AD-A200 318

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

AN ANALYTICAL TREATMENT OF RESONANCE EFFECTS ON SATELLITE ORBITS

M.T. LANE
Group 91

TECHNICAL REPORT 796

19 AUGUST 1988

Approved for public release; distribution unlimited.

LEXINGTON MASSACHUSETTS

Abstract

A first order analytical approximation of the tesseral harmonic resonance perturbations of the Keplerian elements is presented, and the mean elements (the Keplerian elements with the long period portions averaged out) will also be given in closed form. The results of a numerical test, which compares the analytical solution against a numerical integration of the Lagrange equations of motion, will be summarized, and the implementation of the solution in the analytical orbit determination routine ANODE (at Lincoln Laboratory) will be outlined.

Accession For
NTIS GRA&I
DTIC TAB
Unannounced
Justification
By
Distribution/
Availability Codes
Avail and/or
Dist Special
A-1

QUALITY INSPECTED 4

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

AN ANALYTICAL TREATMENT OF RESONANCE EFFECTS ON SATELLITE ORBITS

1. INTRODUCTION

Normally the longitude dependent tesseral harmonics in the expansion of the Earth's potential produce only short period perturbations of satellite orbits, and these are small in comparison to the dominant latitude dependent zonal harmonic J_2. Typically analytical orbit determination routines (such as ANODE at Lincoln Laboratory) carry the short period effects to $o(J_2)$, and so the tesseral harmonic effects are ignored. When the mean motion of the satellite is nearly commensurate with the rotation rate of the earth, however, the trajectory of the satellite repeats itself relative to the earth and the perturbations due to certain 'critical' tesseral harmonics build up at each passage in the same spot. In this case, there can be important long period effects which should not be ignored. In this report, we will present a first order approximation of the isolated resonance effects by integrating the Lagrange equations for the general perturbations directly. This solution is first order in the sense that the elements on the right hand side of the Lagrange differential equations which are not expressed in terms of time explicitly are held fixed and constant. The corresponding mean elements will also be given in closed form, and the results of a test which compares the solution presented here with numerical integration of the equations of motion will be summarized.

The problem of resonance effects on satellite orbits was treated by a number of authors throughout the 1960s. The reader is encouraged to consult Allan [1], Garfinkel [2], and Gedeon [3]. Garfinkel solved the so-called 'ideal resonance problem', which was applicable to a variety of resonant situations and demonstrated how interesting and complicated the notion of resonance can be. In many ways, almost all of the current literature on resonance can be traced back to Allan's paper [1]. Gedeon expanded upon Allan's analysis and presented a general study of resonance effects on satellite orbits by integrating the equations of motion numerically. It is Allan's work, however, which influences the present study more than the others. Indeed, almost all of the work in the literature on resonance effects on artificial satellite orbits consider only the long period change in the 'broken-legged' longitude (i.e., the window of where the satellite is). The changes in the other Keplerian elements are also important in generating an accurate approximate ephemeris, and these changes are influenced directly by the acceleration of this longitude. Allan's suggestion [1, p. 1332] of expressing the longitude as a function of time using the Jacobi elliptic functions has spawned the idea in this report to exploit that representation of the longitude to yield a complete analytical solution of the long period resonance effects on each of the Keplerian elements.

The solution given by Garfinkel [2] applied to the artificial satellite problem should yield the same results as the solution outlined in this report. Recall that Garfinkel applied the von Zeipel method to solve the equations of motion. Our solution will be obtained by integrating the equations of motion directly so that each of the Keplerian elements will be given as explicit functions of time. The mean elements and long periodic corrections can be readily obtained by manipulating the solution directly. Thus, we feel our method is more simple to understand and to implement in existing analytical satellite orbit determination routines where there is assumed to be no interaction between the various harmonics in the geopotential. Our solution will work particularly well with the well-known classes of satellites effected by resonance such as the near circular synchronous and half-synchronous satellites. It is for these objects that this study is intended, and the work presented here is not meant to be an over-all treatment of the universal resonance problem.

It is easiest to describe the appropriate resonance problem by considering Kaula's expression of the geopotential disturbing function in terms of the Keplerian elements [4, p. 37]:

$$V = \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} V_{lmpq} \ ,$$

where

$$V_{lmpq} = \frac{\mu}{a}\left(\frac{a_e}{a}\right)^l F_{lmp}(i) G_{lpq}(e) J_{lm} \begin{cases} \cos(\psi_{lmpq}) & \text{for } l-m \text{ even} \\ \sin(\psi_{lmpq}) & \text{for } l-m \text{ odd} \end{cases} ,$$

and

$$\psi_{lmpq} = (l-2p)\omega + (l-2p+q)M + m(\Omega - \theta - \lambda_{lm}).$$

In the second equation, $F_{lmp}(i)$ is the inclination function, $G_{lpq}(e)$ is the eccentricity function, μ is Newton's gravitational constant times the mass of the earth, $J_{lm} = \sqrt{C_{lm}^2 + S_{lm}^2}$ is the (unnormalized) coefficient of the spherical harmonic of degree l and order m, $\lambda_{lm} = \frac{1}{m}\tan^{-1}(S_{lm}/C_{lm})$ is the corresponding reference longitude along the equator, a is the semimajor axis of the satellite orbit, and a_e is the (mean) radius of the earth. In the equation for the angular argument ψ, M denotes the mean anomaly, ω is the argument of perigee, θ is the mean Greenwich Sidereal time, and Ω is the longitude of the right ascending node.

We will write $s = s_0 + \Delta s$ to denote the mean motion in revolutions per day, where s_0 is integral and $|\Delta s| < 1/2$. Now exact orbital commensurability with the earth's rotation can be expressed as

$$\dot{M} + \dot{\omega} = s_0(\dot{\theta} - \dot{\Omega}) \quad . \tag{1}$$

Of course, we are using the dot over the element to denote differentiation with respect to time. With the above equation as motivation, we introduce the following fundamental quantity of longitude through the differential equation

$$\dot{\lambda} \equiv \frac{\Delta s}{s_0}(\dot{\theta} - \dot{\Omega}) \quad ,$$

and by assuming (1) we write

$$\dot{\lambda} = \frac{1}{s_0}(\dot{M} + \dot{\omega}) - (\dot{\theta} - \dot{\Omega}) \quad .$$

Thus, we can rewrite ψ_{lmpq} as

$$\psi_{lmpq}(t) = \left(l - 2p + q - \frac{m}{s_0}\right)(M + \omega) + m(\lambda - \lambda_{lm}) - q\omega \quad . \tag{2}$$

The underlying *critical indices* will be those sets of l, m, p, q which satisfy

$$l - 2p + q - \frac{m}{s_0} = 0 \quad .$$

The quantity $\lambda(t)$ can be physically interpreted as the osculating value of the longitude of the ascending node of the mean satellite [3].

If we consider only the critical tesseral harmonics in the disturbing function, then the disturbing function will take on the following form:

$$R = \sum_{\text{crit.}} V_{lmpq} \quad , \tag{3}$$

where

$$\psi_{lmpq} = m(\lambda - \lambda_{lm}) - q\omega \quad .$$

To simplify the following discussion, we will temporarily assume that $\dot{\psi}_{lmpq} = m\dot{\lambda}$; that is, we will assume that either $q = 0$ or else that $\dot{\omega} = 0$. One should see [3] for a more general treatment. At any rate, these assumptions can safely be made if the eccentricity is very small or if the inclination is close to the critical inclination ($\cos i \approx \pm 1/\sqrt{5}$). Now from the Lagrange equation of the general perturbation of the semimajor axis $\dot{a} = \frac{2}{na}\frac{\partial R}{\partial M}$, it follows from (3) that

$$\dot{a} = \sum_{\text{crit.}} \frac{2m\mu}{ns_0 a^2}\left(\frac{a_e}{a}\right)^l F_{lmp}(i) G_{lpq}(e) J_{lm} \begin{cases} -\sin(\psi_{lmpq}) & \text{for } l-m \text{ even} \\ \cos(\psi_{lmpq}) & \text{for } l-m \text{ odd} \end{cases} .$$

This change in the semimajor axis will in turn cause an acceleration in the longitude λ

$$\begin{aligned} \ddot{\lambda} &\approx \frac{1}{s_0}\ddot{M} \approx \frac{\partial n}{\partial a}\left(\frac{1}{s_0}\right)\dot{a} \\ &= -\sum_{\text{crit.}} \frac{3m\mu}{s_0^2 a^3}\left(\frac{a_e}{a}\right)^l F_{lmp}(i) G_{lpq}(e) J_{lm} \begin{cases} -\sin(\psi_{lmpq}) & \text{for } l-m \text{ even} \\ \cos(\psi_{lmpq}) & \text{for } l-m \text{ odd} \end{cases} . \end{aligned} \tag{4}$$

We have assumed that the accelerations of ω, Ω, and θ are all negligible. In an attempt to analytically integrate (4), we will assume that the elements a, e, and i on the right hand side remain fixed and constant. Once this assumption is made, a first integral follows easily

$$\dot{\lambda}^2 = C - \sum_{\text{crit.}} \frac{6m^2}{a^2 s_0^2} V_{lmpq} . \tag{5}$$

where C is a constant of integration.

The above differential equation (5) can only be integrated further analytically if we isolate one critical tesseral harmonic in the disturbing function (3). If this assumption is made, a global approximation is subsequently possible under the assumption that no two critical tesseral harmonics interact. For the remainder of this report, we will restrict ourselves to this situation.

2. THE SOLUTION OF THE SIMPLE PENDULUM PROBLEM

The simple pendulum problem in physics can be represented mathematically by the relation

$$\ddot{\psi} = -Q^2 \sin\psi \quad , \tag{6}$$

where Q is positive and constant. The solution is not difficult to obtain, and we will outline the important details for reference.

A first integral is found as in (5) and is

$$\dot{\psi}^2 = C + 2Q^2 \cos\psi \quad .$$

The integration constant C can be expressed in terms of the initial conditions ψ_0 and $\dot{\psi}_0$, and we can write $C = \dot{\psi}_0{}^2 - 2Q^2\cos\psi_0$. Clearly $C \geq -2Q^2\cos\psi$, and so if C is small enough, then ψ will be prevented from making a full cycle from 0 to 2π. If we write $\cos\psi = 1 - 2\sin^2(\frac{\psi}{2})$, then the sine of the maximum deviation from the focal point of libration (if such a value exists) corresponds to $\dot{\psi} = 0$. Explicitly, we have

$$\sin^2(\frac{\psi_m}{2}) = \frac{C + 2Q^2}{4Q^2} \quad , \tag{7}$$

and whether or not ψ_m exists, we can use the right hand side of (7) (which is always defined) to introduce

$$k \equiv 1/\sin(\frac{\psi_m}{2}) \quad .$$

Notice that the representation for C in terms of the initial conditions implies that

$$\sin^2(\frac{\psi_m}{2}) = \frac{\dot{\psi}_0{}^2 + 2Q^2(1 - \cos\psi_0)}{4Q^2} \quad ,$$

and so it will follow that

$$\mathrm{sign}(k) = \mathrm{sign}(\dot{\psi}_0) \quad .$$

Hence,

$$\dot{\psi}^2 = 4Q^2[\sin^2(\frac{\psi_m}{2}) - \sin^2(\frac{\psi}{2})] = \frac{4Q^2}{k^2}[1 - k^2\sin^2(\frac{\psi}{2})] \quad .$$

and so we can separate the variables and integrate

$$\int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{1 - k^2\sin^2(\frac{\psi}{2})}} = \frac{2Q}{k}\int_{t_0}^{t} dt \ .$$

If we separate the integral on the left side into two integrals and make the substitution $\phi = \frac{\psi}{2}$, then we obtain the following relation:

$$2u \equiv \frac{2Q}{k}(t - t_0) + \int_0^{\frac{\psi_0}{2}} \frac{2d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \int_0^{\frac{\psi}{2}} \frac{2d\phi}{\sqrt{1 - k^2\sin^2\phi}} \ ,$$

or

$$u \equiv \frac{Q}{k}(t - t_0) + F(k, \frac{\psi_0}{2}) = \int_0^{\frac{\psi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} \ . \qquad (8)$$

where $F(k, *)$ denotes an incomplete elliptic integral of the first kind.

The solution gets a little complicated at this point, for if there does exist a maximum deviation from the libration focal point, then $|k| > 1$ and we must use the transformation

$$F(k, \phi) = \frac{1}{k}F\left(\frac{1}{k}, \sin^{-1}(k\sin\phi)\right) \ .$$

Hence, in this situation we must transform (8) to

$$u = Q(t - t_0) + F\left(\frac{1}{k}, \sin^{-1}\left(k\sin\frac{\psi_0}{2}\right)\right) \qquad (9)$$

$$= \int_0^{\sin^{-1}\left(k\sin\frac{\psi}{2}\right)} \frac{d\phi}{\sqrt{1 - \frac{1}{k^2}\sin^2\phi}} \ .$$

In the case where $|k| > 1$, we will say that ψ is in *libration*, and when $|k| < 1$ (i.e., the right hand side of (7) > 1), then we will say that ψ is in *circulation*. In this setting, ψ will 'go over the top'. If $|k| = 1$, then ψ will be in the rare situation of being exactly 'stalled' between circulation and libration. We will ignore this possibility since it is highly unlikely that it will happen in a situation involving artificial satellites (and even if such a situation happens, then the nongravitational effects will quickly force the object into either circulation or libration). Garfinkel [2, p. 660] has pointed out that in this case the elliptic functions degenerate into hyperbolic functions, and so an analytical solution is possible in this case as well.

In the case that $|k| < 1$, then there will not exist ψ_m, and ψ will be allowed to circulate from 0 to 2π. The complete solution for ψ can be obtained from (8)

$$\sin\left(\frac{\psi}{2}\right) = \mathrm{sn}(u, k) \quad , \tag{10}$$

where $\mathrm{sn}(u, k)$ is a Jacobi elliptic function. There is a large amount of literature on the elliptic integrals and elliptic functions, and calculation to any desired accuracy is possible by quickly converging series. A good reference to consult for the complete theory of these transcendental functions is Whittaker and Watson [8]. In the libration setting, the complete solution of ψ can be obtained from (9) and is

$$k\sin\left(\frac{\psi}{2}\right) = \mathrm{sn}\left(u, \frac{1}{k}\right) \quad . \tag{11}$$

3. THE SOLUTION OF THE LAGRANGE EQUATIONS OF MOTION

If we become a little less restrictive than we were in the introduction and allow q to be not zero with ω allowed to vary (but having no acceleration), and if we isolate ourselves to one critical tesseral harmonic in the disturbing function, then we establish the following differential equation similar to (4):

$$\ddot{\psi} \approx \frac{m}{s_0}\ddot{M} \approx P\begin{cases} \sin\psi & \text{for } l-m \text{ even} \\ -\cos\psi & \text{for } l-m \text{ odd} \end{cases} ,$$

where

$$P = \frac{3m^2\mu}{s_0{}^2a^3}\left(\frac{a_e}{a}\right)^l F_{lmp}(i)G_{lpq}(e)J_{lm}$$

and we have dropped the subscripts of ψ. It is the product $F_{lmp}(i)G_{lpq}(e)$ which determines the sign of P, and so by adding a multiple of π or of $\pi/2$ to ψ, if necessary, we can set

$$Q = \sqrt{\frac{3m^2\mu}{s_0{}^2a^3}\left(\frac{a_e}{a}\right)^l |F_{lmp}(i)G_{lpq}(e)|J_{lm}}$$

and produce the form $\ddot{\psi} = -Q^2\sin\psi$ as in (6). Our assumptions insure that Q is a positive constant, and so the solution outlined in the previous section is applicable. It is interesting to note that the application of the simple pendulum to resonance problems dates back to Laplace, who in 1800 was investigating the Galilean satellites.

The Lagrange equations of the general perturbations of the Keplerian elements are listed in Kaula [4, p. 29] and are reproduced here for the convenience of the reader.

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M}$$

$$\frac{de}{dt} = \frac{1-e^2}{nea^2}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{nea^2}\frac{\partial R}{\partial \omega}$$

$$\frac{d\omega}{dt} = \frac{-\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{nea^2}\frac{\partial R}{\partial e}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{nea^2}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} \ .$$

The n appearing on the right hand side denotes the mean motion (expressed in the proper units). It is clear that if we remain consistent with our first order development and continue to hold a, e, and i constant on the right hand side, then we can separate the above system of differential equations and integrate directly. Keeping in mind that R in (3) is restricted to one critical tesseral harmonic, it is clear that in each of the Lagrange equations we must either evaluate $\int_{t_0}^{t} \sin\psi dt$ or else $\int_{t_0}^{t} \cos\psi dt$. We will proceed to demonstrate how this is done.

Case 1 ($|k| > 1$) Libration.

Recall that in this case $k\sin\frac{\psi}{2} = \mathrm{sn}\ (u,\frac{1}{k})$. In this regime, we can recenter the longitude, if necessary, so that $\frac{\psi}{2}$ is restricted to the first and fourth quadrants, and so $\cos\ (\frac{\psi}{2}) = \sqrt{1-\sin^2(\frac{\psi}{2})}$. With the representation of $\sin\left(\frac{\psi}{2}\right)$, an identity of elliptic functions yields

$$\cos\ (\frac{\psi}{2}) = \sqrt{1-\frac{1}{k^2}\mathrm{sn}^2\left(u,\frac{1}{k}\right)} = \sqrt{\mathrm{dn}^2\ (u,\frac{1}{k})} = \mathrm{dn}\ (u,\frac{1}{k}) \ .$$

Here we have relied on the fact that $\mathrm{dn}\ (u,\frac{1}{k})$ is always positive when the argument u and the modulus $\frac{1}{k}$ are real. For the elementary properties of elliptic functions that will be assumed, the reader should consult [8].

We can now evaluate

$$\int_{t_0}^{t}\sin\psi dt = \int_{t_0}^{t} 2\sin\ (\frac{\psi}{2})\cos\ (\frac{\psi}{2})\ dt = \int_{t_0}^{t}\frac{2}{k}\mathrm{sn}\ (u,\frac{1}{k})\ \mathrm{dn}\ (u,\frac{1}{k})\ dt$$

$$= \frac{2}{kQ}\int_{u_0}^{u}\mathrm{sn}\ (u,\frac{1}{k})\,\mathrm{dn}\ (u,\frac{1}{k})\ du = \frac{2}{kQ}\left[\mathrm{cn}\left(u_0,\frac{1}{k}\right) - \mathrm{cn}\left(u,\frac{1}{k}\right)\right] \ .$$

where $u = u(t)$ and $u_0 = u(t_0)$. Furthermore,

$$\int_{t_0}^{t} \cos\psi \, dt = \int_{t_0}^{t} 2\cos^2\left(\frac{\Psi}{2}\right) - 1 dt = \int_{t_0}^{t} 2\mathrm{dn}^2\left(u, \frac{1}{k}\right) - 1 dt$$

$$= \frac{1}{Q}\int_{u_0}^{u} 2\mathrm{dn}^2\left(u, \frac{1}{k}\right) - 1 du = \frac{2}{Q}\left[E\left(u, \frac{1}{k}\right) - E\left(u_0, \frac{1}{k}\right)\right] - \frac{1}{Q}\left(u - u_0\right) ,$$

where $E(u, \frac{1}{k})$ is the fundamental elliptic integral $\int_0^u \mathrm{dn}^2 u \, du$ [8, pp. 517-518].

Case 2 ($|k| < 1$) Circulation.

Recall that in this case $\sin\left(\frac{\Psi}{2}\right) = \mathrm{sn}(u, k)$. We must consider u in (8) as a uniformizing variable. As time increases, either u always increases or always decreases (depending upon the sign of k). In a like manner, ψ will always increase or always decrease depending upon the sign of $\dot{\psi}_0$ (which is the sign of k). Hence it will always be the case that u and ψ are moving in the same direction as time increases; that is, $\frac{d\psi}{du}$ is always positive in the circulation regime. Now if we differentiate both sides of the equation $\sin\frac{\Psi}{2} = \mathrm{sn}(u, k)$ by u, we observe that

$$\cos\left(\frac{\Psi}{2}\right)\left[\frac{1 d\psi}{2 du}\right] = \mathrm{cn}(u, k)\mathrm{dn}(u, k) ,$$

and since $\mathrm{dn}(u, k)$ is always positive, it follows that sign $(\cos\frac{\Psi}{2}) = \mathrm{sign}[\mathrm{cn}(u, k)]$. Moreover

$$\cos^2\left(\frac{\Psi}{2}\right) = 1 - \sin^2\left(\frac{\Psi}{2}\right) = 1 - \mathrm{sn}^2(u, k) = \mathrm{cn}^2(u, k) ,$$

and so it follows that $\cos\left(\frac{\Psi}{2}\right) = \mathrm{cn}(u, k)$.

We are now in a position to evaluate the integral of $\sin\psi \, dt$:

$$\int_{t_0}^{t} \sin\psi \, dt = \int_{t_0}^{t} 2\sin\left(\frac{\Psi}{2}\right)\cos\left(\frac{\Psi}{2}\right) dt = \int_{t_0}^{t} 2\mathrm{sn}(u, k)\mathrm{cn}(u, k) dt$$

$$= \frac{2k}{Q}\int_{u_0}^{u} \mathrm{sn}(u, k)\mathrm{cn}(u, k) du = \frac{2}{Qk}[\mathrm{dn}(u_0, k) - \mathrm{dn}(u, k)] .$$

For the integral of $\cos\psi dt$, we begin by writing

$$\int_{t_0}^{t}\cos\psi dt = \int_{t_0}^{t} 1 - 2\sin^2\left(\frac{\psi}{2}\right)dt = \frac{k}{Q}\int_{u_0}^{u} 1 - 2\text{sn}^2(u,k)du \ ,$$

and we consider the identity [8, p. 516] $u - k^2\int_0^u \text{sn}^2 u du = \int_0^u \text{dn}^2 u du$. Hence,

$$\int_{t_0}^{t}\cos\psi dt = \frac{k}{Q}[u - u_0] - \frac{2}{kQ}\{(u - u_0) - [E(u,k) - E(u_0,k)]\}$$

$$= \frac{2}{kQ}[E(u,k) - E(u_0,k)] - \frac{k'^2 + 1}{kQ}(u - u_0) \ ,$$

where $k'^2 = 1 - k^2$ denotes the complementary modulus.

The mean motion, n, on the right hand side of the Lagrange equation for the rate of change of the mean anomaly should not be assumed constant in a first order treatment; for the mean motion appears in the equation for the rate of change of M in the unperturbed situation also. Since $n = \mu^{1/2}a^{-3/2}$, where μ is constant, we can use the solution for a outlined above to compute the integral $\int_{t_0}^{t} n dt$.

In our restricted resonance problem, we have that

$$\dot{a} = \frac{2as_0}{3nm}Q^2\sin\psi \ ,$$

and if we write

$$A = \frac{4as_0Q}{3nmk} \ ,$$

then it will follow that

$$a = a_0 + A\begin{cases}\text{cn}\left(u_0, \frac{1}{k}\right) - \text{cn}\left(u, \frac{1}{k}\right) & \text{for } |k| > 1 \\ \text{dn}\left(u_0, k\right) - \text{dn}\left(u, k\right) & \text{for } |k| < 1\end{cases} \ .$$

We can write $a = a_0 + \delta_a$, and use the binomial series to approximate n. We will write $a = a_0\left[1 + \frac{\delta_a}{a_0}\right]$ and expand

$$n = n_0\left[1 - \frac{3}{2}\left(\frac{\delta_a}{a_0}\right) + \frac{15}{8}\left(\frac{\delta_a}{a_0}\right)^2 - \ \ldots\ \right], \qquad (12)$$

from which point we will truncate the rest upon the assumption that the series drops off rapidly. There are then four integrals that we need to evaluate, and with reference to [8, p. 516], we will list these integrals evaluated:

$$\int_0^u \mathrm{cn}\left(u, \frac{1}{k}\right)\mathrm{d}u = k\tan^{-1}\left[\mathrm{sn}\left(u, \frac{1}{k}\right)/k\mathrm{dn}\left(u, \frac{1}{k}\right)\right] ,$$

$$\int_0^u \mathrm{dn}(u, k)\mathrm{d}u = \mathrm{am}(u, k) = \sin^{-1}\{\mathrm{sn}(u, k)\} + j(u)\pi,\ \ j(u) \text{ integral} ,$$

$$\int_{u_0}^u \mathrm{cn}^2\left(u, \frac{1}{k}\right)\mathrm{d}u = k^2\left[E\left(u, \frac{1}{k}\right) - E\left(u_0, \frac{1}{k}\right) - \bar{k}'^2(u - u_0)\right] ,$$

where $\bar{k}'^2 = 1 - \frac{1}{k^2}$, and

$$\int_{u_0}^u \mathrm{dn}^2(u, k)\mathrm{d}u = E(u, k) - E(u_0, k) .$$

The integral $\int_{t_0}^t n\mathrm{d}t$ can now be readily evaluated with the aid of the above integrals. Hence, each of the Lagrange differential equations of the general perturbations of the Keplerian elements can be separated and integrated to yield the complete solution of the restricted resonance problem to first order.

Suppose that the disturbing function is isolated to one critical tesseral harmonic,

$$R = -\frac{\mu}{a}\left(\frac{a_e}{a}\right)^l |F_{lmp}(i)G_{lpq}(e)| J_{lm}\cos\psi ,$$

where $\psi = m(\lambda - \lambda_{lm}) - q\omega + j\pi$ and $\lambda = \frac{1}{s_0}(M + \omega) - (\theta - \Omega)$. Here $j = 0, 1, 1/2$, or $3/2$ depending upon the sign of $F_{lmp}(i)G_{lpq}(e)$ and whether $l - m$ is even or odd. Because this sign is important in the equations which follow, we will use a symbol,

$$\sigma \equiv \mathrm{sign}[F_{lmp}(i)G_{lpq}(e)] .$$

Upon calculating the rates of change of each of the Keplerian elements given by the Lagrange equations of motion and applying the integration outlined in this section, we arrive at the explicit expressions for the changes of each of the Keplerian elements to first order. For the convenience of the reader, we will list these changes, where the elements n, a, e, and i appearing on the right hand side of each equation are held fixed and constant from the initial osculating elements. To save writing, we will allow $*$ to denote the modulus k if $|k| < 1$ and $1/k$ if $|k| > 1$ and we will write $F'_{lmp}(i)$ and $G'_{lpq}(e)$ to denote the derivative of these functions with respect to their argument.

$$\delta a = A\begin{cases} \mathrm{cn}(u_0, *) - \mathrm{cn}(u, *) & \text{for } |k| > 1 \\ \mathrm{dn}(u_0, *) - \mathrm{dn}(u, *) & \text{for } |k| < 1 \end{cases},$$

$$\delta e = \frac{\mu}{na^3}\left(\frac{a_e}{a}\right)^l J_{lm} F_{lmp}(i) G_{lpq}(e)[\frac{m}{s_0}(\frac{1 - e^2 - \sqrt{1 - e^2}}{e}) + \frac{q\sqrt{1 - e^2}}{e}] \times$$

$$\frac{2}{kQ}\begin{cases} \mathrm{cn}(u_0, *) - \mathrm{cn}(u, *) & \text{for } |k| > 1 \\ \mathrm{dn}(u_0, *) - \mathrm{dn}(u, *) & \text{for } |k| < 1 \end{cases},$$

$$\delta\omega = \frac{\mu}{na^3}\left(\frac{a_e}{a}\right)^l J_{lm}\sigma\left\{\frac{\cos i}{\sqrt{1 - e^2}} G_{lpq}(e)\left(\frac{F'_{lmp}(i)}{\sin i}\right) - \sqrt{1 - e^2} F_{lmp}(i)\left(\frac{G'_{lpq}(e)}{e}\right)\right\} \times$$

$$\begin{cases} \frac{2}{Q}[E(u, *) - E(u_0, *)] - \frac{1}{Q}(u - u_0) & \text{for } |k| > 1 \\ \frac{2}{kQ}[E(u, *) - E(u_0, *)] - \frac{k'^2 + 1}{kQ}(u - u_0) & \text{for } |k| < 1 \end{cases},$$

$$\delta i = \frac{-\mu}{na^3\sqrt{1 - e^2}\sin i}\left(\frac{a_e}{a}\right)^l F_{lmp}(i) G_{lpq}(e) J_{lm}\left[q\cos i + m\left(1 - \frac{1}{s_0}\cos i\right)\right] \times$$

$$\frac{2}{kQ}\begin{cases} \mathrm{cn}(u_0, *) - \mathrm{cn}(u, *) & \text{for } |k| > 1 \\ \mathrm{dn}(u_0, *) - \mathrm{dn}(u, *) & \text{for } |k| < 1 \end{cases},$$

$$\delta\Omega = \frac{-\mu}{na^3\sqrt{1 - e^2}}\left(\frac{a_e}{a}\right)^l J_{lm}\sigma G_{lpq}(e)\left(\frac{F'_{lmp}(i)}{\sin i}\right) \times$$

$$\begin{cases} \frac{2}{Q}[E(u, *) - E(u_0, *)] - \frac{1}{Q}(u - u_0) & \text{for } |k| > 1 \\ \frac{2}{kQ}[E(u, *) - E(u_0, *)] - \frac{k'^2 + 1}{kQ}(u - u_0) & \text{for } |k| < 1 \end{cases},$$

and

$$\delta M = n\left\{\left(1 - \frac{3A}{2a}\begin{cases} \mathrm{cn}(u_0, *) \\ \mathrm{dn}(u_0, *) \end{cases} + \frac{15A^2}{8a^2}\begin{cases} \mathrm{cn}^2(u_0, *) \\ \mathrm{dn}^2(u_0, *) \end{cases}\right)(t - t_0)\right.$$

$$+ \frac{k}{Q}\left(\frac{3A}{2a} - \frac{15A^2}{4a^2}\begin{cases}\mathrm{cn}(u_0, *)\\ \mathrm{dn}(u_0, *)\end{cases}\right)\begin{cases}\tan^{-1}[*\mathrm{sn}(u, *)/\mathrm{dn}(u, *)] - \tan^{-1}[*\mathrm{sn}(u_0, *)/\mathrm{dn}(u_0, *)]\\ \mathrm{am}(u, *) - \mathrm{am}(u_0, *)\end{cases}$$

$$\left. + \frac{15kA^2}{8Qa^2}\begin{cases}k[E(u, *) - E(u_0, *) - \bar{k}'^2(u - u_0)]\\ E(u, *) - E(u_0, *)\end{cases}\right\}$$

$$- \frac{\mu}{na^3}\left(\frac{a_e}{a}\right)^l J_{lm}\left[2(l+1)|F_{lmp}(i)G_{lpq}(e)| - \sigma(1 - e^2)F_{lmp}(i)\left(\frac{G'_{lpq}(e)}{e}\right)\right] \times$$

$$\begin{cases}\frac{2}{Q}[E(u, *) - E(u_0, *)] - \frac{1}{Q}(u - u_0) & \text{for } |k| > 1\\ \frac{2}{kQ}[E(u, *) - E(u_0, *)] - \frac{k'^2 + 1}{kQ}(u - u_0) & \text{for } |k| < 1 \end{cases} .$$

The reader can observe that singularities occur in the above equations as the inclination approaches 0 or as the eccentricity approaches 0 within the precision of the computer. In the case of near zero eccentricity, since the eccentricity function $G_{lpq}(e)$ has order $e^{|q|}$, we can choose $q = 0$ so that the singularity in δe is removable. Moreover, when $q = 0$, it can be seen from Kaula's expression for the eccentricity function [4, p. 37] that $G'_{lp0}(e)$ has order e, and so the case of the singularity with zero eccentricity is of no consequence. Finally, in the case of near zero inclination, the singularity can often be removed with the aid of the inclination function $F_{lmp}(i)$. This is the case with the more dominant tesseral harmonics such as J_{22} and J_{31}, and so this singularity can be handled by avoiding the critical harmonics for which the inclination function does not remove the singularity with $\sin i$, $i \approx 0$.

It was noted earlier that the elliptic functions can be computed by rapidly converging series. A routine was developed for computing the elliptic functions which was based upon the method outlined in the appendix of [6]. This appendix evidently was based upon a section of Whittaker and Watson's development of theta functions [8, p. 486], and while some argue that it is convenient to expand the functions in terms of the modulus $*$, this algorithm transforms the modulus to a smaller parameter q, and so the resulting expansions will always converge rapidly even for the troublesome case where the modulus approaches one. This method was able to compute each of the elliptic functions to an accuracy of 16 digits, and the algorithm was extremely fast. The function $\mathrm{am}(u, *)$, which up until this point has been used but was defined in a somewhat ambiguous fashion, will be expressed more precisely in the next section on mean elements.

4. THE MEAN ELEMENTS

In this section, the mean elements will be isolated from the solution outlined in the last section. The periods of the corrections to the secular changes will be given, and it will shown that a, e, and i all have constant secular change, while the angular arguments ω, Ω, and M all have a nonzero secular rate of change. It should be noted here that Garfinkel [2] expressed his solution in terms of mean elements, which he obtained by integrating the elliptic functions over their periods (as we will do shortly). Also, Gedeon [3] gave an interesting comparative study of the long periods among satellites with varying parameters. The mathematics involved at this point is not difficult due to the friendly nature of the elliptic functions [8].

Let us introduce the number $K = F(*, \pi/2)$, where (as before) the modulus $*$ is assumed to be k if $|k| < 1$ and $1/k$ if $|k| > 1$. The period of the functions $\mathrm{sn}(u, *)$ and $\mathrm{cn}(u, *)$ is $4K$, whereas $\mathrm{dn}(u, *)$ has the smaller period $2K$. Both $\mathrm{sn}(u, *)$ and $\mathrm{cn}(u, *)$ have an average value of 0 over a $4K$ period, and Garfinkel [2, p. 663] has noted that the average value of $\mathrm{dn}(u, *)$ over a $2K$ period is $\pi/2K$. Since this is an important fact and requires the evaluation of $\mathrm{am}(u, *)$ at any value of u, we will present a proof of this fact.

Recall that in the last section we noted that $\int_0^u \mathrm{dn}u\,du = \mathrm{am}u$. Since $-\pi/2 \le \sin^{-1}(\mathrm{sn}u) \le \pi/2$, we must take into account the periodic nature of sn in order to write an exact formula for $\mathrm{am}(u, *)$. Let r denote the integer part of u/K and let $s = r(\mathrm{mod}\ 4)$. Then

$$\mathrm{am}(u,*) = \begin{cases} \sin^{-1}[\mathrm{sn}(u, *)] + \frac{\pi}{2}(r - s) & \text{if } s = 0 \\ \pi - \sin^{-1}[\mathrm{sn}(u, *)] + \frac{\pi}{2}(r - s) & \text{if } s = 1 \text{ or } s = 2 \\ 2\pi + \sin^{-1}[\mathrm{sn}(u, *)] + \frac{\pi}{2}(r - s) & \text{if } s = 3\ . \end{cases}$$

Therefore $\mathrm{am}(u, *) = \frac{\pi u}{2K} + \left\{\mathrm{am}(u, *) - \frac{\pi u}{2K}\right\}$, where the part in braces is the periodic part with period $4K$ and average 0, and $\pi u/2K$ is the secular portion. Hence,

$$\frac{1}{2K}\int_0^{2K} \mathrm{dn}(u, *)du = \frac{1}{2K}\mathrm{am}(2K, *)$$

$$= \frac{\pi}{2K} + \frac{1}{2K}[\pi - \sin^{-1}[\mathrm{sn}(2K, *)] - \pi] = \frac{\pi}{2K}\ .$$

We also require the secular parts of $E(u, *)$ and $\tan^{-1}[*\mathrm{sn}(u, *)/\mathrm{dn}(u, *)]$. Now $E(u, *)$ is not a periodic function, but it can be expressed by the relation $E(u, *) = Z(u, *) + Eu/K$, where $Z(u, *)$ denotes the so-called Zeta function (which has a period of $2K$ and average 0) and E is a complete elliptic integral of the second kind; i.e., $E = E(K, *)$. The function $\tan^{-1}[*\mathrm{sn}(u, *)/\mathrm{dn}(u, *)]$ is periodic with period $4K$, and the average over its period is 0, although this fact is not easy to prove.

In order to write the mean elements analytically, we must have analytical expressions for $<\int_{t_0}^{t} \sin\psi dt>$, $<\int_{t_0}^{t} \cos\psi dt>$, and $<\int_{t_0}^{t} n dt>$, where <•> represents the part of • with the periodic portions removed by averaging. In order to write these explicitly, we must again divide the work into two cases.

Case 1 ($|k| > 1$) Libration.

Since $<\mathrm{cn}\,(u, \frac{1}{k})> = 0$ over a $4K$ period, we have that

$$<\int_{t_0}^{t} \sin\psi dt> = \frac{2\mathrm{cn}\left(u_0, \frac{1}{k}\right)}{kQ} .$$

The function $E(u)$ has a nonzero secular rate of change, and so for the average of the integral of $\cos\psi$, we isolate the linear part and integrate the periodic part over its period of $2K$. This becomes

$$<\int_{t_0}^{t} \cos\psi dt> = \left[\frac{2Eu_0}{KQ} - \frac{2}{Q}E\left(u_0, \frac{1}{k}\right)\right] + \left(\frac{2E}{K} - 1\right)(t - t_0) .$$

The secular part of the integral of the mean motion n is more lengthy because of the binomial series expansion:

$$\begin{aligned}<\int_{t_0}^{t} n dt> = \frac{n_0}{Q}\Bigg\{&\left[\frac{15}{4}A^2\mathrm{cn}\left(u_0, \frac{1}{k}\right) - \frac{3}{2}A\right] k\tan^{-1}\left[\mathrm{sn}\left(u_0, \frac{1}{k}\right)/k\mathrm{dn}\left(u_0, \frac{1}{k}\right)\right] \\ &- \frac{15}{8}k^2A^2\left[E\left(u_0, \frac{1}{k}\right) + \frac{Eu_0}{K}\right]\Bigg\} \\ &+ n_0\Bigg\{1 - \frac{3}{2}A\mathrm{cn}\left(u_0, \frac{1}{k}\right) + \frac{15}{8}A^2\mathrm{cn}^2\left(u_0, \frac{1}{k}\right) \\ &+ \frac{15}{8}k^2A^2\left(\frac{E}{K} - k'^2\right)\Bigg\}(t - t_0) .\end{aligned}$$

Case 2 ($|k| > 1$) Circulation.

Since the average of $\mathrm{dn}(u, k)$ over a $2K$ period is $\pi/2K$, it immediately follows that

$$<\int_{t_0}^{t} \sin\psi dt> = \frac{2}{kQ}\left[\mathrm{dn}(u_0, k) - \frac{\pi}{2K}\right] .$$

Similar to case 1, we have

$$<\int_{t_0}^{t} \cos\psi dt> = \frac{2}{kQ}\left[\frac{Eu_0}{K} - E(u_0, k)\right] + \frac{1}{k}\left[\frac{2E}{K} - (1 + k'^2)\right](t - t_0) ,$$

and finally, we have

$$\begin{aligned}<\int_{t_0}^{t} n dt> = {} & \frac{n_0 k}{Q}\left\{\left[\frac{3}{2}A - \frac{15}{4}A^2\mathrm{dn}(u_0, k)\right]\left[\frac{\pi u_0}{2K} - \mathrm{am}(u_0, k)\right]\right. \\ & \left. + \frac{15}{8}A^2\left[\frac{Eu_0}{K} - E(u_0, k)\right]\right\} \\ & + n_0\left\{1 - \frac{3}{2}A\mathrm{dn}(u_0, k) + \frac{15}{8}A^2\mathrm{dn}^2(u_0, k)\right. \\ & \left. + \left[\frac{3}{2} - \frac{15}{4}A^2\mathrm{dn}(u_0, k)\right]\frac{\pi}{2K} + \frac{15A^2E}{8K}\right\}(t - t_0) .\end{aligned}$$

We are now able to observe from the Lagrange equations for the general perturbations of the Keplerian elements that in each of the two cases, the secular rate of change is 0 for a, e, and i and nonzero for ω, Ω, and M.

5. THE NUMERICAL STUDY

The method described in the preceding sections was used in a FORTRAN computer program to compare the long period propagation of the Keplerian elements with a numerical integration of the Lagrange equations of motion (with which the resonance effects are accounted for automatically). A variable step-size Adams predictor-corrector polynomial integrator was used for this comparison, and the test was performed on a Harris 800 machine with quadruple precision (96 bit arithmetic). The integrator was originally coded by Fred T. Krough at the Jet Propulsion Laboratory in Pasadena, California in April of 1969, and it was adapted for Harris computers by E. Mike Gaposchkin at MIT Lincoln Laboratory in November of 1982. The objectives of this study were:

1. to estimate the accuracy of the analytical theory over the long resonance period,
2. to investigate the parameters which effect the accuracy,
3. to observe the order of magnitude of the perturbation due to a critical tesseral harmonic,
4. to determine the decay in accuracy when two or more critical tesseral harmonics are included in the disturbing function (and it is assumed in the analytical solution that no two tesseral harmonics interact), and
5. to numerically compare the perturbation produced by one dominant critical harmonic with that produced by more than one critical harmonic.

5.1. AN ERROR ANALYSIS

Before the comparative results are presented, it is necessary to optimize the analytical model in the sense of minimizing the absolute error of the analytical solution of the total perturbation from the same perturbation computed by numerically integrating the Lagrange equations of motion. In the analysis of the solution, it is clear that one has complete flexibility in the choice of the elements a, e, and i which are held fixed and constant in the solution. Apart from the use of ψ_0 and $\dot{\psi}_0$ in the statement of the initial value simple pendulum problem, one has freedom to change the choice of a, e, and i in the calculation of the modulus k, the factor Q in the definition of u and in the evaluation of the integrals $\int_{t_0}^{t} \sin\psi dt$ and

$\int_{t_0}^{t} \cos\psi dt$, and in the factors of $\sin\psi$ and $\cos\psi$ in the Lagrange equations for the rates of change of the elements. For notation, we will refer to a generic Keplerian element (except the mean anomaly) by α and write ε to denote the factor of $\sin\psi$ or of $\cos\psi$ in the Lagrange equation for the rate of change of α. Therefore, we can write $\dot{\alpha} = \varepsilon\sin\psi$ or $\dot{\alpha} = \varepsilon\cos\psi$, and one holds $\varepsilon = \varepsilon(a, e, i)$ constant over an entire resonance period and proceeds to integrate $\sin\psi$ or $\cos\psi$ by computing k, Q, and the elliptic functions as outlined in Section 3. It should not be surprising for one to observe that the mean elements can improve the model. Indeed, the mean elements $\langle a\rangle$, $\langle e\rangle$, and $\langle i\rangle$ are constant over an entire resonance period, and we will see that if they are positioned strategically in the analytical solution, then a significant improvement is realized in a comparison with the numerical solution.

We can iterate on the integration procedure by computing the mean values $\langle\delta a(a, e, i)\rangle$, $\langle\delta e(a, e, i)\rangle$, and $\langle\delta i(a, e, i)\rangle$ as outlined in Section 4 and on each successive iteration compute $\delta\alpha(\langle a\rangle, \langle e\rangle, \langle i\rangle)$, where $\langle a\rangle = a_0 + \langle\delta a(a, e, i)\rangle$, $\langle e\rangle = e_0 + \langle\delta e(a, e, i)\rangle$, and $\langle i\rangle = i_0 + \langle\delta i(a, e, i)\rangle$ and the elements a, e, and i are saved from the last iteration. Empirically, this procedure converges rapidly, and in fact, only two iterations are essentially needed to realize the benefit.

Consider the error in the approximation of δa for a synchronous object in libration with the harmonic V_{2200}. In Figure 5-1 we have plotted the error first for the situation where the initial osculating elements a_0, e_0, and i_0 are used to determine k, Q, and ε and then for the situation where $\langle a\rangle$, $\langle e\rangle$, and $\langle i\rangle$ are used to compute k, Q, and ε. One can observe that in the first case the error is growing without bound and in the latter case the error is not only bounded, but the error is bounded by a small amount. Thus, in the libration situation, the use of the mean elements optimizes the model of the motion in a pleasing way.

The above result makes sense physically since the mean values of a, e, and i correspond to the stable equilibrium longitude of the oscillation, about which the motion is completely symmetrical. Unfortunately, the situation is not as nice in the circulation regime since no such point of symmetry exists. In fact, it has been observed that any values of a, e, and i in the analytical solution will cause the error to grow without bound over time, but it has also been observed that it is possible to minimize the growth by using the mean values in strategic places. The optimum is not achieved by simply using the mean values in k, Q, and ε as in the libration situation, but a study of seven synchronous objects in

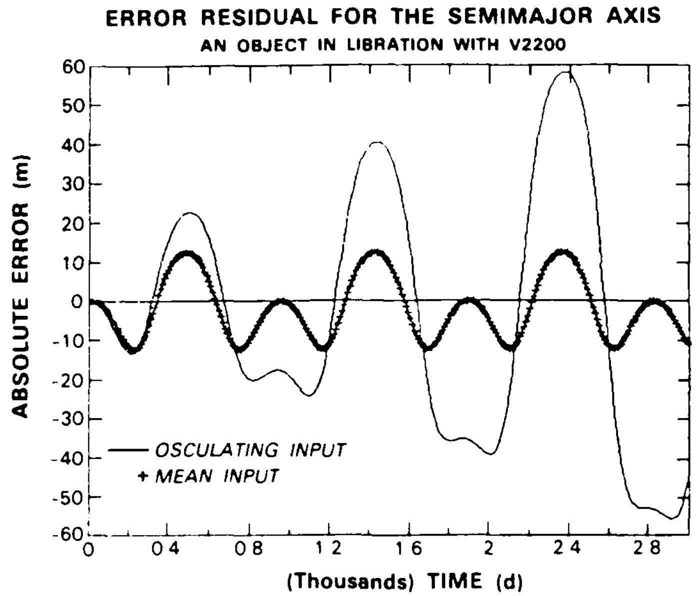


Figure 5-1: *The error curve with initial elements input and then with the mean elements used to compute k, Q, and ε*

circulation with the harmonic V'_{2200} has revealed that there are two cases where the savings can be achieved. It was observed that if the value of $\langle \int_{t_0}^{t} \sin\psi dt \rangle$ is positive, then the smallest growth is achieved by using the initial osculating elements for the computation of k and Q and the mean elements for the computation of ε, and if $\langle \int_{t_0}^{t} \sin\psi dt \rangle$ is negative, then the smallest growth is obtained by using the initial elements for the computation of Q and the mean elements for the computation of k and ε. This phenomenon is not well understood, but it appears to be consistent with every circulation object studied. By using the change in semimajor axis δa again as an example, we demonstrate the first case in Figure 5-2 and the second case in Figure 5-3. Incidentally, the object in Figure 5-2 is very close to the border between circulation and libration, and therefore, the error curve resulting from the mean input appears bounded only because the original is bounded.

5.2. THE ALGORITHM

Armed with the above error analysis, we will outline the algorithm for computing the changes in each of the Keplerian elements due to an isolated resonance harmonic. This algorithm is designed to minimize

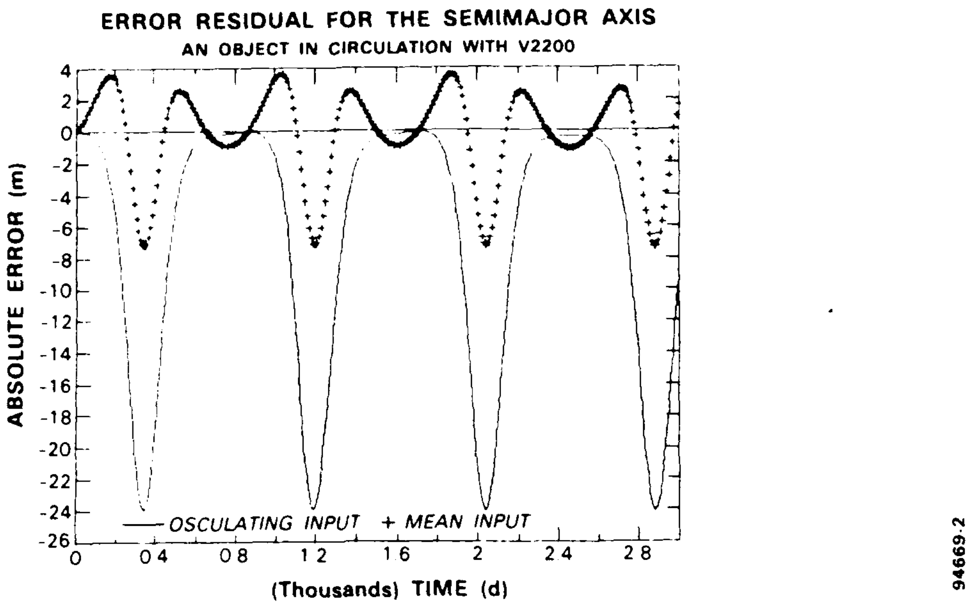


Figure 5-2: *The error curve with the initial elements input and then with the mean elements used to compute ε*

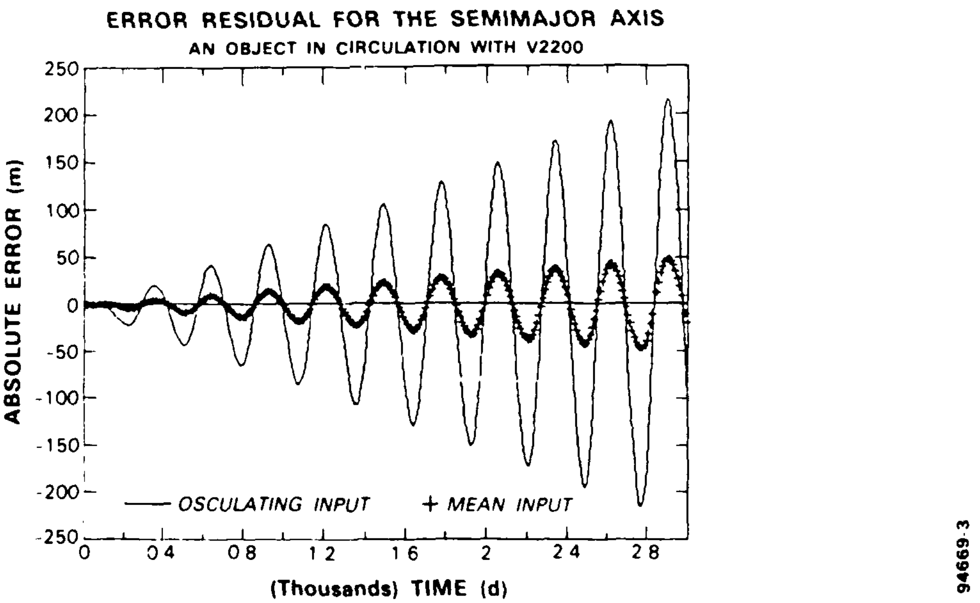


Figure 5-3: *The error curve with the initial elements input and then with the mean elements used to compute k and ε*

the variance from the numerical integration of the equations of motion, and this algorithm was used in the numerical study which follows.

1. The osculating elements a_0, e_0, i_0, ψ_0, and $\dot{\psi}_0$ are assumed given with an epoch time t_0.

2. The dependent variables k, Q, and ε are computed as functions of the initial inputs.

3. The mean elements $\langle a \rangle$, $\langle e \rangle$, and $\langle i \rangle$ are computed as defined above.

4. If $|k| > 1$, then k, Q, and ε are to be recalculated by using $\langle a \rangle$, $\langle e \rangle$, $\langle i \rangle$, ψ_0, and $\dot{\psi}_0$.

5. If $|k| < 1$, then if $\langle \int_{t_0}^{t} \sin\psi dt \rangle < 0$, then k and ε are to be recalculated using $\langle a \rangle$, $\langle e \rangle$, $\langle i \rangle$, ψ_0, and $\dot{\psi}_0$, and if $\langle \int_{t_0}^{t} \sin\psi dt \rangle > 0$, then only ε is to be recalculated by using $\langle a \rangle$, $\langle e \rangle$, and $\langle i \rangle$.

6. For the change in the mean anomaly δM, the integral $\int_{t_0}^{t} n dt$ is to be computed as a function of k, Q, A (as a function of mean elements), and the initial semimajor axis a_0. Recall that the elements a and n appearing are initial elements because of the binomial series expansion which was used.

7. The variable u and the elliptic functions are to be computed so that each of the changes can be computed as listed at the end of Section 3.

5.3. THE RESULTS

To realize the objectives of our study, four satellites were selected; three of the satellites were in a near synchronous orbit, and the other was in a near half-synchronous orbit. Each of the satellites selected was in a deep resonance with at least one critical tesseral harmonic; for it is in this regime where the error is more inclined to be significant. The objects' parameters were taken from an object file maintained by Group 91 at Lincoln Laboratory. The objects are catalogued by the U.S. Space Command catalogue system, and therefore, each object will be referred to by its catalogue number in that file. The objects along with their (osculating) orbit parameters at the epoch cited are listed in Table 5-1.

For each object three plots will be presented. It was observed in each of the situations we considered that, although each of the Keplerian elements was affected by a critical tesseral harmonic, only the mean

TABLE 5-1 Object Parameters				
	Object No.			
	14867	15181	13636	16885
a km	42170.5898	42161.7406	42166.032	26553.963
n rev/day	1.0025	1.0028	1.00267	2.00635
e	2.71×10^{-3}	1.961×10^{-3}	5.714×10^{-4}	0.741
ω deg	348.875	180.467	350.703	288.15
i deg	1.597	1.087	1.816	63.257
Ω deg	85.081	84.648	104.407	79.338
M deg	236.463	179.122	306.277	25.459
λ deg	73.778	116.064	345.24	35.459
epoch yr, day	'87, 140	'87, 138.259	'87, 139.5	'87, 139.9

anomaly M and the semimajor axis a exhibited large changes. We have therefore restricted our discussion to the perturbations of these two elements. For each object we have plotted the following:

1. the absolute error, or *residual* (the perturbation obtained analytically subtracted from the perturbation obtained numerically), for the semimajor axis and the mean anomaly, of the restricted resonance problem where the disturbing function is isolated to one critical tesseral harmonic, and

2. the comparison of the perturbation in the semimajor axis for the situation when the disturbing function is isolated to one dominant critical tesseral harmonic (V_{2200} for example) with that for the situation where the disturbing function includes more than one critical tesseral harmonic.

The last type of plot is generated entirely from numerical data and is intended to show not only the magnitude and amplitude of the perturbations, but also to what extent a dominant critical tesseral harmonic (if such a harmonic exists) is able to approximate the global problem.

The three near synchronous objects were selected on the basis of the perturbation produced by the critical tesseral harmonic V_{2200}. Object No. 14867 is closely commensurate with the rotation rate of the earth and is in a secure libration about a stable equilibrium point. Object No. 15181 is also in libration about a stable equilibrium point, but because of its initial longitude position, it is in a long libration about that point with a large amplitude. Object No. 13636 is presently in circulation, but it is so close to the border between circulation and libration that the presence of other geopotential harmonics or other effects in the disturbing function (such as lunar and solar effects or solar radiation pressure) could cause it to librate about the equilibrium point.

Among the synchronous objects (No. 15181, No. 14867, and No. 13636), the harmonic V_{2200} was considered to be dominant, and the other harmonics included in the study were V_{3110}, V_{3300}, V_{4210}, and V_{4400}. Notice that q is zero in each of the harmonics listed above because all of the synchronous satellites in our study have small eccentricity. It should also be noted that, with each harmonic, the singularities involving small eccentricities and small inclinations were removable. For the Molniya satellite (No. 16885) with high eccentricity, the harmonic V_{220-1} was isolated as dominant, but it will be shown that the harmonics V_{2211} and V_{3210} also produce effects of large amplitude. The harmonics V_{4410} and V_{421-1} were also used in the study of this object.

The local (or isolated) residuals will be presented first. In Figures 5-4 to 5-11, the residuals for the same object, element, and epoch are combined on the same plot. One can see from those plots that, for the most part, the analytical theory has performed satisfactorily in the isolated resonance problem (for which it was designed), with the exception of the residuals for No. 13636 due to the harmonic V_{2200} and for No. 16885 due to the harmonic V_{3210}. Both of these objects are in circulation with the respective harmonic. In order to understand the nature of the residuals a little better, we will consider additional parameters which are indigenous to the analytical solution. For each object, we will list in Table 5-2 the values k, $1/k$, and $\dot{\lambda}$ for the force which produced the largest amplitude in the residuals. For the three synchronous objects, this harmonic is V_{2200}, and for Object No. 16885, this is the harmonic V_{3210}.

Consider the residuals for the synchronous objects (Figures 5-4 to 5-9). Object No. 14867 is librating about the stable equilibrium point 75°E, and its longitude position and velocity indicate that it has

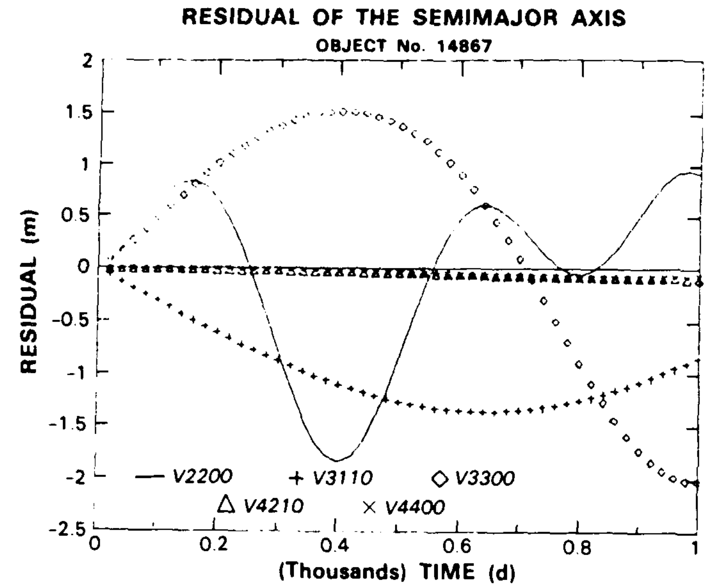


Figure 5-4: *The isolated resonance problem for a synchronous object*

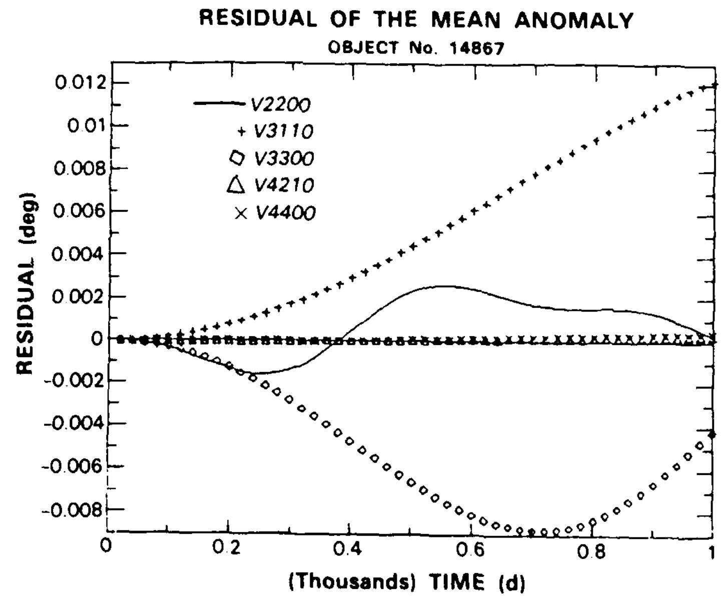
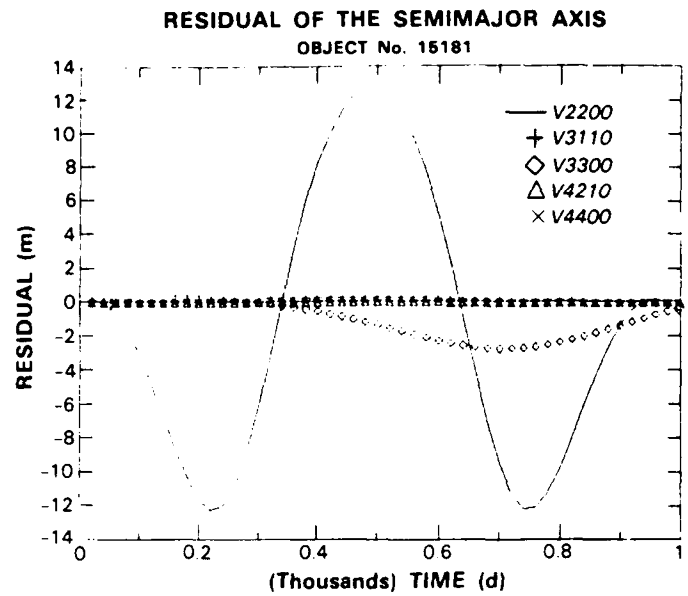
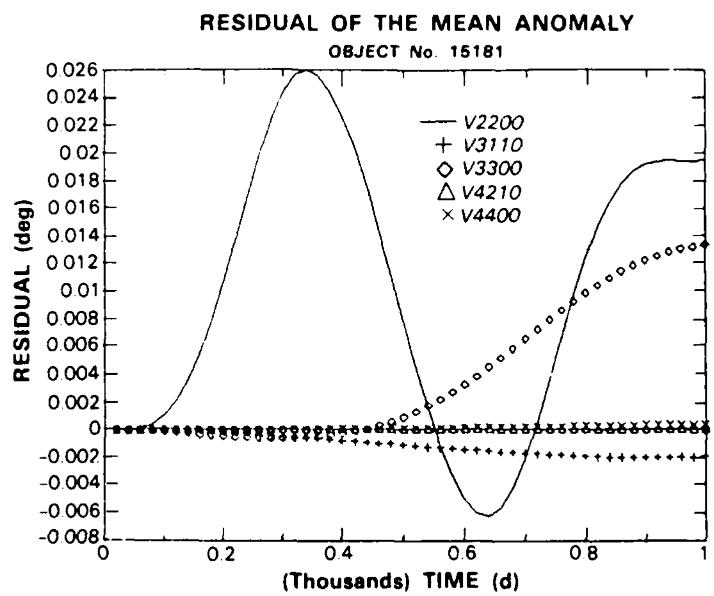


Figure 5-5: *The isolated resonance problem for a synchronous object*



94669-6

Figure 5-6: *The isolated resonance problem for a synchronous object*



94669-7

Figure 5-7: *The isolated resonance problem for a synchronous object*

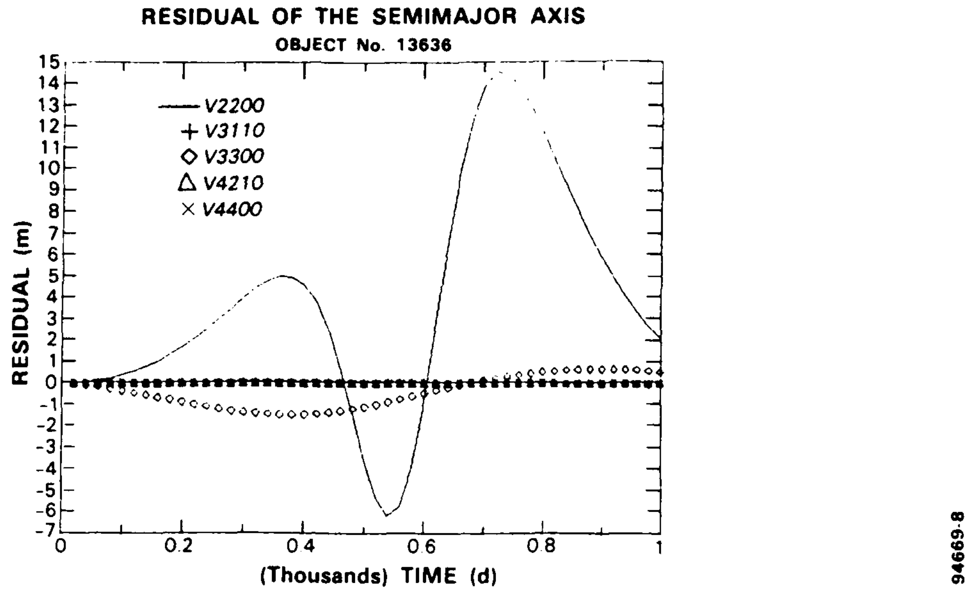


Figure 5-8: *The isolated resonance problem for a synchronous object*

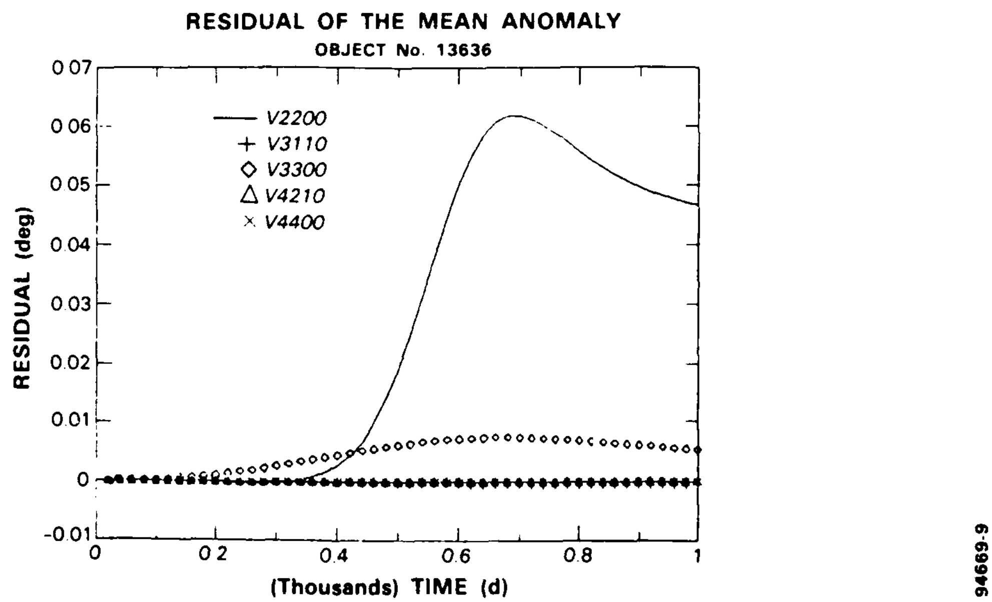


Figure 5-9: *The isolated resonance problem for a synchronous object*

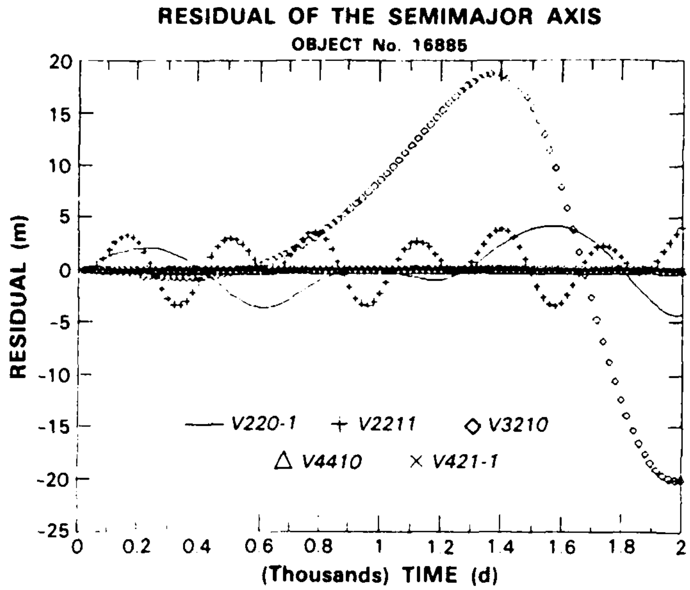


Figure 5-10: *The isolated resonance problem for a semisynchronous object*

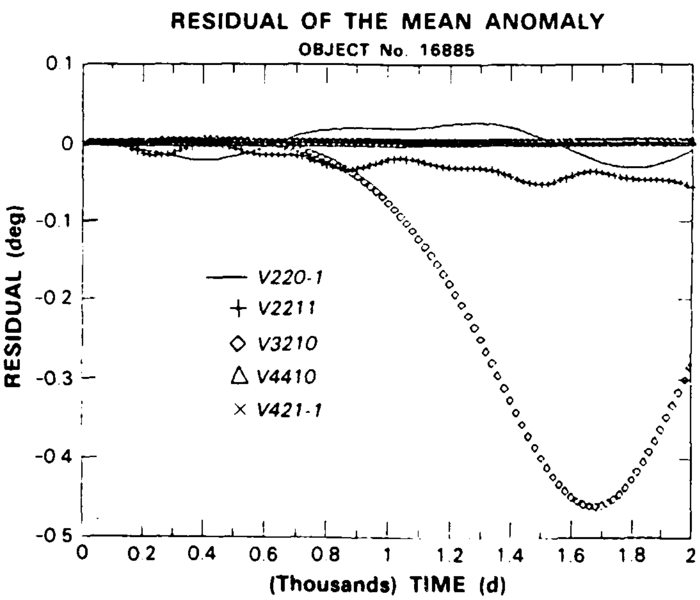


Figure 5-11: *The isolated resonance problem for a semisynchronous object*

TABLE 5-2 **Parameters Associated with the Analytical Solution**				
	Object No.			
	14867	**15181**	**13636**	**16885**
k	-5.295	1.51449	-0.9986	0.9263
1/k	-0.1889	0.66	-1.0014	1.0796
$\dot{\lambda}_0$ deg/day	-8.267×10^{-2}	3.12×10^{-2}	-2.361×10^{-2}	0.159
Harmonic	2200	2200	2200	3210

recently passed this point and is now decelerating west. The larger residuals for No. 15181 and No. 13636 from the harmonic V_{2200} correspond to the larger amplitudes in the perturbations which result from their respective longitude positions and velocities. Object No. 15181 is currently about 41° east of the stable node, and although it is drifting eastward, it will soon be drifting west towards that point. Because of the distance it has to travel, the energy it will attain when it reaches the stable point will cause the amplitudes of the perturbations to be considerably more than those of Object No. 14867. Since Object No. 13636 is drifting west, is in circulation, and is near the unstable equilibrium point 345°E, its energy will also accelerate so that again large amplitudes in the perturbations will result. This analysis places the residuals for the three synchronous objects in an appropriate context (the residuals for No. 14867 are smaller, but so is the perturbation). For the semimajor axis, the difference in the amplitude of the perturbation for these objects can be seen in Figures 5-12 to 5-14.

The amplitudes of the residuals displayed in Figures 5-10 and 5-11 for Object No. 16885 due to the harmonic V_{3210} are larger than expected, and it is not clear why these are much larger than the residuals due to the harmonics V_{220-1} and V_{2211}. The object is in circulation with the harmonic V_{3210}, has a large drift rate, and the resonance period is much longer in this case than those periods from V_{220-1} and V_{2211}. It is possible to blame this on modeling error since the object has high eccentricity and inclination, and so the fundamental assumption $\ddot{\lambda} \approx \ddot{M}/s_0$ may not be valid over this long resonance period. On a more positive note, it is observed that the effects due to the harmonics causing a libration (namely V_{220-1} and V_{2211}) appear to be modeled very well by the first order theory.

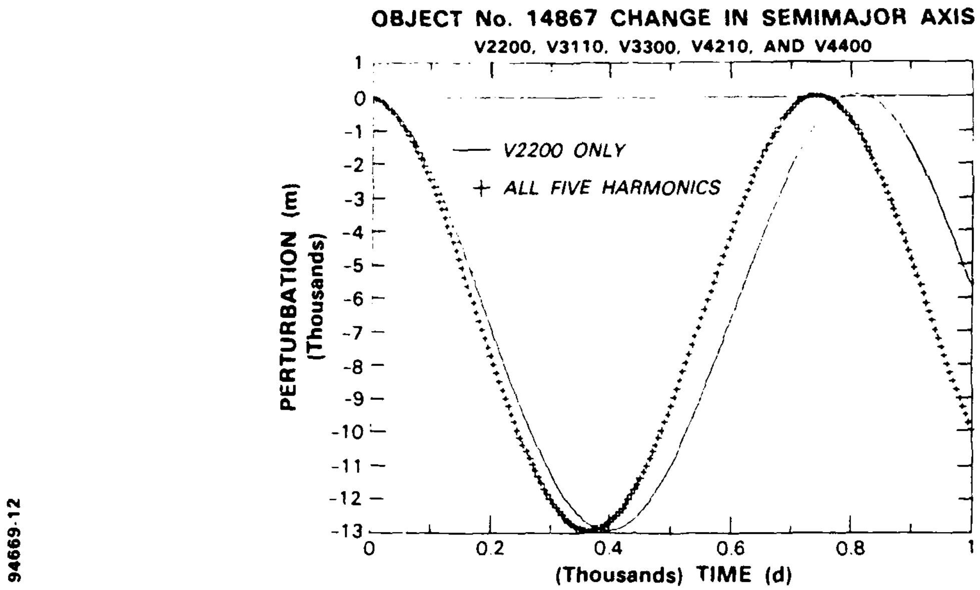


Figure 5-12: *Perturbation comparison of the local and global problem*

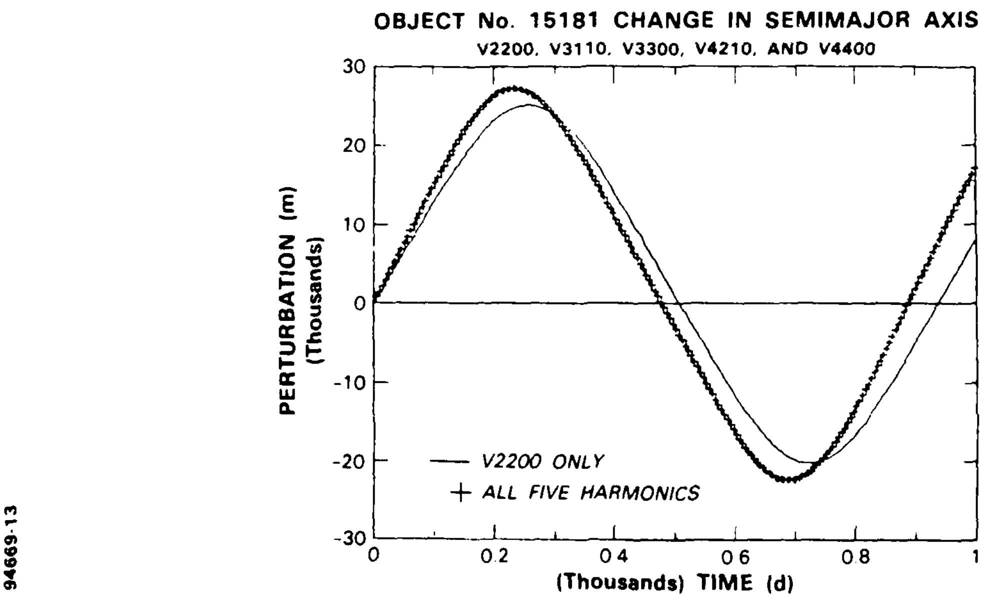


Figure 5-13: *Perturbation comparison of the local and global problem*

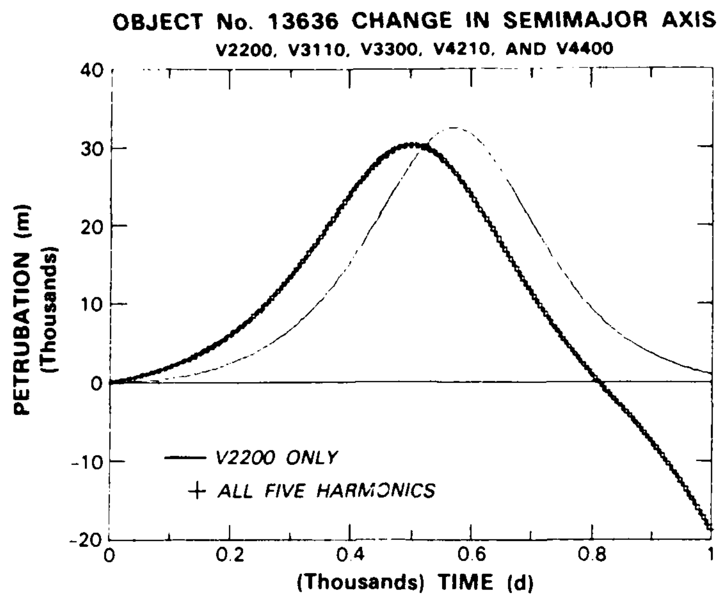


Figure 5-14: *Perturbation comparison of the local and global problem*

Consider now the plots in Figures 5-12 to 5-16. These graphs are generated entirely from data computed from a numerical integration of the equations of motion, and each figure depicts the perturbation in the semimajor axis in the two cases of the disturbing function containing one critical tesseral harmonic and of the disturbing function containing five critical tesseral harmonics. Hence, the interaction of the resonance effects is displayed.

Of the three synchronous objects, one can see that the perturbation from the global resonance problem is best approximated by the isolated harmonic V_{2200} in the case of Object No. 14867. The period and phase differ to a small degree, but the amplitude seems to match up exactly. This is also somewhat true in the case of Object No. 15181, but the amplitude is several kilometers off in the isolated case and the period is different. Still, one can make a case with each of these two objects that the isolated resonance problem describes the perturbation reasonably well for a couple of hundred days. This is far from true in the case of Object No. 13636. Indeed, the global perturbation looks almost as if a long and slow *libration* about the stable equilibrium point is modeled, whereas we have already seen that this object is in circulation when the disturbing function is isolated to one tesseral harmonic. Of course, this problem

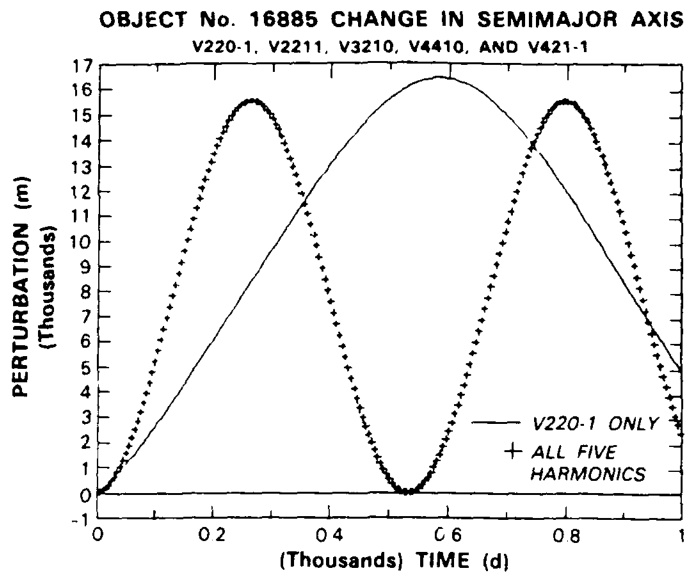


Figure 5-15: *Perturbation comparison of the local and global problem*

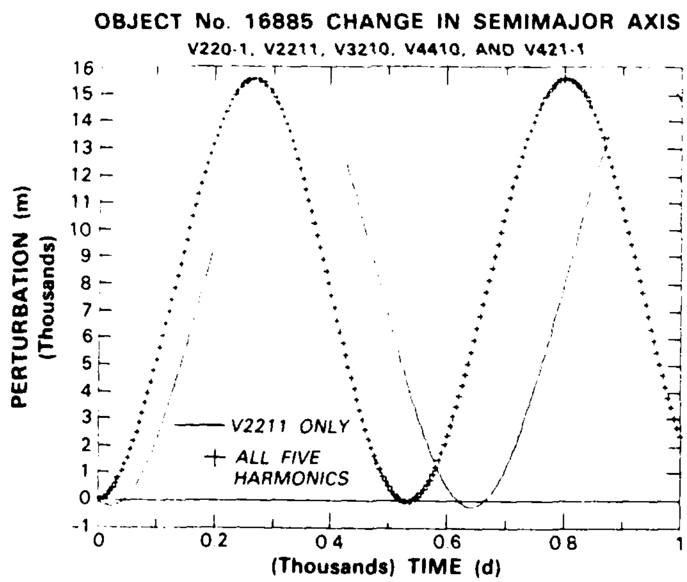


Figure 5-16: *Perturbation comparison of the local and global problem*

arises from the fact that Object No. 13636 is close to the border between circulation and libration and the other effects may cause the object to be perturbed in an entirely different fashion than predicted by one harmonic. Hence, the motion close to the border between circulation and libration is very difficult to predict, even though the object is in close commensurability with the earth's rotation and has favorable parameters such as small eccentricity and inclination.

Finally, the plots in Figures 5-15 and 5-16 show the complexity involved when an object is in resonance and has high eccentricity. Both of the forces V_{220-1} and V_{2211} have similar amplitudes, but the periods are vastly different, and both periods are far from the period exhibited in the global perturbation. Therefore, one has little hope in approximating the global problem by using one isolated critical tesseral harmonic for this situation.

We are led to the question about the performance of the analytical approximation to the global problem by assuming that the critical tesseral harmonics do not interact. Unfortunately, an unsatisfactory model of the problem is made over an entire resonance period. Often the residuals exceed the actual amplitude of the total perturbation. This is displayed in Figure 5-17, where the residual of the semimajor axis is presented. The only positive point that can be made here is that in each situation, a good model of the problem is made over one to two hundred days, and sometimes this is better than if we relied on only one harmonic to dominate and model the global problem over this time span. We conclude that the interaction between the harmonics is significant even in the most favorable situations.

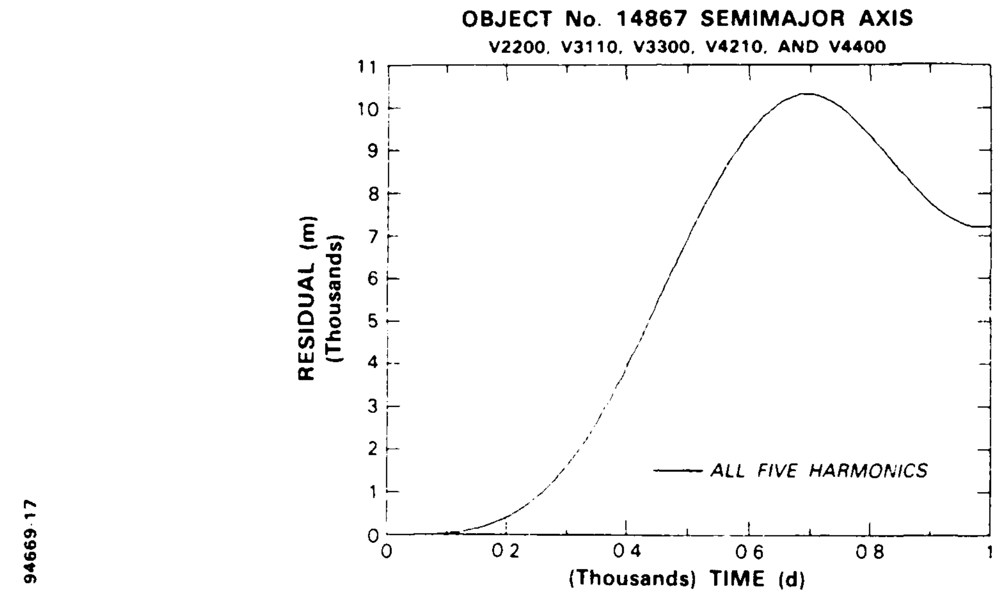


Figure 5-17: *Residual for the global resonance problem*

6. THE RESONANCE ADDITION TO THE ANODE MODEL

The ANalytical Orbit DEtermination routine (ANODE) at Millstone Hill is an entirely analytical propagation model which is tied to a least squares best estimate routine. This software is designed to fit data in near real time for the maintenance and upkeep of the satellite object element sets in our data bases. ANODE has been working on a routine basis for the past seven to eight years, but often the quality of the element sets has been poor for synchronous and half-synchronous satellites because of poor data and the lack of resonance in the propagation model. The original decision to leave resonance out of the propagation was due to fears that resonance would slow down the routine and also because storage space was a fundamental concern at that time. Beside those concerns is the fact that at that time the overall quality of the data required that each object be updated every three to seven days. Since the resonance correction would be very small over that time frame, it was not too difficult to reaquire objects with the instruments that were used. Nevertheless, problems such as mistagging objects or losing (low-priority) objects arose from sparse 'angles-only' optical data, poor quality data, and/or the lack of quality radar range data. While better data would certainly improve this aspect of the problem, there is also a need to improve the ANODE physical model. Moreover, the quality of data now available is much better than ten years ago, and so there can be a longer time span between observation for much of the catalogue. This will be important as an increasing number of objects in space place a greater strain on the space surveillance network, and so it is necessary to improve the propagation model in ANODE to better system performance.

The resonance corrections described in Section 3 have been added to ANODE, and this section will outline how this has been accomplished. Before giving more detail, it is important to understand how the ANODE model works without the corrections. We will use a boldfaced $\mathbf{z}$ to denote a vector containing the six Keplerian elements.

The propagation in ANODE can be described by the following simple stepwise procedure. First, it is assumed that mean elements $\mathbf{z}_0$ with an epoch time t_0 are input, and the secular rates $\dot{\mathbf{z}}(\mathbf{z}_0)$ are calculated. The partial derivatives of these rates with respect to the input mean elements $\mathbf{z}_0$ are also calculated

$$\left[\frac{\partial \dot{\mathbf{z}}(\mathbf{z}_0)}{\partial \mathbf{z}_0}\right] .$$

where the formal notation that is displayed has been adopted from [7]. Now, for propagation to a particular time t, ANODE propagates the mean elements

$$\mathbf{z}_m(t) = \mathbf{z}_0 + \dot{\mathbf{z}}(t - t_0) + \frac{1}{2}(t - t_0)^2 \sum_{j=1}^{6} \left[\frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{z}_0}\right]_j \dot{\mathbf{z}}_j \quad , \tag{13}$$

adds the long periodic corrections

$$\mathbf{z}_L(t) = \mathbf{z}_m(t) + \Delta_L[\mathbf{z}_m(t)] \tag{14}$$

as a function of the mean elements, and adds the short periodic corrections to produce the osculating element set

$$\mathbf{z}_{osc}(t) = \mathbf{z}_L(t) + \Delta_S[\mathbf{z}_L(t)] \quad . \tag{15}$$

The long periodic corrections Δ_L arise from averaging the equations of motion over the period of the moon (28 days) or the period of the motion of the argument of perigee, whichever is shorter, and the short periodic corrections Δ_S come from averaging the equations of motion over the period of the satellite.

Now, in most deep resonance problems (where the order of magnitude of the perturbations is significant), the period of the resonance motion is on the order of three years or more. This is much longer than the long period average described above, and therefore it is decided to add the resonance corrections to the secular (or mean) elements. The resonance corrections are included by the following additional steps. Resonance first must be initialized (as outlined in the first five steps of Section 5.2) as a function of the osculating element set $\mathbf{z}_{osc}(t_0)$ [as computed above in (15)] and a critical index set representing a particular tesseral harmonic. Then, in order to propagate to a specific time t, the vector of resonance corrections $\delta\mathbf{z}$ (as produced at the end of Section 3) is added to the mean element propagation (13)

$$\mathbf{z}_m^*(t) = \mathbf{z}_m(t) + \delta\mathbf{z} \quad , \tag{16}$$

the long periodic corrections are computed as in (14)

$$\mathbf{z}_L^*(t) = \mathbf{z}_m^*(t) + \Delta_L[\mathbf{z}_m^*(t)] \quad , \tag{17}$$

and the short periodic corrections are added to produce the osculating element set as in (15)

$$\mathbf{z}_{osc}^*(t) = \mathbf{z}_L^*(t) + \Delta_S[\mathbf{z}_L^*(t)] \quad . \tag{18}$$

For the least squares fit procedure in ANODE, the analytical Jacobian of the computed pointing $C(t)$ with respect to the input mean elements $\mathbf{z}_0$ is needed [7]. The function $C(t)$ is a vector containing the topocentric coordinates (azimuth, elevation, range, and range rate) which point to the object from a particular site. Since $C(t)$ is merely a geometrical transformation from the inertial position computed from the osculating element set, it is pointed out in [7, pp.9, 10]that the Jacobian can be computed by using the chain rule

$$\left[\frac{\partial C(t)}{\partial \mathbf{z}_0}\right] = \left[\frac{\partial C(t)}{\partial \mathbf{z}_{osc}(t)}\right]\left[\frac{\partial \mathbf{z}_{osc}(t)}{\partial \mathbf{z}_m(t)}\right]\left[\frac{\partial \mathbf{z}_m(t)}{\partial \mathbf{z}_0}\right] , \tag{19}$$

where it is assumed that the matrix

$$\left[\frac{\partial \mathbf{z}_{osc}(t)}{\partial \mathbf{z}_m(t)}\right]$$

is the identity (since it is a function of periodic corrections) and we can write

$$\left[\frac{\partial \mathbf{z}_m(t)}{\partial \mathbf{z}_0}\right] = I_6 + \left[\frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{z}_0}\right](t - t_0) , \tag{20}$$

where I_6 is the 6×6 identity matrix.

After the resonance correction is added, we need to change (19) to

$$\left[\frac{\partial C(t)}{\partial \mathbf{z}_0}\right] = \left[\frac{\partial C(t)}{\partial \mathbf{z}_{osc}{}^*(t)}\right]\left[\frac{\partial \mathbf{z}_m{}^*(t)}{\partial \mathbf{z}_0}\right] \tag{21}$$

and (20) to

$$\left[\frac{\partial \mathbf{z}_m{}^*(t)}{\partial \mathbf{z}_0}\right] = \left[\frac{\partial \mathbf{z}_m(t)}{\partial \mathbf{z}_0}\right] + \left[\frac{\partial \delta \mathbf{z}}{\partial \mathbf{z}_{osc}(t_0)}\right] . \tag{22}$$

The first matrix on the right hand side of (22) is computed as in (20) and the second matrix is adequate to approximate

$$\left[\frac{\partial \delta \mathbf{z}}{\partial \mathbf{z}_0} \right]$$

because the error in so doing is, again, a function of periodic corrections.

Most of the work involved in computing the second matrix on the right hand side of (22) is of a routine nature using the chain rule and product rules (and we will leave this to the reader), but for the convenience of the reader, the partials of those elliptic functions which are relevant to the resonance corrections are listed below in Table 6-1.

TABLE 6-1

Partial Derivatives of the Elliptic Functions

Function	$\frac{\partial}{\partial u}$	$\frac{\partial}{\partial k}$
sn(u, k)	cn(u, k) dn(u, k)	$-\mathrm{cn}(u, k)\,\mathrm{dn}(u, k)\,\Lambda$
cn(u, k)	-sn(u, k) dn(u, k)	$\mathrm{sn}(u, k)\,\mathrm{dn}(u, k)\,\Lambda$
dn(u, k)	$-k^2\mathrm{sn}(u, k)\,\mathrm{cn}(u, k)$	$\frac{k}{\mathrm{dn}(u, k)}[k\mathrm{sn}(u, k)\,\mathrm{cn}(u, k)\,\mathrm{dn}(u, k)\,\Lambda - \mathrm{sn}^2(u, k)]$
E(u, k)	$\mathrm{dn}^2(u, k)$	$\frac{1}{k}\quad [E_I(k, \phi) - F(k, \phi)] - \mathrm{dn}^2(u, k)\,\Lambda$
am(u, k)	dn(u, k)	$-\mathrm{dn}(u, k)\,\Lambda$

A few preliminary comments are needed before the table of partial derivatives can be presented. Suppose the argument u and the modulus k are given and set

$$\phi \equiv \mathrm{am}(u, k)$$

with

$$F(k, \phi) \equiv \int_0^{\phi} \frac{d\xi}{\sqrt{1 - k^2 \sin^2 \xi}}$$

and

$$\Pi(k, \phi) \equiv \int_0^{\phi} \frac{d\xi}{(1 - k^2 \sin^2 \xi)^{3/2}} \quad .$$

Since the integrand in the definition of F is continous, a direct computation (using the fundamental theorem of calculus) yields

$$\frac{\partial F(k, \phi)}{\partial \phi} = (1 - k^2 \sin^2 \phi)^{-1/2}$$

and

$$\Lambda \equiv \frac{\partial F(k, \phi)}{\partial k} = \frac{1}{k}[\Pi(k, \phi) - F(k, \phi)] \quad .$$

Finally, let the symbol $E_I(k, \phi)$ denote the incomplete elliptic integral of the second kind

$$E_I(k, \phi) \equiv \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \xi}\, d\xi \quad .$$

The above procedure can be used for as many critical tesseral harmonics as desired by simply calculating each set of corrections separately and adding them all together, but since no interaction is modeled, the routine will decay in accuracy over time as discussed in Section 5. Nevertheless, for surveillance purposes, where theory is coupled with real data and differential correction, the routine should be trustworthy within 100 or 200 days from the epoch. Since the other aspects of the ANODE propagation model are not as accurate for this long, this should not be too discouraging.

As an example of how the resonance addition has improved ANODE, a simple test has been performed. For an object in a synchronous orbit, we generated 20 days worth of simulated metric data using the precision numerical integration program DYNAMO. Since the data created is essentially free of noise, we were able to ignore the problem of bad data so that we could concentrate on the propagation model in ANODE. For a force model in DYNAMO, we limited the effects to those caused by the geopotential expansion through degree $l = 4$ and order $m = 4$, and this includes all values of p and q for each pair

(l, m). Hence, we only modeled the geopotential effects on the object which ANODE modeled, and we are deferring the effects caused by the moon and the sun to a later study. The data was sampled every quarter of a day so that a thorough representation of the orbit over the 20 day period is gathered. As a test, using the fit routine, we fit this data to the ANODE propagation model for three different cases: I -- a geopotential model consisting of V_{20}, V_{30}, and V_{40}, II -- a geopotential model consisting of V_{20}, V_{30}, V_{40}, and one resonance term V_{2200}, and III -- a geopotential model consisting of V_{20}, V_{30}, V_{40}, and three resonance terms V_{2200}, V_{3110}, and V_{3300}. Case I is the old ANODE model (without the lunar and solar effects), case II contains the dominant resonance effect, and case III attempts to model the complete problem (as much as is possible analytically).

Case I did not fit the data very well. After 10 least squares fit iterations, much of the data near the extreme times was edited out and the root mean square (rms) residual for the azimuth was 0.045° for all the data and 0.017° for the edited data, for the elevation was 0.018° for both sets of data, for the range was 1.068 km for all the data and 0.3878 km for the edited data, and for the range rate was 0.0047 m/s for both sets of data. All of the data fit nicely for case II and case III, and in each case only four iterations were required. The rms residuals for case II were 0.004° for the azimuth, 0.002° for the elevation, 0.0907 km for the range, and 0.0034 m/s for the range rate, and the rms residuals for case III were 0.001° for both the azimuth and the elevation, 0.0049 km for the range, and 0.0033 m/s for the range rate. Hence, the latter model is able to recover the effects to within a millidegree in azimuth and elevation and within 5 meters in range.

7. SUMMARY

A first order analytical theory of resonance effects on satellite orbits has been developed and tested. The motion is modeled by the motion of a simple pendulum with initial conditions, and is thus naturally defined in terms of the elliptic functions of Jacobi. The theory has been tested against a numerical integration of the equations of motion for the general perturbation of the Keplerian elements, and the following conclusions can be drawn.

1. The theory is easy to manipulate directly to produce the periods, the mean elements averaged over the resonance periods, the periodic corrections, and the partial derivatives of the resonance corrections with respect to the input elements.

2. The method works best when one isolates the resonance problem to that of a single critical tesseral harmonic, although this is seldom adequate to represent the 'real world' problem. In fact, there is not guaranteed to be a dominant critical tesseral harmonic even in the case of a circular equatorial orbit which is very near synchronous.

3. The global problem can be modeled analytically by solving the isolated resonance problem locally and adding the effects together, but this is not a good model over a full resonance period, and is only able to describe the motion with reasonable accuracy over a short time span.

4. This method is good for a first approximation to the solution of the resonance problem, but the theory should not be used for accurate predictions without iteration correction by comparison with other data (such as observed data or numerically simulated data). Hence, some applications of this theory are:

    a. as a front end processor for a numerical integrator,

    b. for addition to an entirely analytical theory which fits observed data in real time and attempts to correlate targets quickly (i.e., this algorithm has been implemented in the analytical orbit determination routine ANODE at Lincoln Laboratory),

c. for gaining insight into a physical problem involving resonance such as a station keeping problem or a problem which requires a general notion of the long term effect of a particular Keplerian element from a resonance force. An example of this application is the initial modeling of the resonance effects which were important for the design of a series of station keeping maneuvers for LES-8 [5].

5. An object which is commensurate with the rotation rate of the earth can be in deep resonance with several tesseral harmonics, and the interaction often makes it difficult to analytically model the motion either by assuming no interaction or by hoping one tesseral harmonic will suffice. Two significant parameters which determine the resonance motion are the initial longitude position and the initial longitude rate (along with the direction of the longitude motion). The problem is compounded by the reality that an object may be in libration with one critical tesseral harmonic and in circulation with others. Of course, either libration or circulation will dominate, but the damage to the hope of modeling the global problem by solving the isolated problem separately and adding the effects together is clear. Moreover, it is clear from the magnitude of the amplitudes displayed that the resonance effects on satellite orbits should not be ignored when an object is commensurate with the rotation rate of the earth.

References

[1] R.R. Allan.
Perturbations of a geostationary satellite by the longitude dependent terms in the Earth's gravitational field.
Planet. Space. Sci. 11:1325-1334, 1963.

[2] B. Garfinkel.
Formal solution in the problem of small divisors.
Astron. J. 71:657-669, 1966.

[3] G.S. Gedeon.
Tesseral resonance effects on satellite orbits.
Celestial Mechanics 1:167-189, 1969.

[4] W.M. Kaula.
Theory of Satellite Geodesy.
Blaisdell Publishing Co., Waltham, Massachusetts, 1966.

[5] M.T. Lane and D.E. Whited.
Private communication.

[6] G.W. and R.M. Spenceley.
Smithsonian Elliptic Functions Tables.
The Smithsonian Institution, Washington, D.C., 1947.

[7] R. Sridharan and W.P. Seniw.
ANODE : An Analytical Orbit Determination System.
MIT Lincoln Laboratory Technical Note TN-1980-1, Vol. 1, 24 June 1980, DDC AD-AO9011811.

[8] E.T. Whittaker and G.N. Watson.
A Course of Modern Analysis.
Cambridge University Press, 1927.

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report 796	5. MONITORING ORGANIZATION REPORT NUMBER(S) ESD-TR-87-286
6a. NAME OF PERFORMING ORGANIZATION Lincoln Laboratory, MIT 6b. OFFICE SYMBOL *(If applicable)*	7a. NAME OF MONITORING ORGANIZATION Electronic Systems Division
6c. ADDRESS *(City, State, and Zip Code)* P.O. Box 73 Lexington, MA 02173-0073	7b. ADDRESS *(City, State, and Zip Code)* Hanscom AFB, MA 01731
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Air Force Systems Command, USAF 8b. OFFICE SYMBOL *(If applicable)*	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F19628-85-C-0002
8c. ADDRESS *(City, State, and Zip Code)* Andrews Air Force Base Washington, DC 20334	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO.: 12424F PROJECT NO.: 77 TASK NO.: WORK UNIT ACCESSION NO.:

11. TITLE *(Include Security Classification)*
An Analytical Treatment of Resonance Effects on Satellite Orbits

12. PERSONAL AUTHOR(S)
Mark T. Lane

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT *(Year, Month, Day)*	15. PAGE COUNT
Technical Report	FROM ______ TO ______	19 August 1988	58

16. SUPPLEMENTARY NOTATION
None

17. COSATI CODES			18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
FIELD	GROUP	SUB-GROUP	perturbation theory; tesseral harmonics; resonance; simple pendulum

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

A first order analytical approximation of the tesseral harmonic resonance perturbations of the Keplerian elements is presented, and the mean elements (the Keplerian elements with the long period portions averaged out) will also be given in closed form. The results of a numerical test, which compares the analytical solution against a numerical integration of the Lagrange equations of motion, will be summarized, and the implementation of the solution in the analytical orbit determination routine ANODE (at Lincoln Laboratory) will be outlined.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION
22a. NAME OF RESPONSIBLE INDIVIDUAL Lt. Col. Hugh L. Southall, USAF	22b. TELEPHONE *(Include Area Code)* (617) 981-2330 22c. OFFICE SYMBOL ESD/TML