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EXTENSIONS TO THE MATRIX PENCIL APPROACH FOR DIRECTION FINDING

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This work deals with the problem of direction finding using the matrix-pencil approach. Consider a linear array of m sensor and assume there are d narrowband sources. Having collected the data at the m sensors, the problem is to estimate the direction of arrival of these sources. The matrix-pencil approach is a non-search procedure, thus very easy to use.

In this study, a generalization of the method to a linear array of m identical sensors with some arbitrary beam pattern was performed. It is also shown that the method still works when using different windows. The only restriction is that at least d elements of the window be non-zero to ensure the validity of the algorithm. A perturbation analysis due to equal sensor spacing was also performed. The concept of the chordal metric was introduced. It is shown that the bound derived on the chordal metric is equivalent to the chordal metric itself. The problem of estimating both the angular frequencies and the angles of arrival of the sources was then posed. It is proven that the method still works; i.e., the angular frequencies (over)

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and the locations of the sources were estimated using two matric pencils. The rank values of these matrices is shown to contain both the angles of arrival and the angular frequencies of the sources. A computer simulation was performed each time to ensure the effectiveness of the method.

PREFACE

This work deals with the problem of direction finding using the matrix-pencil approach. Consider a linear array of m sensors and assume there are d narrowband sources. the signal received at the i^{th} sensor is modeled as

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} a_k(\theta_k)s_k(t) + n_i(t) \; ; \; i=1,2,. . .,m.$$

Having collected the data at the m sensors, the problem is to estimate the directions of arrival of these sources. The matrix-pencil approach is a non-search procedure, thus very easy to use.

In this study, a generalization of the method to a linear array of m identical sensors with some arbitrary beam pattern was performed. It is also shown that the method still works when using different windows. The only restriction is that at least d elements of the window be non-zero to ensure the validity of the algorithm. A perturbation analysis due to unequal sensor spacing was also performed. The concept of the chordal metric was introduced. It is shown that the bound derived on the chordal metric is eqivalent to the chordal metric itself. The problem of estimating both the angular frequencies and the angles of arrival of the sources was then posed. It is proven that the method still works; i.e, the angular frequencies and the locations of the sources were estimated using two matix pencils. The rank reducing values of these matrices is shown to contain both the angles of arrival and the angular frequencies of the sources. A computer simulation was performed each time to ensure the effectiveness of the method.

TABLE OF CONTENTS

LIST OF CONVENTIONS

A	matrix A
A^H	transpose complex conjugate of A
$\det(A)$	determinant of A
I	identity matrix
$\underline{x}$	vector $\underline{x}$
$\|\|\underline{x}\|\|$	Euclidean norm of vector $\underline{x}$
$\|\|A\|\|$	Euclidean norm of matrix A
$\|\|A\|\|_f$	Frobenius norm of matrix A
$\langle\underline{x},\underline{y}\rangle$	inner-product of vectors $\underline{x}$ and $\underline{y}$
$E[\underline{X}]$	expected value of random vector $\underline{X}$
$\perp$	orthogonal
C	field of complex numbers
ε	belong to

CHAPTER 1

INTRODUCTION

1.1 GOAL OF THE RESEARCH

High resolution direction of arrival (DOA) estimation is very important in many sensor systems such as radar,sonar,etc... Over the years several methods have been proposed to solve this kind of problem. Our work is closely related to the work done by H. OUIBRAHIM [1]. This approach called the MATRIX PENCIL APPROACH addresses the problem of using a passive array of sensors to find the direction of sources assumed to be in the far field . The array is called passive because the sources generate the signals received at the sensors. The received signal at the i^{th} sensor is modeled as

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)a_i(\theta_k) + n_i(t) \;;i=1,2,. . .m \tag{1-1}$$

where we assume the existence of d sources and an array of m sensors,

$a_i(\theta_k)$ is the relative response of the i^{th} sensor to the k^{th} source,

$s_k(t)$ is the complex envelope of the k^{th} signal,

$n_i(t)$ is the additive noise, considered as the sum of the external and internal noise.

Throughout this work we deal with narrowband signals.A problem is referred to as narrowband if the bandwidth of the impinging signals from the sources is much less than the reciprocal of the propagation time of the wavefronts across the array. Hence, given m measurements collected at the sensors we would like to estimate the angles of arrival of the sources.

1.2 LITERATURE SURVEY

The problem of estimating the location of sources is of great importance and has been approached in many ways [1-3]. Recently several authors have suggested a subspace signal approach [4-8]. This approach is based on an eigenvalue-eigenvector decomposition of the spatial correlation matrix.This makes use of the fact that there is a relationship between the eigenvectors of the spatial correlation matrix and the source angles of arrival. Moreover, C.R. RAO [9] showed that one need only know the first few eigenvectors of the correlation matrix. We now present the background of the eigenstructure approach.

Assume there are d sources emitting signals $s_k(t)$;$k=1,2,\ldots,d$, which are impinging on a linear array composed of m sensors.It is assumed that $d \leq m$.

The received signal vector $\underline{X}$ can be written as

$$\underline{X} = A\,\underline{S} + \underline{N} \qquad (1\text{-}2)$$

where

$\underline{X}^T = \{x_1, x_2, \ . \ . \ . \ , x_m\} =$ (mx1) received vector signals,

$\underline{S}^T = \{s_1, s_2, \ . \ . \ . \ , s_d\} =$ (dx1) impinging signals,

$\underline{N}^T = \{n_1, n_2, \ . \ . \ . \ , n_m\} =$ (mx1) vector noise,

$A = \{\underline{a}_1 \ \underline{a}_2 \ . \ . \ . \ \underline{a}_d \ \} =$ (mxd) direction matrix,

$\underline{a}_i =$ (mx1) i^{th} direction column vector of A.

n_i is the additive noise assumed to have zero mean and an unknown variance σ^2

In all the subspace approaches that have been proposed the noises n_i are assumed to be independent from sensor to sensor and their correlation matrix is the diagonal matrix $\sigma^2 I$ where I is the identity matrix. Let the subscript H denote the Hermitian Transpose. The spatial covariance matrix is

$$\begin{aligned} R &= E[\underline{X} \ \underline{X}^H] = E[(A\underline{S}+\underline{N})(A\underline{S}+\underline{N})^H] \\ &= E[A\underline{S} \ \underline{S}^H A^H] + E[\underline{N} \ \underline{N}^H] \\ &= AE[\underline{SS}^H]A^H + \sigma^2 I \end{aligned} \qquad (1\text{-}3)$$

Let $S = E[\underline{S} \ \underline{S}^H]$. Then R can be written as

$$R = ASA^H + \sigma^2 I \qquad (1\text{-}4)$$

where

R is an (mxm) matrix.

Let $\{\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq, \ldots, \leq \lambda_m\}$ be the set of eigenvalues of R.

Let $\{\underline{V}_1, \underline{V}_2, \underline{V}_3, \ldots, \underline{V}_m\}$ be the set of the corresponding eigenvectors.

If S is non singular and with the assumption that $m \geq d$,we can show that

1) the minimum eigenvalue of R is σ^2 with multiplicity (m-d),i.e,

$$\lambda_{d+1}=\lambda_{d+2}=\lambda_{d+3}=\cdot \ldots =\lambda_m=\lambda_{min}=\sigma^2.$$

2)the eigenvectors associated with the minimum eigenvalue, $\underline{V}_{d+1}$, $\underline{V}_{d+2}$, $\underline{V}_{d+3}$, . . . ,$\underline{V}_m$, are orthogonal to the space spanned by the columns of A. This can be written as

$$\{\underline{V}_{d+1}, \underline{V}_{d+2}, \underline{V}_{d+3}, \ldots, \underline{V}_m\} \perp \{\underline{a}_1, \underline{a}_2, \underline{a}_3, \ldots, \underline{a}_d\} \qquad (1\text{-}5)$$

where

$\perp$, denotes orthogonality,

$\underline{a}_i$ = i^{th} column of A.

This algorithm can be summarized as follows:

1) determine the number of sources d from the multiplicity of λ_{min}.

2) the orthogonality relation (1-5) between the direction vectors of the impinging sources and the eigenvectors corresponding to λ_{min} yields the directions of arrival of the sources. We just have to "search" for those directions vectors that are orthogonal to the eigenvectors corresponding to λ_{min}. For this reason these methods are called search procedures. They assume the eigenvalues and the eigenvectors to

be perfectly known. However,in practice this is not always true. We then need to perform some kind of optimal estimation. Such a procedure was developed by LIGGET,LAWLEY and BARTLETT [10-12]. In such procedures some hypothesis testing is introduced. One difficulty is the subjective judgement required to set the thresholds. An approach, which is considered as one of the best, is the Aikake Information Criterion (AIC),since it does not require any objective judgement on the thresholds. Another approach is the Minimum Description Length (MDL) approach. But one of the most promising techniques is Multiple Signal Classification (MUSIC) proposed by SCHMIDT [5]. This algorithm provides asymptotically unbiased estimates of

1) number of signals,

2) directions of arrival,

3) strengths and crosscorrelation among the directional waveforms,

4) polarizations

5) strength of noise/interference.

More recently, other methods are being developed. Some of them are non-search procedures. These approaches have very importants advantages over search procedures. A. PAULRAJ,R. ROY and T. KAILATH [8], in their approach known as ESPRIT, have shown that their algorithm

1) does not require knowledge of the array geometry and element characteristics (directional pattern,gain/phase),

2) is computationally much less complex because it does not use the search procedure,

3) does not require a calibration of the array, therefore eliminating the need for the associated storage of the array manifold which can be very large for multidimensional problems,

4) simultaneously estimates the number of sources and DOA's .

1.3 OUTLINE OF THE WORK

H. OUIBRAHIM [1] proposed a generalization of the ESPRIT method. This method consists of applying an operator to the received signals in order to form a matrix pencil $M-\lambda N$. The rank reducing values of λ are shown to contain the information needed to estimate the DOA's.

The pencil theorem, presented in chapter 2, establishes the relationship between the rank reducing values of λ and the functional form $f(\phi_i)$ generated by the operator applied to the measurements.

In chapter 3 a generalization of the method to arbitrary but identical beam patterns is presented. Both the cases of deterministic signals and zero-mean random signals are considered.

Previously, only rectangular windows have been applied. In chapter 4 it is shown that the method still works using different windows. As in chapter 3, the cases of deterministic signals and zero-mean random signals are considered. A comparison of the different windows is obtained by means of a computer simulation.

In chapter 5 a perturbation analysis for the case of deterministic signals is performed. The concept of CHORDAL METRIC introduced by STEWART [13] is used. The chordal metric is a very good measure of the perturbation between the perturbed eigenvalue and the true one. A bound is derived which is shown to be effictive by means of a computer simulation.

Chapter 6 is devoted to a new technique for the simultaneous estimation of the angular frequencies and the angle of arrival of d sources assumed to be in the far field. The technique makes use of the decomposition of two (2) matrix pencils.

Finally, a summary and some suggestions for future work are given in chapter 7.

CHAPTER 2

REVIEW OF MATRIX PENCIL

Our problem is the estimation of the angles of arrival of d sources given measurements collected at the m sensors. The expression for the received signal presented in equation (1-1) of chapter 1 shows that the measurements are linear combination of d exponentials whose exponents $j\phi_k$; $k=1,2,\ldots,d$, contain the information needed to determine the locations of the sources. Specifically,

$$\phi_k = \omega/c \; D \sin(\theta_k) \;; k=1,2,\ldots.,d. \qquad (2\text{-}1)$$

The pencil theorem establishes the relationship between the rank reducing values of λ and the functional form $f(\phi_i)$ generated by the measurements.

2.1 PENCIL THEOREM

Denote by C the field of all complex numbers.Consider two matrices M and N of size (kxp).The set

$\{ M-\lambda N \; ; \; \lambda \varepsilon C \}$ is said to be a pencil.

The matrices M and N are required to have the following decompositions

$$M = E\ F$$

$$N = E\ D\ F$$

where

E is a (kxd) matrix and $k \geq d$

F is a (dxp) matrix and $p \geq d$

D is a (dxd) diagonal matrix.

Theorem

If M and N are two matrices which have the decompositions cited above and if E,F and are all of rank d, then the rank of the matrix pencil $M-\lambda N$ is decreased by 1 whenever

$$\lambda_i = (d_{ii})^{-1} \quad ; \; i=1,2,\ldots,d. \qquad (2\text{-}2)$$

Proof

Since $M=EF$ and $N=EDF$,

$$M-\lambda N = EF-\lambda EDF$$
$$= E(I-\lambda D)F.$$

Thus

$$\text{rank}\,(M-\lambda N) = \text{rank}(E(I-\lambda D)F)$$
$$= \min\{\text{rank}(E),\text{rank}(F),\text{rank}(I-\lambda D)\}.$$

However, by assumption

$$\text{rank}(E)=\text{rank}(F)=d$$

and

rank$(I-\lambda D)$ is of rank d as long as

$$1-\lambda_i d_{ii}=0.$$

If $1-\lambda_i d_{ii}=0$ which implies that $\lambda_i=(d_{ii})^{-1}$,rank$(I-\lambda D)=d-1$.

Therefore, the rank$(I-\lambda D)$ is reduced by 1 whenever

$$\lambda_i=(d_{ii})^{-1} \quad ; \; i=1,2,\ldots,d.$$

In our work, the matrix D has all its entries of the form

$$e^{j\phi_i} .$$

Thus, the rank reducing values of λ_i are

$$\lambda_i = (e^{j\phi_i})^{-1} = e^{-j\phi_i} \quad ; i=1,2,\ldots,d.$$

2.2 EVALUATION OF THE RANK REDUCING VALUES

We have assumed previously the existence of d sources and a linear array composed of m sensors with the condition that $d \leq m/2$. We then formed two matrices M and N of size (m-d)xd. We see that two cases may occur. If $m=2d$, M and N are two square matrices. The set of the generalized eigenvalues of the pencil $M-\lambda N$ is defined to be the set of all elements λ_i such that

$$\det(M-\lambda_i N)=0.$$

When the generalized eigenvalues are distinct, the rank of $M-\lambda N$ is reduced by 1 whenever λ equals one of these values. In the case where $d < m/2$, M and N are non square matrices. $\det(M-\lambda_i N)$ no longer exists since the pencil is not square. For this reason we have to "make" the pencil matrix a square one. This can be done by premultiplying the pencil $M-\lambda N$ by either M^H or N^H.

We obtain

$$M^H(M-\lambda N) = M^H M - \lambda M^H N$$

or

$$N^H(M-\lambda N) = N^H M - \lambda N^H N.$$

$M^H(M-\lambda N)$ and $N^H(M-\lambda N)$ are both square matrices (dxd). Notice that

$$M^H(M-\lambda N) = (EF)^H(EF-\lambda EDF) = F^H E^H EF - \lambda F^H E^H EDF$$
$$= F^H E^H E(I-\lambda D)F$$

and

$$N^H(M-\lambda N) = (EDF)^H(EF-\lambda EDF) = F^H D^H E^H EF - \lambda F^H D^H E^H EDF$$
$$= F^H D^H E^H E(I-\lambda D)F.$$

Both equations have the decompositions required by the pencil theorem since

$F^H E^H E$ and F are of rank d

and

$F^H D^H E^H E$ and F are of rank d.

Because $(I-\lambda D)$ arises in all of these equations, we can say that the generalized eigenvalues of $M^H(M-\lambda N)$ and $N^H(M-\lambda N)$ are identical to those obtained for the case where M and N were square matrices.

CHAPTER 3

GENERALIZATION TO ARBITRARY BUT IDENTICAL BEAM PATTERN

The moving window developed by H. Ouibrahim [1] was shown to be a non search procedure. It was applied for azimuth only DOA (direction of arrival) estimation of far field point sources. A generalization of this approch to arbitrary but identical beam pattern is presented here.

3.1 DETERMINISTIC CASE

Assume we have a linear array composed of m identical sensors with uniform spacing D. Assume there are $d \leq m/2$ narrowband sources located at azimuthal angles $\theta_k, k=1,2,\ldots,d$, which are impinging on the array as planar wavefronts and emitting signals whose complex envelopes are denoted by $s_k(t)$, $k=1,2,3,\ldots,d$. The received signal at the i^{th} sensor is modeled as

$$y_i(t,\underline{\theta})=\sum_{k=1}^{d} s_k(t)a_i(\theta_k)+n_i(t) \quad ;i=1,2,\ldots.m, \tag{3-1}$$

where

$a_i(\theta_k)$ is the relative response of the i^{th} sensor to the k^{th} source,

$n_i(t)$ is the additive noise assumed to be zero-mean Gaussian.

$a_i(\theta_k)$ can be written as

$$a_i(\theta_k)=a(\theta_k)\ e^{j(i-1)(\omega/c)\ D\ \sin(\theta\ )}\ ;i=1,2,\ldots,m, \qquad (3\text{-}2)$$

where

ω :center frequency of each of the spatial sources,

c :speed of propagation of the plane waves,

$a(\theta)$: beam pattern of each sensor.

If we let $\phi_k = \omega\ D\ \sin(\theta_k)/c$, $y_i(t,\underline{\theta})$ can be rewritten as

$$y_i(t,\underline{\theta})=\sum_{k=1}^{d} s_k(t)a(\theta_k)e^{j(i-1)\phi_k} + n_i(t)\ ,i=1,2,\ldots,m. \qquad (3\text{-}3)$$

Taking the expected value of equation (3-3), we get

$$x_i(t,\underline{\theta})=E[y_i(t,\underline{\theta})] = \sum_{k=1}^{d} s_k(t)a(\theta_k)e^{j(i-1)\phi_k}\ ;i=1,2,\ldots,m. \qquad (3\text{-}4)$$

Define the rectangular window

$$R_N(n)=\begin{cases} 1\ ;\ i\leq n\leq N \\ 0\ ;\text{elsewhere} \end{cases}$$

Given the number of sources d and the m averaged data points , $x_i(t,\underline{\theta})$, we create (d+1) vectors X_n ; n=1,2,. . . ,d+1 ,where the i^{th} component of $\underline{X}_n$ is

$x_{n+i-1}(t,\underline{\theta})R_{m-d}(i)$; $n=1,2,\ldots,d+1$

and $1 \leq i \leq m-d$.

Specifically, if the argument $(t,\underline{\theta})$ is omitted for simplicity,

$$\underline{X}_1^T = \{x_1, x_2, \ldots, x_{m-d}\}$$
$$\underline{X}_2^T = \{x_2, x_3, \ldots, x_{m-d+1}\}$$
$$\vdots$$
$$\underline{X}_d^T = \{x_d, x_{d+1}, \ldots, x_{m-1}\}$$
$$\underline{X}_{d+1}^T = \{x_{d+1}, x_{d+2}, \ldots, x_m\}.$$

The matrix pencil $M-\lambda N$ is then formed where

$$M = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ | & | & & | \\ \underline{X}_1 & \underline{X}_2 & \cdots & \underline{X}_d \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} ; \quad N = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ | & | & & | \\ \underline{X}_2 & \underline{X}_3 & & \underline{X}_{d+1} \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

Letting $a_k=a(\theta_k)$ and omitting the argument t in $s_k(t)$, $\underline{X}_1$ can be written as

$$\underline{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{m-d} \end{bmatrix} = \begin{bmatrix} a_1 s_1 + a_2 s_2 + \cdots + a_d s_d \\ a_1 s_1 e^{j\phi_1} + a_2 s_2 e^{j\phi_2} + \cdots + a_d s_d e^{j\phi_d} \\ \vdots \\ a_1 s_1 e^{j(m-d-1)\phi_1} + \cdots + a_d s_d e^{j(m-d-1)\phi_d} \end{bmatrix} .$$

$$\underline{X}_1 = \begin{bmatrix} a_1 & a_2 & \cdots & a_d \\ a_1 e^{j\phi_1} & a_2 e^{j\phi_2} & \cdots & a_d e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix} \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix} .$$

In general, $\underline{X}_n$ is given by

$$\underline{X}_n = \begin{bmatrix} x_n \\ x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{m-d+n-1} \end{bmatrix} = \begin{bmatrix} a_1 s_1 e^{j(n-1)\phi_1} + \cdots + a_d s_d e^{j(n-1)\phi_d} \\ a_1 s_1 e^{jn\phi_1} + \cdots + a_d s_d e^{jn\phi_d} \\ \vdots \\ a_1 s_1 e^{j(m-d+n-2)\phi_1} + \cdots + a_d s_d e^{j(m-d+n-2)\phi_d} \end{bmatrix}$$

$$
\underline{X}_n = \begin{bmatrix} a_1 e^{j(n-1)\phi_1} & a_2 e^{j(n-1)\phi_2} & \cdots & a_d e^{j(n-1)\phi_d} \\ a_1 e^{jn\phi_1} & a_2 e^{jn\phi_2} & \cdots & a_d e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ a_1 e^{j(m-d+n-2)\phi_1} & a_2 e^{j(m-d+n-2)\phi_2} & \cdots & a_d e^{j(m-d+n-2)\phi_d} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix} .
$$

$\underline{X}_n$ can also be written as:

$$\underline{X}_n = \begin{bmatrix} e^{j(n-1)\phi_1} & e^{j(n-1)\phi_2} & \cdots & e^{j(n-1)\phi_d} \\ e^{jn\phi_1} & e^{jn\phi_2} & \cdots & e^{jn\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d+n-2)\phi_1} & e^{j(m-d+n-2)\phi_2} & \cdots & e^{j(m-d+n-2)\phi_d} \end{bmatrix} \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix}$$

$$\underline{X}_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix} \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix} \begin{bmatrix} e^{j(n-1)\phi_1} & & & 0 \\ & e^{j(n-1)\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{j(n-1)\phi_d} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix}$$

$$B = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix}$$

$$\Phi = \begin{bmatrix} e^{j\phi_1} & & & 0 \\ & e^{j\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{j\phi_d} \end{bmatrix}$$

$$\underline{S}^T = \{s_1, s_2, \ldots., s_d\}.$$

Then $\underline{X}_n$ can be rewritten as :

$$\underline{X}_n = A\, \Phi^{(n-1)}\, B\underline{S} \qquad (3\text{-}5)$$

and the matrices M and N become

$$M = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ AB\underline{S} & AB\Phi\underline{S} & AB\Phi^2\underline{S} & \cdots\cdots & AB\Phi^{(d-1)}\underline{S} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$N = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ AB\Phi\underline{S} & AB\Phi^2\underline{S} & AB\Phi^3\underline{S} & \cdots\cdots & AB\Phi^{d}\underline{S} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Factoring out AB in M and $AB\Phi$ in N , we get

$$M = AB \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \underline{S} & \Phi\underline{S} & \Phi^2\underline{S} & \cdots\cdots & \Phi^{(d-1)}\underline{S} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$N = AB\Phi \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Phi \underline{S} & \Phi^2 \underline{S} & \cdots\cdots & \Phi^{(d-1)} \underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Let F be the matrix

$$F = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Phi^1 \underline{S} & \Phi^2 \underline{S} & \cdots\cdots & \Phi^{(d-1)} \underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

and E the matrix

$$E = AB.$$

We see that M and N have the decompositions

$$M = E\,F$$

$$N = E\,\Phi\,F.$$

The matrix pencil then becomes

$$\begin{aligned} M - \lambda N &= EF - \lambda E \Phi F \\ &= E(I - \lambda \Phi)F \qquad (3\text{-}6) \end{aligned}$$

which satisfies the requirements of the pencil theorem. Since $E = AB$, $\mathrm{rank}(E) = \min(\mathrm{rank}(A), \mathrm{rank}(B))$. But A is of rank d as long as the directions of arrival of the signals are distinct and $\lambda \leq d/2$. B is of rank d as long as $a_i=a(\theta_i)$,i=1,2,...,d, are different from zero. F is of rank d even in presence of coherent sources.

Hence, the values of λ for which the rank of $M-\lambda N$ decreases by 1 are given by

$$\lambda_k = e^{-j\phi_k} \;;k=1,2,...,d.$$

The angles of arrival are given by

$$\theta_k = \sin^{-1}(jc\ln(\lambda_k)/\omega d) \;;\; k=1,2,...,d.$$

3.2 ZERO-MEAN RANDOM CASE.

As before the signal received at the i^{th} sensor is modeled as follows;

$$y_i(t,\underline{\theta})=\sum_{k=1}^{d} s_k(t)a_i(\theta_k)+n_i(t) \;;i=1,2,....m.$$

Analogous to the previous section, we form (d+1) vectors $\underline{Y}_n$;n=1,2,...,m where

$$\underline{Y}_1^T = \{y_1(t,\underline{\theta}),y_2(t,\underline{\theta}),.\;.\;.\;.,y_{m-d}(t,\underline{\theta})\}$$

$$\underline{Y}_2^T = \{y_2(t,\underline{\theta}),y_3(t,\underline{\theta}),.\;.\;.\;.,y_{m-d+1}(t,\underline{\theta})\}$$

.

.

.

$$\underline{Y}_{d+1}^T= \{y_{d+1}(t,\underline{\theta}),y_{d+2}(t,\underline{\theta}),.\;.\;.\;.,y_m(t,\underline{\theta})\}.$$

Define the inner product

$$m_{h,k} = \langle \underline{Y}_h, \underline{Y}_k \rangle = E[\ \underline{Y}_k^H \underline{Y}_h\ ].$$

We then form the matrices M_1 and N_1 as follows

$$M_1 = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,d} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,d} \\ \vdots & \vdots & & \vdots \\ m_{d,1} & m_{d,2} & \cdots & m_{d,d} \end{bmatrix}$$

$$N_1 = \begin{bmatrix} m_{2,1} & m_{2,2} & \cdots & m_{2,d} \\ m_{3,1} & m_{3,2} & \cdots & m_{3,d} \\ \vdots & \vdots & & \vdots \\ m_{d+1,1} & m_{d+1,2} & \cdots & m_{d+1d} \end{bmatrix}.$$

The vector $\underline{Y}_n$ can be decomposed as

$$\underline{Y}_n = \begin{bmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{m-d+n-1} \end{bmatrix} = \begin{bmatrix} a_1 s_1 e^{j(n-1)\phi_1} + \cdots + a_d s_d e^{j(n-1)\phi_d} \\ a_1 s_1 e^{jn\phi_1} + \cdots + a_d s_d e^{jn\phi_d} \\ \vdots \\ a_1 s_1 e^{j(m-d+n-2)\phi_1} + \cdots + a_d s_d e^{j(m-d+n-2)\phi_d} \end{bmatrix} + \begin{bmatrix} n_n \\ n_{n+1} \\ \vdots \\ n_{n+m-d-1} \end{bmatrix}.$$

Let $\underline{N}_n$ denote the noise vector. $\underline{Y}_n$ can be written as

$$\underline{Y}_n = \begin{bmatrix} e^{j(n-1)\phi_1} & e^{j(n-1)\phi_2} & \cdots & e^{j(n-1)\phi_d} \\ e^{jn\phi_1} & e^{jn\phi_2} & \cdots & e^{jn\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d+n-1)\phi_1} & e^{j(m-d+n-2)\phi_2} & \cdots & e^{j(m-d+n-2)\phi_d} \end{bmatrix} \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix} + \underline{N}_n$$

$$\underline{Y}_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix} \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix} \begin{bmatrix} e^{j(n-1)\phi_1} & & & 0 \\ & e^{j(n-1)\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{j(n-1)\phi_d} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix} + \underline{N}_n .$$

Let $\underline{S}$ be the vector

$$\underline{S}^T = \{s_1, s_2, \ldots, s_d\}$$

Then $\underline{Y}_n$ can be rewritten as

$$\underline{Y}_n = A\,\Phi^{(n-1)}\,B\underline{S} + \underline{N}_n \qquad (3\text{-}8)$$

where

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix}$$

$$B = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix}$$

$$\Phi = \begin{bmatrix} e^{j\phi_1} & & & 0 \\ & e^{j\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{j\phi_d} \end{bmatrix} .$$

We know that

$$m_{h,k} = \langle \underline{Y}_h, \underline{Y}_k \rangle = \langle AB\phi^{(h-1)}\underline{S}+\underline{N}_h \, , \, AB\phi^{(k-1)}\underline{S}+\underline{N}_k \rangle$$

$$=E[\underline{Y}_k^H . \underline{Y}_h] = E[(AB\phi^{(k-1)}\underline{S}+\underline{N}_k)^H(AB\phi^{(h-1)}\underline{S}+\underline{N}_h)]. \qquad (3\text{-}9)$$

Assuming the signals and noise to be statistically independent, we can write

$$m_{h,k} = E[\underline{S}^H\phi^{H(k-1)}B^HA^HAB\phi^{(h-1)}\underline{S}] + E[\underline{N}^H{}_k\underline{N}_h]. \qquad (3\text{-}10)$$

Suppose the noise components are statistically independent Gaussian random variables with zero-mean and variance σ^2. Then

$$E[\underline{N}_k{}^H\underline{N}_h] = \begin{cases} 0 & ;k\neq h \\ (m-d)\sigma^2 & ;k=h \end{cases} \qquad (3\text{-}11)$$

and $m_{h,k}$ is

$$m_{h,k} = \begin{cases} E[\underline{S}^H\phi^{H(k-1)}B^HA^HAB\phi^{(h-1)}\underline{S}] & ;\ k\neq h \\ E[\underline{S}^H\phi^{H(k-1)}B^HA^HAB\phi^{(h-1)}\underline{S}] + (m-d)\sigma^2 & ;\ k=h. \end{cases} \qquad (3\text{-}12)$$

A more convenient expression for $m_{h,k}$ is now derived. Note that

$$A^HA = \begin{bmatrix} 1 & e^{-j\phi_1} & \ldots & e^{-j(m-d-1)\phi_1} \\ 1 & e^{-j\phi_2} & \ldots & e^{-j(m-d-1)\phi_2} \\ \vdots & & & \vdots \\ 1 & e^{-j\phi_d} & \ldots & e^{-j(m-d-1)\phi_d} \end{bmatrix} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \ldots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \ldots & e^{j(m-d-1)\phi_d} \end{bmatrix}.$$

Let $F_{pq} = \sum_{i=1}^{m-d} e^{j(i-1)(\phi_p-\phi_q)}$. Then

$$A^H A = \begin{bmatrix} (m-d) & F_{21} & \cdots & F_{d1} \\ F_{12} & (m-d) & \cdots & F_{d2} \\ \vdots & \vdots & & \vdots \\ F_{1d} & F_{2d} & \cdots & (m-d) \end{bmatrix} .$$

Next we compute

$$\Phi^{H(k-1)} A^H A = \begin{bmatrix} e^{-j(k-1)\phi_1} & & & 0 \\ & e^{-j(k-1)\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{-j(k-1)\phi_d} \end{bmatrix} \begin{bmatrix} (m-d) & F_{21} & \cdots & F_{d1} \\ F_{12} & (m-d) & \cdots & F_{d2} \\ \vdots & \vdots & & \vdots \\ F_{1d} & F_{2d} & \cdots & (m-d) \end{bmatrix}$$

$$
= \begin{bmatrix} (m-d)e^{-j(k-1)\phi_1} & F_{21}e^{-j(k-1)\phi_1} & \cdots & F_{d1}e^{-j(k-1)\phi_1} \\ F_{12}e^{-j(k-1)\phi_2} & (m-d)e^{-j(k-1)\phi_2} & \cdots & F_{d2}e^{-j(k-1)\phi_2} \\ \vdots & \vdots & & \vdots \\ F_{1d}e^{-j(k-1)\phi_d} & F_{2d}e^{-j(k-1)\phi_d} & \cdots & (m-d)e^{-j(k-1)\phi_d} \end{bmatrix}.
$$

Multiplying $\phi^{H(k-1)}A^HA$ by $\phi^{(h-1)}$ we get

$$
\phi^{H(k-1)}A^HA\phi^{(h-1)} = \begin{bmatrix} (m-d)e^{j(h-k)\phi_1} & F_{21}e^{-j(k-1)\phi_1}e^{j(h-1)\phi_2} & \cdots & F_{d1}e^{-j(k-1)\phi_1}e^{j(h-1)\phi_d} \\ F_{12}e^{-j(k-1)\phi_2}e^{j(h-1)\phi_1} & (m-d)e^{j(h-k)\phi_2} & \cdots & F_{d2}e^{-j(k-1)\phi_2}e^{j(h-1)\phi_d} \\ \vdots & \vdots & & \vdots \\ F_{1d}e^{-j(k-1)\phi_d}e^{j(h-1)\phi_1} & F_{2d}e^{-j(k-1)\phi_d}e^{j(h-1)\phi_2} & \cdots & (m-d)e^{j(h-k)\phi_d} \end{bmatrix}.
$$

Since

$$
B = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix},
$$

$$\underline{BS} = \begin{bmatrix} a_1 & & & & \\ & a_2 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & a_d \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_d \end{bmatrix} = \begin{bmatrix} a_1 s_1 \\ a_2 s_2 \\ \cdot \\ \cdot \\ \cdot \\ a_d s_d \end{bmatrix}$$

and

$\underline{S}^* B^* =$

$$[s_1^* \; s_2^* \; . \; . \; . \; . \; s_d^*] \begin{bmatrix} a_1^* & & & \\ & a_2^* & & \\ & & \cdot & \\ & & & \cdot \\ & & & & a_d^* \end{bmatrix} = [a_1^* s_1^* \; a_2^* s_2^* \; . \; . \; . \; a_d^* s_d^*].$$

Thus, $\phi^{H(k-1)} A^H A \phi^{(h-1)} B\underline{S}$ is equal to

$$
\begin{bmatrix}
a_1 s_1 (m-d) e^{j(h-k)\phi_1} + a_2 s_2 F_{21} e^{-j(k-1)\phi_1} \, e^{j(h-1)\phi_2} + \cdots \\
\cdots + a_d s_d F_{d1} e^{-j(k-1)\phi_1} \, e^{j(h-1)\phi_d} \\
\\
a_1 s_1 F_{12} e^{-j(k-1)\phi_2} \, e^{j(h-1)\phi_1} + a_2 s_2 (m-d) e^{j(h-k)\phi_2} + \cdots \\
\cdots + a_d s_d F_{d2} e^{-j(k-1)\phi_2} \, e^{j(h-1)\phi_d} \\
\vdots \\
a_1 s_1 F_{12} e^{-j(k-1)\phi_d} \, e^{j(h-1)\phi_1} + a_2 s_2 F_{2d} e^{-j(k-1)\phi_d} \, e^{j(h-1)\phi_2} + \cdots \\
\cdots + a_d s_d (m-d) e^{j(h-k)\phi_d}
\end{bmatrix}
$$

and $\underline{S}^H B^H \Phi^{H(l-1)} A^H A \Phi^{(k-1)} B \underline{S}$ becomes

$$a_1^* s_1^* [\ a_1 s_1 (m-d) e^{j(h-k)\phi_1} + a_2 s_2 F_{21} e^{-j(k-1)\phi_1}\ e^{j(h-1)\phi_2} + \dots\dots$$
$$\dots\dots + a_d s_d F_{d1} e^{-j(k-1)\phi_1}\ e^{j(h-1)\phi_d}\ ]$$

$+$

$$a_2^* s_2^* [\ a_1 s_1 F_{12} e^{-j(k-1)\phi_2}\ e^{j(h-1)\phi_1} + a_2 s_2 (m-d) e^{j(h-k)\phi_2} + \dots\dots$$
$$\dots\dots + a_d s_d F_{d2} e^{-j(k-1)\phi_2}\ e^{j(h-1)\phi_d}\ ]$$

$+$

.

.

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$+$

$$a_d^* s_d^* [\ a_1 s_1 F_{12} e^{-j(k-1)\phi_d}\ e^{j(h-1)\phi_1} + a_2 s_2 F_{2d} e^{-j(k-1)\phi_d}\ e^{j(h-1)\phi_2} + .$$
$$\dots\dots + a_d s_d (m-d) e^{j(h-k)\phi_d}\ ].$$

Noting that $e^{j(h-k)\phi_i}$ can be written as

$$e^{j(h-k)\phi_i} = e^{-j(k-1)\phi_i}\ e^{j(h-1)\phi_i} \quad \text{for all } i=1,2,\dots,d$$

and

$$F_{ii} = (m-d) \quad \text{for all } i=1,2,\dots,d,$$

we obtain

$$\underline{S}^H B^H \Phi^{H(k-1)} A^H A \Phi^{(h-1)} B \underline{S} = \sum_{q=1}^{d} \sum_{p=1}^{d} F_{pq} s_q^* s_p a_q^* a_p e^{-j(k-1)\phi_q} e^{j(h-1)\phi_p}.$$

If we let

$$S_{pq} = E[s_q^* s_p]$$

$$a_{pq} = a_q^* s_p,$$

then

$$E[\underline{S}^H \Phi^{H(k-1)} B^H A^H A B \Phi^{(h-1)} \underline{S}] = \sum_{q=1}^{d} \sum_{p=1}^{d} S_{pq} a_{pq} F_{pq} e^{-j(k-1)\phi_q} e^{j(h-1)\phi_p}.$$

$m_{h,k}$ becomes

$$m_{h,k} = \begin{cases} E[\underline{S}^H \Phi^{H(k-1)} B^H A^H A B \Phi^{(h-1)} \underline{S}] & ; h \neq k \\ E[\underline{S}^H \Phi^{H(k-1)} B^H A^H A B \Phi^{(h-1)} \underline{S}] + (m-d)\sigma^2 & ; h = k \end{cases} \tag{3-13}$$

$$= \begin{cases} \sum_{q=1}^{d} \sum_{p=1}^{d} S_{pq} a_{pq} F_{pq} e^{-j(k-1)\phi_q} e^{j(h-1)\phi_p} & ; h \neq k \\ \sum_{q=1}^{d} \sum_{p=1}^{d} S_{pq} a_{pq} F_{pq} e^{-j(k-1)\phi_q} e^{j(h-1)\phi_p} + (m-d)\sigma^2 & ; h = k. \end{cases} \tag{3-14}$$

Let I be the identity matrix and I_1 the matrix defined as follows

$$I_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Finally we, define the matrices M and N as follows

$$\begin{aligned} M &= M_1 - (m-d)\sigma^2 I \\ N &= N_1 - (m-d)\sigma^2 I_1. \end{aligned} \qquad (3\text{-}15)$$

The matrix pencil is

$$M-\lambda N = (M_1-(m-d)\sigma^2 I)-\lambda(N_1-(m-d)\sigma^2 I_1).$$

Define the matrices U, V and Φ as follows :

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(d-1)\phi_1} & e^{j(d-1)\phi_2} & \cdots & e^{j(d-1)\phi_d} \end{bmatrix}$$

$$V = \begin{bmatrix} S_{11}a_{11}F_{11} & S_{12}a_{12}F_{12} & \cdots & S_{1d}a_{1d}F_{1d} \\ S_{21}a_{21}F_{21} & S_{22}a_{22}F_{22} & \cdots & S_{2d}a_{2d}F_{2d} \\ \vdots & \vdots & & \vdots \\ S_{d1}a_{d1}F_{d1} & S_{d2}a_{d2}F_{d2} & \cdots & S_{dd}a_{dd}F_{dd} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} e^{j\phi_1} & & & & 0 \\ & e^{j\phi_2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & e^{j\phi_d} \end{bmatrix} .$$

It can be shown that M and N have the following decompositions :

$$M = U\ V\ U^H$$

$$N = U\ V\ \Phi^H\ U^H. \tag{3-16}$$

Hence, as required by the pencil theorem, the matrix decomposition of the pencil $M-\lambda N$ is given by

$$\begin{aligned} M-\lambda N &= (UVU^H)-\lambda(UV\Phi^H U^H) \\ &= UV(I-\lambda\Phi)U^H. \end{aligned} \tag{3-17}$$

The matrices UV and U^H are of rank d as long as the directions of arrival of the signals are distinct. Therefore the values of λ for which the rank of the pencil $M-\lambda N = UV(I-\lambda\Phi^H)U^H$ is decreased by 1 are given by

$$\lambda_i = e^{j\phi_i} \ ;i=1,2,\ldots,d. \tag{3-18}$$

The angles of arrival are given by

$$\theta_i = \sin^{-1}(-j\ln(\lambda_i)/\omega D) \quad ;i=1,2,\ldots,d. \tag{3-19}$$

CHAPTER 4

WINDOWS

In chapter 3 the rectangular window was the logical and obvious choice. It was used to form the sequence $x_{n+i-1}(t,\underline{\theta})R_{m-d}(i)$. In this chapter we show that any shaped window will work; i.e, the angles of arrival are obtained from the matrix decomposition of a matrix pencil. A computer simulation and a comparative performance of the different windows used are also presented.

4.1 DETERMINISTIC CASE :

Consider a linear array composed of m identical sensors with uniform spacing D. Assume there are $d \leq m/2$ sources emitting signals whose complex envelopes are denoted by $s_k(t)$; $k=1,2,\ldots,d$.
As before the received signal at the i^{th} sensor is modeled as

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\, a_i(\theta_k) + n_i(t) \; ; \; i=1,2,\ldots,m \qquad (4\text{-}1)$$

where

$a_i(\theta_k)$ is the relative response of the i^{th} sensor to the k^{th} source,

$n_i(t)$ is the additive noise assumed to be zero-mean Gaussian.

$a_i(\theta)$ can be written as

$$a_i(\theta)=a(\theta)e^{j(i-1)\phi}$$

where

$$\phi=(\omega D/c)\ \sin(\theta).$$

For simplicity, the arguments $(t,\underline{\theta})$ in $y_i(t,\underline{\theta})$ and t in both $s_k(t)$ and $n_i(t)$ are dropped.

Taking the expected value of (1) we get

$$x_i = E[y_i] = \sum_{k=1}^{d} s_k\ a(\theta_k)e^{j(i-1)\phi_k} \quad ;i=1,2,\ldots,m. \qquad (4\text{-}2)$$

Consider the sequence $x_{n+i-1}W_{m-d}(i)$;n=1,2,...,d+1 and i=1,2,....,m-d, where $W_{m-d}(i)$ is the value of the window of width (m-d) evaluated at the point (i). We then form (d+1) vectors $\underline{X}_n$ where

$$\underline{X}_n = \begin{bmatrix} x_n W_{m-d}(1) \\ x_{n+1}W_{m-d}(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n+m-d-1}W_{m-d}(m-d) \end{bmatrix}.$$

If, for simplicity, we let $c_i = W_{m-d}(i)$, $\underline{X}_n$ can be written as

$$\underline{X}_n = \begin{bmatrix} c_1 & & & & 0 \\ & c_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & c_{m-d} \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n+m-d-1} \end{bmatrix} .$$

However, it is shown in chapter 3 that

$$\begin{bmatrix} x_n \\ x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n+m-d-1} \end{bmatrix} = A\Phi^{(n-1)}B\underline{S} \qquad (4\text{-}3)$$

where

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} e^{j\phi_1} & & & & 0 \\ & e^{j\phi_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & e^{j\phi_d} \end{bmatrix}$$

$$B = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \cdot & \\ 0 & & & a_d \end{bmatrix}$$

and

$$\underline{S}^T = \{s_1\ s_2\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ s_d\}.$$

Let C be the matrix given by

$$C = \begin{bmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \cdot & \\ 0 & & & c_{m-d} \end{bmatrix}.$$

Then $\underline{X}_n$ can be written as

$$\underline{X}_n = CA\Phi^{(n-1)}B\underline{S}. \qquad (4\text{-}4)$$

This expression holds for various choices of the c_i corresponding to different windows.

The matrices M and N which are formed by

$$M = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ | & | & & | \\ \underline{X}_1 & \underline{X}_2 & \cdots & \underline{X}_d \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} ; \quad N = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ | & | & & | \\ \underline{X}_2 & \underline{X}_3 & & \underline{X}_{d+1} \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

become

$$M = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ CAB\underline{S} & CAB\Phi\underline{S} & CAB\Phi^2\underline{S} & \cdots\cdots & CAB\Phi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$N = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ CAB\Phi\underline{S} & CAB\Phi^2\underline{S} & CAB\Phi^3\underline{S} & \cdots\cdots & CAB\Phi^{d}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

If we factor out CAB from M and CABΦ from N, we get

$$M = CAB \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Phi\underline{S} & \Phi^2\underline{S} & \cdots\cdots & \Phi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$N = CAB\Phi \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Phi\underline{S} & \Phi^2\underline{S} & \cdots\cdots & \Phi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Let F be the matrix

$$F = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Phi\underline{S} & \Phi^2\underline{S} & \cdots\cdots & \Phi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Since M and N have the decompositions

$M = C\,A\,B\,F$

$$N = C\,A\,B\,\Phi\,F, \tag{4-5}$$

the matrix pencil becomes

$$M-\lambda N = CABF-\lambda CAB\Phi F$$

$$= CAB(I-\lambda\Phi)F. \tag{4-6}$$

Let E denote the matrix CAB. Then

$$M-\lambda N = E(I-\lambda\Phi)F \tag{4-7}$$

which satisfies the requirements of the pencil theorem. Note that $\mathrm{rank}(E)=\min\{\mathrm{rank}(A),\mathrm{rank}(B),\mathrm{rank}(C)\}$.

We have seen that A is of rank d as long as the directions of arrival are distinct and the separation D is less than $\lambda/2$. B is of rank D as seen earlier. When choosing the elements of the matrix C, it is necessary that at least d of the diagonal elements be non zero. This will ensure that $\mathrm{rank}(C) \geq d$. Therefore the rank of the pencil $M-\lambda N$ is decreased by 1 whenever

$$\lambda_i = e^{j\phi_i} \quad ;i=1,2,\ldots,d. \tag{4-8}$$

4.2 ZERO-MEAN RANDOM CASE :

Again the signal received at the i^{th} sensor is modeled as

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\, a_i(\theta_k) + n_i(t) \;;\; i=1,2,\ldots,m. \tag{4-9}$$

$$y_i(t,\underline{\theta}) = x_i(t,\underline{\theta}) + n_i(t) \;;i=1,2,\ldots,m. \tag{4-10}$$

Consider the sequence

$$y_{n+i-1}(t,\underline{\theta})W_{m-d}(i) \quad ;n=1,2,\ldots,d+1 \text{ and } 1\leq i \leq m-d.$$

For simplicity, the arguments $(t,\underline{\theta})$ and t are dropped. (d+1) vectors $\underline{Y}_n$ are then formed where

$$\underline{Y}_n = \begin{bmatrix} y_n W_{m-d}(1) \\ y_{n+1} W_{m-d}(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n+m-d-1} W_{m-d}(m-d) \end{bmatrix} .$$

As before, let $c_j=W_{m-d}(j)$. Then

$$\underline{Y}_n = \begin{bmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_{m-d} \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n+m-d-1} \end{bmatrix} + \begin{bmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_{m-d} \end{bmatrix} \begin{bmatrix} n_n \\ n_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ n_{n+m-d-1} \end{bmatrix} .$$

Let C be the matrix

$$C = \begin{bmatrix} c_1 & & & & \\ & c_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & c_{m-d} \end{bmatrix} .$$

Noting that

$$\begin{bmatrix} x_n \\ x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n+m-d-1} \end{bmatrix} = A\Phi^{(n-1)}B\underline{S},$$

$\underline{Y}_n$ can be written as

$$\underline{Y}_n = CAB\Phi^{(n-1)}\underline{S} + C\underline{N}_n \qquad (4\text{-}11)$$

where

$$\underline{N}_n^T = \{n_n \; n_{n+1} \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; n_{n+m-d-1} \}.$$

This expression holds for various choices of c_i corresponding to different windows.

Define the inner product

$$m_{h,k} = \langle \underline{Y}_h, \underline{Y}_k \rangle = E[\underline{Y}_k^H \underline{Y}_h].$$

Define the matrices M_1 and N_1 as follows

$$M_1 = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots\cdots & m_{1,d} \\ m_{2,1} & m_{2,2} & \cdots\cdots & m_{2,d} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ m_{d,1} & m_{d,2} & \cdots\cdots & m_{d,d} \end{bmatrix}$$

$$N_1 = \begin{bmatrix} m_{2,1} & m_{2,2} & \cdots\cdots & m_{2,d} \\ m_{3,1} & m_{3,2} & \cdots\cdots & m_{3,d} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ m_{d+1,1} & m_{d+1,2} & \cdots\cdots & m_{d+1,d} \end{bmatrix} .$$

Since $\underline{Y}_n = CAB\Phi^{(n-1)}\underline{S} + C\underline{N}_n$, it follows that

$$m_{h,k} = E[\underline{Y}_k{}^H \cdot \underline{Y}_h]$$

$$= E[(\underline{S}^H\Phi^{H(k-1)}B^HA^HC^H + \underline{N}_k{}^HC^H)(CAB\Phi^{(h-1)}\underline{S} + C\underline{N}_h)].$$

Assuming that the signals and noise are statistically independent,

$$m_{h,k} = E[\underline{S}^H\Phi^{H(k-1)}B^HA^HC^HCAB\Phi^{(h-1)}\underline{S}\ ] + E[\underline{N}_k{}^HC^HC\underline{N}_h)].$$

Let the noise components be statistically independent Gaussian random variables with zero-mean and variance σ^2. Then

$$E[\underline{N}_k{}^HC^HC\underline{N}_h)] = \begin{cases} 0 & ;h \neq k \\ \sigma^2\left(\sum_{i=1}^{m-d} |c_i|^2\right) & ;h = k. \end{cases} \qquad (4\text{-}12)$$

Assuming all c_i's to be real, $|c_i|^2 = c_i^2$. Therefore, $C^H C$ is the matrix

$$C^H C = \begin{bmatrix} c_1^2 & & & & \\ & c_2^2 & & & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & c_{m-d}^2 \end{bmatrix} .$$

It follows that

$$C^H C A = \begin{bmatrix} c_1^2 & & & & \\ & c_2^2 & & & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & c_{m-d}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix}$$

$$= \begin{bmatrix} c_1^2 & c_1^2 & & c_1^2 \\ c_2^2 e^{j\phi_1} & c_2^2 e^{j\phi_2} & & c_2^2 e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ c_{m-d}^2 e^{j(m-d-1)\phi_1} & c_{m-d}^2 e^{j(m-d-1)\phi_2} & \cdots & c_{m-d}^2 e^{j(m-d-1)\phi_d} \end{bmatrix} .$$

If we define $F_{pq} = \sum_{i=1}^{m-d} c_i^2 \, e^{j(i-1)(\phi_p - \phi_q)}$, then $A^H C^H C A$ becomes

$$A^H C^H C A = \begin{bmatrix} F_{11} & F_{21} & \cdots & F_{d1} \\ F_{12} & F_{22} & \cdots & F_{d2} \\ \vdots & \vdots & & \vdots \\ F_{1d} & F_{2d} & \cdots & F_{dd} \end{bmatrix} .$$

Following the same procedure as in section 3.2, it can be shown that

$$E[\underline{S}^H \Phi^H (k-1) B^H A^H C^H C A B \Phi (h-1) \underline{S}] = \sum_{q=1}^{d} \sum_{p=1}^{d} S_{pq} a_{pq} F_{pq} e^{-j(k-1)\phi_q} e^{j(h-1)\phi_p}$$

where

$$S_{pq} = E[s_q^* s_p]$$

$$a_{pq} = a_q^* a_p$$

$$F_{pq} = \sum_{i=1}^{m-d} c_i^2 \, e^{j(i-1)(\phi_p - \phi_q)} .$$

Therefore,

$$m_{h,k} = \begin{cases} E[\underline{S}^H \Phi^H (k-1) B^H A^H C^H C A B \Phi (h-1) \underline{S}] & ; h \neq k \\ E[\underline{S}^H \Phi^H (k-1) B^H A^H C^H C A B \Phi (h-1) \underline{S}] + \sigma^2 \left(\sum_{i=1}^{m-d} |c_i|^2 \right) & ; h = k \end{cases}$$

$$
=\begin{cases}
\sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}a_{pq}F_{pq}e^{-j(k-1)\phi_q}\, e^{j(h-1)\phi_p} & ;h\neq k \\
\sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}a_{pq}F_{pq}e^{-j(k-1)\phi_q}\, e^{j(h-1)\phi_p} + \sigma^2\left(\sum_{i=1}^{m-d} |c_i|^2\right) & ;h=k.
\end{cases}
\tag{4-13}
$$

Let I be the identity matrix and I_1 the matrix defined as

$$
I_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix} .
$$

Also, define the matrices M and N as follows :

$$
M=M_1 - \left(\sum_{i=1}^{m-d} c_i^2\right) \sigma^2 I
$$

$$
N=N_1 - \left(\sum_{i=1}^{m-d} c_i^2\right) \sigma^2 I_1 .
$$

The matrix pencil becomes

$$
M-\lambda N = [M_1 - \left(\sum_{i=1}^{m-d} c_i^2\right) \sigma^2 I]-\lambda[N_1 - \left(\sum_{i=1}^{m-d} c_i^2\right) \sigma^2 I_1].
$$

If we define the matrices U, V and Φ by

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(d-1)\phi_1} & e^{j(d-1)\phi_2} & \cdots & e^{j(d-1)\phi_d} \end{bmatrix}$$

$$V = \begin{bmatrix} S_{11}a_{11}F_{11} & S_{12}a_{12}F_{12} & \cdots & S_{1d}a_{1d}F_{1d} \\ S_{21}a_{21}F_{21} & S_{22}a_{22}F_{22} & \cdots & S_{2d}a_{2d}F_{2d} \\ \vdots & \vdots & & \vdots \\ S_{d1}a_{d1}F_{d1} & S_{d2}a_{d2}F_{d2} & \cdots & S_{dd}a_{dd}F_{dd} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} e^{j\phi_1} & & & 0 \\ & e^{j\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{j\phi_d} \end{bmatrix},$$

it can be shown that M and N have the following decompositions

$$M = U\, V\, U^H$$

$$N = U\, V\, \Phi^H\, U^H.$$

The matrices UV and U^H are of rank d as required by the pencil theorem. Therefore the rank of the pencil $M-\lambda N = UV(I-\lambda\Phi^H)U^H$ is decreased whenever

$$\lambda_i = e^{j\phi_i} \quad ;i=1,2,\ldots,d.$$

The angles of arrival are given by

$$\theta_i = \sin^{-1}(-jc\ln(\lambda_i)/\omega D) \quad ;i=1,2,\ldots,d. \qquad (4\text{-}14)$$

Hence, theoretically any shaped window would give the same directions of arrival of the sources.

4.3 COMPUTER SIMULATION

In this section the comparative performance of the rectangular, Hamming, Hanning and Blackman windows is evaluated by means of a computer simulation.

The different windows are defined as follows:

Rectangular

$$R_N(n) = \begin{cases} 1 \; ; \; 0 \le n \le N \\ 0 \; ; \; \text{elsewhere} \end{cases}$$

Hanning

$$R_N(n) = \begin{cases} .5(1-\cos(2\pi n/N)) & ; \; 0 \le n \le N \\ 0 & ; \; \text{elsewhere} \end{cases}$$

Hamming

$$R_N(n) = \begin{cases} .54-.46\cos(2\pi n/N) & ; \; 0 \le n \le N \\ 0 & ; \; \text{elsewhere} \end{cases}$$

Blackman

$$R_N(n) = \begin{cases} .42-.5\cos(2\pi n/N)+.08\cos(4\pi n/N) & ; \; 0 \le n \le N \\ 0 & ; \; \text{elsewhere.} \end{cases}$$

The scenario used for this simulation consisted of two coherent sources ($d=2$) which are incident on a linear array consisting of eight sensors ($m=8$). The sources are assumed to be located at $\theta_1=18°$ and $\theta_2=22°$.

The received signal at the i^{th} sensor was modeled as

$$y_i(t,\underline{\theta}) = \sum_{i=1}^{d} s_k(t)\, a(\theta_k)e^{j(i-1)\phi_k} + n_i(t) \quad ; \; i=1,2,\ldots,m,$$

where

$\phi_k = \omega D/c \sin(\theta_k)$.

In this simulation the sensors were chosen to be omnidirectional. Consequently,

$$a(\theta_k) = 1$$

Also the noise was simulated to be white Gaussian with zero-mean and unit variance. The sensors were positioned at half wavelength apart such that $\omega D/c = \pi$. Finally, the complex envelopes were selected to be $s_1 = s_2 = s$ where s is a constant. The signal to noise ratio is defined as

$$SNR = \frac{P_s}{P_n} = \frac{2|s|^2}{\sigma^2} = 2|s|^2.$$

The cases considered in this simulation are shown in table (4-1)

Table (4-1)

SNR	$\lvert s \rvert$
30 dB	22.36
10 dB	2.24

In this simulation, equation (4-15) which gives the angles of arrival, was not used because it assumes that magnitude of λ is unity. Because this was not the case in actual practice due to numerical inaccuracies, the method did not perform well. To overcome this situation the following approach was used.

Let $\lambda = a + jb$ where a and b are two real numbers. Using an exponential notation λ can be written as

$$\lambda = \{a^2+b^2\}^{1/2}\exp\{j\tan^{-1}(b/a)\}.$$

From equation (4-14) $\lambda = \exp\{j\phi\}$. Thus, ignoring the magnitude of λ

$$\phi = \tan^{-1}(b/a)$$

and the angles of arrival are given by

$$\theta = \sin^{-1}\{(c\tan^{-1}(b/a))/\omega D\}.$$

The results of the simulation are shown in tables (4-2) and (4-3).

Table (4-2)

SNR	Rectangular θ_1	Rectangular θ_2	Hanning θ_1	Hanning θ_2	Hamming θ_1	Hamming θ_2	Blackman θ_1	Blackman θ_2
30 dB	17.982	21.995	17.882	21.948	17.887	21.999	18.217	22.418
10 dB	18.050	22.142	17.295	23.672	18.294	23.324	14.529	21.065

Mean of θ_1 and θ_2

(500 snapshots/run 10 runs)

Table (4-3)

SNR	Rectangular θ_1	θ_2	Hanning θ_1	θ_2	Hamming θ_1	θ_2	Blackman θ_1	θ_2
30 dB	0.018	0.072	0.126	0.078	0.177	0.090	0.138	0.317
10 dB	4.385	4.960	11.418	14.654	7.843	9.552	20.639	4.595

Variance of θ_1 and θ_2

(500 snapshots/run 10 runs)

Theoretically, the results should be independent of the choice of the window. In practice, however, different windows result in different entries in the matrices M and N. As we can see, the rectangular window performed best both at low and high signal to noise ratio. The Hanning and the Hamming windows also gave acceptable results. The Blackman window gave the most biased estimates with the largest variances.

CHAPTER 5

PERTURBATION ANALYSIS

In this chapter we want to investigate the behavior of the moving window in the presence of perturbations due to sensor spacing. We restrict ourselves to first order perturbations. STEWART has shown [13] that a good measure of this perturbation is the chordal metric which is introduced next.

5.1 CHORDAL METRIC

Let C denote the field of all complex numbers. Consider two matrices M and N and let λ be an eigenvalue of

$$M\,\underline{x} = \lambda\, N\, \underline{x}. \tag{5-1}$$

$\underline{x}$ is called right the eigenvector of equation (5-1). Also let $\underline{y}$ be a left eigenvector of the matrix pencil. $\underline{y}$ satisfies

$$\underline{y}^H M = \lambda\, \underline{y}^H N. \tag{5-2}$$

For convenience, $\underline{x}$ and $\underline{y}$ are usually normalized. Thus, we set

$$||\underline{x}|| = 1 \text{ and } ||\underline{y}|| = 1.$$

We also introduce the Euclidean matrix norm defined as

$$||M|| = \sup_{||\underline{x}||=1} ||M\underline{x}||.$$

We are interested in the generalized eigenvalue problem

$$\tilde{M}\,\tilde{\underline{x}} = \tilde{\lambda}\,\tilde{N}\,\tilde{\underline{x}}. \qquad (5\text{-}3)$$

where

$$\begin{aligned} \tilde{M} &= M + \Delta M = M + E \\ \tilde{N} &= N + \Delta N = N + F \end{aligned} \qquad (5\text{-}4)$$

Let $\qquad \alpha = \underline{y}^H M \underline{x} \qquad (5\text{-}5)$

and

$$\beta = \underline{y}^H N \underline{x} \qquad (5\text{-}6)$$

From equation (5-1) it follows

$$\lambda = \alpha/\beta. \qquad (5\text{-}7)$$

Stewart [13] showed that small perturbations in E and F result in

$$\tilde{\lambda} = \frac{\alpha + \underline{y}^H E \underline{x} + 0(\varepsilon^2)}{\beta + \underline{y}^H F \underline{x} + 0(\varepsilon^2)} = \frac{\alpha' + 0(\varepsilon^2)}{\beta' + 0(\varepsilon^2)} \qquad (5\text{-}8)$$

where

$$\lim_{\varepsilon \to 0} \frac{0(\varepsilon^2)}{\varepsilon} = 0.$$

Define the chordal metric as

$$\chi(\lambda, \tilde{\lambda}) = \frac{|\ \lambda - \tilde{\lambda}\ |}{\sqrt{1 + |\lambda|^2}\ \sqrt{1 + |\tilde{\lambda}|^2}}\ . \qquad (5\text{-}9)$$

Geometrically $\chi(\lambda,\tilde{\lambda})$ is half the length of the chord connecting λ and $\tilde{\lambda}$ when they have been projected in the usual way onto a Rieman sphere of unit radius. The maximum value of the chordal metric is unity.

With this definition it was shown [13] that

$$\chi(\lambda,\tilde{\lambda}) \leq \varepsilon/\gamma + O(\varepsilon^2) \tag{5-10}$$

where

$$\varepsilon = \sqrt{||E||^2 + ||F||^2} \tag{5-11}$$

$$\gamma = \sqrt{\alpha^2 + \beta^2} \tag{5-12}$$

and

α and β have been defined earlier.

5.2 APPLICATION TO THE MOVING WINDOW

Again assume a linear array composed of m identical sensors spaced at $D+\Delta D_i$ where $\Delta D_1=0$. Assume there are d narrowband sources. The received signal at the i^{th} sensor is modeled as

$$\tilde{y}_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\tilde{a}_i(\theta_k) + n_i(t) \quad ;i=1,2,\ldots,m \tag{5-13}$$

where the "~" denotes the response of the perturbed array,

$n_i(t)$ is the additive noise assumed to be zero-mean gaussian,

and

$\tilde{a}_i(\theta_k)$ is the perturbed relative response of the i^{th} sensor to the k^{th} source.

Taking the expected value of equation (5-13) we have

$$\tilde{x}_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t)\tilde{a}_i(\theta_k) \quad ;i=1,2,\ldots,m. \tag{5-14}$$

Note that

$$\tilde{a}_i(\theta) = a(\theta)\exp\{j(i-1)D(\omega/c)\sin(\theta)+j(\omega/c)\Delta D_i\sin(\theta)\}$$

$$= a(\theta)e^{j(i-1)D(\omega/c)\sin(\theta)}\,e^{j(\omega/c)\Delta D_i\sin(\theta)}.$$

To a first order approximation

$$e^{j(\omega/c)\Delta D_i\sin(\theta)} \approx 1+j(\omega/c)\Delta D_i\sin(\theta) = 1+j(2\pi\Delta D_i/\delta)\sin(\theta)$$

where δ is the wavelength of the signal wavefront.

Thus $\tilde{a}_i(\theta)$ can be written as

$$\tilde{a}_i(\theta) = a(\theta)e^{j(i-1)D(\omega/c)\sin(\theta)}+j(2\pi\Delta D_i/\delta)\,e^{j(i-1)D(\omega/c)\sin(\theta)}\sin(\theta)a(\theta). \tag{5-15}$$

Equation (5-14) becomes

$$\tilde{x}_i(t,\underline{\theta}) = \sum_{k=1}^{d} \{a(\theta_k)s_k(t)(e^{j(i-1)D(\omega/c)\sin(\theta_k)} +j(2\pi\Delta D_i/\delta)\,e^{j(i-1)D(\omega/c)\sin(\theta_k)}\sin(\theta_k)a(\theta_k))\}. \tag{5-16}$$

For simplicity, denote $a_k=a(\theta_k)$. Then

$$\tilde{x}_i(t,\underline{\theta}) = \sum_{k=1}^{d} a_k s_k(t)(e^{j(i-1)D(\omega/c)\sin(\theta_k)}$$

$$+j(2\pi\Delta D_i/\delta)\sum_{k=1}^{d} a_k s_k(t)(e^{j(i-1)D\omega/c\sin(\theta_k)}\sin(\theta_k). \qquad (5\text{-}17)$$

Notice that the first part of equation (5-17) is just the non-perturbed quantity x_i which appeared in chapters 3 and 4. Dropping the argument $(t,\underline{\theta})$ in equation (5-17) , it can be written as

$$\tilde{x}_i = x_i + \Delta x_i = x_i + e_i. \qquad (5\text{-}18)$$

(d+1) vectors $\tilde{\underline{X}}_n$ are then formed where $\tilde{\underline{X}}_n$ is given by

$$\tilde{\underline{X}}_n^T = \{ \tilde{x}_n \ \tilde{x}_{n+1} \ . \ . \ . \ . \ . \ \tilde{x}_{n+m-d-1} \}.$$

$\tilde{\underline{X}}_n$ can be written as

$$\tilde{\underline{X}}_n = \begin{bmatrix} x_n \\ x_{n+1} \\ . \\ . \\ . \\ x_{n+m-d-1} \end{bmatrix} + \begin{bmatrix} e_n \\ e_{n+1} \\ . \\ . \\ . \\ e_{n+m-d-1} \end{bmatrix} = \underline{X}_n + \underline{E}_n \qquad (5\text{-}19)$$

where $\underline{E}_n^T = \{ e_n \ e_{n+1} \ . \ . \ . \ . \ . \ e_{n+m-d-1} \}$. In chapter 3 and 4 it is shown that $\underline{X}_n$ can be expressed as

$$\underline{X}_n = AB\Phi^{(n-1)}\underline{S} \qquad (5\text{-}20)$$

where B is now the identity matrix and A, Φ and $\underline{S}$ are :

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} e^{j\phi_1} & & & 0 \\ & e^{j\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{j\phi_d} \end{bmatrix}$$

$$\underline{s}^T = \{ s_1 \; s_2 \; \cdots\cdots \; s_d \}.$$

The matrices $\tilde{M}$ and $\tilde{N}$ and then formed in the usual way. Specifically,

$$\tilde{M} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ | & | & & | \\ \tilde{\underline{X}}_1 & \tilde{\underline{X}}_2 & \cdots\cdots & \tilde{\underline{X}}_d \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} ; \tilde{N} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ | & | & & | \\ \tilde{\underline{X}}_2 & \tilde{\underline{X}}_3 & \cdots\cdots & \tilde{\underline{X}}_{d+1} \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Knowing that $\tilde{\underline{X}}_i = \underline{X}_i + \underline{E}_i$, $\tilde{M}$ can be written as

$$\tilde{M} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ | & | & & | \\ \underline{X}_1 & \underline{X}_2 & \cdots\cdots & \underline{X}_d \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} + \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ | & | & & | \\ \underline{E}_1 & \underline{E}_2 & \cdots\cdots & \underline{E}_d \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Since

$$\tilde{M} = M + \Delta M = M + E \qquad (5\text{-}21)$$

Where M is the unperturbed matrix, it follows that $\Delta M = E$.

Similarly, it can be shown that $\tilde{N}$ can be written as

$$\tilde{N} = N + \Delta N = N + F \qquad (5\text{-}22)$$

where $\Delta N = F$ and N and F are given by

$$N = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ | & | & & | \\ \underline{X}_2 & \underline{X}_3 & \cdots\cdots & \underline{X}_{d+1} \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$F = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ | & | & & | \\ \underline{E}_2 & \underline{E}_3 & \cdots\cdots & \underline{E}_{d+1} \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

To apply equations (5-10) through (5-12), one has to get the Euclidean norms of E and F.

The Euclidean matrix norm is defined as

$$||E|| = \sup_{||\underline{x}||=1} ||E\underline{x}||.$$

However

$$||E\underline{x}||=\{(E\underline{x})^H(E\underline{x})\}^{1/2} = \{\underline{x}^H E^H E\underline{x}\}^{1/2}. \qquad (5\text{-}23)$$

For simplicity, let $D=E^HE$. Notice that maximizing $||E\underline{x}||$ is the same as maximizing the quadratic form $\underline{x}^H D\underline{x}$. The maximization of $\underline{x}^H D\underline{x}$ involves obtaining the eigenvector of

$$D\underline{x} = \lambda\underline{x}$$

which corresponds to the largest eigenvalue of D. Then

$$\max(\underline{x}^H D\underline{x})=\lambda^E{}_{max}x^H\underline{x}. \qquad (5\text{-}24)$$

Since $\underline{x}^H\underline{x}=1.$

$$\max(\underline{x}^H D\underline{x})=\lambda^E{}_{max}$$

and

$$||E||=(\lambda^E{}_{max})^{1/2} \qquad (5\text{-}25)$$

where $\lambda^E{}_{max}$ is the largest eigenvalue of E^HE. Similarly,

$$||F||=(\lambda^F{}_{max})^{1/2} \qquad (5\text{-}26)$$

where $\lambda^F{}_{max}$ is the largest eigenvalue of F^HF.

It follows that

$$\varepsilon = \sqrt{||E||^2 + ||F||^2} = \sqrt{\lambda^E_{max} + \lambda^F_{max}} \, . \qquad (5\text{-}27)$$

We know that $D=E^HE$ is a symmetric matrix. Let d_{ii} be the ii^{th} element on the diagonal of D where d_{ii} is real. Define the TRACE of D as the sum of all the elements on the digonal of D which also equals the sum of all eigenvalues of D. Hence $\lambda^E_{max} \leq trace(D)=sum(d_{ii})$. To obtain the trace, the diagonal elements of $D=E^HE$ must be computed. Recall that E is the matrix

$$E = \begin{bmatrix} e_1 & e_2 & \cdots & e_d \\ e_2 & e_3 & \cdots & e_{d+1} \\ \vdots & \vdots & & \vdots \\ e_{m-d} & e_{m-d+1} & \cdots & e_{m-1} \end{bmatrix} .$$

Therefore,

$$E^H = \begin{bmatrix} e_1^* & e_2^* & \cdots & e_{m-d}^* \\ e_2^* & e_3^* & \cdots & e_{m-d+1}^* \\ \vdots & \vdots & & \vdots \\ e_d^* & e_{d+1}^* & \cdots & e_{m-1}^* \end{bmatrix} .$$

D is then the matrix

$$
D = \begin{bmatrix} e_1 & e_2 & \cdots & e_d \\ e_2 & e_3 & \cdots & e_{d+1} \\ \vdots & \vdots & & \vdots \\ e_{m-d} & e_{m-d+1} & \cdots & e_{m-1} \end{bmatrix} \begin{bmatrix} e_1^* & e_2^* & \cdots & e_{m-d}^* \\ e_2^* & e_3^* & \cdots & e_{m-d+1}^* \\ \vdots & \vdots & & \vdots \\ e_d^* & e_{d+1}^* & \cdots & e_{m-1}^* \end{bmatrix} .
$$

Since we are interested in the trace of D, only its diagonal elements are enumerated below, where use is made of the fact that $e_1=0$.

$$
D = \begin{bmatrix} \sum_{i=2}^{m-d} |e_i|^2 & & & \\ & \sum_{i=2}^{m-d+1} |e_i|^2 & & \\ & & \ddots & \\ & & & \sum_{i=m-d}^{m-1} |e_i|^2 \end{bmatrix}
$$

Thus,

$$\begin{aligned}\operatorname{trace}(E^H E) = & |e_2|^2 + |e_3|^2 + \ldots + |e_{m-d}|^2 \\ & + |e_2|^2 + |e_3|^2 + \ldots + |e_{m-d}|^2 + |e_{m-d+1}|^2 \\ & + |e_3|^2 + \ldots + |e_{m-d}|^2 + |e_{m-d+1}|^2 + |e_{m-d+2}|^2 \\ & + \\ & \cdot \\ & \cdot \\ & \cdot \\ & + |e_d|^2 + |e_{d+1}|^2 + \ldots + |e_{m-1}|^2 .\end{aligned} \tag{5-28}$$

For $i \neq 1$, an element e_i of E can be written as

$$e_i = j \frac{2\pi}{\delta} \Delta D_i \sum_{k=1}^{d} s_k a_k e^{j(i-1)\phi_k} \sin(\theta_k). \tag{5-29}$$

Note that

$$|e_i| = \frac{2\pi}{\delta} |\Delta D_i| \left| \sum_{k=1}^{d} s_k a_k e^{j(i-1)\phi_k} \sin(\theta_k) \right|$$

$$\leq \frac{2\pi}{\delta} |\Delta D_i| \sum_{k=1}^{d} \left| s_k a_k e^{j(i-1)\phi_k} \sin(\theta_k) \right|$$

$$\leq \frac{2\pi}{\delta} |\Delta D_i| \sum_{k=1}^{d} |a_k s_k|$$

$$\leq \frac{2\pi}{\delta} |\Delta D_{max}| \sum_{k=1}^{d} |a_k s_k|. \qquad (5\text{-}30)$$

Hence,

$$\mathrm{trace}(E^H E)^{1/2} \leq [(m-d-1)+(m-d)(d-1)]^{1/2} \frac{2\pi}{\delta} |\Delta D_{max}| \sum_{k=1}^{d} |a_k s_k|$$

$$\mathrm{trace}(E^H E)^{1/2} \leq [d(m-d)-1]^{1/2} \frac{2\pi}{\delta} |\Delta D_{max}| \sum_{k=1}^{d} |a_k s_k|. \qquad (5\text{-}31)$$

Using exactly the same procedure, it can be shown that

$$\mathrm{trace}(F^H F)^{1/2} \leq [d(m-d)]^{1/2} \frac{2\pi}{\delta} |\Delta D_{max}| \sum_{k=1}^{d} |a_k s_k|. \qquad (5\text{-}32)$$

Since

$$\lambda^E_{max} \leq \mathrm{trace}(E^H E)$$

and $\lambda^F_{max} \leq \mathrm{trace}(F^H F)$,

$$\varepsilon = \sqrt{||E||^2 + ||F||^2} = \sqrt{\lambda^E_{max} + \lambda^F_{max}}$$

$$\leq \frac{2\pi}{\delta} |\Delta D_{max}| \sqrt{2d(m-d)-1} \left(\sum_{k=1}^{d} |a_k s_k| \right).$$

Finally, with reference to equations (5-10) through (5-12), the bound on the chordal metric becomes

$$\tilde{\chi}(\lambda,\lambda) \leq \frac{\frac{2\pi}{\delta}\,|\Delta D_{max}|\,\sqrt{2d(m-d)-1}\;\left(\sum_{k=1}^{d} |a_k s_k|\right)}{\sqrt{(\underline{y}^H M \underline{x})^2 + (\underline{y}^H N \underline{x})^2}} \tag{5-33}$$

where $\underline{y}^H$ and $\underline{x}$ are, respectively, the right and left eigenvectors corresponding to λ.

A second approach for obtaining a bound on the chordal metric makes use of the Frobenius norm which is defined for an (mxn) matrix A by

$$||A||_f = \{ \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \}^{1/2} . \tag{5-34}$$

It can be shown [14] that the Euclidean norm is always less than or equal to the Frobenius norm. Specifically,

$$||A||_f \geq ||A||$$

Recall that E is the matrix defined as

$$E = \begin{bmatrix} e_1 & e_2 & \cdots\cdots & e_d \\ e_2 & e_3 & \cdots\cdots & e_{d+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ e_{m-d} & e_{m-d+1} & \cdots\cdots & e_{m-1} \end{bmatrix}$$

where e_1 is equal to zero.

Also, from equation (5-30),

$$|e_i| \le \frac{2\pi}{\delta} |\Delta D_{max}| \sum_{k=1}^{d} |a_k s_k|.$$

It follows that

$$||E||_f \le \{ (2\pi/\delta)^2 \mid \Delta D_{max} \mid^2 ((m-d)d-1)(\sum_{k=1}^{d} |a_k s_k|)^2\}^{1/2}. \qquad (5\text{-}35)$$

Similarly, we have

$$||F||_f \le \{ (2\pi/\delta)^2 \mid \Delta D_{max} \mid^2 ((m-d)d)(\sum_{k=1}^{d} |a_k s_k|)^2\}^{1/2} \qquad (5\text{-}36)$$

Therefore,

$$e = \sqrt{\;||E||^2 + ||F||^2} \le \sqrt{\;||E||_f^2 + ||F||_f^2}$$

$$\le \frac{2\pi}{\delta} |\Delta D_{max}| \sqrt{2d(m-d)-1}\,(\sum_{k=1}^{d} |a_k s_k|).$$

The bound on the chordal metric then becomes

$$\chi(\lambda,\tilde{\lambda}) \le \frac{\frac{2\pi}{\delta}\,|\Delta D_{max}|\,\sqrt{2d(m-d)-1}\,\left(\sum_{k=1}^{d}|a_k s_k|\right)}{\sqrt{(\underline{y}^H M\underline{x})^2+(\underline{y}^H N\underline{x})^2}}. \qquad (5\text{-}37)$$

Notice that this bound is exactly the same as the one obtained while using the Euclidean norm.

5.3 COMPUTER SIMULATION

To evaluate the usefulness of the derived bound for the chordal metric a computer simulation was performed. The model used consisted of two deterministic sources (d=2) with constant and equal complex envelopes incident on a linear array consisting of eight sensors (m=8). The sources are assumed to be located at θ_1=18° and θ_2=22°. For simplicity, the case of omnidirectional sensors was assumed. In the simulation the case of perfect sensor spacing was first considered. 500 snapshots were used to obtain the matrices M and N. The process was repeated 10 times and the results averaged to obtain nominal values for λ_i, $\underline{x}_i$ and $\underline{y}_i$;i=1,2. A pertubation ΔD =.001D was then introduced and the procedure used in the unperturbed case was repeated. The unperturbed signal received at the i^{th} sensor was modeled as

$$y_i(t,\underline{\theta})=\sum_{k=1}^{d} s_k(t)e^{j(i-1)\omega/c\ D\sin(\theta_k)}+n_i(t)\ ;i=1,2,\ldots,m$$

whereas the model for the perturbed signal was

$$\bar{y}_i(t,\underline{\theta}) = \sum_{k=1}^{d} s_k(t) e^{j((i-1)D+\Delta D_i)\omega/c \sin(\theta_k)} + n_i(t) \quad ; i=1,2,\ldots,m.$$

$n_i(t)$ was simulated as white Gaussian noise with zero-mean and variance $\sigma^2=1$. D was assumed to be equal to $\delta/2$ (half the wavelength).

Because $s_1=s_2=s$, the signal to noise ratio is is defined to be

$$SNR=2|s|^2.$$

The cases considered are listed in table (5-1)

Table 5-1

SNR	$\lvert s\rvert$
30 dB	22.36
10 dB	2.24

The computed results are shown in table (5-2)

Table 5-2

SNR	$\chi(\lambda_1,\lambda_1)$ Bound	$\chi(\lambda_1,\lambda_1)$ Exact	$\chi(\lambda_2,\lambda_2)$ Bound	$\chi(\lambda_2,\lambda_2)$ Exact
30 dB	0.1127	0.0318	0.1245	0.0651
10 dB	0.1873	0.1053	0.0633	0.0148

Observe that the bounds for the chordal metric are of the same order of magnitude as the chordal metric itself.

CHAPTER 6

SIMULTANEOUS ESTIMATION OF DIRECTIONS OF ARRIVAL AND ANGULAR FREQUENCIES

In the previous chapters the sources were assumed to be emitting at a common angular frequency ω with complex envelopes denoted by $s_k(t)$;$k=1,2,...,d$. In this section we show that the matrix pencil method still works for the case where the signals have different center frequencies. The angles of arrival and the angular freqencies are estimated simultaneously.

6.1 DETERMINISTIC CASE

Consider a linear array of m identical sensors uniformly spaced at a distance D. Assume there are $d \leq m/2$ narrowband sources located at azimuthal angles θ_k ;$k=1,2,...,d$, which are impinging on the array as planar wavefronts and emitting signals whose complex envelopes are denoted by $s_k(t)$;$k=1,2,...,d$. The first sensor is followed by an equally spaced tapped delay line consisting of m taps with successive delays of T seconds (see fig 6-1).

With reference to the first sensor, the signal received at the i^{th} sensor is modeled as

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} a(\theta_k)s_k(t)\, e^{j(i-1)D(\omega_k/c)\sin(\theta_k)} + n_i(t) \quad ;i=1,2,...,m. \tag{6-1}$$

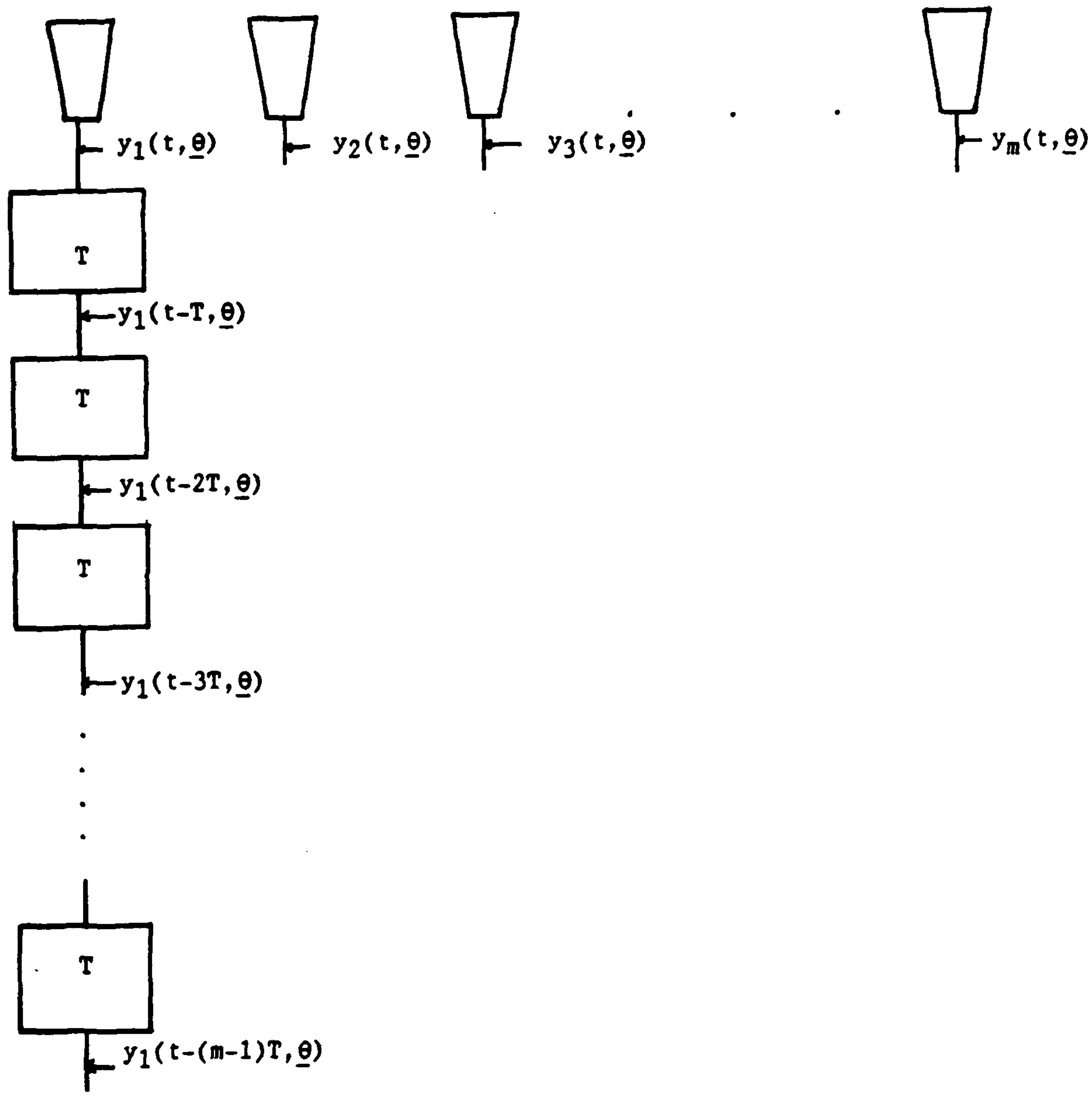

Fig. 6-1 Array Configuration

The signal received at the $(h+1)^{th}$ delay line tap is

$$y_1(t-hT,\underline{\theta}) = \sum_{k=1}^{d} a(\theta_k)s_k(t)e^{jhT\omega_k} + n_1(t-hT) \;;h=0,1,\ldots,m-1. \qquad (6\text{-}2)$$

Assume the noise to be a zero-mean stationary Gaussian process. Then taking the expected values of equation (6-1), we get

$$x_i(t,\underline{\theta})=E[y_i(t,\underline{\theta})]= \sum_{k=1}^{d} a(\theta_k)s_k(t)\, e^{j(i-1)D(\omega_k/c)\sin(\theta_k)};i=1,2,\ldots,m. \qquad (6\text{-}3)$$

Similarly, the expected value equation (6-2) is

$$z_h(t-hT,\underline{\theta})=E[y_1(t-hT,\underline{\theta})]= \sum_{k=1}^{d} a(\theta_k)s_k(t)e^{jhT\omega_k} \;;h=0,1,\ldots,m-1. \qquad (6\text{-}4)$$

Let $\phi_k=(\omega_k D/c)\sin(\theta_k)$; $k=1,2,\ldots,d$. Equation (6-3) becomes

$$x_i(t,\underline{\theta})= \sum_{k=1}^{d} a(\theta_k)s_k(t)\, e^{j(i-1)\phi_k} \;;i=1,2,\ldots,m. \qquad (6\text{-}5)$$

For simplicity, the arguments $(t,\underline{\theta})$ in $x_i(t,\underline{\theta})$, $(t-hT,\underline{\theta})$ in $z_h(t-hT,\underline{\theta})$ and (t) in $s_k(t)$ are dropped. Denote $a_k=a(\theta_k)$. Equations (6-5) and (6-4) become

$$x_i= \sum_{k=1}^{d} a_k s_k\, e^{j(i-1)\phi_k} \;;i=1,2,\ldots,m, \qquad (6\text{-}6)$$

and

$$z_h = \sum_{k=1}^{d} a_k s_k e^{jhT\omega_k} \quad ;h=0,1,\ldots,m-1. \qquad (6\text{-}7)$$

We then form (d+1) vectors $\underline{X}_n$ and (d+1) vectors $\underline{Z}_n$, where

$$\underline{X}_n^T = \{x_n \ x_{n+1} \ \ldots \ x_{n+m-d-1}\};\ n=1,2,\ldots,d+1,$$

and

$$\underline{Z}_r^T = \{z_{r-1} \ z_r \ \ldots \ z_{r+m-d-2}\};\ r=1,2,\ldots,d+1.$$

$\underline{X}_n$ can be expressed as

$$\underline{X}_n = \begin{bmatrix} x_n \\ x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{m-d+n-1} \end{bmatrix} = \begin{bmatrix} a_1 s_1 e^{j(n-1)\phi_1} + \ldots + a_d s_d e^{j(n-1)\phi_d} \\ a_1 s_1 e^{jn\phi_1} + \ldots + a_d s_d e^{jn\phi_d} \\ \cdot \\ \cdot \\ \cdot \\ a_1 s_1 e^{j(m-d+n-2)\phi_1} + \ldots + a_d s_d e^{j(m-d+n-2)\phi_d} \end{bmatrix}$$

$$\underline{X}_n = \begin{bmatrix} a_1 e^{j(n-1)\phi_1} & a_2 e^{j(n-1)\phi_2} & \ldots & a_d e^{j(n-1)\phi_d} \\ a_1 e^{jn\phi_1} & a_2 e^{jn\phi_2} & \ldots & a_d e^{j\phi_d} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_1 e^{j(m-d+n-2)\phi_1} & a_2 e^{j(m-d+n-2)\phi_2} & \ldots & a_d e^{j(m-d+n-2)\phi_d} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ s_d \end{bmatrix}$$

$\underline{X}_n$ can also be written as:

$$\underline{X}_n = \begin{bmatrix} e^{j(n-1)\phi_1} & e^{j(n-1)\phi_2} & \cdots & e^{j(n-1)\phi_d} \\ e^{jn\phi_1} & e^{jn\phi_2} & \cdots & e^{jn\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d+n-2)\phi_1} & e^{j(m-d+n-2)\phi_2} & \cdots & e^{j(m-d+n-2)\phi_d} \end{bmatrix} \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ & & & a_d \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix}$$

$$\underline{X}_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix} \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix} \begin{bmatrix} e^{j(n-1)\phi_1} & & & 0 \\ & e^{j(n-1)\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{j(n-1)\phi_d} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_d \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)\phi_1} & e^{j(m-d-1)\phi_2} & \cdots & e^{j(m-d-1)\phi_d} \end{bmatrix},$$

$$B = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_d \end{bmatrix},$$

$$\Phi = \begin{bmatrix} e^{j\phi_1} & & & 0 \\ & e^{j\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{j\phi_d} \end{bmatrix},$$

$$\underline{S}^T = \{s_1, s_2, \ldots, s_d\}.$$

Then $\underline{X}_n$ can be revritten as

$$\underline{X}_n = A\,\Phi^{(n-1)}\,B\underline{S}. \qquad (6\text{-}8)$$

Following the same procedure, it can be shown that $\underline{Z}_n$ can be written as

$$\underline{Z}_n = A_1 \Psi^{(n-1)}\,B\underline{S} \qquad (6\text{-}9)$$

where B and $\underline{S}$ have been defined previously and A_1 and Ψ are the matrices

$$A_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{jT\omega_1} & e^{jT\omega_2} & \cdots & e^{jT\omega_d} \\ \vdots & \vdots & & \vdots \\ e^{j(m-d-1)T\omega_1} & e^{j(m-d-1)T\omega_2} & \cdots & e^{j(m-d-1)T\omega_d} \end{bmatrix},$$

$$\Psi = \begin{bmatrix} e^{jT\omega_1} & & & 0 \\ & e^{jT\omega_2} & & \\ & & \ddots & \\ 0 & & & e^{jT\omega_d} \end{bmatrix}.$$

Four (4) matrices M, N, P and Q are then formed, where

$$M = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ | & | & & | \\ \underline{X}_1 & \underline{X}_2 & \cdots & \underline{X}_d \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} ; \quad N = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ | & | & & | \\ \underline{X}_2 & \underline{X}_3 & & \underline{X}_{d+1} \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$P = \begin{bmatrix} \uparrow & \uparrow & \cdots\cdots & \uparrow \\ | & | & & | \\ \underline{Z}_1 & \underline{Z}_2 & \cdots\cdots & \underline{Z}_d \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \; ; \quad Q = \begin{bmatrix} \uparrow & \uparrow & \cdots\cdots & \uparrow \\ | & | & & | \\ \underline{Z}_2 & \underline{Z}_3 & & \underline{Z}_{d+1} \\ | & | & & | \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Using the explicit expression for $\underline{X}_n$, M and N become

$$M = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ AB\underline{S} & AB\Phi\underline{S} & AB\Phi^2\underline{S} & \cdots\cdots & AB\Phi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} ,$$

$$N = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ AB\Phi\underline{S} & AB\Phi^2\underline{S} & AB\Phi^3\underline{S} & \cdots\cdots & AB\Phi^{d}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Factoring out AB in M and $AB\Phi$ in N , we get

$$M = AB \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Phi\underline{S} & \Phi^2\underline{S} & \cdots\cdots & \Phi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$N = AB\Phi \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Phi\underline{S} & \Phi^2\underline{S} & \cdots\cdots & \Phi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} .$$

Let F be the matrix

$$F = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Phi^1\underline{S} & \Phi^2\underline{S} & \cdots\cdots & \Phi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

and E the matrix

$$E = AB.$$

M and N have the decompositions

$$M = E\ F$$

$$N = E\ \Phi\ F. \qquad (6\text{-}10)$$

The matrix pencil $M-\lambda N$ can then be written as

$$M-\lambda N = EF-\lambda E\Phi F$$

$$= E(I-\lambda\Phi)F \qquad (6\text{-}11)$$

which satisfies the requirements of the pencil theorem. Hence, the values of λ for which the rank of $M-\lambda N$ decreases by 1 are given by

$$\lambda_k = e^{-j\phi_k} \; ; k=1,2,\ldots,d. \tag{6-12}$$

The same procedure is now applied for the decomposition of the matrices P and Q. It follows

$$P = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ A_1B\underline{S} & A_1B\Psi\underline{S} & A_1B\Psi^2\underline{S} & \cdots\cdots & A_1B\Psi^{(d-1)}\underline{S} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix},$$

$$Q = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ A_1B\Psi\underline{S} & A_1B\Psi^2\underline{S} & A_1B\Psi^3\underline{S} & \cdots\cdots & A_1B\Psi^{d}\underline{S} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Factoring out A_1B in P and $A_1B\Psi$ in Q, we get

$$P = A_1B \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ \underline{S} & \Psi\underline{S} & \Psi^2\underline{S} & \cdots\cdots & \Psi^{(d-1)}\underline{S} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$Q = A_1 B \Psi \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Psi\underline{S} & \Psi^2\underline{S} & \cdots\cdots & \Psi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Define the matrix G as

$$G = \begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ | & | & | & & | \\ \underline{S} & \Psi\underline{S} & \Psi^2\underline{S} & \cdots\cdots & \Psi^{(d-1)}\underline{S} \\ | & | & | & & | \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Let E_1 be the matrix A_1B. P and Q have the following expressions,

$$P = E_1 G$$

$$Q = E_1 \Psi G. \qquad (6\text{-}13)$$

The matrix pencil $P-\delta Q$ becomes

$$P-\delta Q = E_1 G - \delta E_1 \Psi G$$

$$= E_1(I-\delta\Psi)G, \qquad (6\text{-}14)$$

which satisfies the requirements of the pencil theorem. Hence, the rank reducing numbers of the pencil $P-\delta Q$ are given by

$$\delta_k = e^{-jT\omega_k} \;; k=1,2,\ldots,d. \qquad (6\text{-}15)$$

Note that equation (6-15) gives us an estimate of ω_k. This estimate is used in equation (6-12) to get an estimate of θ_k. Therefore, this method, simultaneously estimates the angular frequencies and the angles of arrival of the sources; i.e

$$\omega_i = j\ln(\delta_i)/T$$
$$\theta_i = \sin^{-1}\{(j\ln(\lambda_i))/(\omega_i\ D)\};\ i=1,2,\ldots,d. \qquad (6\text{-}16)$$

6-2 ZERO-MEAN RANDOM CASE

In this section we assume that the complex envelopes of the emitted signals are stationary random processes with zero-mean. It is shown that the matrix pencil method still works;i.e, the angular frequencies and the angles of arrival of the sources are estimated simultaneously using matrix decompositions of two (2) matrix pencils. Again, let the received signal at the i^{th} sensor be

$$y_i(t,\underline{\theta}) = \sum_{k=1}^{d} a(\theta_k)s_k(t)\ e^{j(i-1)D(\omega_k/c)\sin(\theta_k)} + n_i(t)\ ;i=1,2,\ldots,m. \qquad (6\text{-}17)$$

The signal received at the $(h+1)^{th}$ delay line tap is

$$y_1(t-hT,\underline{\theta}) = \sum_{k=1}^{d} a(\theta_k)s_k(t)e^{jhT\omega_k} + n_1(t-hT)\ ;h=0,1,\ldots,m-1. \qquad (6\text{-}18)$$

Let $\phi_k=(\omega_k D/c)\sin(\theta_k)$; $k=1,2,\ldots,d$. Equation (6-17) becomes

$$y_i(t,\underline{\theta})=\sum_{k=1}^{d} a(\theta_k)s_k(t)\, e^{j(i-1)\phi_k} + n_i(t); \; i=1,2,\ldots,m. \qquad (6\text{-}19)$$

For simplicity, the arguments $(t,\underline{\theta})$ in $x_i(t,\underline{\theta})$, $(t-hT,\underline{\theta})$ in $z_h(t-hT,\underline{\theta})$ and (t) in $s_k(t)$ and in $n_i(t)$ are dropped. Denote $a_k=a(\theta_k)$. Equations (6-19) and (6-18) become

$$y_i=\sum_{k=1}^{d} a_k s_k\, e^{j(i-1)\phi_k} + n_i \quad ; \; i=1,2,\ldots,m,. \qquad (6\text{-}20)$$

and

$$z_h=\sum_{k=1}^{d} a_k s_k e^{jhT\omega_k} + n_h \quad ; h=0,1,\ldots,m-1. \qquad (6\text{-}21)$$

Analogous to the previous section (d+1) vectors $\underline{Y}_n$ and (d+1) vectors $\underline{Z}_n$ are formed, where

$$\underline{Y}_n^T = \{y_n \; y_{n+1} \; \ldots \; y_{n+m-d-1}\}; \; n=1,2,\ldots,d+1,$$

and

$$\underline{Z}_n^T = \{z_{n-1} \; z_n \; \ldots \; z_{n+m-d-2}\}; \; n=1,2,\ldots,d+1.$$

Define the inner products $m_{h,k}$ and $v_{i,n}$ to be

$$m_{h,k}=< \underline{Y}_h, \underline{Y}_k >,$$

and

$$v_{i,n} = \langle \underline{Z}_i, \underline{Z}_n \rangle .$$

Four matrices M_1, N_1, P_1 and Q_1 are then formed as follows

$$M_1 = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,d} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,d} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ m_{d,1} & m_{d,2} & \cdots & m_{d,d} \end{bmatrix} ,$$

$$N_1 = \begin{bmatrix} m_{2,1} & m_{2,2} & \cdots & m_{2,d} \\ m_{3,1} & m_{3,2} & \cdots & m_{3,d} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ m_{d+1,1} & m_{d+1,2} & \cdots & m_{d+1d} \end{bmatrix} ,$$

$$P_1 = \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,d} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,d} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ v_{d,1} & v_{d,2} & \cdots & v_{d,d} \end{bmatrix} ,$$

and

$$Q_1 = \begin{bmatrix} w_{2,1} & w_{2,2} & \cdots & w_{2,d} \\ w_{3,1} & w_{3,2} & \cdots & w_{3,d} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_{d+1,1} & w_{d+1,2} & \cdots & w_{d+1,d} \end{bmatrix} .$$

Following the same procedure as in chapters 3 and 4, it can be shown that $\underline{Y}_n$ and $\underline{Z}_r$ can be expressed as

$$\underline{Y}_n = AB\Phi^{(n-1)}\underline{S} + \underline{N}_n$$

and

$$\underline{Z}_r = A_1 B\Psi^{(r-1)}\underline{S} + \underline{N}_r .$$

A, A_1, B, Φ, Ψ, and $\underline{S}$ have been defined earlier, $\underline{N}_n$ and $\underline{N}_r$ are given by

$$\underline{N}_n^T = \{ n_n \; n_{n+1} \; \cdots \; n_{n+m-d-1} \},$$

and

$$\underline{N}_r^T = \{ n_r \; n_{r+1} \; \cdots \; n_{r+m-d-1} \}.$$

For simplicity, denote $\psi_k = T\omega_k$; k=1,2,. . .,d. The matrix Ψ then becomes

$$\Psi = \begin{bmatrix} e^{j\psi_1} & & & & 0 \\ & e^{j\psi_2} & & & \\ & & \ddots & & \\ & & & & \\ 0 & & & & e^{j\psi_d} \end{bmatrix} .$$

We know that

$$m_{h,k} = \langle \underline{Y}_h, \underline{Y}_k \rangle = \langle AB\Phi^{(h-1)}\underline{S}+\underline{N}_h \; , \; AB\Phi^{(k-1)}\underline{S}+\underline{N}_k \rangle$$
$$= E[\; \underline{Y}_k^{H} \cdot \underline{Y}_h] = E[(AB\Phi^{(k-1)}\underline{S}+\underline{N}_k)^{H}(AB\Phi^{(h-1)}\underline{S}+\underline{N}_h)]. \qquad (6\text{-}22)$$

Assuming the signals and noise to be statistically independent, we can write

$$m_{h,k} = E[\underline{S}^{H}\Phi^{H(k-1)}B^{H}A^{H}AB\Phi^{(h-1)}\underline{S}] + E[\underline{N}^{H}_k\underline{N}_h]. \qquad (6\text{-}23)$$

Suppose the noise components are statistically independent Gaussian random variables with zero-mean and variance σ^2. Then

$$E[\underline{N}_k{}^{H}\underline{N}_h] = \begin{cases} 0 & ;k\neq h \\ (m-d)\sigma^2 & ;k=h \end{cases} \qquad (6\text{-}24)$$

and $m_{h,k}$ is

$$m_{h,k} = \begin{cases} E[\underline{S}^{H}\Phi^{H(k-1)}B^{H}A^{H}AB\Phi^{(h-1)}\underline{S}] & ;\; k\neq h \\ E[\underline{S}^{H}\Phi^{H(k-1)}B^{H}A^{H}AB\Phi^{(h-1)}\underline{S}] + (m-d)\sigma^2 & ;\; k=h. \end{cases} \qquad (6\text{-}25)$$

We obtain

$$\underline{S}^{H}B^{H}\Phi^{H(k-1)}A^{H}A\Phi^{(h-1)}B\underline{S} = \sum_{q=1}^{d}\sum_{p=1}^{d} F_{pq}s_q^{*}s_p a_q^{*}a_p e^{-j(k-1)\phi_q}\, e^{j(h-1)\phi_p}$$

where

$$F_{pq} = \sum_{i=1}^{m-d} e^{j(i-1)(\phi_p - \phi_q)} .$$

Hence,

$$E[\underline{S}^H \Phi^{H(k-1)} B^H A^H A B \Phi^{(h-1)} \underline{S}] = \sum_{q=1}^{d} \sum_{p=1}^{d} S_{pq} a_{pq} F_{pq} e^{-j(k-1)\phi_q} e^{j(h-1)\phi_p}$$

where

$$S_{pq} = E[s_q^* s_p]$$

$$a_{pq} = a_q^* a_p \ .$$

$$m_{h,k} = \begin{cases} E[\underline{S}^H \Phi^{H(k-1)} B^H A^H A B \Phi^{(h-1)} \underline{S}] & ;h \neq k \\ E[\underline{S}^H \Phi^{H(k-1)} B^H A^H A B \Phi^{(h-1)} \underline{S}] + (m-d)\sigma^2 & ;h = k \end{cases} \qquad (6\text{-}26)$$

$$= \begin{cases} \sum_{q=1}^{d} \sum_{p=1}^{d} S_{pq} a_{pq} F_{pq} e^{-j(k-1)\phi_q} e^{j(h-1)\phi_p} & ;h \neq k \\ \sum_{q=1}^{d} \sum_{p=1}^{d} S_{pq} a_{pq} F_{pq} e^{-j(k-1)\phi_q} e^{j(h-1)\phi_p} + (m-d)\sigma^2 & ;h = k. \end{cases} \qquad (6\text{-}27)$$

Let I be the identity matrix and I_1 the matrix defined by

$$I_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Finally, we define the matrices M and N as follows

$$\begin{aligned} M &= M_1 - (m-d)\sigma^2 I \\ N &= N_1 - (m-d)\sigma^2 I_1. \end{aligned} \qquad (6\text{-}28)$$

The matrix pencil is

$$M-\lambda N = (M_1-(m-d)\sigma^2 I)-\lambda(N_1-(m-d)\sigma^2 I_1).$$

Define the matrices U, V and Φ as follows :

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\phi_1} & e^{j\phi_2} & \cdots & e^{j\phi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(d-1)\phi_1} & e^{j(d-1)\phi_2} & \cdots & e^{j(d-1)\phi_d} \end{bmatrix},$$

$$V = \begin{bmatrix} S_{11}a_{11}F_{11} & S_{12}a_{12}F_{12} & \cdots & S_{1d}a_{1d}F_{1d} \\ S_{21}a_{21}F_{21} & S_{22}a_{22}F_{22} & \cdots & S_{2d}a_{2d}F_{2d} \\ \vdots & \vdots & & \vdots \\ S_{d1}a_{d1}F_{d1} & S_{d2}a_{d2}F_{d2} & \cdots & S_{dd}a_{dd}F_{dd} \end{bmatrix},$$

$$
\Phi = \begin{bmatrix} e^{j\phi_1} & & & & & 0 \\ & e^{j\phi_2} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & e^{j\phi_d} \end{bmatrix} .
$$

It can be shown that M and N have the following decompositions

$$M = U\ V\ U^H$$

$$N = U\ V\ \Phi^H\ U^H. \qquad (6\text{-}29)$$

Hence, as required by the pencil theorem, the matrix decomposition of the pencil $M-\lambda N$ is given by

$$
\begin{aligned}
M-\lambda N &= (UVU^H)-\lambda(UV\Phi^H U^H) \\
&= UV(I-\lambda\Phi^H)U^H. \qquad (6\text{-}30)
\end{aligned}
$$

Therefore, the values of λ for which the rank of the pencil $M-\lambda N = UV(I-\lambda\Phi^H)U^H$ is decreased by 1 are given by

$$\lambda_i = e^{j\phi_i}\ ; i=1,2,\ldots,d. \qquad (6\text{-}31)$$

As it was stated earlier, assuming the noise components to be stationary and statically independent, it can be shown that $w_{i,n}$ has a similar form; i.e,

$$
v_{i,n}=\begin{cases} E[\underline{S}^H\Psi^{H(n-1)}B^HA_1{}^HA_1B\Psi^{(i-1)}\underline{S}] & ;i\neq n \\ E[\underline{S}^H\Psi^{H(n-1)}B^HA_1{}^HA_1B\Psi^{(i-1)}\underline{S}] + (m-d)\sigma^2 & ;i=n \end{cases} \tag{6-32}
$$

$$
=\begin{cases} \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}a_{pq}F'_{pq}e^{-j(n-1)\psi_q}\, e^{j(i-1)\psi_p} & ;i\neq n \\ \sum_{q=1}^{d}\sum_{p=1}^{d} S_{pq}a_{pq}F'_{pq}e^{-j(n-1)\psi_q}\, e^{j(i-1)\psi_p} + (m-d)\sigma^2 & ;i=n \end{cases} \tag{6-33}
$$

where

$$S_{pq}=E[s_q^* s_p]$$

$$a_{pq}=a_q^* a_p$$

$$F'_{pq} = \sum_{i=1}^{m-d} e^{j(i-1)(\psi_p - \psi_q)} .$$

Let then P and Q be the matrices

$$
\begin{aligned} P &= P_1 - (m-d)\sigma^2 I \\ Q &= Q_1 - (m-d)\sigma^2 I_1. \end{aligned} \tag{6-34}
$$

The matrix pencil P-δN becomes

$$P-\delta Q = (P_1-(m-d)\sigma^2 I)-\delta(Q_1-(m-d)\sigma^2 I_1).$$

It can be shown that P and Q can be decomposed as

$$
\begin{aligned} P &= U'V'U'^H \\ Q &= U'V'\Psi^H U'^H, \end{aligned} \tag{6-35}
$$

where Ψ was defined previously and U' and V' are the matrices

$$U' = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\psi_1} & e^{j\psi_2} & \cdots & e^{j\psi_d} \\ \vdots & \vdots & & \vdots \\ e^{j(d-1)\psi_1} & e^{j(d-1)\psi_2} & \cdots & e^{j(d-1)\psi_d} \end{bmatrix},$$

$$V' = \begin{bmatrix} S_{11}a_{11}F'_{11} & S_{12}a_{12}F'_{12} & \cdots & S_{1d}a_{1d}F'_{1d} \\ S_{21}a_{21}F'_{21} & S_{22}a_{22}F'_{22} & \cdots & S_{2d}a_{2d}F'_{2d} \\ \vdots & \vdots & & \vdots \\ S_{d1}a_{d1}F'_{d1} & S_{d2}a_{d2}F'_{d2} & \cdots & S_{dd}a_{dd}F'_{dd} \end{bmatrix}.$$

The matrix pencil $P-\delta Q$ then becomes

$$\begin{aligned} P-\delta Q &= (U'V'U'^H)-\delta(U'V'\Psi^H U'^H) \\ &= U'V'(I-\delta\Psi^H)U'^H, \qquad (6\text{-}36) \end{aligned}$$

which satisfies the requirements of the pencil theorem provided all ω_k are distincts. Hence, the rank reducing numbers of the pencil $P-\delta Q$ are given by

$$\delta_i = e^{j\psi_i} = e^{jT\omega_i}\ ;\ i=1,2,\ldots,d. \qquad (6\text{-}37)$$

Equation (6-37) together with equation (6-31) allows us to estimate simultaneously the angular frequencies and the angles of arrival of the sources; i.e,

$$\omega_i = -j\ln(\delta_i)/T$$

$$\theta_i = \sin^{-1}\{(-j\ln(\lambda_i))/(\omega_i\ D)\};\ i=1,2,\ldots,d. \qquad (6\text{-}38)$$

6.3 COMPUTER SIMULATION

The model used in the simulation consisted of two coherent sources (d=2) incident on a linear array of eight uniformly spaced sensors (m=8). For convenience, the sensors are assumed to be omnidirectional. The noise was simulated as white Gaussian with zero-mean and variance $\sigma^2 = 1$. The complex envelopes of the signals emitted by the sources are assumed to be constant and equal; i.e,

$$s_1(t) = s_2(t) = s.$$

The sources were assumed to be located at $\theta_1=18°$ and $\theta_2=22°$ with center frequencies given by $\omega_1=0.2\text{x}2\pi$ rd/s and $\omega_2=0.25\text{x}2\pi$ rd/s respectively. D and T were assumed to be equal to 6 10^8 meters and 1 second respectively.

With the above definition, the signal to noise ratio is given by

$$\text{SNR} = \frac{2|s|^2}{\sigma^2} = 2|s|^2$$

The cases considered in this simulation are shown in table (6-1)

Table (6-1)

SNR	$\|s\|$
30 dB	22.36
10 dB	2.24

Equation (6-16) was not used because it assumes that both magnitudes of λ and δ are unity. Because this was not the case in actual practice due to numerical inaccuracies, the method did not perform well. One way to solve the problem is to use the technique as in chapter 4.

Let $\lambda = a + jb$ and $\delta = c + jd$ where a, b, c and d are real numbers. Using polar form, we have

$$\lambda = \{a^2+b^2\}^{1/2}\exp\{j\tan^{-1}(b/a)\}$$

and

$$\delta = \{c^2+d^2\}^{1/2}\exp\{j\tan^{-1}(d/c)\}$$

From equations (6-12) and (6-15), $\lambda=\exp\{-j\phi\}$ and $\delta=\exp\{-j\psi\}$. Thus, ignoring the magnitudes of λ and δ

$$\phi = -\tan^{-1}(b/a)$$

and

$$\psi = -\tan^{-1}(d/c).$$

The results of the simulation are shown in tables (6-2) and (6-3).

Table (6-2)

SNR	ω_1	ω_2	θ_1	θ_2
30 dB	0.2001x2π	0.2495x6π	18.0012	22.0233
10 dB	0.1925x2π	0.2448x6π	18.5130	22.7785

Mean of θ_i and ω_i

(500 snapshots/run , 10 runs)

Table (6-3)

SNR	ω_1	ω_2	θ_1	θ_2
30 dB	0.3687 10^{-4}	0.0546 10^{-4}	0.0111	0.0045
10 dB	0.0033	0.0004	1.0504	0.4662

Variance of θ_i and ω_i

(500 snapshots/run , 10 runs)

Observe that the bias in the estimates is very small at both high and low signal to noise ratio. The technique can thus, be classified as giving very good results.

CHAPTER 7

CONCLUSION

In this study the objective was to extend the matrix pencil approach proposed by H. OUIBRAHIM [1]. The flexibility, the effictiveness and the ease of use of the method are once again shown. A generalization of the method to arbitrary but identical beam patterns was derived in chapter 3. This was done to prove that the choice of omnidirectional sensors is not the only choice possible. In chapter 4 we used different windows to generate the sequences needed in the formulation of the matrix pencil. It was shown that the rectangular window performed best compared to the Hanning, Hamming and Blackman windows. Errors arise when the assumption of uniformly spaced elements is not adhered to in practice. For this reason, a perturbation analysis due to sensor spacing was performed in chapter 5. We made use of the chordal metric introduced by Stewart [13]. The chordal metric provides a good measure between the perturbed and the unperturbed eigenvalues. It was shown in a computer simulation that a useful bound was found for the chordal metric. In chapter 6 a new technique was introduced to simultaneously estimate the angular frequencies of the signals emitted by the sources and the directions of arrival of these sources.

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