIREVS, FIRST GET T(MIN) AS BASIS FOR STARTER
        XM = 1D0/(1.5D0*(M + 5D-1)*PI)
        IF (THR2.LT.5D-1) XM = D8RT(2D0*THR2)*XM
        IF (THR2.GT.5D-1) XM = (2D0 - D8RT(2D0 - 2D0*THR2,))*XM
C          (STARTER FOR TMIN)
        DO 1 I=1,12
        CALL TLAMB (M, Q, QSQFM1, XM, 3, TMIN, DT, D2T, D3T)
        IF (D2T.EQ.0D0) GO TO 2
        XMOLD = XM
        XM = XM - DT*D2T/(D2T*D2T - DT*D3T/2D0)
        XTEST = DABS(XMOLD/XM - 1D0)
        IF (XTEST.LE.TOL) GO TO 2
    1   CONTINUE
        N = -1
        RETURN
C          (BREAK OFF & EXIT IF TMIN NOT LOCATED - SHOULD NEVER HAPPE.
C          NOW PROCEED FROM T(MIN) TO FULL STARTER
    2   CONTINUE
        TDIFFM = TIN - TMIN
        IF (TDIFFM.LT.0D0) THEN
          N = 0
          RETURN
C           (EXIT IF NO SOLUTION WITH THIS M,
         ELSE IF (TDIFFM.EQ.0D0) THEN
          X = XM
          N = 1
          RETURN
C           (EXIT IF UNIQUE SOLUTION ALREADY FROM X(TMIN))
         ELSE
```


027

```
            N = 3
            IF (D2T.EQ.0D0) D2T = 6D0*M*PI
            X = DSQRT(TDIFFM/(D2T/2D0 + TDIFFM/(1D0 - XM)**2))
            W = XM + X
            W = W*4D0/(4D0 + TDIFFM) + (1D0 - W)**2
            X = X*(1D0 - (1D0 + M + C41*(THR2 - 0.5D0))/(1D0 + C3*M)*
     A                                  X*(C1*W + C2*X*DSQRT(W))) + XM
            D2T2 = D2T/2D0
            IF (X.GE.1D0) THEN
              N = 1
              GO TO 3
            END IF
C            (NO FINITE SOLUTION WITH X > XM)
          END IF
C
        END IF
C           (NOW HAVE A STARTER, SO PROCEED BY HALLEY)
      5 CONTINUE
        DO 4 I=1,3
        CALL TLAMB (M, Q, QSQFM1, X, 2, T, DT, D2T, D3T)
        T = TIN - T
        IF (DT.NE.0D0) X = X - T*DT/(DT*DT + T*D2T/2D0)
      4 CONTINUE
        IF (N.NE.3) RETURN
C           (EXIT IF ONLY ONE SOLUTION, NORMALLY WHEN M = 0)
        N = 2
        XPL = X
C           (SECOND MULTI-REV STARTER)
      3 CALL TLAMB (M, Q, QSQFM1, 0D0, 0, T0, DT, D2T, D3T)
        TDIFF0 = T0 - TMIN
        TDIFF = TIN - T0
        IF (TDIFF.LE.0) THEN
          X = XM - DSQRT(TDIFFM/(D2T2 - TDIFFM*(D2T2/TDIFF0
     A                                          - 1D0/XM**2)))
        ELSE
          X = -TDIFF/(TDIFF + 4D0)
          IJ = 200
          W = X + C0*DSQRT(2D0*(1D0 - THR2))
          IF (W.LT.0D0) X =
     A     X - DSQRT(D8RT(-W))*(X + DSQRT(TDIFF/(TDIFF + 1.5D0*T0)))
          W = 4D0/(4D0 + TDIFF)
          X = X*(1D0 + (1D0 + M + C42*(THR2 - 0.5D0))/(1D0 + C3*M)*
     A                                    X*(C1*W - C2*X*DSQRT(W)))
          IF (X.LE.-1D0) THEN
            N = N - 1
C             (NO FINITE SOLUTION WITH X < XM)
            IF (N.EQ.1) X = XPL
          END IF
        END IF
        GO TO 5
        END
```

Appendix E

FORTRAN-77 SUBROUTINE VLAMB

```
      SUBROUTINE VLAMB (GM, R1, R2, TH, TDELT, N, VR11, VT11,
     1 VR12, VT12, VR21, VT21, VR22, VT22)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (PI = 3.14159265358979D0, TWOPI = 2D0*PI)
      M = TH/TWOPI
      THR2 = TH/2D0 - M*PI
      DR = R1 - R2
      R1R2 = R1*R2
      R1R2TH = 4D0*R1R2*DSIN(THR2)**2
      CSQ = DR**2 + R1R2TH
      C = DSQRT(CSQ)
      S = (R1 + R2 + C)/2D0
      GMS = DSQRT(GM*S/2D0)
      QSQFM1 = C/S
      Q = DSQRT(R1R2)*DCOS(THR2)/S
      IF (C.NE.0D0) THEN
        RHO = DR/C
        SIG = R1R2TH/CSQ
       ELSE
        RHO = 0D0
        SIG = 1D0
      END IF
      T = 4D0*GMS*TDELT/S**2
      CALL XLAMB (M, Q, QSQFM1, T, N, X1, X2)
C           PROCEED FOR SINGLE SOLUTION, OR A PAIR
      DO 1 I=1,N
      IF (I.EQ.1) THEN
        X = X1
       ELSE
        X = X2
      END IF
      CALL TLAMB (M, Q, QSQFM1, X, -1, UNUSED, QZMINX, QZPLX, ZPLQX)
      VT2 = GMS*ZPLQX*DSQRT(SIG)
      VR1 = GMS*(QZMINX - QZPLX*RHO)/R1
      VT1 = VT2/R1
      VR2 = -GMS*(QZMINX + QZPLX*RHO)/R2
      VT2 = VT2/R2
      IF (I.EQ.1) THEN
        VR11 = VR1
        VT11 = VT1
        VR12 = VR2
        VT12 = VT2
       ELSE
        VR21 = VR1
        VT21 = VT1
        VR22 = VR2
        VT22 = VT2
      END IF
    1 CONTINUE
      RETURN
      END
```

LIST OF SYMBOLS

(Principal usage in the main text - some symbols have different meanings in the Appendices)

a	semi-major axis
c	chord of triangle CP_1P_2
C	centre of force
e	eccentricity
E	eccentric anomaly
k	arbitrary non-negative integer
m	number of complete revolutions included in the flight path
N	number of solutions of a particular Lambert problem
p	semi-latus section, $a(1 - e^2)$
P_1, P_2	end-points of flight path
q	$\tan \frac{1}{4}(\pi - \Theta_r)$, whence also $q^2 = 1 - c/s$
r_1, r_2	distances CP_1 and CP_2
s	semi-perimeter of triangle CP_1P_2
t_1, t_2	times such that $\Delta t (= t_2 - t_1)$ is the flight time
T	non-dimensionalized Δt , defined by equation (2)
T_o	value of $T(x, \Theta)$, or $T(x, q, m)$, when $x = 0$
T_M	minimum value of T for given $\Theta > 2\pi$
T', *etc*	partial derivatives $\partial T/\partial x$ *etc*
V_R	radial velocity
V_T	transverse velocity
w, W	empirical quantities used in starter weighting
x	fundamental parameter (iteration variable) such that $x^2 = 1 - s/2a$
x_i	value of x estimated after iteration i (x_0 = starter)
x_M	value of x associated with T_M
x^+	second solution for x , when relevant
z	$\sqrt{(1 - q^2 + q^2x^2)}$

LIST OF SYMBOLS (concluded)

α	μ/a
γ	$\sqrt{(\mu s/2)}$, introduced at equation (26)
δr, $\delta\theta$	absolute values of residual differences in r_2 and θ
δ_r, δ_θ	$\delta r/r_2$ and $\delta\theta/\theta$
ε	smaller of relative errors in (i) solved-for x and (ii) corresponding T
θ	angle P_1CP_2
θ_r	θ reduced to $(0, 2\pi)$ range
Θ	value of θ for the Lambert-equivalent isosceles triangle
λ	empirical quantity used in starting formulae
μ	strength of gravitational force centre (C)
ρ	$(r_1 - r_2)/c$
σ	$\sqrt{(1 - \rho^2)}$
τ	time from pericentre

REFERENCES

No.	Author	Title, etc
1	O. Volk	Johann Heinrich Lambert and the determination of orbits for planets and comets. *Celestial Mech.*, 21, 237-250 (1980)
2	R.H. Battin	*Astronautical guidance.* New York, etc, McGraw-Hill (1964)
3	P.R. Escobal	*Methods of orbit determination.* New York, etc, Wiley (1965)
4	S. Herrick	*Astrodynamics.* Vol.1, London, etc, Van Nostrand Reinhold (1971)
5	J.V. Breakwell R.W. Gillespie S. Ross	Researches in interplanetary transfer. *J. Amer. Rocket Soc.*, 31, 201-208 (1961)
6	G.S. Gedeon	Lambertian mechanics. In: *Proc. XII International Astronautical Congress*, 1961, pp 172-190, Vienna, Springer Verlag, and New York, Academic Press (1963)
7	G.S. Gedeon	A practical note on the use of Lambert's equation. *AIAA J.*, 3, 149-150 (1965)
8	E.R. Lancaster R.C. Blanchard R.A. Devaney	A note on Lambert's theorem. *J. Spacecraft and Rockets*, 3, 1436-1438 (1966)
9	E.T. Pitkin	A general solution of the Lambert problem. *J. Astronaut. Sci.*, 15, 270-271 (1968)
10	E.R. Lancaster R.C. Blanchard	A unified form of Lambert's theorem. NASA Technical Note D-5368 (1969)
11	L.A. D'Amario S.P. Synnott	A comparison of solutions of Kepler's and Lambert's problems. Instrumentation Lab. Report T-520 (1969)
12	R.H. Battin	A new solution for Lambert's problem. In: *Proc. XIX International Astronautical Congress*, 1968, Vol 2, pp 131-150, Oxford, Pergamon, and Warsaw, PWN (1970)

REFERENCES (continued)

No.	Author	Title, etc
13	E.R. Lancaster	Solution of Lambert's problem for short arcs. *Celestial Mech.*, 2, 60-63 (1970)
14	S. Herrick	A universal, singularity-free determination of an orbit for two positions and time interval. In: *Proc. XX International Astronautical Congress, 1969*, pp 507-510, Oxford, Pergamon, and Warsaw, PWN (1972)
15	E.E. Burniston C.E. Siewert	Exact analytical solutions basic to a class of two-body orbits. *Celestial Mech.*, 7, 225-235 (1973)
16	E.E. Burniston C.E. Siewert	Further results concerning exact analytical solutions basic to two-body orbits. *Celestial Mech.*, 10, 5-15 (1974)
17	J. Kriz	A uniform solution of the Lambert problem. *Celestial Mech.*, 14, 509-513 (1976)
18	R.H. Battin	Lambert's problem revisited. *AIAA J.*, 15, 707-713 (1977)
19	R.H. Battin T.J. Fill S.W. Shepperd	A new transformation invariant in the orbital boundary-value problem. *J. Guid. and Control*, 1, 50-55 (1978)
20	J.E. Prussing	Geometrical interpretation of the angles α and β in Lambert's problem. *J. Guid. and Control*, 2, 442-443 (1979)
21	F.T. Sun	A new treatment of Lambertian mechanics. *Acta Astronautica*, 8, 105-122 (1981)
22	F.T. Gun N.X. Vinh	Lambertian invariance and application to the problem of optimal fixed-time impulsive orbital transfer. *Acta Astronautica*, 10, 319-330 (1983)
23	R.H. Battin R.M. Vaughan	An elegant Lambert algorithm. *J. Guidance*, 7, 662-670 (1984)
24	F.W. Boltz	Second-order p-iterative solution of the Lambert/Gauss problem. *J. Astronaut. Sci.*, 32, 475-485 (1984)

REFERENCES (concluded)

No.	Author	Title, etc
25	L.G. Taff P.M.S. Randall	Two locations, two times, and the element set. *Celestial Mech.*, 37, 149-159 (1985)
26	F.T. Sun N.X. Vinh T.J. Chern	Analytic study of the solution families of the extended Godal's time equation for Lambert's problem. *J. Astronaut. Sci.*, 35, 213-234 (1987)
27	R.H. Gooding	Universal procedures for conversion of orbital elements to and from position and velocity (unperturbed orbits). RAE Technical Report 87043 (1987)
28	A.J. Sarnecki	'Minimal' orbital dynamics. *Acta Astronautica*, 17 (1988)
29	R.H. Gooding	Solution of the hyperbolic Kepler's equation. RAE Technical Report 87042 (1987)
30	A.W. Odell R.H. Gooding	Procedures for solving Kepler's equation. *Celestial Mech.*, 38, 307-334 (1986) RAE Technical Memorandum Space 356 (1986)
31	R.H. Gooding A.W. Odell	A monograph on Kepler's equation. RAE Technical Report 85080 (1985)
32	F.R. Moulton	*An introduction to celestial mechanics.* New York, Macmillan (1914 and 1958)
33	H.C. Plummer	*An introductory treatise on dynamical astronomy.* New York, Dover (1960)
34	K.F. Gauss	Theoria motus corporum coelestium in sectionibus conicis solem ambientium. Hamburg (1809) English Transl., by C.H. Davis, as Theory of motion of the heavenly bodies moving about the sun in conic sections. New York, Dover (1963)

Fig 1

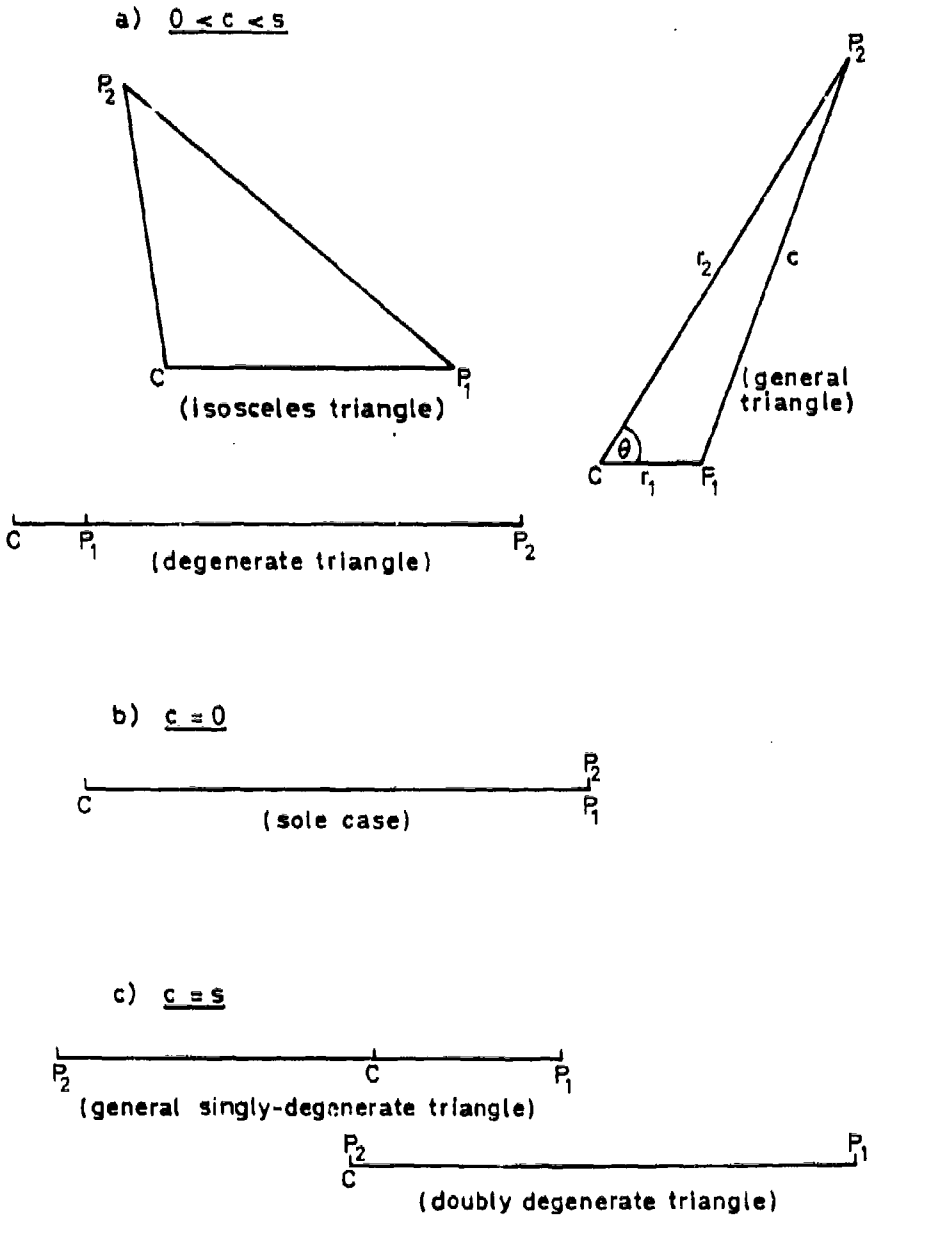

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Fig 1 Examples of Lambert-congruent equivalent triangles - three classes illustrated (same s , different c)

Fig 2

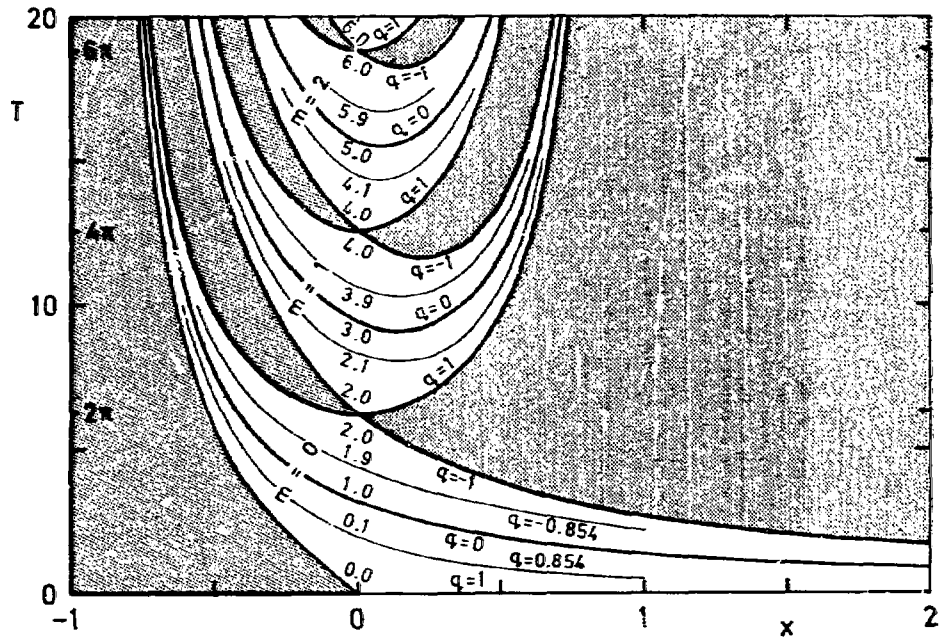

Fig 2 Plot of T vs x for selected values of Θ/π , corresponding to q and m as indicated

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Fig 3

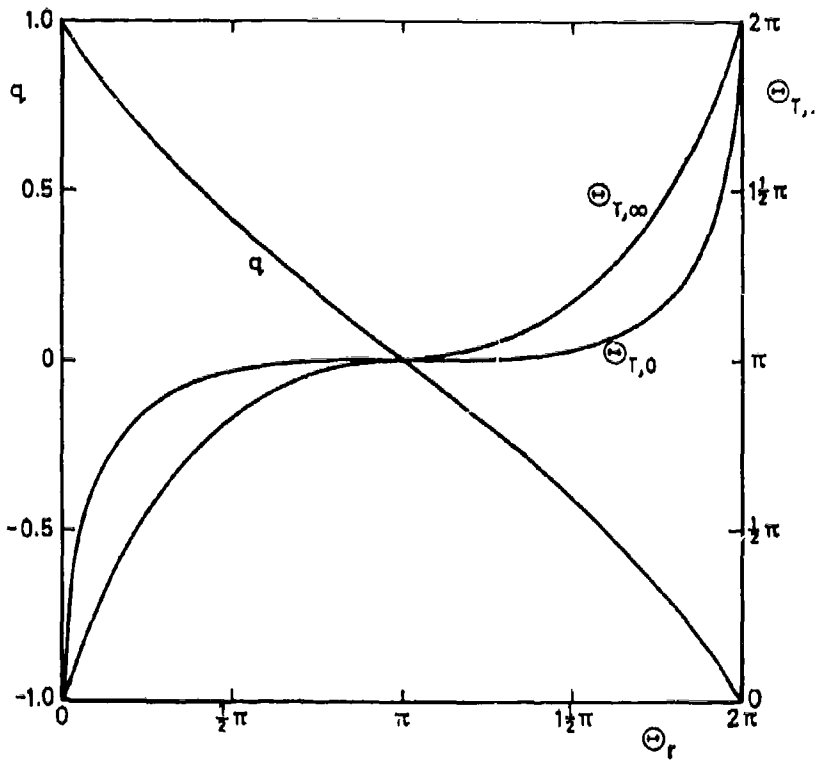

Fig 3 Variation of q, $\Theta_{T,0}$ and $\Theta_{T,\infty}$ with Θ_r

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Fig 4

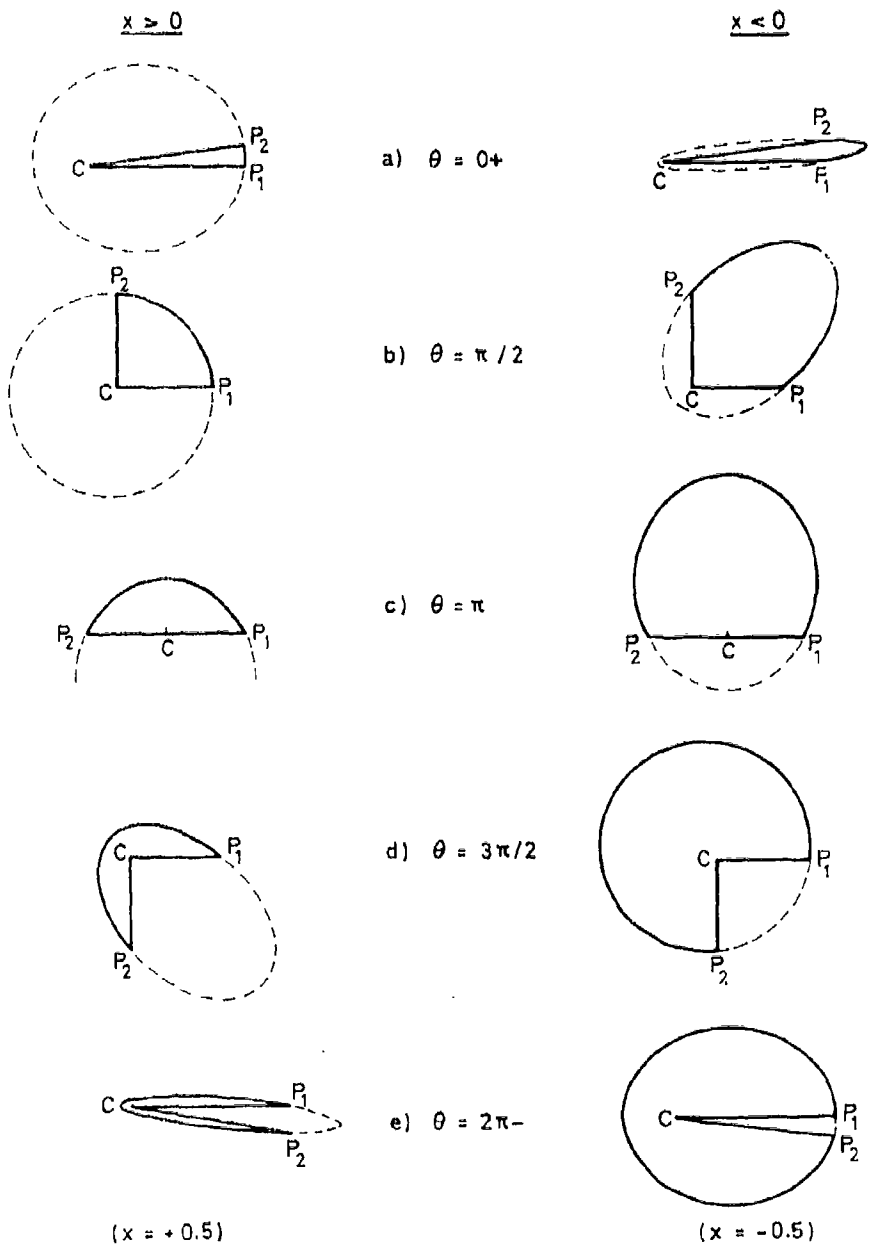

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Fig 4 Orbital paths associated with an evolving isosceles triangle (fixed value of s , with |x| = 0.5)

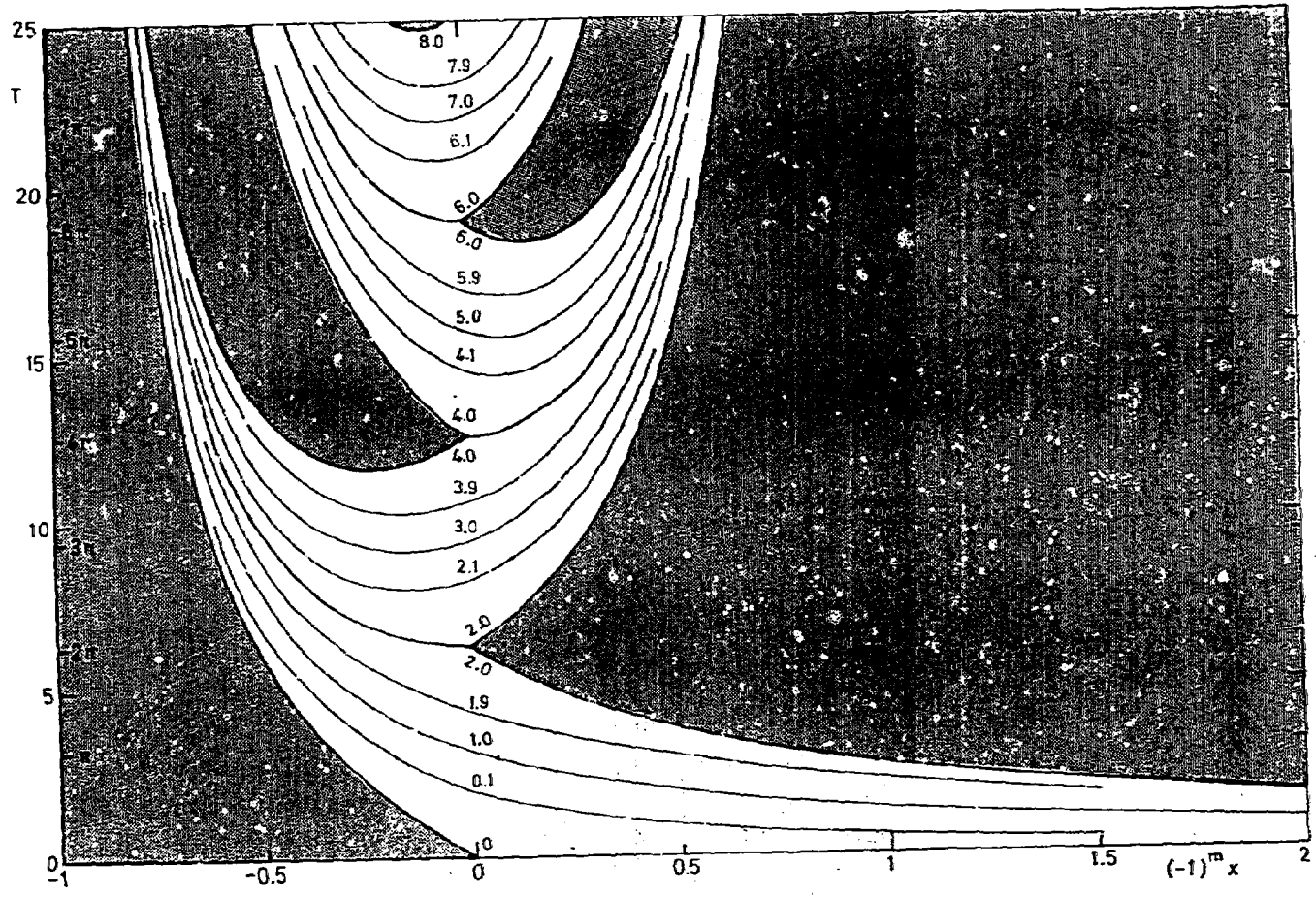

Fig 5

Fig 5 T plotted against $(-1)^m x$ instead of x (Θ/π contours)

Fig 6

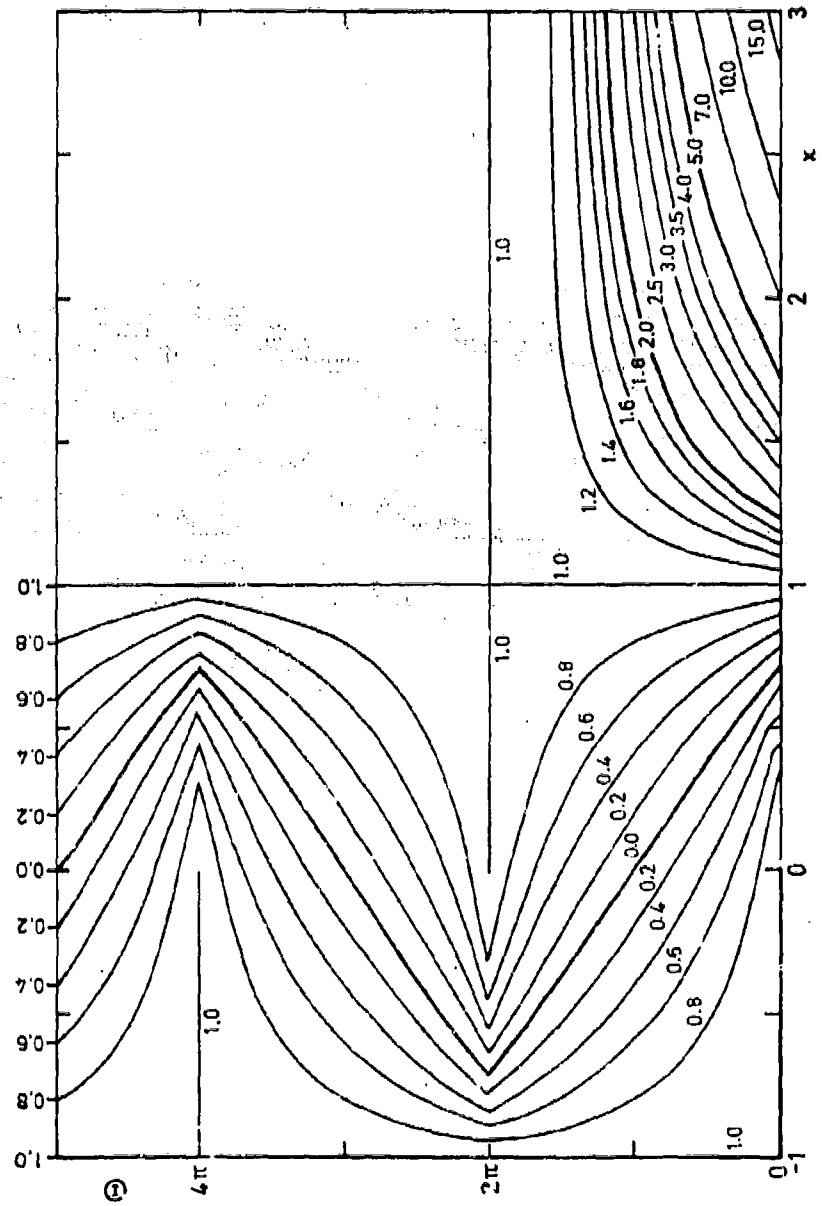

Fig 6 Contours of e for Lambert solutions in (x,Θ)-space (sign of x reversed for $2\pi < \Theta < 4\pi$)

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Orbits*. Celestial mechanics*. Lambert's problem. Great Britain. Halley

17. Abstract

During the past 30 years there has been a resurgence of interest in the classical orbital boundary-value problem of Lambert, largely because of its relevance to space rendezvous and interception. The most notable contribution to the subject was by Lancaster, Blanchard and Devaney, in 1966, but more recent researchers have failed to build on that work. The present Report is aimed at ~~remedying this neglect~~ by providing details of a universal solution of Lambert's problem, based on the approach of Lancaster *et al.* In particular, the Report presents starting formulae for Halley's cubic iteration process, used for evaluation of the unknown parameter, x , at the heart of the approach; this process always gives highly accurate values of x after three iterations.

A Fortran-77 computing procedure for a general solution of Lambert's problem has been developed, and its three main subroutines are listed. Details are given of the testing of this procedure.

Much of the Report is devoted to a classification of the set of all Lambert problems, and to a discussion of various geometric and physical aspects. Keywords:

FS9101/1