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Calculation of the Power Density in the Fredholm Equation That Yields a Weibull pdf

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The Fredholm equation has been solved for the power density that yields a Weibull distribution for all values of parameter alpha. The closed-form analytical solution for the power level density function pdf of the averaged clutter return is found by using an asymptotic expansion. The Fredholm equation is used to generate a Weibull distribution, and the results are compared to the actual Weibull distribution. There is particularly good agreement in the tail region of the distribution.

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CONTENTS

CALCULATION OF THE POWER DENSITY IN THE FREDHOLM EQUATION THAT YIELDS A WEIBULL pdf

1. INTRODUCTION

Detection in non-Gaussian noise is a very difficult subject to study in radar. To maximize detection probability, the optimum filter is the Neyman-Pearson likelihood ratio test. It is a difficult problem to obtain the multivariate probability density function required for generating the optimum filter. Several detectors for detecting a signal in non-Gaussian noise have been found by Cantrell [1,2] and by Martinez et al. [3]. These detectors were obtained by applying the Neyman-Pearson test after a closed-form solution of the multivariate probability density function had been found.

This report considers clutter that has a non-Gaussian distribution. Specifically, clutter has a greater probability of having a large value than the Rayleigh model obtained from a Gaussian distribution, and thus it requires a higher tailed distribution to model it properly. This higher tail must be properly accounted for in the design of a detector to avoid false alarms. Generally, the Rayleigh pdf underestimates the amplitudes obtained from real clutter. One physical model that has been proposed to describe the non-Rayleigh nature of sea clutter amplitude statistics is the composite surface scattering model [4-6] that describes the fluctuation of clutter amplitudes by a conditional Rayleigh pdf, conditioned on a varying clutter power level. The overall non-Rayleigh pdf for the clutter amplitude is given by

$$g(|x|) = \int_0^\infty \frac{|x|}{\sigma^2} \exp\left[-\left(\frac{|x|^2}{e\sigma^2}\right)\right] f(\sigma^2) d\sigma^2, \tag{1}$$

where $g(|x|)$ is the non-Rayleigh pdf of clutter amplitude return $|x|$ and $f(\sigma^2)$ is the pdf of clutter power level σ^2. This is a Fredholm equation of the first kind for the unknown pdf $f(\sigma^2)$. Equation (1) will be shown to be in the form of a Laplace transform.

In principle, the averaged clutter pdf $f(\sigma^2)$ can be found by inverting the transform if $g(|x|)$ is known. Selecting $g(|x|)$ to be a Weibull distribution is expected to better represent real clutter. Inverting the transform when $g(|x|)$ is a Weibull density, is very difficult, so an approximate solution was found. The inverse was approximated by employing the method of steepest descent to evaluate the Laplace inverse contour integral.

In Section 1, we discuss the nature of a spherically invariant random process (**SIRP**) and derive Eq. (1) by noting that the clutter envelope is a **SIRP**. An asymptotic expansion technique is applied to the inversion of the Laplace transform. In Section 2, we discuss the difference between our results, which is a mixture of Rayleigh pdf's, and the Weibull pdf. Also we generate random samples from the mixture of Rayleigh pdf's and the Weibull pdf, and we compare the result of these two pdfs.

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2. NATURE OF A SIRP AND DERIVATION OF EQ. (1)

A SIRP can be expressed as a random mixture of n-dimensional Gaussian-probable densities. Given a random vector $x = (x_1, x_2, \ldots, x_n)$ from the random process, $RP = [X(t), t \epsilon T]$ with mean value μ and covariance matrix K, a necessary and sufficient condition for the random vector to be from a SIRP is for the probability density to be of the form

$$p_n(x) = (2\pi)^{-\frac{n}{2}} \, |K|^{\frac{-1}{2}} \, h_n[(x-\mu)^T K^{-1} (x-\mu)],$$

where $h_n(\,.\,)$ is a quadratic form. It can be shown [6] that the pdf of the clutter envelope associated with a SIRP must satisfy

$$g(|x|) = \int_0^\infty \frac{|x|}{(\sigma^2)} e^{-\frac{|x|^2}{2(\sigma^2)}} f(\sigma^2)\, d\sigma^2. \tag{2}$$

We now assume a Weibull pdf for the clutter envelope; i.e.,

$$g(|x|) = \alpha \ln(2) \left[\frac{|x|}{M}\right]^{\alpha-1} \frac{1}{M} e^{-\left[\ln(2)\left(\frac{|x|}{M}\right)^{\alpha}\right]}. \tag{3}$$

Substituting Eq. (3) into Eq. (2) and letting $t = \frac{1}{\sigma^2}$ and $s = \frac{|x|^2}{2}$ yields

$$\int_0^\infty \frac{\sqrt{2s}}{t} e^{-st} f\left(\frac{1}{t}\right) d\left(\frac{1}{t}\right) = \alpha \ln(2) \left[\frac{1}{M}\right]^{\alpha} [2s]^{\frac{\alpha-1}{2}} e^{-\left[\frac{\ln(2)(2s)^{\frac{\alpha}{2}}}{(M)^{\alpha}}\right]}$$

Equating $h(t) = \frac{1}{t} f\left(\frac{1}{t}\right)$, $A = \alpha \ln(2) \left[\frac{1}{M}\right]^{\alpha} 2^{\frac{\alpha-2}{2}}$, and $B = \frac{\ln(2)}{(M)^{\alpha}} 2^{\frac{\alpha}{2}}$ yields

$$\int_0^\infty e^{-st} h(t)dt = A\, s^{\frac{\alpha-2}{2}} \exp(-Bs^{\frac{\alpha}{2}}) \tag{4}$$

This is in the form of a Laplace transform. An inversion of the Laplace transform in Eq. (4) yields

$$h(t) = \begin{cases} \frac{A}{2\pi i}\int_{c-i\infty}^{c+i\infty} s^{\frac{\alpha-2}{2}} \exp(-(Bs^{\frac{\alpha}{2}} - st))ds & 0 < \alpha < 2 \\ \frac{A}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(-(Bs - st))ds & \alpha = 2 \end{cases} \tag{5}$$

We can easily show that the clutter amplitude pdf Eq. (3) is Rayleigh for $\alpha = 2$. In this report, we are interested in the cases $0 < \alpha < 2$, since we are interested in a heavier tailed distribution than the Rayleigh distribution.

To solve the inversion of the Laplace transform (Eq. (5)), we derived an asymptotic expansion of the integral to find an approximation solution. Some sufficient conditions that should be met are discussed in the Appendix. Since in our problem all sufficient conditions are met, we can generate an approximation solution. In Eq. (5) for the case $0 < \alpha < 2$, let

$$q(s) = s^{\frac{\alpha - 2}{2}}, \quad p(s) = \frac{Bs^{\frac{\alpha}{2}}}{t} - s. \tag{6}$$

Then

$$h(t) = \frac{A}{2\pi i}\int_{c - i\infty}^{c + i\infty} q(s) \exp(-tp(s))ds. \qquad 0 < \alpha < 2 \tag{7}$$

If the singularities of $q(s)$ all take the form of isolated poles and/or isolated branch points, then by a suitable deformation of the inversion contour we may reduce the integral to the sum of residues at the poles plus a sum of loop integrals around the branch points. A contour is shown in Fig. 1. Equation (7) reduces to an integral on the interval $(-\infty, +\infty)$.

$$h(t) = \frac{A}{2\pi i}\int_{-\infty}^{+\infty} q(s) \exp(-tp(s))ds, \qquad 0 < \alpha < 2.$$

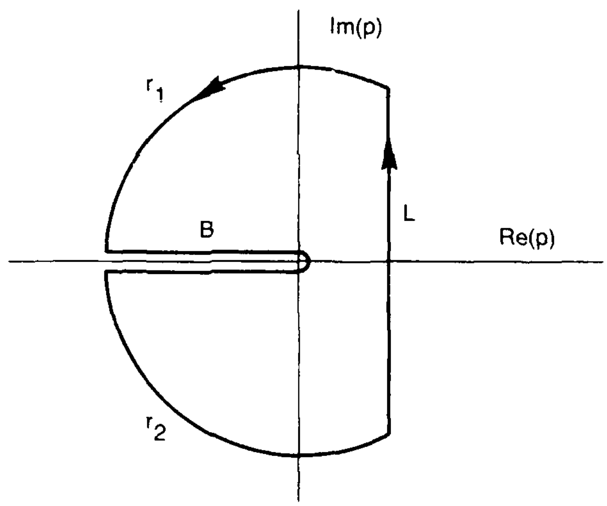

Fig. 1 — Contour

Letting

$$I(t) = \int_{-\infty}^{\infty} q(s)e^{-tp(s)}ds. \tag{8}$$

$$h(t) = \frac{AI(t)}{2\pi i}.$$

The peak value of the factor $e^{-tp(s)}$ is located at s_0 the minimum of $p(s)$. When t is large this peak is very sharp, and the overwhelming contribution to the integral comes from the neighborhood of s_0. It is therefore reasonable to approximate $p(s)$ and $q(s)$ by the leading terms in their power series expansion at s_0. $I(t)$ is evaluated by extending the integration limits to $(-\infty,\infty)$. The result is Laplace's approximation to $I(t)$. This solution will also be used for small t and any discrepancies in the resulting pdf, and the assumed Weibull one will be noted.

We will expand $p(s)$ in a Taylor-series. The first two terms will be sufficient since $e^{-tp(s)}$ has a sharp peak value at s_0. One obtains

$$I(t) \approx q(s_0)e^{-tp(s_0)} \int_{-\infty}^{\infty} e^{\frac{-t}{2}(s - s_0)^2 p''(s_0)} ds \tag{9}$$

We will now evaluate the right-hand integral. Let $J = \frac{t}{2}p''(s_0)$ and $x = s - s_0$. Since

$$\int_{-\infty}^{\infty} e^{-Jx^2} dx = \sqrt{\frac{\pi}{J}},$$

the integral in Eq. (9) equals

$$\int_{-\infty}^{\infty} e^{-\frac{t}{2}(s - s_0)^2 p''(s_0)} ds = \left(\frac{2\pi}{tp''(s_0)}\right)^{\frac{1}{2}}.$$

Since $p'(s) = 0$ yields the minimum, from Eq. (6) $s_0 = \left(\frac{2t}{B\alpha}\right)^{\frac{2}{\alpha - 2}}$.

At s_0 we have $q(s_0) = \left(\frac{2t}{B\alpha}\right)$. After substituting for the value of s_0 and $p''(s)$ in Eq. (9), the general solution becomes

$$I(t) = i\left(\frac{2t}{\alpha B}\right)\left(\frac{8\pi}{\alpha(2 - \alpha)B}\left(\frac{2t}{\alpha B}\right)^{\left(\frac{4 - \alpha}{\alpha - 2}\right)}\right)^{\frac{1}{2}}$$

$$\times \exp\left(-\left(B\left(\frac{2t}{\alpha B}\right)^{\frac{\alpha}{\alpha - 2}} - t\left(\frac{2t}{\alpha B}\right)^{\frac{2}{\alpha - 2}}\right)\right).$$

The probability density function is given by

$$f(\sigma^2) = th(t) = \frac{At}{2\pi i} I(t) = \frac{A}{2\pi\sigma^2 i} I\left(\frac{1}{\sigma^2}\right).$$

Letting $U = \frac{2}{\alpha B}$, the probability density function of clutter power that yields a Weibull density becomes

$$f(\sigma^2) = \frac{AU}{2\pi(\sigma^2)^2} \left[\frac{8\pi}{\alpha(2-\alpha)B} \left(\frac{U}{\sigma^2} \right)^{\left(\frac{4-\alpha}{\alpha-2}\right)} \right]^{\frac{1}{2}}$$

$$\times \exp \left[- \left[B \left(\frac{U}{\sigma^2} \right)^{\frac{\alpha}{\alpha-2}} - \frac{1}{\sigma^2} \left(\frac{U}{\sigma^2} \right)^{\frac{2}{\alpha-2}} \right] \right],$$

where

$$A = \alpha \ln(2) \left[\frac{1}{M} \right]^{\alpha} 2^{\frac{\alpha-2}{2}}, \quad B = \frac{\ln(2)}{M^{\alpha}} 2^{\frac{\alpha}{2}}.$$

Given $U = \frac{1}{A}$, one more step to simplify the equation becomes

$$f(\sigma^2) = \frac{1}{\sigma^2} \left[\frac{1}{\pi(2-\alpha)U} \left(\frac{U}{\sigma^2} \right)^{\frac{\alpha}{\alpha-2}} \right]^{1/2} \exp \left[\frac{\alpha-2}{\alpha} \frac{1}{\sigma^2} \left(\frac{U}{\sigma^2} \right)^{\frac{2}{\sigma^2}} \right]. \tag{10}$$

3. DISCUSSION

In Section 1, we obtained the probability density function of the clutter power level that allows Weibull clutter to be modeled as a SIRP. The SIRP may be used to describe the non-Rayleigh distribution including the correlation properties.

The approximation solution from Section 1 shows that the Weibull pdf may be modeled as an infinite mixture of Rayleigh pdfs. To examine the accuracy of this result, we compare the pdf generated by numerically solving Eq. (1) to the Weibull pdf. Specifically, we compare the probability of exceeding $|x|$, for the two pdfs, for various values of parameter alpha between 0 and 2, any value of median, and any value of $|x|$.

The actual values of Weibull and the Rayleigh mixture are observed to be slightly different at the beginning of the curves as shown in Fig. 2. We integrated the closed-form solution (Eq. (1)) from zero to infinity. This integration was performed numerically by subdividing the integral into small intervals and summing the appropriately weighted values.

For alpha = 0.5, the mixture of Rayleigh pdfs is lower than the Weibull pdf, since a large enough upper limit was not used for the numerical integration. However, the curve can be made to approach the Weibull pdf if we use a good upper limit. Even though this closed-form solution is based on an approximation, it is seen to be very close to the Weibull pdf, especially when $\alpha = 1$, where it matched almost perfectly. The pdf is particularly good at the tail of the distribution. For values of 3 or greater of $|x|$, the two curves overlapped each other all the way to a probability of 10^{-7}.

We also generated histograms based on the two pdfs. To generate a histogram, we used a transformation method to generate the desired distribution of σ^2, solve the differential equation, and take the inverse of the function ζ. It is necessary first to generate a sample from the power level pdf

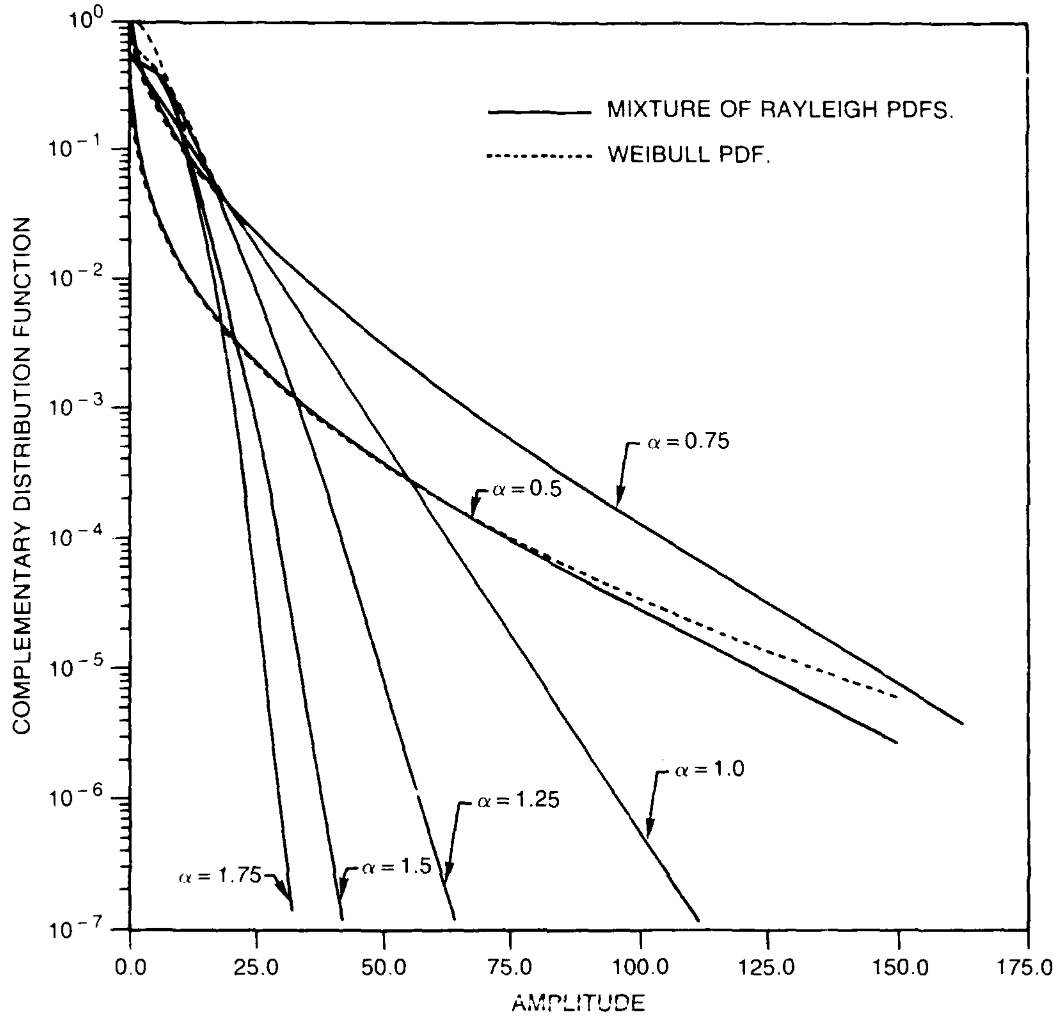

Fig. 2 — Probabilities of mixture of Rayleigh pdfs and Weibull pdf

$f\ (\sigma^2)$, then to generate a sample from the Rayleigh pdf by using the same σ^2 of the power level pdf. The histogram that results from these samples is the histogram of the mixture of Rayleigh pdfs.

Histograms with 10,000 samples, are shown in Figs 3 and 4 for cases of $\alpha = 0.75$ and $\alpha = 1.0$ respectively, along with the Weibull pdf histogram for 10,000 samples. The mixture of Rayleigh pdfs histogram is seen to be very similar to the Weibull pdf histogram. Also, we calculated the cumulative distribution function of power level pdf, which indicates that the area under the curve is 0.999302. It is very close to 1. In Figs. 5 to 7, we show the density function for the closed-form solution and the Weibull pdf.

These results could be used for the problem of detecting targets in non-Gaussian noise, as described by Cantrell [1,2], and by Martinez [3]. To obtain similar results, a closed-form non-Gaussian multivariate pdf is required. This requires a closed-form expression for the density that is obtained by averaging the Gaussian multivariate over the power density function found in this report. This problem is still open for solution.

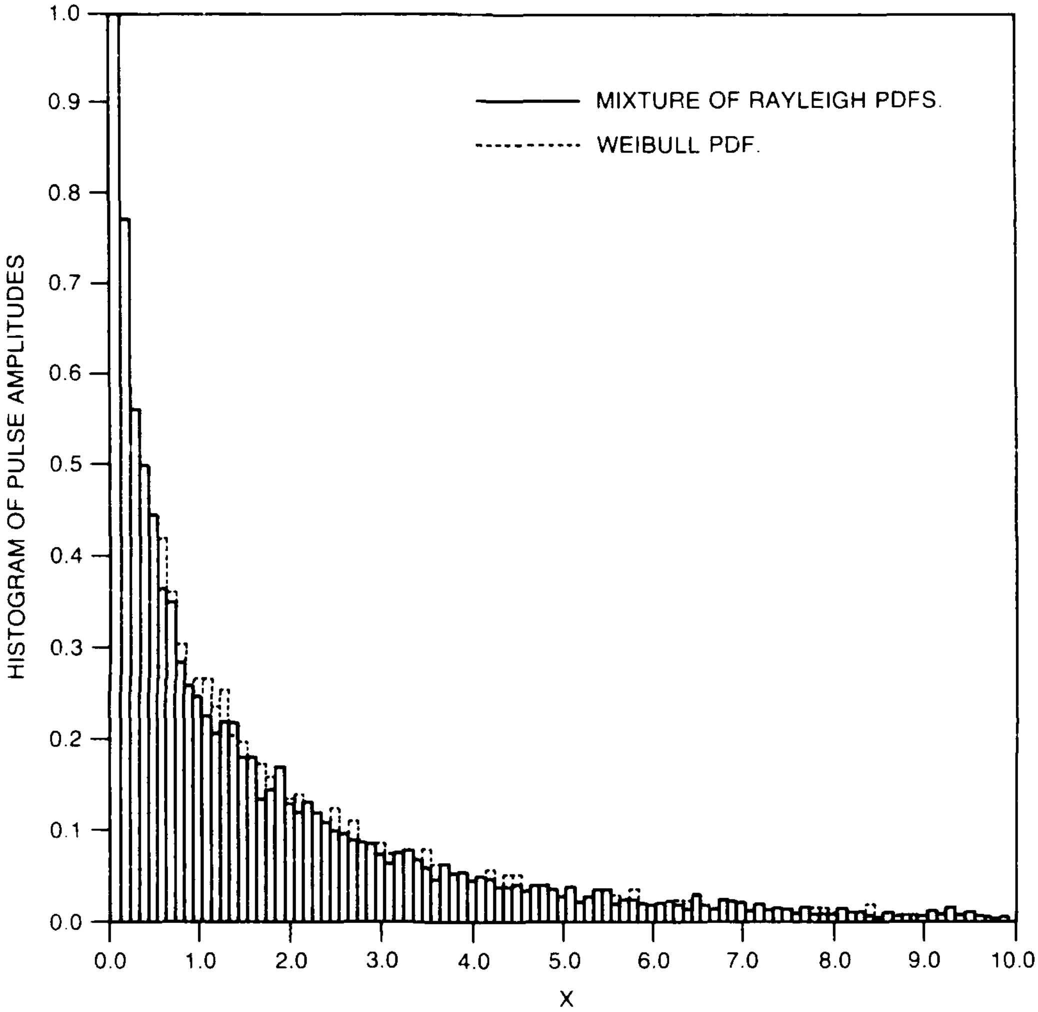

Fig. 5 — Histograms of Rayleigh pdfs and Weibull bdf when parameter alpha = 0.75, and median = 1.0

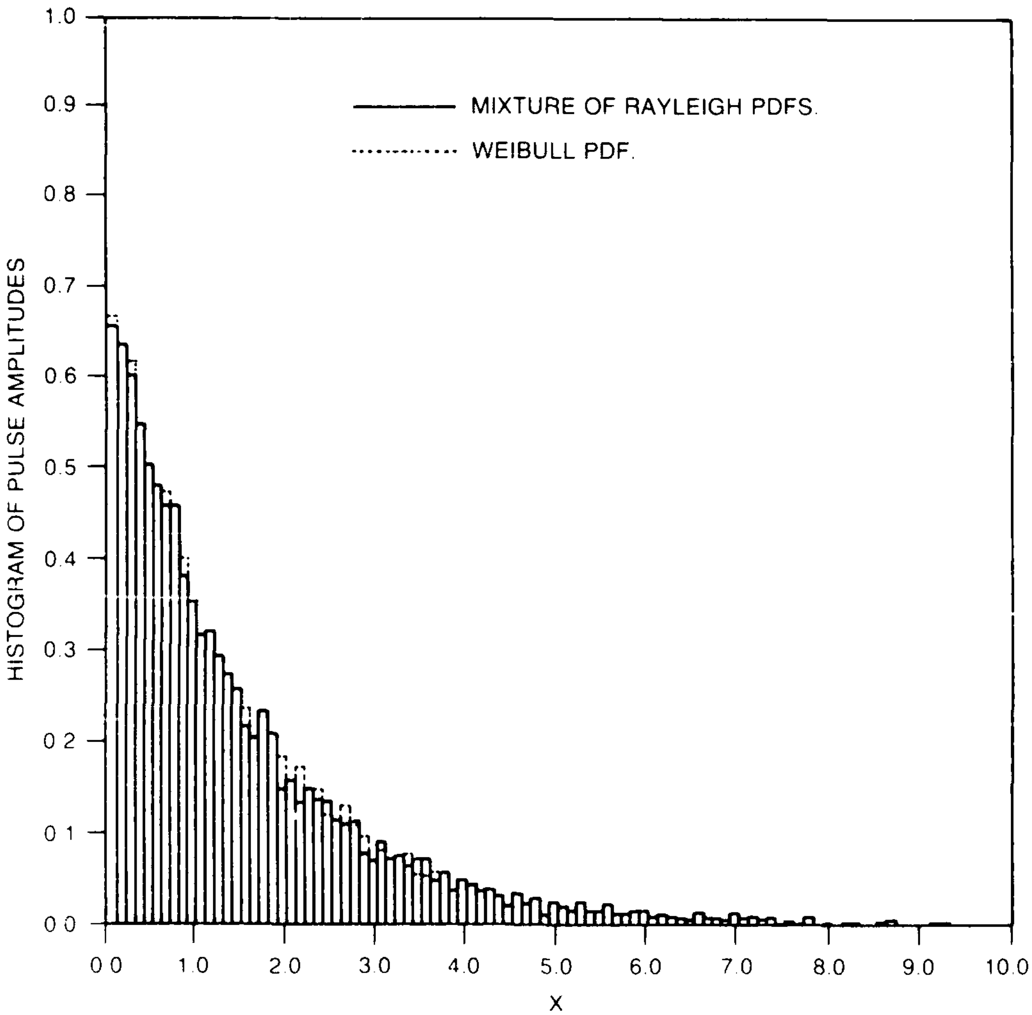

Fig. 4 Histograms of Rayleigh pdfs. and Weibull pdf. when parameter alpha = 1.0, and median = 1.0

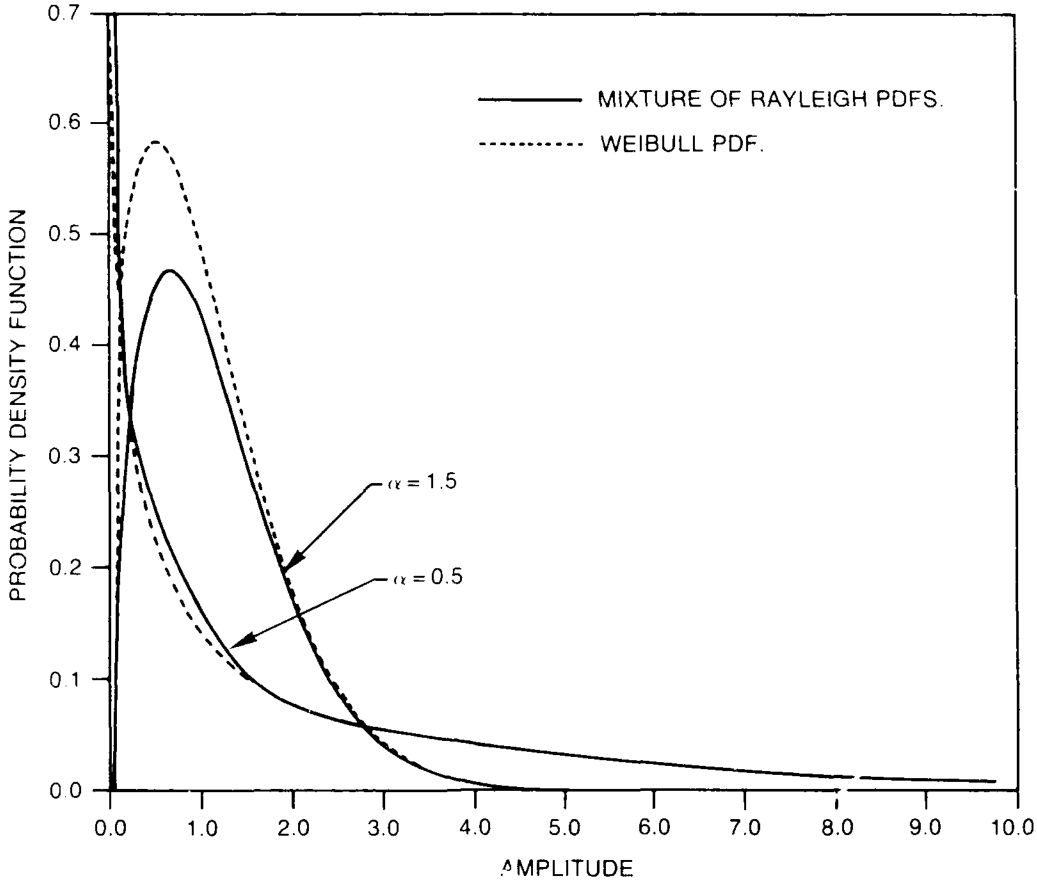

Fig. 5 — Probabilities of sum of 2 pdfs and Weibull pdf

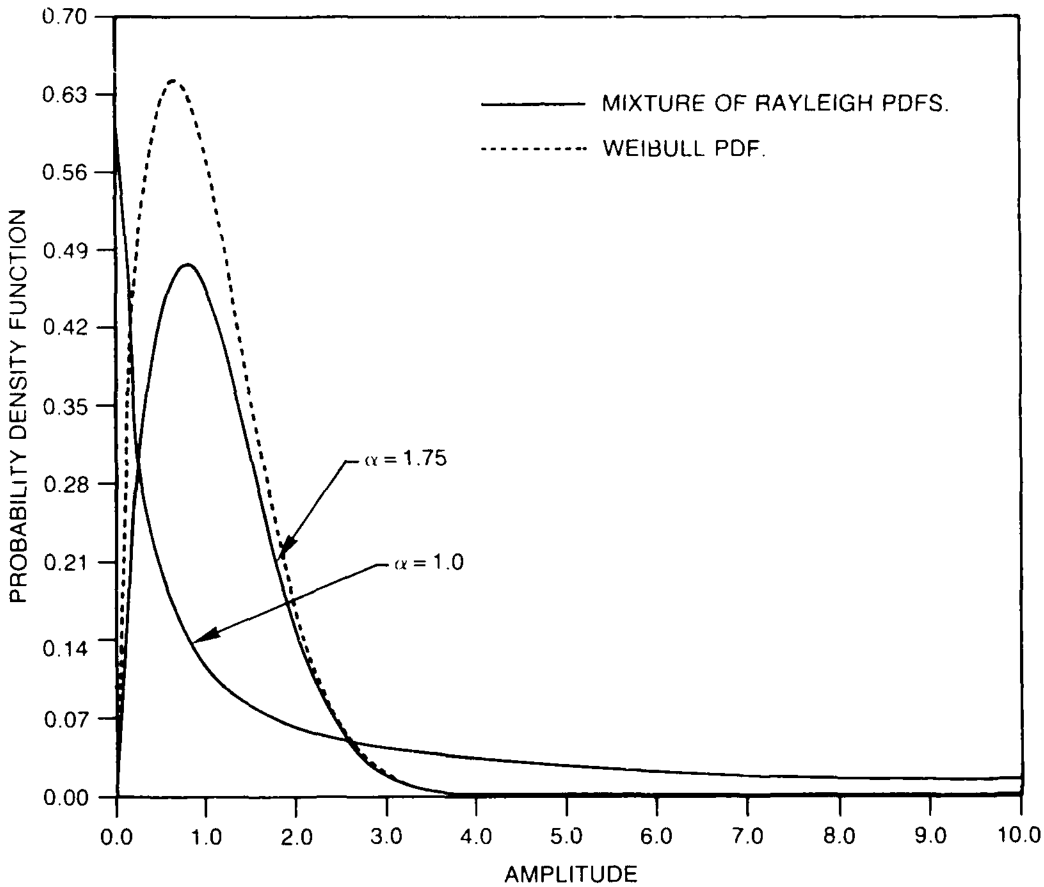

Fig. 6 — Probabilities of Sum of 2 pdfs and Weibull pdf

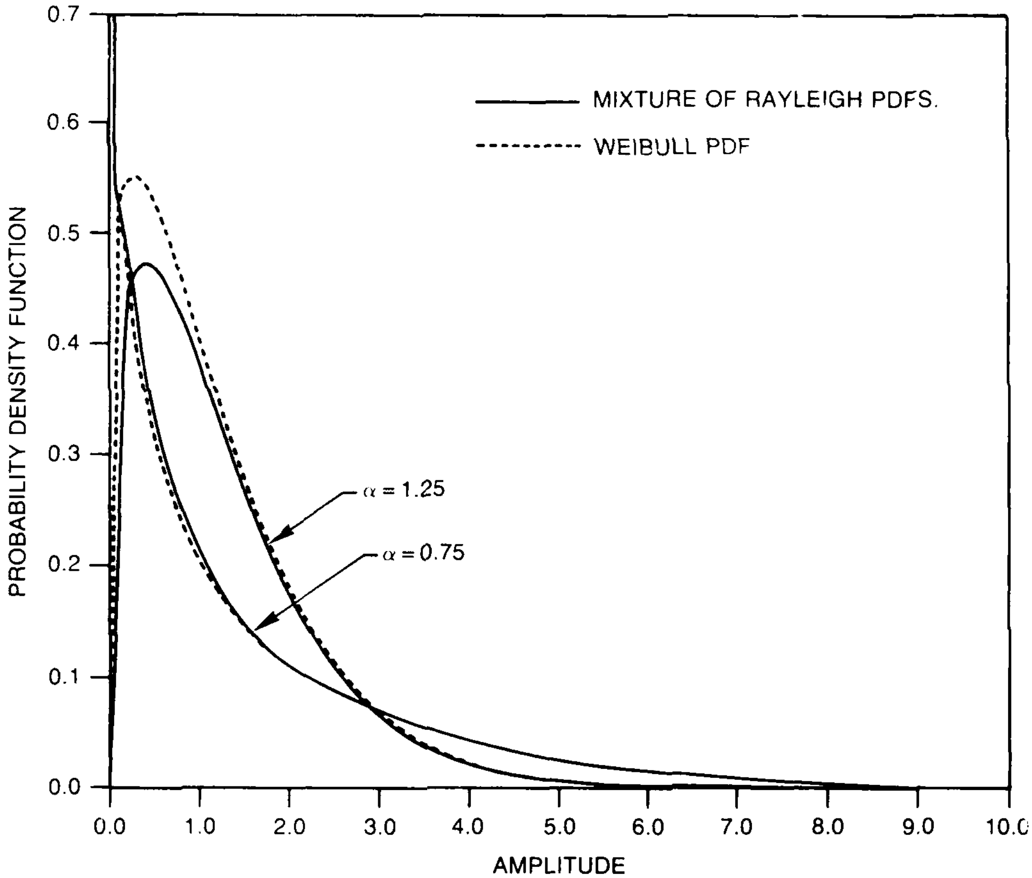

Fig. 7 — Probabilities of sum of 2 pdfs and Weibull pdf

4. SUMMARY

An analytical solution for a mixture of Rayleigh pdfs of the averaged clutter return is found by using an asymptotic expansion method. The mixture of Rayleigh pdfs fits the amplitude fluctuation of the Weibull density. It is a very good fit at the tail of the distribution, a fact that allows a detection threshold to be set for a probability of false alarm equal to 10^{-7}.

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APPENDIX

The expansion technique described in Ref. 16 is an effective way of deriving the asymptotic expansion of an integral containing a parameter. A general type of integral amenable to the method of approximation is given by

$$I(x) = \int_{\infty}^{\infty} \exp\,(-xp(t))q(t)dt\ , \tag{A-1}$$

where x is a positive parameter and the function $q(t)$ is independent of the parameter x. The peak value of the factor $e^{-xp(t)}$ is located at the minimum t_0, and almost all the contribution to the integral comes from the neighborhood of t_0. Equation (8) is exactly the same as Eq. (A-1). Thus it is reasonable to approximate $p(t)$ and $q(t)$ by leading terms in their ascending power series expansions at t_0. Also, we need to show that $q(t)$ is infinitely differentiable by taking the derivative in $[\,0,\infty\,)$ for the sufficient conditions. By taking the derivative infinitely from Eq. (6), we still get the function of g that satisfies one sufficient condition.

$$q^{(n)}(t) = O(e^{\lambda t}), \quad 0 \leq t < \infty$$

where $O(e^{\lambda t})$ is an expression of infinitely differentiable function. Also λ is a real constant that is independent of t. The integral converges when $x > \lambda$ and equals

$$I(x) = \frac{q(t_0)}{x} + \frac{q^1(t_0)}{x^2} + \ldots + \frac{q^{(n-1)}(t_0)}{x^n} + \epsilon_n(x)$$

where

$$\epsilon_n(x) = \frac{1}{x^n}\int_0^{\infty} \exp\,(-xp(t))q^n(t)dt = O\left\{\frac{1}{n^2(x-\lambda)}\right\}$$

and n is an arbitrary nonnegative integer. Therefore

$$I(x) \approx \sum_{s=0}^{\infty} \frac{q^{(s)}(t_0)}{x^{s+1}}, \quad x \to \infty$$

since the maximum value of $|q^{(n)}(t)|$ is attained at t= t_0

$$|\epsilon_n(x)| = |q^{(n)}(t_0)|\,x^{-n-1}.$$

Calculating the error terms by using the above equation yields

$$\epsilon_n(x) - \epsilon_{n+1}(x) = q^{(n)}x^{-n-1}.$$