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PERCENTAGE POINTS FOR THE FISHER-COCHRAN TEST FOR EQUALITY OF VARIANCES

BY

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1. Introduction.

A well known test for equality of normal population variances, based on sample variances, was introduced by Cochran (1941). Suppose σ_i^2, $i = 1, \cdots, n$ represents the population variances of n normal populations, and let independent sample variances, each based on k degrees of freedom, be s_i^2, $i = 1, \cdots, n$. Suppose the s^2 are ranked, so that the *ordered* variables are $s_{(1)}^2, s_{(2)}^2, \cdots, s_{(n)}^2$. To test H_0: the σ_i^2 are all equal (suppose the common value is σ^2), Cochran (1941) introduced the test statistic

$$Z = \frac{s_{(n)}^2}{\sum_{i=1}^{n} s_i^2}, \tag{1}$$

which compares the largest sample variance with the sum of the sample variances. Clearly the intent is to discover if one variance is an outlier (too large), and, in general, H_0 will be rejected for large values of Z. However, excessively small values of Z could be useful also: they will suggest that in some way the sample variances are more homogeneous than should be expected from a set of normal variances all from equi-variable populations, an event which might occur if samples with widely disperse variances have been omitted from a set of data. Such an event would be important, for example, in monitoring quality or process control.

Cochran gave some distributional results for Z, and these were used by Eisenhart and Solomon (1947) to construct tables for the upper 5% and upper 1% points, 95*th* and 99*th* percentiles to supplement the very brief table presented by Cochran in his article. Another table with 5% and 1% points, for $n \leq 20$ and for slightly different values of k, was given by Yamauti (1972). The Eisenhart and Solomon tables have been frequently reproduced and we now augment those tables in what follows by producing the additional percentiles listed in Table 1 for $n = 2(1)10, 12(2)20, 25(5)50, 60, 120$ and $k = 1(1)20, 30$. When $k = 2$, the test statistic serves as the basis of a significance test for any particular term in the harmonic analysis of a series as was demonstrated by Fisher (1929) who also provided a brief table of 95% and 99% points. In another paper, Fisher (1940) provided another brief table that also gave 95% values for the largest and second largest fractions. Tables for the Fisher problem, namely $k = 2$ were supplemented extensively by Nowroozi (1967) but to our knowledge more extensive tables for other values of k have not been produced. It might be useful to have such points, and we give them in the tables below.

An obvious application would be to process control or quality control. Historically, this was mostly examined using, say, the mean of a sample taken daily, but it has become increasingly the practice to examine the variance also, for stability of the process. Thus, for example, the seven variances in a week might be examined to see if these were homogeneous, using the test given below. One might not always wish to be limited to comparing the largest variance with the total, and, for example, the two largest of the week or the four largest of a monthly set of variances, might be tested as too large. The theory given below can be adapted to provide such a test, and work is in progress to provide tables for this more general situation. The test procedure is given in Section 2, followed by theoretical results in Section 3.

2. Test for Equality of n Normal Population Variances.

The test of H_0 thus proceeds as follows.

1. Suppose for the ith population the sample variance s_i^2 is given, (the unbiased estimator of σ_i^2), based on k degrees of freedom.
2. Calculate Z from (1).
3. Reject H_0 given in Section 1 at significance level α if $Z > Z_\alpha$, where Z_α is given in Table 1, for appropriate values n, k, and α.

3′. On occasion, H_0 might be rejected at level $1 - \alpha$ if Z is *smaller* than Z_α.

Table 1 has been constructed, for values $Z_\alpha > 0.5$, from an exact formula for upper tail probabilities, given by Cochran (1941) and used by Eisenhart and Solomon (1947): for critical values $Z_\alpha < 0.5$ Pearson curve approximations have been used. The techniques used, and comments on the accuracy of the tables, are given in Section 3.

3. Theory of the Tests

3.1. Calculation of critical points $Z_\alpha > 0.5$. Define $r_j = s_j^2/\sum_i s_i^2$. Cochran (1941) showed that the probability $P(Z > g)$ is given by

$$P(g) = nP_1(g) - \frac{n(n-1)}{2}P_2(g) + \frac{n(n-1)(n-2)}{3!}P_3(g)\cdots \qquad (2)$$

where $P_1(g)$ is the probability that any one ratio r_j exceeds g, $P_2(g)$ is the probability that two of the ratios both exceed g, etc., and observed that the upper tail probabilities $P(g)$ will be exactly $nP_1(g)$ when g exceeds 0.5. Eisenhart and Solomon (1947) showed limits for the accuracy of approximating $P(g)$ by $nP_1(g)$ for lower values of g. $P_1(g)$ is given by an incomplete Beta function (Cochran, 1941):

$$P_1(g) = \frac{\int_g^1 X^{k/2-1}(1-x)^{\frac{k(n-1)}{2}-1}dx}{B(\frac{k}{2}, \frac{k(n-1)}{2})} \qquad 0 \le g \le 1 \qquad (3)$$

where $B(\cdot,\cdot)$ is the Beta function. $P_1(g)$ can also be evaluated from tables of the F distribution. In Table 1, $P_1(Z_\alpha)$ has been used to give critical values Z_α, when these are greater than 0.5.

3.2. Pearson curve approximations for $Z_\alpha < 0.5$. For smaller values of Z_α corresponding to higher significance levels, we have approximated the distribution of Z by Pearson curves. For this, the first four moments of $Z^{1/2}$ are used.

Suppose Z is constructed as follows:

(a) Let $y_1, y_2, \cdots, y_n$ be i.i.d. random variables, each with the distribution $\sigma^2\chi_k^2$, where σ^2 is any positive value; let $y_{(1)} < y_{(2)} < \cdots < y_{(n)}$ be the order statistics of the set y_i.

(b) Let $Y = \Sigma_j y_j$.

(c) Then $Z = y_{(n)}/Y$.

It is clear that the distribution of Z is independent of σ, the scale parameter of y_i; also Y is a completely sufficient statistic for σ^2. Thus, by the Basu/Hogg/Craig Theorem,

Z and Y are independently distributed. We can henceforth assume that $\sigma = 1$. Then $ZY = y_{(n)}$, and we have

$$E(Z^r) = \frac{E(y_{(n)}^r)}{E(Y^r)}. \tag{4}$$

where $E(\cdot)$ denotes expectation. The denominator of (5) is easy to find, since Y is a χ^2-variable with kn degrees of freedom: then

$$E(Y^r) = \frac{2^r\Gamma\{(kn+2r)/2\}}{\Gamma(kn/2)} \tag{5}$$

For the distribution of $y_{(n)}$ suppose $G(t)$ is the distribution of χ_k^2; the distribution of $y_{(n)}$ is then $[G(t)]^n = P(y_{(n)} < t)$, and moments are given by

$$E(y_{(n)}^r) = \int_0^\infty t^r n(G(t))^{n-1} g(t)dt \tag{6}$$

where $g(t)$ is the density of χ_k^2.

Thus the moments of Z or of $Z^{1/2}$ are very easy to calculate, from (4) using (5) and (6). The first four moments of $Z^{1/2}$ have been found and used to fit Pearson curves (see Solomon and Stephens, 1978) to the distribution of $Z^{1/2}$ and hence to obtain significance points Z_α for Z.

3.3. The case when $k = 1$.

For $k = 1$, when each sample variance has only one degree of freedom, there is an interesting connection with a distribution in the statistics of directions. Suppose P is a point uniformly distributed on the n-sphere with center 0 and radius 1. It is well-known (Marsaglia, 1972) that a method to generate such points P is as follows. Generate $w_1, w_2, \cdots, w_n$ i.i.d. from $N(0,1)$, and calculate $X_i = w_i/Y^{1/2}$ where $Y = \sum_j w_j^2$; then X_i, $i = 1, \cdots, n$ are the components of vector OP. Let $S_i = |X_i|$; then $Z = S_{(n)}^2$, that is, Z, when $k = 1$, is the square of the largest component of a random unit vector on an n-sphere. We can then get some distributional results for Z by finding the distribution of $S_{(n)}$.

$k = 1, n = 2$. For example, when $n = 2$, OP can be fixed by the angle θ it makes with the x-axis, and θ is uniformly distributed, between 0 and 2π, written $U(0, 2\pi)$. For the distribution of $S_{(n)}$ we can confine attention to the first quadrant, $0 < \theta < \frac{\pi}{2}$; then when $0 < \theta < \frac{\pi}{4}$, $S_{(n)} = \cos\theta$ and when $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, $S_{(n)} = \sin\theta$. It is easily shown that the density $f_S(x)$ of $S_{(n)}$ is given by

$$f_S(x) = \frac{4}{\pi\sqrt{1-x^2}}, \qquad \frac{1}{\sqrt{2}} \le x \le 1. \tag{7}$$

The moments $\mu'_r = E(S^r_{(n)})$ are $\int_c^1 x^r f_S(x)dx$, where $c = 1/\sqrt{2}$; these can be easily calculated to give

$$\mu'_1 = 0.90032, \quad \mu'_2 = 0.81831, \quad \mu'_3 = 0.75026, \quad \mu'_4 = 0.69331.$$

Check with formula (2). From (2) and (3), with $k = 1$, $n = 2$, we have for $g > 0.5$,

$$P(g) = \frac{2}{\pi}\int_g^1 x^{-1/2}(1-x)^{-1/2}dx,$$

since $B(\frac{1}{2}, \frac{1}{2}) = \pi$. Let $x = t^2$ and we have

$$P(Z > g) = P(S_{(2)} > \sqrt{g})$$
$$= \frac{2}{\pi}\int_{\sqrt{g}}^2 2(1-t^2)^{-1/2}dt = 2 - \frac{4}{\pi}\sin^{-1}\sqrt{g}, \quad g \ge 0.5.$$

This is the same result as obtained by integrating (7).

Of course, for $n = 2$ there is a simple correspondence with the F-test for equality of two variances, since $Z^{-1} = 1 + s^2_{(1)}/s^2_{(2)}$. It quickly follows that z_α, the critical value of Z at level α, is related to $F_{k,k}(\alpha/2)$.

For $n = 3$, the algebra is more complicated. Let vector OP, where P is now uniformly distributed on the first orthant of the unit sphere (that is, all coordinates of OP are positive), have usual spherical coordinates θ, ϕ. Then $z = \cos\theta$ is $U(0,1)$, and $\phi = U(0, \frac{\pi}{2})$, so that, if we use rectangular axes for (ϕ, z), probability is uniform on the rectangle R : $0 < z < 1, 0 < \phi < \frac{\pi}{2}$. We now want Pr(maximum component of $OP < t) = P(t)$. This is found as follows.

$k = 1$, $n = 3$, $t \ge \frac{1}{\sqrt{2}}$. If $z > t$, and $t \ge \frac{1}{\sqrt{2}}$, it is clear that z must be the maximum component, for all ϕ. Thus $P(z > t) = 1 - t$. By symmetry, $x = \sin\theta\cos\phi$ or $y = \sin\theta\sin\phi$ (the other two components of OP) could be the maximum component with equal probability, so that P(maximum component $> t) = 3(1-t)$.

Equations (2) and (3), for $n = 3$, $k = 1$, $g > 0.5$, give

$$P(Z > g) = 3\int_g^1 x^{-1/2}dx/B(\frac{1}{2}, 1) = 3(1-\sqrt{g});$$

thus $P(S_{(3)} > \sqrt{g}) = 3(1 - \sqrt{g})$, in agreement with the result above.

$k = 1$, $n = 3$, $\frac{1}{\sqrt{3}} \leq t \leq \frac{1}{\sqrt{2}}$. Then $P(S_{(3)} < t) = P(\text{all 3 components} < t)$; this is given by the probability over an area bounded by (1) $z = t$; (2) $\sqrt{1-z^2}\cos\phi = t$; (3) $\sqrt{1-z^2}\sin\phi = t$, roughly in the middle of rectangle R. Thus

$$P(S_{(3)} < t) = \frac{4}{\pi}\int_{\phi_1}^{\pi/4}(t - z)d\phi$$

where $\cos\phi_1 = t/\sqrt{1-t^2}$ and where $\sqrt{(1-z^2)}\cos\phi = t$; so

$$P(S_{(3)} < t) = \frac{4}{\pi}\int_{\phi_1}^{\pi/4}\{t - \sqrt{1 - t^2\sin^2\phi}\}d\phi, \qquad \frac{1}{\sqrt{3}} \leq t \leq \frac{1}{\sqrt{2}}.$$

This last expression must be evaluated numerically.

Similar ideas can be used to give upper tail results for higher values of n, for $k = 1$, but again they lead in the end to integrals which must be evaluated numerically.

3.4. Accuracy of Table 1.

Various checks on accuracy have been made for the points in Table 1. In fitting the Pearson curves, the fit was made to $Z^{1/2}$, which, when $k = 1$, is $S_{(n)}$. The numerator of $S_{(n)}$ is the largest absolute value of a set of n standard normals, and the expectation of this variable is known (*Biometrika Tables for Statisticians*, Vol. 2), since absolute values of standard normals are used in analysis of experiments when main effects and interactions are plotted on half-normal plots (see, for example, Bennett and Franklin, 1954). These known values enabled a check to be made on the accuracy of (6), for $k = 1$ and $r = \frac{1}{2}$. Also, for $k = 1$, $n = 2$, the moments of $S_{(n)}$ could be compared with the exact values given above.

In addition, the exact distributions given above have been used, for $n = 2$ and $n = 3$, to check the percentage points. A further check was made, for $k = 2$, by comparing the values with those given by Nowroozi (1967). Finally, the most extensive check on the Pearson curve points was made by comparing the points Z_α with those given by the exact formula (2), when $Z_\alpha > 0.5$. It was found that the Pearson curve fits were very accurate, with occasional differences from the exact values in the fourth, or sometimes the third, decimal place: however, these differences will make negligible error in the α-value corresponding to the given point Z_α.

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TABLE 1

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
2	1	0.5078	0.5196	0.5392	0.578[illegible]	0.8536	0.9938	0.9985	0.9996	0.9999
2	2	0.5050	0.5125	0.5250	0.5500	0.7500	0.9500	0.9750	0.9875	0.9950
2	3	0.5039	0.5098	0.5196	0.5393	0.7020	0.9027	0.9392	0.9619	0.9794
2	4	0.5033	0.5083	0.5167	0.5334	0.6737	0.8647	0.9057	0.9340	0.9586
2	5	0.5030	0.5074	0.5147	0.5295	0.6545	0.8347	0.8773	0.9083	0.9373
2	6	0.5027	0.5067	0.5133	0.5267	0.6406	0.8107	0.8534	0.8857	0.9172
2	7	0.5024	0.5061	0.5123	0.5246	0.6298	0.7911	0.8332	0.8659	0.8988
2	8	0.5023	0.5057	0.5114	0.5229	0.6212	0.7747	0.8159	0.8486	0.8823
2	9	0.5021	0.5054	0.5107	0.5215	0.6140	0.7607	0.8010	0.8335	0.8674
2	10	0.5020	0.5051	0.5102	0.5204	0.6080	0.7486	0.7880	0.8200	0.8540
2	11	0.5019	0.5048	0.5097	0.5194	0.6029	0.7381	0.7765	0.8080	0.8118
2	12	0.5019	0.5046	0.5092	0.5185	0.5984	0.7287	0.7662	0.7972	0.8307
2	14	0.5017	0.5043	0.5085	0.5171	0.5910	0.7129	0.7487	0.7785	0.8113
2	16	0.5016	0.5040	0.5080	0.5160	0.5850	0.7000	0.7341	0.7629	0.7948
2	18	0.5015	0.5038	0.5075	0.5150	0.5801	0.6892	0.7219	0.7497	0.7807
2	20	0.5014	0.5035	0.5071	0.5142	0.5759	0.6799	0.7114	0.7382	0.7684
2	25	0.5013	0.5032	0.5063	0.5127	0.5678	0.6616	0.6904	0.7152	0.7435
2	30	0.5012	0.5029	0.5058	0.5116	0.5619	0.6480	0.6747	0.6978	0.7243
2	40	0.5010	0.5025	0.5050	0.5100	0.5535	0.6286	0.6522	0.6728	0.6966
2	50	0.5009	0.5022	0.5045	0.5089	0.5478	0.6153	0.6366	0.6553	0.6771

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
3	1	0.4195	0.4291	0.4444	0.4739	0.6945	0.9344	0.9669	0.9834	0.9933
3	2	0.3850	0.3955	0.4096	0.4337	0.5918	0.8174	0.8709	0.9087	0.9423
3	3	0.3712	0.3813	0.3942	0.4150	0.5452	0.7430	0.7977	0.8405	0.8832
3	4	0.3630	0.3728	0.3849	0.4038	0.5173	0.6934	0.7457	0.7883	0.8335
3	5	0.3594	0.3684	0.3793	0.3963	0.4990	0.6578	0.7070	0.7483	0.7933
3	6	0.3555	0.3643	0.3747	0.3906	0.4847	0.6307	0.6770	0.7166	0.7606
3	7	0.3544	0.3624	0.3719	0.3864	0.4731	0.6093	0.6531	0.6909	0.7335
3	8	0.3519	0.3598	0.3690	0.3828	0.4642	0.5918	0.6333	0.6695	0.7107
3	9	0.3498	0.3576	0.3666	0.3799	0.4567	0.5772	0.6167	0.6514	0.6912
3	10	0.3498	0.3569	0.3652	0.3776	0.4501	0.5647	0.6025	0.6358	0.6743
3	11	0.3477	0.3549	0.3632	0.3754	0.4449	0.5540	0.5902	0.6223	0.6595
3	12	0.3477	0.3544	0.3622	0.3737	0.4399	0.5446	0.5794	0.6103	0.6463
3	14	0.3462	0.3526	0.3599	0.3706	0.4320	0.5289	0.5613	0.5902	0.6241
3	16	0.3451	0.3512	0.3581	0.3682	0.4256	0.5161	0.5465	0.5738	0.6058
3	18	0.3445	0.3502	0.3567	0.3662	0.4202	0.5056	0.5343	0.5601	0.5906
3	20	0.3440	0.3494	0.3555	0.3645	0.4157	0.4966	0.5239	0.5484	0.5775
3	25	0.3412	0.3466	0.3526	0.3611	0.4072	0.4787	0.5036	0.5256	0.5518
3	30	0.3395	0.3449	0.3506	0.3587	0.4008	0.4656	0.4877	0.5086	0.5327
3	40	0.3386	0.3433	0.3483	0.3552	0.3917	0.4477	0.4668	0.4844	0.5057
3	50	0.3385	0.3425	0.3468	0.3529	0.3854	0.4355	0.4526	0.4683	0.4874

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
4	1	0.3549	0.3664	0.3825	0.4106	0.5981	0.8533	0.9065	0.9406	0.9676
4	2	0.3175	0.3287	0.3427	0.3648	0.5000	0.7076	0.7679	0.8158	0.8643
4	3	0.3022	0.3126	0.3249	0.3437	0.4555	0.6286	0.6839	0.7305	0.7814
4	4	0.2929	0.3028	0.3141	0.3309	0.4277	0.5787	0.6287	0.6721	0.7212
4	5	0.2880	0.2970	0.3072	0.3224	0.4085	0.5438	0.5894	0.6297	0.6761
4	6	0.2835	0.2922	0.3018	0.3159	0.3945	0.5178	0.5598	0.5973	0.6410
4	7	0.2815	0.2893	0.2982	0.3111	0.3834	0.4973	0.5365	0.5716	0.6129
4	8	0.2783	0.2861	0.2947	0.3070	0.3747	0.4807	0.5175	0.5506	0.5897
4	9	0.2761	0.2837	0.2919	0.3037	0.3675	0.4670	0.5018	0.5330	0.5702
4	10	0.2755	0.2824	0.2900	0.3010	0.3612	0.4556	0.4878	0.5181	0.5536
4	11	0.2733	0.2802	0.2878	0.2985	0.3560	0.4455	0.4761	0.5052	0.5392
4	12	0.2728	0.2793	0.2864	0.2965	0.3513	0.4370	0.4663	0.4931	0.5266
4	14	0.2707	0.2769	0.2835	0.2930	0.3436	0.4225	0.4496	0.4744	0.5154
4	16	0.2692	0.2750	0.2813	0.2902	0.3375	0.4109	0.4361	0.4593	0.5054
4	18	0.2681	0.2736	0.2795	0.2879	0.3324	0.4014	0.4250	0.4469	0.4736
4	20	0.2667	0.2721	0.2779	0.2859	0.3281	0.3931	0.4155	0.4362	0.4616
4	25	0.2643	0.2694	0.2747	0.2821	0.3198	0.3773	0.3971	0.4155	0.4382
4	30	0.2642	0.2684	0.2729	0.2793	0.3134	0.3661	0.3841	0.4007	0.4209
4	40	0.2619	0.2657	0.2697	0.2754	0.3048	0.3499	0.3653	0.3795	0.3969
4	50	0.2616	0.2646	0.2680	0.2728	0.2988	0.3392	0.3529	0.3654	0.3805

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
5	1	0.3124	0.3242	0.3398	0.3656	0.5354	0.7745	0.8360	0.8831	0.9293
5	2	0.2744	0.2852	0.2981	0.3181	0.5319	0.7783	0.8413	0.8868	0.9279
5	3	0.2584	0.2682	0.2795	0.2963	0.3932	0.6239	0.6838	0.7341	0.7885
5	4	0.2488	0.2580	0.2682	0.2832	0.3666	0.5462	0.5981	0.6437	0.6957
5	5	0.2434	0.2517	0.2609	0.2743	0.3483	0.4653	0.5440	0.5850	0.6329
5	6	0.2386	0.2466	0.2553	0.2677	0.3350	0.4408	0.5063	0.5435	0.5875
5	7	0.2361	0.2433	0.2513	0.2627	0.3245	0.4220	0.4557	0.5123	0.5531
5	8	0.2329	0.2400	0.2477	0.2585	0.3162	0.4067	0.4380	0.4670	0.5259
5	9	0.2306	0.2374	0.2448	0.2551	0.3094	0.3940	0.4234	0.4507	0.5038
5	10	0.2295	0.2358	0.2426	0.2523	0.3034	0.3835	0.4113	0.4371	0.4688
5	11	0.2273	0.2337	0.2404	0.2498	0.2986	0.3743	0.4006	0.4251	0.4555
5	12	0.2266	0.2325	0.2388	0.2477	0.2941	0.3665	0.3916	0.4150	0.4438
5	14	0.2244	0.2299	0.2358	0.2441	0.2869	0.3533	0.3764	0.3978	0.4244
5	16	0.2224	0.2278	0.2334	0.2412	0.2811	0.3427	0.3641	0.3841	0.4088
5	18	0.2212	0.2262	0.2315	0.2388	0.2763	0.3340	0.3541	0.3728	0.3959
5	20	0.2194	0.2244	0.2297	0.2368	0.2724	0.3265	0.3454	0.3631	0.3851
5	25	0.2180	0.2222	0.2267	0.2329	0.2644	0.3126	0.3292	0.3448	0.3640
5	30	0.2165	0.2203	0.2244	0.2300	0.2586	0.3022	0.3173	0.3313	0.3486
5	40	0.2142	0.2176	0.2211	0.2260	0.2506	0.2878	0.3006	0.3125	0.3272
5	50	0.2138	0.2164	0.2192	0.2233	0.2449	0.2783	0.2897	0.3001	0.3127

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
6	1	0.2814	0.2930	0.3078	0.3316	0.4843	0.7141	0.7807	0.8324	0.8828
6	2	0.2435	0.2538	0.2657	0.2838	0.3902	0.5591	0.6162	0.6658	0.7218
6	3	0.2277	0.2367	0.2470	0.2620	0.3475	0.4842	0.5321	0.5752	0.6258
6	4	0.2179	0.2263	0.2356	0.2490	0.3223	0.4388	0.4791	0.5182	0.5635
6	5	0.2122	0.2199	0.2282	0.2401	0.3049	0.4080	0.4439	0.4771	0.5195
6	6	0.2074	0.2147	0.2225	0.2336	0.2923	0.3852	0.4176	0.4478	0.4852
6	7	0.2048	0.2114	0.2185	0.2286	0.2824	0.3677	0.3975	0.4253	0.4597
6	8	0.2015	0.2079	0.2148	0.2244	0.2747	0.3534	0.3811	0.4069	0.4390
6	9	0.1991	0.2054	0.2119	0.2210	0.2682	0.3418	0.3676	0.3918	0.4220
6	10	0.1979	0.2036	0.2097	0.2182	0.2627	0.3321	0.3565	0.3792	0.4076
6	11	0.1957	0.2015	0.2075	0.2157	0.2581	0.3236	0.3466	0.3682	0.3952
6	12	0.1947	0.2001	0.2058	0.2136	0.2539	0.3164	0.3383	0.3589	0.3845
6	14	0.1925	0.1975	0.2028	0.2101	0.2472	0.3043	0.3244	0.3432	0.3666
6	16	0.1906	0.1954	0.2004	0.2072	0.2416	0.2947	0.3133	0.3306	0.3524
6	18	0.1895	0.1940	0.1986	0.2049	0.2372	0.2868	0.3042	0.3204	0.3406
6	20	0.1879	0.1923	0.1968	0.2029	0.2335	0.2801	0.2964	0.3117	0.3308
6	25	0.1858	0.1896	0.1936	0.1990	0.2261	0.2673	0.2816	0.2950	0.3117
6	30	0.1837	0.1874	0.1911	0.1962	0.2208	0.2578	0.2707	0.2828	0.2979
6	40	0.1808	0.1842	0.1876	0.1921	0.2134	0.2447	0.2556	0.2658	0.2788
6	50	0.1804	0.1831	0.1858	0.1895	0.2082	0.2362	0.2458	0.2548	0.2659

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
7	1	0.2575	0.2687	0.2826	0.3048	0.4438	0.6599	0.7270	0.7814	0.8376
7	2	0.2201	0.2298	0.2408	0.2574	0.3532	0.5074	0.5612	0.6090	0.6644
7	3	0.2044	0.2128	0.2222	0.2359	0.3125	0.4360	0.4788	0.5203	0.5685
7	4	0.1944	0.2023	0.2109	0.2230	0.2886	0.3930	0.4297	0.4639	0.5080
7	5	0.1891	0.1961	0.2037	0.2144	0.2721	0.3641	0.3965	0.4268	0.4644
7	6	0.1843	0.1909	0.1980	0.2080	0.2602	0.3427	0.3719	0.3993	0.4335
7	7	0.1812	0.1874	0.1939	0.2030	0.2509	0.3264	0.3531	0.3781	0.4094
7	8	0.1783	0.1842	0.1904	0.1990	0.2435	0.3132	0.3379	0.3611	0.3901
7	9	0.1759	0.1816	0.1876	0.1957	0.2374	0.3024	0.3254	0.3471	0.3742
7	10	0.1749	0.1800	0.1855	0.1930	0.2321	0.2935	0.3152	0.3355	0.3609
7	11	0.1726	0.1778	0.1832	0.1905	0.2279	0.2856	0.3060	0.3253	0.3494
7	12	0.1714	0.1764	0.1815	0.1885	0.2240	0.2789	0.2983	0.3166	0.3395
7	14	0.1693	0.1739	0.1786	0.1850	0.2176	0.2678	0.2855	0.3022	0.3231
7	16	0.1674	0.1717	0.1762	0.1822	0.2125	0.2590	0.2753	0.2907	0.3100
7	18	0.1663	0.1702	0.1744	0.1800	0.2083	0.2517	0.2670	0.2813	0.2993
7	20	0.1647	0.1686	0.1726	0.1780	0.2048	0.2456	0.2599	0.2734	0.2903
7	25	0.1623	0.1658	0.1694	0.1742	0.1980	0.2338	0.2464	0.2581	0.2729
7	30	0.1608	0.1640	0.1671	0.1715	0.1929	0.2253	0.2366	0.2471	0.2603
7	40	0.1577	0.1607	0. 636	0.1676	0.1861	0.2133	0.2228	0.2318	0.2431
7	50	0.1572	0.1595	0.1618	0.1650	0.1812	0.2056	0.2140	0.2218	0.2314

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
8	1	0.2383	0.2491	0.2622	0.2829	0.4108	0.6138	0.6798	0.7352	0.7945
8	2	0.2016	0.2107	0.2210	0.2362	0.3236	0.4654	0.5157	0.5613	0.6152
8	3	0.1861	0.1940	0.2027	0.2152	0.2847	0.3972	0.4367	0.4735	0.5210
8	4	0.1765	0.1838	0.1916	0.2027	0.2618	0.3566	0.3902	0.4217	0.4610
8	5	0.1710	0.1775	0.1844	0.1942	0.2462	0.3293	0.3588	0.3866	0.4213
8	6	0.1663	0.1725	0.1790	0.1880	0.2349	0.3092	0.3357	0.3607	0.3920
8	7	0.1633	0.1689	0.1749	0.1831	0.2261	0.2939	0.3181	0.3408	0.3694
8	8	0.1606	0.1659	0.1715	0.1793	0.2191	0.2817	0.3040	0.3250	0.3514
8	9	0.1586	0.1636	0.1689	0.1761	0.2133	0.2717	0.2924	0.3120	0.3365
8	10	0.1569	0.1616	0.1666	0.1734	0.2084	0.2633	0.2827	0.3011	0.3241
8	11	0.1547	0.1595	0.1644	0.1710	0.2044	0.2559	0.2743	0.2916	0.3135
8	12	0.1535	0.1580	0.1627	0.1690	0.2008	0.2497	0.2671	0.2836	0.3043
8	14	0.1513	0.1555	0.1598	0.1657	0.1948	0.2395	0.2553	0.2702	0.2891
8	16	0.1494	0.1534	0.1575	0.1629	0.1900	0.2312	0.2458	0.2596	0.2770
8	18	0.1484	0.1520	0.1557	0.1608	0.1860	0.2246	0.2382	0.2510	0.2671
8	20	0.1468	0.1504	0.1540	0.1589	0.1827	0.2189	0.2317	0.2437	0.2588
8	25	0.1447	0.1478	0.1509	0.1552	0.1763	0.2081	0.2193	0.2297	0.2429
8	30	0.1430	0.1458	0.1487	0.1525	0.1716	0.2003	0.2103	0.2196	0.2313
8	40	0.1402	0.1428	0.1453	0.1488	0.1651	0.1892	0.1977	0.2056	0.2156
8	50	0.1391	0.1412	0.1433	0.1463	0.1606	0.1821	0.1895	0.1964	0.2050

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
9	1	0.2225	0.2328	0.2452	0.2645	0.3832	0.5742	0.6385	0.6936	0.7544
9	2	0.1865	0.1951	0.2047	0.2188	0.2991	0.4304	0.4761	0.5209	0.5727
9	3	0.1713	0.1787	0.1868	0.1983	0.2619	0.3653	0.4020	0.4362	0.4789
9	4	0.1622	0.1689	0.1761	0.1862	0.2401	0.3268	0.3577	0.3869	0.4235
9	5	0.1565	0.1625	0.1690	0.1779	0.2252	0.3009	0.3281	0.3536	0.3858
9	6	0.1523	0.1579	0.1637	0.1719	0.[illegible]144	0.2821	0.3064	0.3293	0.3581
9	7	0.1489	0.1541	0.1596	0.1672	0.2061	0.2677	0.2897	0.3106	0.3368
9	8	0.1463	0.1512	0.1563	0.1634	0.1995	0.2562	0.2765	0.2957	0.3199
9	9	0.1443	0.1489	0.1537	0.1603	0.1940	0.2468	0.2657	0.2835	0.3060
9	10	0.1426	0.1470	0.1515	0.1577	0.1894	0.2390	0.2566	0.2733	0.2944
9	11	0.1406	0.1449	0.1494	0.1554	0.1856	0.2321	0.2488	0.2645	0.2844
9	12	0.1393	0.1434	0.1477	0.1535	0.1822	0.2264	0.2421	0.2570	0.2758
9	14	0.1372	0.1410	0.1449	0.1502	0.1765	0.2168	0.2311	0.2446	0.2617
9	16	0.1356	0.1391	0.1428	0.1476	0.1720	0.2092	0.2224	0.2348	0.2505
9	18	0.1342	0.1375	0.1409	0.1455	0.1682	0.2030	0.2152	0.2268	0.2413
9	20	0.1329	0.1361	0.1393	0.1437	0.1651	0.1977	0.2092	0.2200	0.2337
9	25	0.1307	0.1335	0.1363	0.1401	0.1591	0.1877	0.1977	0.2071	0.2189
9	30	0.1289	0.1315	0.1341	0.1375	0.1547	0.1804	0.1894	0.1978	0.2083
9	40	0.1261	0.1285	0.1308	0.1339	0.1486	0.1702	0.1777	0.1848	0.1938
9	50	0.1257	0.1273	0.1291	0.1315	0.1443	0.1637	0.1704	0.1765	0.1840

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
10	1	[illegible]	0.2189	0.2308	0.2490	0.3596	0.5399	0.6020	0.6563	0.7175
10	2	[illegible]	0.1820	0.1911	0.2042	0.2786	0.4008	0.4438	0.4838	0.5358
10	3	0.1590	0.1660	0.1735	0.1843	0.2429	0.3385	0.3727	0.4048	0.4450
10	4	0.1501	0.1564	0.1631	0.1724	0.2220	0.3019	0.3306	0.3577	0.3919
10	5	0.1445	0.1502	0.1561	0.1644	0.2079	0.2774	0.3024	0.3261	0.3560
10	6	0.1400	0.1453	0.1509	0.1585	0.1976	0.2595	0.2818	0.3030	0.3298
10	7	0.1371	0.1420	0.1470	0.1540	0.1897	0.2460	0.2663	0.2854	0.3097
10	8	0.1345	0.1391	0.1439	0.1503	0.1834	0.2352	0.2538	0.2714	0.2938
10	9	0.1326	0.1369	0.1413	0.1474	0.1781	0.2264	0.2437	0.2600	0.2807
10	10	0.1308	0.1349	0.1391	0.1448	0.1738	0.2190	0.2351	0.2504	0.2698
10	11	0.1293	0.1332	0.1372	0.1427	0.1701	0.2127	0.2279	0.2423	0.2605
10	12	0.1276	0.1315	0.1354	0.1407	0.1669	0.2071	0.2215	0.2352	0.2525
10	14	0.1256	0.1291	0.1327	0.1376	0.1616	0.1982	0.2113	0.2236	0.2393
10	16	0.1242	0.1274	0.1307	0.1351	0.1572	0.1911	0.2032	0.2145	0.2288
10	18	0.1227	0.1258	0.1289	0.1330	0.1537	0.1853	0.1964	0.2070	0.2203
10	20	0.1214	0.1243	0.1273	0.1313	0.1508	0.1804	0.1908	0.2007	0.2131
10	25	0.1192	0.1218	0.1244	0.1279	0.1451	0.1710	0.1801	0.1887	0.1994
10	30	0.1176	0.1199	0.1222	0.1254	0.1409	0.1642	0.1723	0.1800	0.1896
10	40	0.1148	0.1169	0.1191	0.1219	0.1352	0.1547	0.1615	0.1680	0.1761
10	50	0.1138	0.1155	0.1172	0.1196	0.1312	0.1486	0.1546	0.1602	0.1671

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
12	1	0.1874	0.1966	0.2074	0.2237	0.3216	0.4836	0.5410	0.5927	0.6528
12	2	0.1540	0.1613	0.1693	0.1810	0.2458	0.3532	0.3916	0.4277	0.4727
12	3	0.1398	0.1460	0.1526	0.1620	0.2128	0.2959	0.3259	0.3544	0.3903
12	4	0.1313	0.1368	0.1427	0.1508	0.1936	0.2626	0.2876	0.3114	0.3415
12	5	0.1259	0.1309	0.1361	0.1433	0.1806	0.2404	0.2621	0.2827	0.3089
12	6	0.1216	0.1263	0.1311	0.1377	0.1713	0.2244	0.2436	0.2619	0.2852
12	7	0.1189	0.1231	0.1275	0.1335	0.1640	0.2122	0.2296	0.2462	0.2672
12	8	0.1164	0.1204	0.1245	0.1301	0.1583	0.2025	0.2185	0.2336	0.2529
12	9	0.1145	0.1182	0.1220	0.1273	0.1536	0.1946	0.2094	0.2235	0.2413
12	10	0.1129	0.1164	0.1200	0.1249	0.1496	0.1880	0.2018	0.2149	0.2316
12	11	0.1113	0.1147	0.1182	0.1229	0.1462	0.1824	0.1954	0.2077	0.2233
12	12	0.1097	0.1131	0.1165	0.1211	0.1434	0.1775	0.1897	0.2014	0.2162
12	14	0.1077	0.1109	0.1140	0.1181	0.1385	0.1695	0.1806	0.1911	0.2045
12	16	0.1061	0.1090	0.1119	0.1158	0.1346	0.1632	0.1733	0.1830	0.1953
12	18	0.1051	0.1077	0.1103	0.1139	0.1314	0.1581	0.1675	0.1765	0.1878
12	20	0.1037	0.1063	0.1088	0.1122	0.1288	0.1537	0.1625	0.1709	0.1815
12	25	0.1016	0.1038	0.1061	0.1091	0.1236	0.1454	0.1531	0.1603	0.1695
12	30	0.0995	0.1017	0.1039	0.1067	0.1199	0.1393	0.1461	0.1526	0.1608
12	40	0.0974	0.0993	0.1011	0.1035	0.1147	0.1311	0.1368	0.1422	0.1491
12	50	0.0964	0.0979	0.0994	0.1014	0.1111	0.1257	0.1307	0.1354	0.1413

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
14	1	0.1707	0.1792	0.1891	0.2040	0.2919	0.4391	0.4903	0.5407	0.5986
14	2	0.1388	0.1454	0.1527	0.1631	0.2208	0.3165	0.3511	0.3839	0.4251
14	3	0.1252	0.1308	0.1368	0.1452	0.1900	0.2635	0.2903	0.3157	0.3481
14	4	0.1169	0.1219	0.1272	0.1345	0.1722	0.2329	0.2550	0.2761	0.3031
14	5	0.1120	0.1165	0.1211	0.1274	0.1602	0.2127	0.2318	0.2500	0.2732
14	6	0.1078	0.1120	0.1163	0.1222	0.1516	0.1980	0.2149	0.2310	0.2517
14	7	0.1054	0.1091	0.1130	0.1182	0.1449	0.1870	0.2023	0.2168	0.2354
14	8	0.1029	0.1064	0.1101	0.1150	0.1396	0.1782	0.1921	0.2054	0.2224
14	9	0.1011	0.1044	0.1078	0.1124	0.1353	0.1710	0.1840	0.1963	0.2119
14	10	0.0995	0.1026	0.1058	0.1101	0.1316	0.1651	0.1771	0.1886	0.2031
14	11	0.0980	0.1010	0.1041	0.1082	0.1286	0.1600	0.1712	0.1820	0.1957
14	12	0.0966	0.0996	0.1026	0.1065	0.1259	0.1555	0.1662	0.1763	0.1893
14	14	0.0947	0.0974	0.1002	0.1038	0.1215	0.1483	0.1580	0.1671	0.1788
14	16	0.0931	0.0956	0.0982	0.1016	0.1180	0.1426	0.1515	0.1599	0.1705
14	18	0.0921	0.0944	0.0967	0.0998	0.1150	0.1381	0.1463	0.1540	0.1638
14	20	0.0907	0.0930	0.0953	0.0983	0.1126	0.1341	0.1417	0.1490	0.1582
14	25	0.0888	0.0908	0.0927	0.0953	0.1079	0.1267	0.1333	0.1396	0.1475
14	30	0.0868	0.0888	0.0907	0.0931	0.1046	0.1212	0.1271	0.1327	0.1398
14	40	0.0848	0.0864	0.0881	0.0901	0.0998	0.1138	0.1188	0.1235	0.1294
14	50	0.0838	0.0851	0.0864	0.0881	0.0965	0.1090	0.1133	0.1174	0.1225

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
16	1	0.1573	0.1653	0.1744	0.1881	0.2680	0.4028	0.4504	0.4947	0.5527
16	2	0.1268	0.1329	0.1395	0.1490	0.2009	0.2872	0.3188	0.3487	0.3866
16	3	0.1138	0.1189	0.1243	0.1319	0.1721	0.2380	0.2621	0.2851	0.3145
16	4	0.1058	0.1104	0.1152	0.1218	0.1555	0.2096	0.2294	0.2484	0.2728
16	5	0.1012	0.1052	0.1094	0.1151	0.1443	0.1910	0.2081	0.2244	0.2453
16	6	0.0975	0.1012	0.1050	0.1101	0.1362	0.1776	0.1927	0.2071	0.2255
16	7	0.0946	0.0981	0.1016	0.1063	0.1301	0.1674	0.1809	0.1939	0.2105
16	8	0.0925	0.0957	0.0989	0.1033	0.1252	0.1593	0.1717	0.1836	0.1987
16	9	0.0908	0.0937	0.0967	0.1008	0.1211	0.1528	0.1643	0.1752	0.1891
16	10	0.0892	0.0920	0.0949	0.0987	0.1178	0.1473	0.1580	0.1682	0.1811
16	11	0.0878	0.0905	0.0932	0.0969	0.1149	0.1427	0.1527	0.1622	0.1744
16	12	0.0864	0.0891	0.0918	0.0953	0.1125	0.1386	0.1480	0.1570	0.1685
16	14	0.0846	0.0871	0.0895	0.0928	0.1084	0.1321	0.1406	0.1487	0.1590
16	16	0.0832	0.0855	0.0877	0.0907	0.1051	0.1269	0.1347	0.1421	0.1515
16	18	0.0820	0.0841	0.0862	0.0890	0.1024	0.1226	0.1299	0.1367	0.1455
16	20	0.0809	0.0829	0.0849	0.0876	0.1002	0.1191	0.1259	0.1323	0.1404
16	25	0.0791	0.0808	0.0825	0.0848	0.0958	0.1124	0.1182	0.1237	0.1307
16	30	0.0771	0.0788	0.0806	0.0827	0.0928	0.1074	0.1126	0.1175	0.1238
16	40	0.0753	0.0767	0.0781	0.0799	0.0884	0.1007	0.1051	0.1092	0.1144
16	50	0.0742	0.0753	0.0765	0.0781	0.0854	0.0963	0.1001	0.1037	0.1082

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
18	1	0.1463	0.1538	0.1623	0.1749	0.2482	0.3726	0.4171	0.4587	0.5136
18	2	0.1169	0.1226	0.1287	0.1374	0.1846	0.2634	0.2922	0.3198	0.3548
18	3	0.1045	0.1092	0.1142	0.1211	0.1575	0.2172	0.2392	0.2602	0.2871
18	4	0.0967	0.1010	0.1054	0.1114	0.1420	0.1908	0.2088	0.2260	0.2482
18	5	0.0925	0.0961	0.0999	0.1051	0.1314	0.1736	0.1890	0.2038	0.2227
18	6	0.0890	0.0923	0.0957	0.1004	0.1239	0.1611	0.1747	0.1878	0.2045
18	7	0.0863	0.0894	0.0925	0.0968	0.1181	0.1517	0.1639	0.1756	0.1906
18	8	0.0841	0.0870	0.0900	0.0939	0.1136	0.1443	0.1554	0.1661	0.1797
18	9	0.0825	0.0852	0.0879	0.0916	0.1098	0.1382	0.1485	0.1584	0.1709
18	10	0.0810	0.0836	0.0861	0.0896	0.1067	0.1332	0.1428	0.1519	0.1636
18	11	0.0797	0.0821	0.0846	0.0879	0.1040	0.1289	0.1379	0.1464	0.1574
18	12	0.0783	0.0807	0.0832	0.0864	0.1018	0.1252	0.1336	0.1417	0.1520
18	14	0.0766	0.0788	0.0810	0.0840	0.0979	0.1191	0.1267	0.1340	0.1433
18	16	0.0754	0.0774	0.0794	0.0820	0.0949	0.1144	0.1214	0.1280	0.1365
18	18	0.0741	0.0760	0.0779	0.0804	0.0924	0.1105	0.1169	0.1231	0.1309
18	20	0.0730	0.0748	0.0767	0.0791	0.0904	0.1072	0.1132	0.1190	0.1263
18	25	0.0712	0.0728	0.0744	0.0764	0.0863	0.1010	0.1062	0.1112	0.1174
18	30	0.0694	0.0710	0.0725	0.0745	0.0835	0.0965	0.1011	0.1055	0.1111
18	40	0.0679	0.0691	0.0703	0.0719	0.0794	0.0905	0.0943	0.0980	0.1025
18	50	0.0670	0.0679	0.0688	0.0701	0.0766	0.0865	0.0899	0.0930	0.0968

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
20	1	0.1370	0.1440	0.1520	0.1637	0.2315	0.3470	0.3886	0.4278	0.4768
20	2	0.1088	0.1140	0.1197	0.1277	0.1711	0.2434	0.2701	0.2956	0.3281
20	3	0.0969	0.1012	0.1058	0.1121	0.1454	0.2001	0.2202	0.2396	0.2643
20	4	0.0899	0.0937	0.0976	0.1030	0.1307	0.1754	0.1919	0.2077	0.2280
20	5	0.0853	0.0886	0.0921	0.0968	0.1209	0.1592	0.1733	0.1868	0.2042
20	6	0.0819	0.0850	0.0881	0.0924	0.1138	0.1476	0.1600	0.1719	0.1871
20	7	0.0793	0.0822	0.0851	0.0890	0.1084	0.1388	0.1500	0.1606	0.1743
20	8	0.0773	0.0799	0.0826	0.0862	0.1041	0.1319	0.1421	0.1518	0.1642
20	9	0.0757	0.0782	0.0807	0.0840	0.1005	0.1263	0.1357	0.1446	0.1560
20	10	0.0743	0.0766	0.0790	0.0821	0.0976	0.1216	0.1303	0.1386	0.1492
20	11	0.0730	0.0753	0.0775	0.0805	0.0951	0.1176	0.1258	0.1336	0.1435
20	12	0.0717	0.0739	0.0762	0.0791	0.0930	0.1142	0.1218	0.1291	0.1385
20	14	0.0701	0.0721	0.0741	0.0768	0.0894	0.1086	0.1155	0.1221	0.1305
20	16	0.0689	0.0707	0.0725	0.0749	0.0866	0.1042	0.1105	0.1166	0.1242
20	18	0.0677	0.0695	0.0712	0.0734	0.0843	0.1006	0.1064	0.1120	0.1191
20	20	0.0665	0.0683	0.0699	0.0721	0.0824	0.0975	0.1030	0.1082	0.1148
20	25	0.0649	0.0664	0.0678	0.0697	0.0786	0.0918	0.0966	0.1010	0.1067
20	30	0.0631	0.0646	0.0660	0.0678	0.0760	0.0876	0.0918	0.0957	0.1009
20	40	0.0612	0.0625	0.0638	0.0653	0.0722	0.0819	0.0854	0.0887	0.0930
20	50	0.0609	0.0617	0.0626	0.0637	0.0695	0.0784	0.0815	0.0843	0.0878

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
25	1	0.1189	0.1251	0.1320	0.1421	0.1993	0.2973	0.3332	0.3673	0.4104
25	2	0.0931	0.0976	0.1025	0.1092	0.1453	0.2055	0.2278	0.2493	0.2769
25	3	0.0823	0.0860	0.0898	0.0951	0.1226	0.1677	0.1844	0.2005	0.2212
25	4	0.0760	0.0792	0.0824	C.0869	0.1096	0.1464	0.1599	0.1730	0.1898
25	5	0.0718	0.0746	0.0775	0.0814	0.1010	0.1324	0.1439	0.1550	0.1693
25	6	0.0687	0.0713	0.0739	0.0774	0.0949	0.1224	0.1326	0.1423	0.1548
25	7	0.0664	0.0687	0.0711	0.0743	0.0901	0.1149	0.1240	0.1327	0.1439
25	8	0.0645	0.0667	0.0689	0.0719	0.0864	0.1090	0.1172	0.1252	0.1353
25	9	0.0631	0.0652	0.0672	0.0699	0.0833	0.1042	0.1118	0.1191	0.1284
25	10	0.0619	0.0638	0.0657	0.0682	0.0808	0.1003	0.1073	0.1140	0.1227
25	11	0.0607	0.0625	0.0643	0.0668	0.0786	0.0968	0.1034	0.1097	0.1178
25	12	0.0597	0.0614	0.0632	0.0655	0.0768	0.0939	0.1001	0.1061	0.1137
25	14	0.0582	0.0598	0.0614	0.0635	0.0737	0.0892	0.0948	0.1001	0.1069
25	16	0.0567	0.0583	0.0598	0.0618	0.0713	0.0854	0.0905	0.0953	0.1016
25	18	0.0559	0.0573	0.0587	0.0605	0.0692	0.0824	0.0871	0.0916	0.0973
25	20	0.0548	0.0562	0.0576	0.0594	0.0676	0.0798	0.0842	0.0883	0.0937
25	25	0.0532	0.0544	0.0556	0.0572	0.0644	0.0750	0.0787	0.0823	0.0869
25	30	0.0519	0.0531	0.0542	0.0556	0.0621	0.0715	0.0748	0.0780	0.0821
25	40	0.0499	0.0511	0.0521	0.0534	0.0589	0.0666	0.0694	0.0721	0.0755
25	50	0.0501	0.0506	0.0512	0.0520	0.0565	0.0638	0.0662	0.0685	0.0711

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
30	1	0.1059	0.1114	0.1175	0.1263	0.1759	0.2610	0.2926	0.3227	0.3611
30	2	0.0820	0.0860	0.0901	0.0959	0.1269	0.1785	0.1977	0.2163	0.2402
30	3	0.0720	0.0751	0.0784	0.0830	0.1065	0.1448	0.1591	0.1729	0.1906
30	4	0.0661	0.0689	0.0717	0.0755	0.0948	0.1260	0.1375	0.1486	0.1630
30	5	0.0623	0.0647	0.0672	0.0705	0.0871	0.1137	0.1234	0.1329	0.1450
30	6	0.0591	0.0615	0.0638	0.0668	0.0817	0.1048	0.1134	0.1216	0.1323
30	7	0.0573	0.0593	0.0614	0.0641	0.0775	0.0983	0.1060	0.1133	0.1228
30	8	0.0556	0.0575	0.0594	0.0619	0.0741	0.0932	0.1001	0.1068	0.1154
30	9	0.0544	0.0561	0.0578	0.0601	0.0714	0.0890	0.0954	0.1015	0.1094
30	10	0.0532	0.0548	0.0564	0.0586	0.0691	0.0855	0.0914	0.0971	0.1044
30	11	0.0521	0.0537	0.0552	0.0573	0.0673	0.0825	0.0881	0.0934	0.1002
30	12	0.0512	0.0527	0.0542	0.0562	0.0656	0.0800	0.0852	0.0902	0.0966
30	14	0.0498	0.0512	0.0525	0.0543	0.0629	0.0759	0.0805	0.0850	0.0907
30	16	0.0485	0.0499	0.0512	0.0528	0.0607	0.0725	0.0768	0.0809	0.0861
30	18	0.0478	0.0490	0.0501	0.0516	0.0589	0.0699	0.0739	0.0776	0.0824
30	20	0.0466	0.0479	0.0491	0.0506	0.0575	0.0677	0.0713	0.0748	0.0793
30	25	0.0455	0.0465	0.0474	0.0487	0.0546	0.0635	0.0667	0.0697	0.0735
30	30	0.0442	0.0451	0.0460	0.0472	0.0526	0.0604	0.0632	0.0659	0.0693
30	40	0.0429	0.0436	0.0443	0.0453	0.0498	0.0564	0.0587	0.0609	0.0636
30	50	0.0422	0.0427	0.0432	0.0440	0.0478	0.0537	0.0558	0.0576	0.0599

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
60	1	0.0669	0.0703	0.0739	0.0791	0.1072	0.1554	0.1735	0.1912	0.2140
60	2	0.0498	0.0521	0.0544	0.0577	0.0746	0.1025	0.1130	0.1232	0.1365
60	3	0.0427	0.0445	0.0464	0.0488	0.0614	0.0818	0.0894	0.0968	0.1064
60	4	0.0386	0.0402	0.0417	0.0438	0.0540	0.0703	0.0764	0.0823	0.0899
60	5	0.0360	0.0373	0.0387	0.0404	0.0492	0.0630	0.0681	0.0730	0.0794
60	6	0.0341	0.0353	0.0364	0.0380	0.0457	0.0577	0.0622	0.0664	0.0720
60	7	0.0326	0.0337	0.0348	0.0362	0.0431	0.0538	0.0578	0.0616	0.0665
60	8	0.0315	0.0325	0.0334	0.0347	0.0410	0.0508	0.0543	0.0578	0.0622
60	9	0.0305	0.0314	0.0323	0.0336	0.0394	0.0483	0.0516	0.0547	0.0587
60	10	0.0299	0.0307	0.0315	0.0326	0.0380	0.0463	0.0493	0.0522	0.0559
60	11	0.0290	0.0298	0.0307	0.0317	0.0368	0.0446	0.0474	0.0501	0.0535
60	12	0.0284	0.0292	0.0300	0.0310	0.0358	0.0431	0.0457	0.0482	0.0515
60	14	0.0275	0.0282	0.0289	0.0298	0.0341	0.0407	0.0430	0.0453	0.0482
60	16	0.0266	0.0273	0.0280	0.0289	0.0328	0.0388	0.0409	0.0430	0.0456
60	18	0.0261	0.0267	0.0273	0.0281	0.0318	0.0373	0.0392	0.0411	0.0435
60	20	0.0256	0.0261	0.0267	0.0274	0.0309	0.0360	0.0378	0.0396	0.0418
60	25	0.0246	0.0251	0.0256	0.0262	0.0292	0.0336	0.0351	0.0366	0.0385
60	30	0.0237	0.0242	0.0247	0.0253	0.0280	0.0318	0.0332	0.0345	0.0362
60	40	0.0227	0.0231	0.0235	0.0240	0.0263	0.0295	0.0306	0.0317	0.0331
60	50	0.0221	0.0225	0.0228	0.0232	0.0251	0.0280	0.0289	0.0299	0.0311

n	df	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
120	1	0.0413	0.0432	0.0454	0.0483	0.0637	0.0897	0.0996	0.1093	0.1219
120	2	0.0297	0.0310	0.0323	0.0340	0.0430	0.0577	0.0632	0.0686	0.0756
120	3	0.0250	0.0259	0.0269	0.0282	0.0348	0.0454	0.0493	0.0531	0.0581
120	4	0.0223	0.0231	0.0239	0.0250	0.0303	0.0387	0.0418	0.0448	0.0487
120	5	0.0206	0.0212	0.0219	0.0229	0.0273	0.0344	0.0370	0.0395	0.0428
120	6	0.0193	0.0199	0.0205	0.0213	0.0253	0.0314	0.0336	0.0358	0.0386
120	7	0.0183	0.0189	0.0195	0.0202	0.0237	0.0291	0.0311	0.0330	0.0355
120	8	0.0176	0.0181	0.0186	0.0193	0.0225	0.0274	0.0292	0.0309	0.0331
120	9	0.0170	0.0175	0.0179	0.0185	0.0215	0.0260	0.0276	0.0292	0.0312
120	10	0.0165	0.0169	0.0173	0.0179	0.0206	0.0248	0.0263	0.0278	0.0296
120	11	0.0160	0.0164	0.0168	0.0174	0.0199	0.0238	0.0252	0.0266	0.0283
120	12	0.0157	0.0160	0.0164	0.0170	0.0193	0.0230	0.0243	0.0256	0.0272
120	14	0.0150	0.0154	0.0157	0.0162	0.0184	0.0216	0.0228	0.0239	0.0254
120	16	0.0145	0.0149	0.0152	0.0156	0.0176	0.0206	0.0216	0.0226	0.0239
120	18	0.0142	0.0145	0.0148	0.0151	0.0170	0.0197	0.0207	0.0216	0.0228
120	20	0.0138	0.0141	0.0144	0.0148	0.0164	0.0190	0.0199	0.0208	0.0219
120	25	0.0132	0.0135	0.0137	0.0140	0.0155	0.0176	0.0184	0.0192	0.0201
120	30	0.0127	0.0129	0.0131	0.0134	0.0148	0.0166	0.0173	0.0180	0.0188
120	40	0.0120	0.0122	0.0124	0.0127	0.0138	0.0153	0.0159	0.0164	0.0171

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT

A well known test for equality of normal population variances, based on sample variances, was introduced by Cochran (1941). Suppose σ_i^2, $i=1,\ldots,n$ represents the population variances of n normal populations, and let independent sample variances, each based on k degrees of freedom, be s_i^2, $i=1,\ldots,n$. Suppose the s^2 are ranked, so that the ordered variables are $s_{(1)}^2, s_{(2)}^2, \ldots, s_{(n)}^2$. To test H_o: the σ_i^2 are are all equal (suppose the common value is σ^2), Cochran (1941) introduced the test statistic

$$Z = \frac{s_{(n)}^2}{\sum_{i=1}^n s_i^2},$$

which compares the largest sample variance with the sum of the sample variances. Clearly the intent is to discover if one variance is an outlier (too large), and, in general, H_0 will be rejected for large values of Z. Tables of various percentiles are given for various values of n and k.