CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

A COUNTEREXAMPLE CONCERNING THE EXTREMAL INDEX

by

Richard L. Smith

Technical Report No. 237

August 1988

163. M. O'Sullivan and T.R. Fleming, Statistics for the two-sample survival analysis problem based on product limit estimators of the survival functions, Nov. 86.

164. F. Avram, On bilinear forms in Gaussian random variables, Toeplitz matrices and Parseval's relation, Nov. 86.

165. D.B.H. Cline, Joint stable attraction of two sums of products, Nov. 86. *J. Multivariate Anal.*, 25, 1988, 272-285.

166. R.J. Wilson, Model fields in crossing theory-a weak convergence perspective, Nov. 86.

167. D.B.H. Cline, Consistency for least squares regression estimators with infinite variance data, Dec. 86.

168. L.L. Campbell, Phase distribution in a digital frequency modulation receiver, Nov. 86.

169. B.G. Nguyen, Typical cluster size for 2-dim percolation processes, Dec. 86. *J. Statist. Physics*, to appear.

170. H. Oodaira, Freidlein-Wentzell type estimates for a class of self-similar processes represented by multiple Wiener integrals, Dec. 86.

171. J. Nolan, Local properties of index-β stable fields, Dec. 86. *Ann. Probability*, to appear.

172. R. Menich and R.F. Serfozo, Optimality of shortest queue routing for dependent service stations, Dec. 86.

173. F. Avram and M.S. Taqqu, Probability bounds for M-Skorohod oscillations, Dec. 86.

174. F. Moricz and R.L. Taylor, Strong laws of large numbers for arrays of orthogonal random variables, Dec. 86.

175. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations driven by nuclear space valued martingales, Apr. 87. *Appl. Math. Optimization*, 17, 1988, 237-272.

176. E. Merzbach, Point processes in the plane, Feb. 87.

177. Y. Kasahara, M. Maejima and W. Vervaat, Log fractional stable processes, March 87.

178. G. Kallianpur, A.G. Miamee and H. Niemi, On the prediction theory of two parameter stationary random fields, March 87. *J. Multivariate Anal.*, 1988, to appear.

179. R. Brigola, Remark on the multiple Wiener integral, Mar. 87.

180. R. Brigola, Stochastic filtering solutions for ill-posed linear problems and their extension to measurable transformations, Mar. 87.

181. G. Samorodnitsky, Maxima of symmetric stable processes, Mar. 87.

182. H.L. Hurd, Representation of harmonizable periodically correlated processes and their covariance, Apr. 87.

183. H.L. Hurd, Nonparametric time series analysis for periodically correlated processes, Apr. 87.

184. T. Mori and H. Oodaira, Freidlin-Wentzell estimates and the law of the iterated logarithm for a class of stochastic processes related to symmetric statistics, May 87.

185. R.F. Serfozo, Point processes, May 87. *Operations Research Handbook on Stochastic Processes*, to appear.

186. Z.D. Bai, W.Q. Liang and W. Vervaat, Strong representation of weak convergence, June 87.

187. O. Kallenberg, Decoupling identities and predictable transformations in exchangeability, June, 87.

188. O. Kallenberg, An elementary approach to the Daniell-Kolmogorov theorem and some related results, June 87. *Math. Narch.*, to appear.

189. G. Samorodnitsky, Extrema of skewed stable processes, June 87.*Stochastic Proc. Appl.*, to appear.

190. D. Nualart, M. Sanz and M. Zakai, On the relations between increasing functions associated with two-parameter continuous martingales, June 87.

191. F. Avram and M. Taqqu, Weak convergence of sums of moving averages in the α-stable domain of attraction, June 87.

192. M.R. Leadbetter, Harald Cramér (1893-1985), July 87. *ISI Review*, 56, 1988, 89-97.

193. R. LePage, Predicting transforms of stable noise, July 87.

194. R. LePage and B.M. Schreiber, Strategies based on maximizing expected log, July 87.

195. J. Rosinski, Series representations of infinitely divisible random vectors and a generalized shot noise in Banach spaces, July 87.

196. J. Szulga, On hypercontractivity of α-stable random variables, $0<\alpha<2$, July 87.

197. I. Kuznezova-Sholpo and S.T. Rachev, Explicit solutions of moment problems I, July 87. *Probability Math. Statist.*, 10, 1989, to appear.

198. T. Hsing, On the extreme order statistics for a stationary sequence, July 87.

199. T. Hsing, Characterization of certain point processes, Aug. 87. *Stochastic Proc. Appl.* 26, 1987, 297-316.

200. J.P. Nolan, Continuity of symmetric stable processes, Aug. 87.

201. M. Marques and S. Cambanis, Admissible and singular translates of stable processes, Aug. 88.

202. O. Kallenberg, One-dimensional uniqueness and convergence results for exchangeable processes, Aug. 87.

203. R.J. Adler, S. Cambanis and G. Samorodnitsky, On stable Markov processes, Sept. 87.

204. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations driven by nuclear space valued martingales, Sept. 87. *Appl. Math. Optimization*, 17, 1988, 237-272.

205. R.L. Smith, Approximations in extreme value theory, Sept. 87.

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b. RESTRICTIVE MARKINGS:

2a. SECURITY CLASSIFICATION AUTHORITY: NA

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: NA

3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for Public Release; Distribution Unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 237

5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 88-1008

6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina

6b. OFFICE SYMBOL (If applicable):

6c. ADDRESS (City, State and ZIP Code): Statistics Department, CB #3260, Phillips Hall, Chapel Hill, NC 27599-3260

7a. NAME OF MONITORING ORGANIZATION: AFOSR/NM

7b. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, DC 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR

8b. OFFICE SYMBOL (If applicable): NM

8c. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, DC

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620 85 C 0144

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
6.1102F	2304	A6	leave blnk

11. TITLE (Include Security Classification): A counterexample concerning the extremal index

12. PERSONAL AUTHOR(S): Smith, R.L.

13a. TYPE OF REPORT: preprint

13b. TIME COVERED: FROM 9/1/87 TO 8/31/88

14. DATE OF REPORT (Yr., Mo., Day): 1988, August

15. PAGE COUNT: 6

16. SUPPLEMENTARY NOTATION:

17. COSATI CODES

FIELD	GROUP	SUB. GR.
XXX	XXXXXXXXXX	XX

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Key words and phrases: N/A

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The concept of an extremal index, which is a measure of local dependence amongst the exceedances over a high threshold by a stationary sequence, has a natural interpretation as the reciprocal of mean cluster size. ~~We exhibit~~ a counterexample which shows that this interpretation is not necessarily correct.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL: Major Brian W. Woodruff

22b. TELEPHONE NUMBER (Include Area Code): (202) 767-5026

22c. OFFICE SYMBOL: AFOSR/NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

A COUNTEREXAMPLE CONCERNING THE EXTREMAL INDEX

Richard L. Smith

University of Surrey*

and

University of North Carolina at Chapel Hill

Summary

The concept of an extremal index, which is a measure of local dependence amongst the exceedances over a high threshold by a stationary sequence, has a natural interpretation as the reciprocal of mean cluster size. We exhibit a counterexample which shows that this interpretation is not necessarily correct.

December 1987

Supported by Air Force Office of Scientific Research;

Grant Number: F 49620 85C 0144

* Permanent address: Department of Mathematics, University of Surrey, Guildford, GU2 5XH, U.K.

Let $\{\xi_n, \ n=0,1,2,\ldots\}$ denote a stationary sequence and define $M_n = \max(\xi_1,\ldots,\xi_n)$. Under suitable conditions it is possible to prove results of the form

$$nP(\xi_1 > u_n) \to \tau \quad \Longleftrightarrow \quad P\{M_n \le u_n\} \to e^{-\theta\tau} \quad (0<\tau<\infty) \tag{1}$$

where $0 \le \theta \le 1$. The parameter θ was termed the extremal index by Leadbetter (1983), though the concept had occured earlier in papers of Newell (1964), Loynes (1965), O'Brien (1974a, 1974b) and Davis (1982). For a general overview of extremes in stationary sequences, see Leadbetter, Lindgren and Rootzén (1983).

It is possible to define an exceedance point process N_n on $(0,1]$, such that $N_n(s,t]$ is the number of exceedances of the level u_n among $\{\xi_r: ns < r \le nt\}$. Convergence of $\{N_n\}$ as $n\to\infty$ is studied by Hsing, Hüsler and Leadbetter (1988). One of their main results is that, if a limiting point process exists, then it must be compound Poisson. The atoms of this limiting process correspond to clusters of exceedances. Somewhat parallel results have also been obtained by Alpuim (1987).

A natural interpretation of θ is that $1/\theta$ is the main cluster size in the limiting point process. Hsing et al. were not, however, able to prove this without making additional assumptions. The following example shows that the result is false without such assumptions.

The example is a regenerative sequence of the form

$$\xi_n = \zeta_j \quad \text{for} \quad \sum_{0}^{j-1} N_i \le n < \sum_{0}^{j} N_i \quad (j \ge 1) \tag{2}$$

where

(i) ζ_j, $j\ge1$ are independent with a common distribution function F satisfying $F(1) = 0$,

(ii) For $j>1$, given $N_1, \ldots, N_{j-1}$, $\zeta_1, \ldots, \zeta_j$ with $m \le \zeta_j < m+1$, the probability of the event $N_j = i$ is q_{mi}. Here $\{q_{mi}, m \ge 1, i \ge 1\}$ is a sequence of probabilities with $q_{mi} \ge 0$, $\Sigma_i \, q_{mi} = 1$ for each m.

In words, the process remains in state ζ_i for a random number of time epochs determined by the probability distribution $q_{mi}(i \ge 1)$ with $m = [\zeta_i]$, and then moves to a new state which is independently chosen from F.

Let $p_m = P(m \le \zeta_j < m-1)$, $\mu_m = \Sigma_i \, i q_{mi}$ and suppose $\mu = \Sigma \, p_m \, \mu_m < \infty$. Then μ is the mean recurrence time of the process. The process may be made stationary by a suitable choice of distribution of N_1. It may also be regarded as a function of a Harris chain, and may therefore be treated by extreme value arguments of O'Brien (1987) and Rootzén (1987).

Now let us specify $\{q_{mi}\}$ to be

$$q_{mi} = \begin{cases} (m-1)/m, & i=1, \\ 1/m, & i=m+1, \\ 0, & \text{otherwise}. \end{cases} \qquad (3)$$

Then $\mu_m = 2$ for all m, and so $\mu = 2$ also. Let $\{u_n, n \ge 1\}$ be a sequence of thresholds such that $n \, P\{\zeta_1 > u_n\} \to \tau$, $0 < \tau < \infty$.

<u>Proposition 1</u> $\theta = \frac{1}{2}$.

<u>Proof</u>. By Theorem 3.1 of Rootzén (1987), for $\delta > 0$ and $\delta' = \delta + 1/n$ we have

$$\left| P\{M_n \le x\} - P\{\zeta_1 \le x\}^{n/\mu} \right| \le 2\delta' + P\{ \left| \nu_n/n - 1/2 \right| \ge \delta\} \qquad (4)$$

where ν_n is the number of regenerations up to time n. By choosing $\delta = \delta_n \to 0$

appropriately and setting $x = u_n$, we deduce

$$P\{M_n \leq x\} \to e^{-\tau/2}$$

as required.

Proposition 2. The exceedance point process N_n converges to a simple Poisson process of intensity $\frac{1}{2}$, as $n \to \infty$.

Proof. This follows from Theorem 4.2 of Hsing, Hüsler and Leadbetter (1988). Positive recurrent Harris chains are strong mixing (cf. O'Brien 1987), and hence satisfy the mixing condition $\Delta(u_n)$ used by Hsing et al. For a suitable sequence $\{r_n\}$ satisfying $r_n \to \infty$, $r_n/n \to 0$, let

$$\pi_n(j) = P\{\sum_{i=1}^{r_n} X_{n,i} = j \mid \sum_{i=1}^{r_n} X_{n,i} > 0\},\ j=1,2,\ldots \tag{5}$$

where $X_{n,i}$ is 1 if $\xi_i > u_n$, 0 otherwise. Let $\pi(j) = \lim_{n\to\infty} \pi_n(j)$ for $j=1,2,.$. The theorem of Hsing, Hüsler and Leadbetter asserts that the point process N_n converges to a compound Poisson process with compounding distribution $\pi(\cdot)$.

However, under (3) it is easy to see that $\pi(j) = 1$ for $j=1$, 0 for $j>1$. Hence the limiting process in this case is simple Poisson, with a mean cluster size of 1. This completes our description of the example.

From a statistical point of view, the most natural way to estimate the extremal index is via the point process of high-level exceedances. Such a procedure was in fact proposed by Smith (1984). The example here reveals a possible fallacy in this procedure, though it may not be possible to do much about it in practice.

Remarks.

1. What is going wrong is the lack of tightness of the sequence $\{q_{mi}, i\geq 1\}$ as $m\to\infty$. In a similar way it is possible to construct an example of the same phenomenon for any $\theta<1$. Hsing et al. show that the extremal index is in general given by $\theta = \lim_n \{\Sigma_j \pi_n(j)\}^{-1}$, so what is at issue is whether

$$\Sigma\, j\, \pi_n(j) \to \Sigma\, j\, \pi(j) \quad \text{as} \quad n\to\infty . \tag{6}$$

This is false for the example considered here.

2. It is possible to exhibit other kinds of extremal behaviour by taking other sequences $\{q_{mi}\}$. For instance, if $\mu_m \to \infty$ we very easily obtain an example of a process with extremal index 0. Taking this one step further, if $q_{mi} \to 0$ as $m\to\infty$ for each i but the distribution $\{q_{mi}, i\geq 1\}$ converges under some renormalisation to the distribution of a continuous random variable as $m\to\infty$ (example: take $q_{mi} = 1/m$ for $i=1,2,\dots,m$), then the point process N_n does not converge but a suitably renormalised sequence converges to a compound Poisson point process with continuous compounding distribution. Such behaviour is admitted in the general theory of Hsing, Hüsler and Leadbetter, but they do not give any examples.

Acknowledgement. This work was carried out during a visit to the Center for Stochastic Processes, University of North Carolina. I am grateful to Ross Leadbetter, Tailen Hsing and Teresa Alpuim for helpful discussions.

REFERENCES

Alpuim, M.T. (1987), High-level exceedances in stationary processes with index. Preprint, University of Lisbon.

Davis, R.A. (1982), Limit laws for the maximum and minimum of stationary sequences. Z. Wahrsch.v.Geb. 61, 31-42.

Hsing, T., Hüsler, J. and Leadbetter, M.R. (1988), Limits for exceedance point processes. To appear, Prob.Thy.Rel.Fields.

Leadbetter, M.R. (1983), Extremes and local dependence in stationary sequences. Z.Wahrsch.v.Geb. 65, 291-306.

Leadbetter, M.R., Lindgren, G. and Rootzén, H. (1983), Extremes and Related Properties of Random Sequences and Series. Springer Verlag, New York.

Loynes, R.M. (1965), Extreme values in uniformly mixing stationary stochastic processes. Ann.Math.Statist. 36, 993-999.

Newell, G.F. (1964), Asymptotic extremes for m-dependent random variables. Ann.Math.Statist. 35, 1322-1325.

O'Brien, G.L. (1974), Limit theorems for the maximum term of a stationary process. Ann.Probab. 2, 540-545.

O'Brien, G.L. (1974b), The maximum term of uniformly mixing stationary processes. Z.Wahrsch.v.Geb. 30, 57-63.

O'Brien, G.L. (1987), Extreme values for stationary and Markov sequences. Ann.Probab. 15, 281-291.

Rootzén, H. (1987), Maxima and exceedances of stationary Markov chains. Adv.Appl.Prob., to appear.

Smith, R.L. (1984), Threshold methods for sample extremes. In Statistical Extremes and Applications, ed. J. Tiago de Oliveira, Reidel, Dordrecht, 621-638

206. E. Willekens, Estimation of convolution tails, Sept. 87.

207. J. Rosinski, On path properties of certain infinitely divisible processes, Sept. 87.

208. A.H. Korezlioglu, Computation of filters by sampling and quantization, Sept. 87.

209. J. Bather, Stopping rules and observed significance levels, Sept. 87.

210. S.T. Rachev and J.E. Yukich, Convolution metrics and rates of convergence in the central limit theorem, Sept. 87. *Ann. Probability*, to appear.

211. M. Fujisaki, Normed Bellman equation with degenerate diffusion coefficients and its applications to differential equations, Oct. 87.

212. G. Simons, Y.C. Yao and X. Wu, Sequential tests for the drift of a Wiener process with a smooth prior, and the heat equation, Oct. 87.

213. R.L. Smith, Extreme value theory for dependent sequences via the Stein-Chen method of Poisson approximation, Oct. 87.

214. C. Houdré, A vector bimeasure integral with some applications, June 88 (Revised).

215. M.R. Leadbetter, On the exceedance random measures for stationary processes, Nov. 87.

216. M. Marques, A study on Lebesgue decomposition of measures induced by stable processes, Nov. 87 (*Dissertation*).

217. M.T. Alpuim, High level exceedances in stationary sequences with extremal index, Dec. 87. *Stochastic Proc. Appl.*, to appear.

218. R.F. Serfozo, Poisson functionals of Markov processes and queueing networks, Dec. 87.

219. J. Bather, Stopping rules and ordered families of distributions, Dec. 87.

220. S. Cambanis and M. Maejima, Two classes of self-similar stable processes with stationary increments, Jan. 88.

221. H.P. Hucke, G. Kallianpur and R.L. Karandikar, Smoothness properties of the conditional expectation in finitely additive white noise filtering, Jan. 88. *J. Multivariate Anal.*, to appear.

222. I. Mitoma, Weak solution of the Langevin equation on a generalized functional space, Feb. 88.

223. L. de Haan, S.I. Resnick, H. Rootzén and C. de Vries, Extremal behaviour of solutions to a stochastic difference equation with applications to arch-processes, Feb. 88.

224. O. Kallenberg and J. Szulga, Multiple integration with respect to Poisson and Lévy processes, Feb. 88.

225. D.A. Dawson and L.G. Gorostiza, Generalized solutions of a class of nuclear space valued stochastic evolution equations, Feb. 88.

226. G. Samorodnitsky and J. Szulga, An asymptotic evaluation of the tail of a multiple symmetric α-stable integral, Feb. 88.

227. J.J. Hunter, The computation of stationary distributions of Markov chains through perturbations, Mar. 88.

228. H.C. Ho and T.C. Sun, Limiting distribution of nonlinear vector functions of stationary Gaussian processes, Mar. 88.

229. R. Brigola, On functional estimates for ill-posed linear problems, Apr. 88.

230. M.R. Leadbetter and S. Nandagopalan, On exceedance point processes for stationary sequences under mild oscillation restrictions, Apr. 88.

231. S. Cambanis, J. P. Nolan and J. Rosinski, On the oscillation of infinitely divisible processes, Apr. 88.

232. G. Hardy, G. Kallianpur and S. Ramasubramanian, A nuclear space-valued stochastic differential equation driven by Poisson random measures, Apr. 88.

233. D.J. Daley, T. Rolski, Light traffic approximations in queues (II), May 88.

234. G. Kallianpur, I. Mitoma, R.L. Wolpert, Diffusion equations in duals of nuclear spaces, July 88.

235. S. Cambanis, Admissible translates of stable processes: A survey and some new models, July 88.

236. E. Platen, On a wide range exclusion process in random medium with local jump intensity, Aug. 88.

237. R.L. Smith, A counterexample concerning the extremal index, Aug. 88.

238. G. Kallianpur and I. Mitoma, A Langevin-type stochastic differential equation on a space of generalized functionals, Aug. 88.