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ON TURING DEGREES OF WALRASIAN MODELS AND A GENERAL IMPOSSIBILITY RESULT IN THE THEORY OF DECISION MAKING

by

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0. Introduction

In von Neumann [1928] N-person games in normal form were characterized as triplets

$$\Gamma = \langle N, \{S_j\}_{j \varepsilon N}, \{\phi_j\}_{j \varepsilon N} \rangle$$

where N is a finite set of players. The sets S_j are finite sets for all $j \varepsilon N$ and the functions $\{\phi_j\}_{j \varepsilon N}$ are real-valued with $\phi_j\colon \prod_{j \varepsilon N} S_j \to \mathbb{R}$ for all $j \varepsilon N$. In a more general setting, one can view the game Γ abstractly as a relational structure and the theory of games is comprised of several types of relational structures that are used to model diverse kinds of game-theoretic phenomena. We consider game-theoretic structures to be of the form: $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ where A is a nonempty set and the R_j's are relations of finite arity on A for all $j = 1, \ldots, n$.

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The use of relational structures of the algebraic kind of $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ is most likely unfamiliar to most readers in the theory of games. However, it is not very difficult to show that the normal form games of von Neumann [1932] can be reexpressed as relational structures of the form $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$. Formally, we will state this simple fact as the following result:

Representation Lemma: If $\Gamma = \langle N, \{S_j\}_{j \varepsilon N}, \{\phi_j\}_{j \varepsilon N} \rangle$ is an N-person von Neumann game in normal form, then Γ can be reexpressed as a relational structure $\mathfrak{A}(\Gamma) = \langle A(\Gamma), R_1(\Gamma), R_2(\Gamma) \rangle$ where $\mathrm{Dom}(R_i(\Gamma)) = A(\Gamma)$ for $i = 1,2$.

Proof: Suppose we are given a game of the form $\Gamma = \langle N, \{S_j\}_{j \varepsilon N}, \{\phi_j\}_{j \varepsilon N} \rangle$. We can then define the domain $A(\Gamma) \mathrm{df:} = N \times T$, where $T = \{t\}$ and $t = \langle t(1), \ldots, t(n) \rangle = \prod_{j \varepsilon N} S_j$. Next, let $\langle R_1(\Gamma), R_2(\Gamma) \rangle$ be given a pair of relations $\langle \tilde{\Phi}, \tilde{\Phi} \rangle$ such that $\mathrm{Dom}(\tilde{\Phi}) = \mathrm{Dom}(\tilde{\Phi}) = A(\Gamma)$ with the defining conditions for all pairs $(j,t) \varepsilon N \times T = A(\Gamma)$ given by

$$\begin{cases} \tilde{\Phi}(j,t) = \phi_j \text{ and } \mathrm{Dom}(\phi_j) = t \\ \text{and} \\ \tilde{\Phi}(j,t) = S_j = t(j) \end{cases}$$

Then clearly, the structure $\mathfrak{A}(\Gamma) = \langle A(\Gamma), R_1(\Gamma), R_2(\Gamma) \rangle$ with $R_1(\Gamma) = \tilde{\Phi}$ and $R_2(\Gamma) = \tilde{\Phi}$ recovers algebraically the structure $\Gamma = \langle N, \{S_j\}_{j \varepsilon N}, \{\phi_j\}_{j \varepsilon N} \rangle$ and hence $\mathfrak{A}(\Gamma)$ is the desired reexpression of Γ as claimed.

From the representation lemma, our use of generic relational structures of the form $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ merely serves to provide us with a very convenient way to code the components of an N-person game algebraically. From well-known results of von Neumann/Morgenstern [1944], every N-person von Neumann game, cooperative or non-cooperative, can be expressed in von Neumann's normal form $\Gamma = \langle N, \{S_j\}_{j\varepsilon N}, \{\phi_j\}_{j\varepsilon N} \rangle$. It follows trivially that every N-person game, cooperative or noncooperative can be expressed generically as a relational structure of the form $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ and our use of this model-theoretic framework is vindicated game-theoretically.

We will assume Church's theses and identify those functions that are computable with the recursive functions; and identify with every recursive function $g: \mathbb{N} \to \mathbb{N}$ a Turing Machine TM_g that realizes the computations of g on $\mathbb{N}$, i.e. TM_g reads the coded inputs of $Dom(g)$ and prints the coded output of $Rng(g)$ after some finite time period. In Appendix I, we discuss Turing machines in a formal way.

Within this formalism, we show that it is possible to obtain computable representations of any game-theoretic structure $\mathfrak{A}$ by approximating the various components of the model with recursive functions. Intuitively speaking a recursive function $g: \mathbb{N} \to \mathbb{N}$ is said to index a recursive presentation of a game-theoretic structure $\mathfrak{A}$ if $A = Rng(g)$ and if each relation R_j is an effectively enumerated subset of the appropriate product space generated by powers of $g(\mathbb{N})$, i.e. if R_j is m-ary on A, then $dom(R_j)$ is an r.e. subset of $g(\mathbb{N})^m$.

Given an arbitrary recursive presentation of a game-theoretic structure $\mathfrak{A}_g$ with index g, we ask whether or not $\mathfrak{A}_g$ can be recursively realized. When we say that $\mathfrak{A}_g$ can be recursively realized we mean that the task of $\mathfrak{A}_g$ can be executed by a Turing machine. Associated with every model $\mathfrak{A}_g$ is some task of $\mathfrak{A}_g$, i.e. generate winning strategies, equilibrium points, optimal choices, stable outcomes, etc. Typically, the task of $\mathfrak{A}_g$ is the form of a correspondence

$$\Phi: \ \text{Alt}\,\mathfrak{A}_g \to \text{Out}\,\mathfrak{A}_g$$

Φ acts a space of alternatives to a space of outcomes for $\mathfrak{A}_g$. Let $\deg(\mathfrak{A}_g)$ be the Turing degree of computational complexity of the realization of Φ. Then $\mathfrak{A}_g$ is a recursively realizable game-theoretic structure if and only if $\deg(\mathfrak{A}_g) := \deg(\text{graph}(\Phi))$ is recursive, in which case the task of $\mathfrak{A}_g$ is realizable by a Turing machine.

By Rabin's theorem (Rabin [1957]) there exists a recursively presented Gale-Stewart game Γ_g with no effectively computable winning strategy, and thus with no Turing realization since $\deg(\Gamma_g)$ is not recursive. In Lewis [1985a] we have shown that single person games against nature having the form of representable choice functions such that

$$\mathcal{C}(A) = \{x \in A: \ \forall y \in A \quad x \succ y\}$$

for $\succ: X^2 \to \{1,0\}$ and $A \in P(X)$ for some compact convex set in $X \subseteq \mathbb{R}^n$, are such that $\deg(\mathcal{C}_g)$ is not recursive when g is an index of a presentation of $\mathcal{C}$ that is recursive. In the present paper, we show that if $\mathcal{O}\!\mathcal{C}_g$ is a recursive presentation of a non-trivial Walrasian model of general equilibrium, then $\deg(\mathcal{O}\!\mathcal{C}_g)$ is not recursive. It follows that non-trivial Walrasian models of general equilibrium are not realizable by Turing machines.

The natural partial ordering associated with the Turing degrees of unsolvability gives a usable concept of rank for the computational complexity of the game-theoretic structures we deal with. In terms of the Turing degrees we rank the minimal Turing degrees of unsolvability associated with recursive presentations of the following game-theoretic structures as a measure of the least extent to which the game is to be considered non-effective.

(i) $\mathcal{C}_g$: Single Player Choice Functions

(ii) $\mathcal{O}\!\mathcal{C}_g$: Walrasian Models of General Equilibrium

(iii) $\mathcal{L}_g$: N-person Noncooperative Games in the sense of Nash.

For $\{\mathcal{O}\!\mathcal{C}_{g_j}\}_{j<\omega}$ the class of recursive presentations of a fixed game theoretic structure $\mathcal{O}\!\mathcal{C}_g$, let $\min(\deg(\mathcal{O}\!\mathcal{C}_g))$ denote the set of <u>minimal degrees</u> for the class $\{\mathcal{O}\!\mathcal{C}_{g_j}\}_{j<\omega}$. A degree $\underset{\sim}{d} \in \mathcal{D}$ is minimal for the class $\{\mathcal{O}\!\mathcal{C}_{g_j}\}_{j<\omega}$ iff

(i) $\exists j' < \omega \lfloor \deg(\mathcal{O}\!\mathcal{C}_{g_{j'}}) = \underset{\sim}{d} \rfloor$ and (ii) $\forall j < \omega \lfloor \deg(\mathcal{O}\!\mathcal{C}_{g_j}) \not< \underset{\sim}{d} \rfloor$.

Specifically, our results show that for the complete r.e. set $K = \{e\colon\ e \in W_e\}$

$$\deg(K) = \underset{\sim}{0}' < \inf\{\min(\deg(\mathcal{C}_g)),\ \min(\deg(\mathcal{O}\!\mathcal{C}_g))\}.$$

Every pure exchange Walrasian model is comprised of a set of single-player choice function games, and is equivalent in task to an N-person non-cooperative game in the sense of Nash. As a corollary, we have the following therefore,

$$\underset{\sim}{0}' < \min(\deg(\mathcal{L}_g)).$$

These classification results were announced in an abstract that appeared in The Recursive Function Newsletter, No. 33, July 1985 entitled: "Some Turing Degrees of Complexity of Certain Game-Theoretic Structures". The paper itself is a generalization and extension of results first obtained in Lewis [1985a] and [1985b] where Turing reducibilities were introduced to determine the extent to which recursively representable choice functions can be effectively realized.

The paper is organized in the following way. Section 1 is a preliminaries section and covers the basic definitions and concepts used from the theory of recursive functions and the theory of Turing degrees of unsolvability. We fix the notation of the paper in this section and give the definitions for a recursive presentation of a relational structure and its Turing degree. In section 2, we deal with the specific structure associated with single-person choice function games against nature. In this specific instance we obtain bounds for the minimal Turing degree of unsolvability (i.e. $\neq \underset{\sim}{0}$) of recursive presentations of the model of the game. The main theorem is Theorem 2.2, the proof of which is a Kleene-Post reduction-style argument. In section 2. at the end, we briefly indicate the application of the result on single-person choice function games to the degrees of Walrasian models of general equilibrium and N-person non-cooperative games in the sense of Nash. In section 3 we generalize the impossibility result of Kramer [1974] and obtain the result of Kramer as a special instance of Theorem 2.2. Persons having some acquaintence with classical recursion theory as presented in Rogers [1967] or Soare [1986] may skip the first section and proceed directly to sections 2 and 3 of the paper. Two Appendices are provided at the end of the paper. The first Appendix gives a formal description of Turing machines and finite state machines, while the second Appendix presents results that characterize recursive metric spaces in the sense of Moschovakis [1965].

1. Preliminaries and Notation from Recursion Theory.

The notation and terminology from recursion theory that we use is standard and can be found in several places, e.g. Lerman [1983], Rogers [1967], Shoenfield [1971], Simpson [1977] or Soare [1986].

Informally, a number theoretic function

$$f: \omega \to \omega$$

where $\omega = \{0,1,2,\ldots n\ldots\}$ is the first infinite ordinal, is said to be <u>recursive</u> if there is a programme for a Turing machine such that following the instructions of the programme, the machine can compute $f(n)$ for each input $n \in \omega$. This mechanistic model for the recursive functions was developed by Turing [1936].

Let ${}^{\omega}2$ be the set of all functions $f\colon \omega \to \{0,1\}$ and let ${}^{\omega}\omega$ be the set of all number-theoretic functions $f\colon \omega \to \omega$. If f and g are elements of ${}^{\omega}2$, we let $f \oplus g$ denote the unique function $h \in {}^{\omega}2$ such that for all $n \in \omega$ $h(2n) = f(n)$ and $h(2n+1) = g(n)$. Identifying sets in ω with their characteristic functions, the notation $A \oplus B$ gives a set C whose characteristic function is $\chi_C = \chi_A \oplus \chi_B$.

Definition 1.1: A function $f \in {}^{\omega}\omega$ is recursive in a function $g \in {}^{\omega}\omega$ (or f is Turing reducible to g) if there is an algorithm that computes $f(n)$ from the input n using an oracle for the function g for all $n \in \omega$. If $f \in {}^{\omega}\omega$ is Turing reducible to $g \in {}^{\omega}\omega$ then we write $f \leq_T g$ to indicate this.

Assume a fixed Gödel numbering of the algorithms used to compute the recursive functions $\{\phi_e\}_{e<\omega}$.

Definition 1.2. For every function $f \in {}^{\omega}2$ define the function $f^* \in {}^{\omega}2$ by the rule:

$$f^*(e) = \begin{cases} 1 \text{ if } \phi_e^f(e)\downarrow & (\text{i.e. } \phi_e^f \text{ halts on } e) \\ 0 \text{ if } \phi_e^f(e)\uparrow & (\text{i.e. } \phi_e^f \text{ diverges on } e) \end{cases}$$

where ϕ_e^f is the e^{th} recursive function using f as an oracle.

The following proposition is not difficult to prove, and the details can be found in Rogers [1967].

Proposition 1.1

(i) $f \leq_T f$

(ii) $(f \leq_T g$ and $g_T\leq h) \Rightarrow f_T\leq h$

(iii) $f \oplus g \leq_T h$ iff $f \leq_T h$ and $g \leq_T h$

(iv) $f \leq_T f^*$ and $f^* \nleq f$

(v) $(f \leq_T g) \Rightarrow f^* \leq_T g^*$.

We define the set of degrees of unsolvability using the following equivalence relation

Definition 1.3: If f and g are both in ${}^{\omega}\omega$ define the relation $f \equiv_T g$ as

$$f \equiv_T g \text{ df: } = f \leq_T g \text{ and } g \leq_T f.$$

Definition 1.4: For a fixed $f \varepsilon\, {}^{\omega}\omega$. The degree of unsolvability of f, written $\underset{\sim}{f}$ is the set

$$\underset{\sim}{f} = \{g \varepsilon\, {}^{\omega}\omega\colon\ g \equiv_T f\}.$$

We let $\mathcal{D}$ denote the set $\{\underset{\sim}{f}\colon\ f \varepsilon\, {}^{\omega}\omega\}$ and often use $\deg(f)$ for $\underset{\sim}{f}$. By convention, the degree of a set $A \subseteq \omega$ is $\deg(f)$ where $f \varepsilon\, {}^{\omega}2$ and $f = \chi_A$.

Definition 1.5: Let $\underset{\sim}{a} = \deg(f)$ and $\underset{\sim}{b} = \deg(g)$. The relation $\leq$ on $\mathcal{D}$ is defined by the rule:

$$\underset{\sim}{a} \leq \underset{\sim}{b} \text{ iff } f \leq_T g.$$

The binary operator $\cup$ on $\mathcal{D}$ is defined by the rule:

$$\underset{\sim}{a} \cup \underset{\sim}{b} \text{ df: } = \deg(f \oplus g).$$

The unary operator $j\colon \mathcal{D} \to \mathcal{D}$ is called the <u>jump operator</u> and is defined by the rule:

$$j(\underset{\sim}{a}) = \underset{\sim}{a}' \text{ df: } = \deg(f^{*}) \quad \text{if} \quad f \in {}^{\omega}2 \quad \text{and} \quad \deg(f) = \underset{\sim}{a}.$$

We let $\underset{\sim}{0}$ be a distinguished element of $\mathcal{D}$ defined by $\underset{\sim}{0} = \deg(\lambda n \cdot 0)$. Since no recursive function requires an oracle to be computed and thus $f \leq_T (\lambda n \cdot 0)$ if f is recursive, $\underset{\sim}{0}$ is the degree of recursive functions.

<u>Proposition 1.2</u>:

(i) The cardinality of $\mathcal{D}$ is $2^{\aleph_0}$.

(ii) $\mathcal{D}$ is partially ordered by the relation $\leq$.

(iii) Subsets of $\mathcal{D}$ have upper bounds under $\leq$ if and only if the subset is bounded.

(iv) $\underset{\sim}{0}$ is the least element of $\mathcal{D}$.

(v) $\underset{\sim}{a} \cup \underset{\sim}{b}$ in the least upper bound of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(vi) $\underset{\sim}{a} \leq \underset{\sim}{a}'$ for all $a \in \mathcal{D}$.

(vii) $\underset{\sim}{a} \leq \underset{\sim}{b} \Rightarrow \underset{\sim}{a}' \leq \underset{\sim}{b}'$.

If $\phi_e \in {}^{\omega}\omega$ and $\phi_e(x)$ converges then we write $\phi_e(x)\downarrow$; if not, we write $\phi_e(x)\uparrow$. Let $\{\phi_e\}_{e<\omega}$ be an acceptable listing of the partial recursive functions on ω. A function ϕ_e is partially recursive if there is an algorithm to compute its values and if $\mathrm{Dom}(\phi_e) \subseteq \omega$, i.e., ϕ_e is not necessarily total.

<u>Definition 1.6</u>: If $S \subseteq \omega$ then S is R.E. or recursively enumerable if either $S = \emptyset$ or $S = \mathrm{Rng}(\phi_e)$ for some partial recursive function ϕ_e. S is R.E. in a degree $\underset{\sim}{a}$ if $S = \emptyset$ or $S = \mathrm{Rng}(\phi_e)$ for some partial $\underset{\sim}{a}$-recursive function ϕ_e, i.e. $\deg(\phi_e) \leq \underset{\sim}{a}$, and a degree $\underset{\sim}{b}$ is R.E. in $\underset{\sim}{a}$ if $\underset{\sim}{b} = \deg(S)$ and S is R.E. in $\underset{\sim}{a}$, where $\deg(S) = \deg(\chi_S)$ if S is a set.

For any degree $\underset{\sim}{a}$, $\underset{\sim}{a}' = j(a)$ is the largest degree that is R.E. in $\underset{\sim}{a}$. Jumps of degrees are important for determining the complexity of relatively recursive functions.

<u>Definition 1.7</u>: A function $f \in {}^{\omega}\omega$ is <u>limit recursive</u> if there is a recursive function $f\colon \omega \times \omega \to \omega$ such that $f(m) = \lim_{n\to\infty} f(m,n)$ for all $m \in \omega$. If $\underset{\sim}{a} = \deg(f)$, then $\underset{\sim}{a} \leq \underset{\sim}{0}'$ if and only if f is limit recursive.

The degree $\underset{\sim}{0}'$ is significant for the fact that if $S \subseteq \omega$ is an R.E. set, then $\deg(S) \leq \underset{\sim}{0}'$. $\underset{\sim}{0}'$ is also the degree of the set

$$K = \{e\colon\ \phi_e(e)\downarrow\}.$$

If $A \subseteq \omega$ is any set, then the jump of A can be expressed in terms of K relativized to A, i.e.

$$A' = K^A = \{e: \phi_e^A(e)\downarrow\}$$

where ϕ_e^A is the eth A-partially recursive function, i.e., ϕ_e^A is the e^{th} partial recursive function using the set A as an oracle.

The jump operation can be iterated.

Definition 1.8: If $\underset{\sim}{a} = \deg(f)$ for some $f \in {}^{\omega}2$, then $a^{(n)} = \deg(f^{(n)})$ for $f^{(0)} = f$ and $f^{(n+1)} = (f^{(n)})^*$ inductively for any $n \in \omega$. Thus, $\underset{\sim}{a}^{(0)} = \underset{\sim}{a}$ and $\underset{\sim}{a}^{(n+1)} = (\underset{\sim}{a}^{(n)})' = j(\underset{\sim}{a}^{(n)})$. In particular, $j(\underset{\sim}{0}^{(1)}) = (\underset{\sim}{0}^{(2)}) = \underset{\sim}{0}''$ and $j(\underset{\sim}{0}) = \underset{\sim}{0}^{(1)} = \underset{\sim}{0}'$.

Starting with $\underset{\sim}{0}$ one can form the chain of jumps:

$$\underset{\sim}{0} < \underset{\sim}{0}' < \underset{\sim}{0}'' < \underset{\sim}{0}''' < \cdots < \underset{\sim}{0}^{(n)} < \underset{\sim}{0}^{(n+1)} < \cdots .$$

Let <x,y> be the value of the pairing function (Rogers [1967] p. 64)

$$\tau(x,y) = \frac{1}{2}\left(x^2 + 2xy + y^2 + 3x + y\right).$$

The function τ gives a bijection from $\omega \times \omega$ to ω.

The jump function can be iterated into the transfinite as follows.

Definition 1.9: The set

$$0^{(\omega)} = \{\langle x,y\rangle: \ x \in 0^{(y)}\}$$

is termed the ω-jump of 0 or the ω-completion of 0.

Obviously, $\underset{\sim}{0}^{(n)} < \underset{\sim}{0}^{(\omega)}$ for any $n \in \omega$, and $0^{(\omega)} \not\leq_T 0^{(n)}$ for any $n \in \omega$. Clearly, $0^{(\omega)}$ is not an arithmetic degree.

Finally, the degree of $0^{(2)}$ is important for the fact that $\underset{\sim}{0}^{(2)} = \deg(K')$ and the fact that the degrees of the sets:

(a) $\{e: \phi_e$ is total on $\omega\}$ (b) $\{e:$ $\mathrm{Rng}(\phi_e)$ is infinite$\}$ and (c) $\{e: |\mathrm{Rng}(\phi_e)| < \infty\}$ are also equivalent to $\deg(K') = \underset{\sim}{0}^{(2)} = \underset{\sim}{0}''$.

The degrees $\underset{\sim}{0}'$ and $\underset{\sim}{0}''$ figure prominently in assessing the complexity of game-theoretic structures in later sections.

Further results on the structure of $\mathcal{D}$ as an upper semi-lattice with least element $\underset{\sim}{0}$ can be found in the comprehensive study of Lerman [1983] or the shorter set of lectures by Shore [1982]. Such results are of considerable mathematical interest, but of little use in the present paper.

We need to characterize arithematical sets in ω and relations on $\omega \times \omega$.

By elementary number theory we mean the first-order theory of Peano Arithmetic with the structure (cf. Rogers [1967] Ch. 14)

$$\mathcal{L} = \langle \omega, +, x, 0, 1 \rangle \; .$$

<u>Definition 1.10</u>: A relation $R \subseteq {}^n\omega$ for some $n \in \omega$ $(n > 0)$ is said to be arithmetic if R is explicitly definable by a formula Φ of elementary number theory. If R is arithmetic then R has the expression

$$\{\langle x_1, \ldots, x_n \rangle : \;\; Q_{1y1}, \ldots, Q_{mym}[\langle y_1, \ldots, y_m, x_1, \ldots, x_n \rangle \in R]\}$$

where $Q_1,\ldots,Q_n$ are quantifiers of the form $\exists$ or $\forall$ and where $R \subseteq {}^{n+m}\omega$ is a recursive relation, and where '$R(y_1,\ldots,x_n)$' is abbreviated by '$\langle y_1,\ldots,x_n\rangle \in R$'.

The following is obvious.

Proposition 1.4: If $A \subseteq \omega$, then A is arithmetic iff there is a recursive relation $R \subseteq {}^{n+1}\omega$ for some $n \in \omega$ such that

$$A = \{x: Q_{1y1},\ldots,Q_{nyn}R(y_1,\ldots,y_n,x)\}$$

for some (possibly empty) choice of quantifiers $Q_1,\ldots,Q_n$.

Given a choice of quantifiers $Q_1,\ldots,Q_n$ and some recursive R we say that

$$Q_{1y1}\ldots Q_{nyn}R(y_1,\ldots,y_n,x)$$

is a predicate form and that it expresses the set A.

Definition 1.11: Let $\sum_n^0$ df: = the collection of all sets in ω expressible by a predicate form with n alternating quantifiers of which the first is existential, and let Π_n^0 df: = the collection of all sets in ω expressibly by a predicate form with n alternating quantifiers of which the first is universal. Note that $\sum_0^0 = \Pi_0^0$ df: = the recursive subsets of ω.

The proofs of the assertions in the next proposition can be found in Rogers [1967] Ch. 14.

Proposition 1.6:

(i) $A \varepsilon \Sigma^0_n$ iff $\bar{A} \varepsilon \Pi^0_n$

(ii) $\Sigma^0_n \subseteq \Sigma^0_{n+1}$, $\Pi^0_n \subseteq \Sigma^0_{n+1}$, $\Sigma^0_n \subseteq \Pi^0_{n+1}$ and $\Pi^0_n \subseteq \Pi^0_{n+1}$.

(iii) $\Sigma^0_n \nsubseteq \Pi^0_n$ and $\Pi^0_n \nsubseteq \Sigma^0_n$ for all $n \bullet 0$

(iv) $\Sigma^0_0 = \Pi^0_0 = \Sigma^0_1 \cap \Pi^0_1$

(v) $\Sigma^0_n \cup \Pi^0_n \quad \Sigma^0_{n+1} \cap \Pi^0_{n+1}$

The Σ^0_n and Π^0_n classes form a hierarchy by Proposition 1.6(iii) and it is referred to as the Kleene-Mostowski Arithmetic Hierarchy.

There are important relationships between levels in the Kleene-Mostowski Hierarchy and the Turing degrees of unsolvability, $\mathcal{D}$. Proposition 1.7 summarizes the important ones. Again, proofs may be found in Rogers [1967] Ch. 14.

Proposition 1.7:

(i) A is r.e. in $0^{(n)}$ iff $A \varepsilon \Sigma^0_{n+1}$ iff $\deg(A) \leq \underset{\sim}{0}^{(n+1)}$.

(ii) $A \varepsilon \Sigma^0_{n+1} \cap \Pi^0_{n+1}$ iff $A \leq_T 0^{(n)}$.

(iii) A is arithmetic iff $A \varepsilon \Sigma^0_n \cup \Pi^0_n$ for some $n \varepsilon \omega$ iff $A \leq_T 0^{(n)}$ for some $n \varepsilon \omega$.

In terms of the hierarchy, if a set $A \subseteq \omega$ is such that $\deg(A) = \underset{\sim}{a} < \underset{\sim}{b} = \deg(B)$ for some other set $B \subseteq \omega$, then $A \leq_T B$ but $B \nleq_T A$ and A is to be considered less complex than B in terms of

Turing equivalences. Thus to determine the relative complexity of a set $A \subseteq \omega$ it suffices to determine the parity or non-parity of $\deg(A)$ with a known degree of $\mathcal{D}$, say $\underset{\sim}{0}'$ or $\underset{\sim}{0}''$, and this is roughly the kind of procedure we will employ in the paper to rank the complexity of structures.

We will conclude the preliminaries section with the following set of definitions for the recursively presented structures in terms of Turing degrees. All of the required material from recursion theory for this paper has been covered thus far in this section.

In this paper we take a model-theoretic approach to game-theoretic structures and let game-theoretic structures have the form:

$$\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$$

where A is an arithmetic set, so that $\deg(A) < \underset{\sim}{0}^{(\omega)}$, and each relation R_j is finitary on A for $j = 1, \ldots, n$.

<u>Definition 1.12:</u> Let $= \langle A, R_1, \ldots, R_n \rangle$ be a game-theoretic structure. A function $g \in {}^{\omega}\omega$ is said to index $\mathfrak{A}$ if

$$A \subseteq \mathrm{Rng}(g) = \{j \in \omega \colon \exists i \in \omega[g(i) = j]\}.$$

If $\mathfrak{A}$ is a structure with index g, then we write $\mathfrak{A}_g$ to indicate this and say that $\mathfrak{A}$ is indexed by g.

<u>Definition 1.13:</u> Let $\mathfrak{A}_g$ be an indexed game-theoretic structure. The degree of presentation of $\mathfrak{A}_g$ is denoted by $D(\mathfrak{A}_g)$ and is defined as

$$D(\mathfrak{A}_g) \text{df:} = \deg(A \bigoplus_{j=1}^{n} \text{Dom}(R_j)).$$

The symbol $\bigoplus_{j=1}^{n}$ denotes the iterated join over $\{R_1,\ldots,R_n\}$ and note that in general we allow the R_j to be partial on A so that $\text{Dom}(R_j) \neq A$ may occur for some $j = 1,\ldots,n$.

<u>Definition 1.14</u>: If $\mathfrak{A}_g$ is an indexed game-theoretic structure. Then $\mathfrak{A}_g$ is <u>recursively presented</u> if

$$D(\mathfrak{A}_g) \leq \deg(B)$$

for B some R.E. set. Then if $\mathfrak{A}_g$ is recursively presented, $D(\mathfrak{A}_g) \leq \underset{\sim}{0}'$, since $\deg(B) \leq \underset{\sim}{0}'$ for any R.E. set.

We will make use of the distinction that the degree of presentation of a game-theoretic model is different from the degree of the model. For the latter concept, we use the fact that associated with every game-theoretic model is some task of the model, i.e. generate winning strategies, equilibrium points, optimal or "best" choice, or outcomes that are stable in some sense.

Let $\text{Alt}(\mathfrak{A}_g)$ be a space of alternatives for an indexed game-theoretic structure, and let $\text{Out}(\mathfrak{A}_g)$ be a space of outcomes of $\mathfrak{A}_g$. Typically, the task of $\mathfrak{A}_g$ is in the form of a correspondence:

$$\Phi\colon \ \text{Alt}(\mathfrak{A}_g) \to \text{Out}(\mathfrak{A}_g).$$

Implicitly we will assume that both $\text{Alt}(\mathfrak{A}_g)$ and $\text{Out}(\mathfrak{A}_g)$ are arithmetically presented in the form of a Σ_1^0-class of objects. Thus, we can obtain R.E. indices for both $\text{Alt}(\mathfrak{A}_g)$ and $\text{Out}(\mathfrak{A}_g)$.

Definition 1.15: If $\mathcal{O}_g$ is a game-theoretic structure with index g, then the degree of the structure $\mathcal{O}_g$ is denoted by $\deg(\mathcal{O}_g)$ and is defined as

$$\deg(\mathcal{O}_g) \text{ df: } = \deg(\text{graph})(\Phi)).$$

The distinction between the degree of presentation of a structure $\mathcal{O}_g$ and the Turing degree of the realization of the task associated with the structure $\mathcal{O}_g$ is a necessary distinction. The specific structrures we treat subsequently are such that $D(\mathcal{O}_g) \leqslant \underset{\sim}{0}'$ but in general we will have $\deg(\mathcal{O}_g) \neq \underset{\sim}{0}$. The two notions of degree that are used here refer to different forms of complexity.

2. Single-Person Choice Function Games.

In this section we are concerned with certain single-player games against "nature". There is only one player and "nature" is represented by a topological space of alternatives. Given a particular state of nature, i.e. a configuration of subsets of alternatives, the player does the "best" that he can according to some rule of preference. Typically, a model of such a game is a pair $\langle F, \mathcal{C} \rangle$ where F is a subfamily of subsets of $\mathbb{R}^n$ and $\mathcal{C}$ is a sub-invariant set-function; that is for all $A \in F$, $\mathcal{C}(A) \subseteq A$.

To obtain an effective presentation of such a game we will introduce the notion a recursive metric space and consider recursive subsets, F_R, of compact subspaces for a specific recursive metric space for $\mathbb{R}^n$. We then consider the degree of an effectively representatble choice function $\zeta: F_R \to F_R$. Since every recursive metric space is an R-space in the sense of Lachlan [1964] the choice functions we construct can be alternatively characterized as representable Banach-Mazur operators on R-spaces. Our construction uses the specific framework for recursive metric spaces due in origin to the work of Moschovakis [1964]. A different approach to effectivizing the Euclidean space $\mathbb{R}^n$ is the topological framework of Kalantrai and Retzlaff [1979] which is more general in nature.

The results of this section are proved in Lewis [1985a] and [1985b].

<u>Definition 2.1</u>: Let $r: \omega \to Q$ where

$$r(x) = (-1)^{\mathrm{sign}(x)}\mathrm{num}(x)/\mathrm{den}(x)$$

for sign(x), num(x) and den(x) primitive recursive values. A real number $\alpha \in \mathbb{R}$ is a <u>recursive real number</u> if there exists a total recursive function $f: \omega \to \omega$ such that

(i) $\forall x, y \in \omega(|r(f(x)) - r(f(x + y))| < 2^{-x}$

and

(ii) $\alpha = \lim_{x\to\infty} r(f(x))$.

<u>Definition 2.2</u>: If α is a recursive real number determined by a total recursive function $f: \omega \to \omega$ with Gödel number n(f), then we term n(f) an <u>R-index of</u> α and denote α as $\alpha_{n(f)}$. The set $N(\mathbb{R})$

of natural numbers that are R-indices of recursive real numbers is characterized as

$$n(f) \in N(\mathbb{R}) \text{ iff } \forall x \exists z\, T_1(n(f),x,z) \cdot \wedge \cdot$$
$$\forall x \forall y \forall z \forall t \left[T_1(n(f),x,z) \cdot \wedge \cdot \right.$$
$$T_1(n(f),x+y,t) \Rightarrow$$
$$\left. |r(U(z)) - r(U(t))| < 2^{-x}\right]$$

The functions T_1 and U occur in Kleene's Normal Form Theorem.

Each element $n(f) \in N(\mathbb{R})$ determines a recursive real $\alpha_{n(f)}$, but the correspondence is not one-one for the reason that different Gödel numbers may determine the same function, and different functions may determine the same recursive real. Taking this into consideration, we reduce $N(\mathbb{R})$ modulo the equivalence relation

$$f \sim g \text{ iff } \alpha_f = \alpha_g.$$

The ordered pair $\langle N(\mathbb{R}), \sim\rangle$ is called a notation system for the set of recursive real numbers. The extension to a notation system for the recursive elements of $\mathbb{R}^n (n > 0)$ is straightforward: let

$$N(\mathbb{R}^n) = \{\langle n(f_1),\ldots,n(f_n)\rangle : n(f_1),\ldots,n(f_n) \in N(\mathbb{R})\}$$

for $\langle n(f_1),\ldots,n(f_n)\rangle df := \prod_{j=0}^{n-1} P_j^{n(f_{j+1})}$ where $\{P_0,\ldots,P_{n-1}\}$ is the set of n initial prime numbers. The equivalence relation $\sim_n$ on $N(\mathbb{R})$ is defined as

$$\langle n(f)\rangle \sim_n \langle n(g)\rangle \text{ iff } \{[\langle n(f)\rangle, \langle n(g)\rangle \varepsilon N(\mathbb{R}^n)] \cdot \Lambda \cdot \forall i < n[n(f_i) \sim n(g_i)]\}.$$

Definition 2.3: A recursive metric space in the sense of Moschovakis [1964] is an abstract notation system $\langle T,\sim\rangle$ together with a binary recursive operator $D\colon T \to \mathbb{R}$ such that:

(i) $\forall\alpha, \beta\ \varepsilon\ T[D(\alpha,\beta) = 0 \text{ iff } (\alpha = \beta)]$

(ii) $\forall\alpha, \beta\ \varepsilon\ T[D(\alpha,\beta) = D(\beta,\alpha)]$

(iii) $\forall\alpha, \beta, \gamma\ \rho\ T[D(\alpha,\gamma) \leq D(\alpha,\beta) + D(\beta,\gamma)]$.

The convention of Moschovakis [1964] is to treat the equivalence classes in $\langle T,\sim\rangle$ as points in the metric space. The operator D is obtained by extending the appropriate partial function on ω to the equivalence classes of $\langle T,\sim\rangle$ in the usual way.

Consider now the notation systems $\langle N(\mathbb{R}^n,\sim\rangle$ and $\langle N(\mathbb{R}^n,\sim_n\rangle$. We construct the recursive metric spaces

$$M(\mathbb{R}) = \langle\langle N(\mathbb{R}),\sim\rangle,D_{\mathbb{R}}\rangle$$

and

$$M(\mathbb{R}^n) = \langle\langle N(\mathbb{R}^n),\sim_n\rangle,D_{\mathbb{R}^n}\rangle$$

by taking the operators $D_{\mathbb{R}}$ and $D_{\mathbb{R}^n}$ to be defined as

$$D_{\mathbb{R}}\ df: = |o|$$

and

$$D_{\mathbb{R}n}\ df: = [\sum_{j<n} (|o|j)2]1/2.$$

It can be shown that the operators $D_{\mathbb{R}}$ and $D_{\mathbb{R}^n}$ are in fact recursive on ω with restrictions partially recursive on $N(\mathbb{R})$ and $N(\mathbb{R}^n)$; and the recursive metric spaces $M(\mathbb{R})$ and $M(\mathbb{R}^n)$ can be given a concrete representation by topological means (cf. Lewis [1985a] and the discussion of Appendix II for details). Specifically, the rationals Q are each recursive real numbers and are dense in $\mathbb{R}$. One first forms the recursive metric spaces $QM(\mathbb{R})$ and $QM(\mathbb{R}^n)$ consisting of the rational elements only and then takes the recursive completion of both to obtain the spaces $M(\mathbb{R})$ and $M(\mathbb{R}^n)$ respectively.

Definition 2.4: Let X be a compact, convex subset of $\mathbb{R}^n_+ (n > 0)$. Denote by $R(X)$ that subset of the recursive metric space $M(\mathbb{R}^n)$ given by

$$R(X) = \mathrm{rcl}\{\alpha \in QM(\mathbb{R}^n) \colon \exists x \in X \cdot \Lambda \cdot t(x) \in \alpha\}$$

for $t \colon \mathbb{R}^n \to Q^n(\mathbb{R})$ the function that associates n-tuples of real numbers with Gödel numbers of a notation derived from a fixed approximation by members of Q^n and where rcl denotes the recursive closure in the natural topology induced by the metric on $QM(\mathbb{R}^n)$. Let $\mathbb{F}_R$ be the class of recursive subsets of $R(X)$, i.e.

$$\mathbb{F}_R = \{A \in P(R(X)) \text{ and } A \text{ is recursive}\}.$$

We term the pair $\langle R(X), \mathbb{F}_R\rangle$ a recursive space of alternatives in $M(\mathbb{R}^n)$. A set-function $\mathcal{C} \colon \mathbb{F}_R \to P(R(X))$ such that

$$\forall A \in \mathbb{F}_R[\mathcal{C}(A) \subseteq A]$$

is termed a recursive choice on $\langle R(X), \mathbb{F}_R \rangle$.

Definition 2.5: If $\langle R(X), \mathbb{F}_N \rangle$ is a recursive space of alternatives in $M(\mathbb{R}^n)$, a recursive choice $\mathcal{C}$ on $\langle R(X), \mathbb{F}_R \rangle$ is recursively rational if the following two items exist:

(1) $\geq: \quad R(X) \times R(X) \to \{0,1\}$

(2) $f: \quad R(X) \to \omega$ such that:

(a) f is potentially partially recursive

(b) $\forall \alpha, \beta \in R(X)\ [\alpha \geq \beta \Rightarrow f(\alpha) \geq f(\beta)]$

(c) $\forall A \in \mathbb{F}_R[\mathcal{C}(A) = \{\alpha: \ \forall \beta \in A(f(\alpha) \geq f(\beta))\}$.

A recursive choice on $\langle R(X), \mathbb{F}_R \rangle$ is recursively rational if the action of $\mathcal{C}$ on $\mathbb{F}_R$ can be explained in terms of a binary relation on $R(X)$ such that the relaton has a computable representation. The requirement that the function f be p.p.r. (potentially partial recursive) is the weakest restriction possible, since f is not required to be recursive, i.e. f is p.p.r. if for some $g \in {}^{\omega}\omega$ partial recursive, $g \upharpoonright Dom(f) = f$.

Definition 2.6: Let $\langle R(X), \mathbb{F}_R \rangle$ be a recursive space of alternatives and let $\mathcal{C}$ be a recursive choice on $\langle R(X), \mathbb{F}_R \rangle$. If g is an index for a Σ_1^0-subfamily $\{\mathbb{F}_{R_j}\}_{j<\omega}$ of $\mathbb{F}_R$, then we say that

$\mathcal{C}_g = \mathcal{C}\restriction\{\mathbb{F}_{R_j}\}_{j<\omega}$ is a single-player choice-function game on $\langle R(X),\mathbb{F}_R\rangle$ indexed by g.

Definition 2.7: Let $\mathcal{C}_g$ be an indexed single-player choice-function game on a recursive space of alternatives $\langle R(X),\mathbb{F}_R\rangle$. $\mathcal{C}_g$ is recursively presentable if and only if $\mathcal{C}\restriction\{\mathbb{F}_{R_j}\}_{j<\omega}$ is a Banach-Mazur operator, i.e. given the index g for $\mathrm{Dom}(\mathcal{C}_g) = \{\mathbb{F}_{R_j}\}_{j<\omega}$, there is an index $h \in {}^{\omega}\omega$ that makes $\mathrm{Rng}(\mathcal{C}_g) = \{\mathbb{F}_{R_j})\}_{j<\omega}$ a Σ^0_1-subfamily of $\mathbb{F}_R$.

The next theorem shows that in order to get a recursively presented single person choice function game on $\langle R(X),\mathbb{F}_{R},\rangle$, it is sufficient to let $\mathcal{C}_g$ be recursively rational.

Theorem 2.1: If $\mathcal{C}$ is recursively rational on $\langle R(X),\mathbb{F}_R\rangle$ and if $\mathcal{C}_g = \mathcal{C}\restriction\{\mathbb{F}_{R_j}\}_{j<\omega}$ is an indexed single-player choice-function game, then $\mathcal{C}_g$ is Banach-Mazur and thus recursively presentable.

Proof: We use the index g for $\{\mathbb{F}_{R_j}\}_{j<\omega}$ and the lemma below, the proof for which is not difficult and therefore is omitted.

Lemma 2.1.1: If $\mathcal{C}$ is recursively rational on $\langle R(X),\mathbb{F}_R\rangle$, then for any A in $\mathbb{F}_R$, $\mathcal{C}(A) \in \mathbb{F}_R$.

Definition 2.8: Let $\mathcal{C}_g$ be an indexed recursively presentable single-player choice-function game with $\mathrm{Dom}(\mathcal{C}_g) = \{\mathbb{F}_{R_j}\}_{j<\omega}$ indexed by g and $\mathrm{Rng}(\mathcal{C}_g) = \{\mathcal{C}_g(\mathbb{F}_{R_j})\}_{j<\omega}$ indexed by h. The graph of $\mathcal{C}_g$ is the set of pairs $\{\langle\mathbb{F}_{R_j}, \mathcal{C}_g(\mathbb{F}_{R_j})\rangle\}_{j\in N}$ in the space

$P(R(X)) \times P(R(X))$ indexed by $f = g \times h$. We will say that graph $(\mathcal{C}_g)$ has <u>full domain</u> if for some $k \in \omega$ and all pairs $i \neq j > k$, $\mathbb{F}_{R_j} \Delta \mathbb{F}_{R_i} \neq \emptyset$.

It follows that a full domain in $\mathbb{F}_R$ is not a null sequence of sets and contains infinitely many distinct elements that are enumerated effectively.

We have shown elsewhere that any non-trivial recursively presented single-person choice-function game with full domain cannot have a recursive graph (Lewis [1985a] Theorem 3.1). In fact more is true. Such games do not even have an R.E. graph. We show this in a theorem that places a lower bound on $\deg(\text{graph}(\mathcal{C}_g))$ at $\underset{\sim}{0}''$ (cf. Lewis [1985b]). For the proof of the first theorem, we use a reduction-type of argument for determining the degree of bounded sets of recursive real numbers due in origin to N.Z. Shapiro [1956].

<u>Theorem 2.2</u>: Let $\langle R(X), \mathbb{F}_R\rangle$ be a recursive space of alternatives and let $\mathcal{C}$ be a non-trivial recursively rational choice function. If g is a Σ_1^0-index of some full domain $\{\mathbb{R}_{R_j}\}_{j<\omega}$, then $\deg(\text{graph}(\mathcal{C}_g)) = \deg(\text{graph}(\mathcal{C} \upharpoonright \{\mathbb{F}_{R_j}\}_{j<\omega}))$ is bounded weakly below by $\underset{\sim}{0}''$, i.e. $\underset{\sim}{0}'' \leq \deg(\text{graph}(\mathcal{C}_g))$.

<u>Proof</u>: (abbreviated) The first step in the proof is to reduce the notation for $M(\mathbb{R}^n)$ to a sub-notaton for the recursive reals.

Lemma 2.2.1: For an appropriate choice of Gödel numbering, there exists a notation $\hat{\mathbb{N}}(\mathbb{R})$ of R-indices for the recursive real numbers and an injective mapping γ such that $\gamma((\mathbb{N}(\mathbb{R}^n)) \subseteq \hat{\mathbb{N}}(\mathbb{R})$.

The effect of the lemma is that we may consider n-tuples of R-indices in $M(\mathbb{R}^n)$ as R-indices in the notation $\hat{\mathbb{N}}(\mathbb{R})$ of the metric space: $\hat{M}(\mathbb{R}) = \langle(\mathbb{N}\hat{}(\mathbb{R}), \sim_{\hat{\mathbb{N}}(\mathbb{R})}), D_{\mathbb{R}}\rangle$. Since γ is injective, the set $R(X)$ in $M(\mathbb{R}^n)$ and members in the class $\mathbb{F}_R$ have well-define images in $M\hat{}(\mathbb{R})$, e.g. for any $\mathbb{F}_j \in \mathbb{F}_R$ we have its image in $M\hat{}(\mathbb{R})$ given by

$$\gamma(\mathbb{F}_j) = \{\alpha \in \hat{M}(\mathbb{R}): \exists\beta \in \mathbb{F}_j \subseteq M(\mathbb{R}^n).\hat{}.\ \gamma(\beta) = \alpha\}$$

Recall that a relation is potentially partially Σ^0_n (or Π^0_n) if it has an extension which is partially Σ^0_n (or Π^0_n). The key result used in the proof is the Shapiro Extension Lemma.

Lemma 2.2.2: (Shapiro [1956]) Let $\bigcup_{j<\omega} P(\mathbb{N}^j)$ denote the set of all finitary relations on $\mathbb{N}$ and assume that $\Gamma \in \bigcup_{j<\omega} P(\mathbb{N}^j)$ is potentially partially Σ^0_n (or Π^0_n). Then Γ is the restriction of some $\Phi \in \bigcup_{j<\omega} P(\mathbb{N}^j)$ such that Φ is Σ^0_n (or Π^0_n).

This significance of the Shapiro Extension Lemma is that it allows the extension of the taxonomy of complexity provided by the Kleen-Mostowski Hierarchy to the domain of the partial recursive predicates.

Lemma 2.2.3: The image, under the mapping γ, of the co-domain of a non-trivial recursive rational choice with full domain is contained within a bounded interval of $\hat{M}(\mathbb{R})$.

Since bounded intervals in $\hat{M}(\mathbb{R})$ are effectively indicated in the sense of Shapiro [1956], if the assumption that the co-domain of a recursive rational choice function with full domain is itself recursive a contradiction is reached using Lemma 2.2.2 since the characteristic function of finitely many recursive reals within an effectively indicated interval $\hat{M}(\mathbb{R})$ cannot be extended to be recursive, and therefore cannot be the restriction of any recursive function. Since the co-domain of a recursive rational choice function is a projection of the graph of such a function the theorem follows by corollary XI of Rogers [1967] p. 66, and Theorems II.15 and II.5 of Shapiro [1956]. □

Corollary 2.3: Let $\langle R(X), \mathbb{F}_R \rangle$ be a recursive space of alternatives and let $\mathcal{C}$ be a non-trivial recursively rational choice function. If g is a Σ_1^0-index of some full domain $\{\mathbb{F}_{R_j}\}_{j<\omega}$, then $\deg(\mathrm{graph}(\mathcal{C}_g))$ is not R.E.

Proof: If A is R.E., then $\deg(A) \leq \underset{\sim}{0}'$. □

Theorem 2.2 and Corollary 2.3 have a very useful application to assessing the degrees of Walrasian models of general equilibrium. We give a brief sketch of this application.

Definition 2.9: A Walrasian model of general equilibrium is a two-sorted structure:

$$\mathfrak{A} = \langle \mathbb{R}^{(m+n)\ell}\ ,\ I.J.\ \{(X_i,\xi_i)\}_{i\varepsilon I},\{(Y_j,\eta_j)\}_{j\varepsilon J}\rangle.$$

It is assumed that ℓ is the dimension of the commodity space and the structure has two sorts of variables: I, of cardinality m for consumers, and J, of cardinality n for producers. The pairs $\{(X_i,\xi_i(p,w))\}_{i\varepsilon I}$ and $\{(Y_j,\eta_j(p)\}_{j\varepsilon J}$ represent the feasible space of alternatives along with the agent's criterion function for both sorts. The sets $\{X_i\}_{i\varepsilon I}$ are assumed to be compact and convex in $\mathbb{R}^n$. The reader who is interested in more detail on these matters is referred to the treatise of Debreu [1959].

Given a structure of the form $\mathfrak{A}$, it is clear how one obtains recursive presentations. First, one constructs a recursive metric space for $\mathbb{R}^{(m+n)\ell}$, $M(\mathbb{R}^{(m+n)\ell})$ and then next allows $R(X_i)$ and $R(Y_j)$ to be the appropriate image in $M(\mathbb{R}^{(m+n)\ell})$ for each $i \varepsilon I$ and each $j \varepsilon J$, respectively. The response functions for each sort of agent are choice functions and so to obtain recursive presentations we use the earlier framework and select Σ_1^0-indexed sequences of recursive sets in $R(X_i)$ and $R(Y_j)$, $\{\mathbb{F}^i_{R_t}\}_{t<\omega}$ and $\{G^j_{R_t}\}_{t<\omega}$ and note that from Debreu [1959] the real-valued functions that are order-preserving homorphisms of preference structures on X_i and Y_j can be taken to be recursive on $R(X_i)$ and $R(Y_j)$ for all $i \varepsilon I$ and $j \varepsilon J$.

The task of a Walrasian model of general equilibrium is to generate prices $p \varepsilon \mathbb{R}^\ell_+$ such the choices made from the correspondences $\xi_i(p.w)$ and $\eta_j(p)$ are optimal in terms of order-preserving homomorphisms of the preference structures on X_i and Y_j for all $i \varepsilon I$ and $j \varepsilon J$.

Clearly, this task can be no less in complexity for a recursively presented $\mathfrak{A}$ with Σ_1^0-indices for sequences of recursive sets in $R(X_i)$ and $R(Y_j)$ for all $i \in I$ and $j \in J$, than the complexity of realizing the single-person choice function games associated with the two sorts of agents. And so, making the obvious applications of Theorem 2.2 we have the straightforward result below.

Let $\mathbb{F}_R^i$ and G_R^j denote the class of recursive subsets of $R(X_i)$ and $R(Y_j)$ in $M(R^{(m+n)\ell})$ respectively, for all $i \in I$ and $j \in J$.

Theorem 2.4: Let $M(\mathbb{R}^{(m+n)\ell})$ be a recursive metric space and let $\mathfrak{A}_{g\times h}$ be a recursively presented model of Walrasian general equilibrium such that $g: I \times \omega \to \omega$ and $h: J \times \omega \to \omega$ provide uniform Σ_1^0-indices for full sequences of recursive subsets $\{\mathbb{F}_{R_t}^i\}_{t<\omega} \subseteq R(X_i)$ and $\{G_{R_t}^j\}_{t<\omega} \subseteq R(Y_j)$ for all $i \in I$ and $j \in J$ and such that the choice functions $\mathcal{C}^{\xi_i}$ and $\mathcal{C}^{\eta_j}$ are non-trivial and recursively rational for all $i \in I$ and $j \in J$ on $\mathbb{F}_R^i$ and G_R^j respectively. Then $\underset{\sim}{0}'' \leq \deg(\mathfrak{A}_{g\times h})$ and thus $\deg(\mathfrak{A}_{g\times h})$ is not R.E.

Proof: It is possible to formalize the remarks immediately preceding the statement of the theorem into the following inequality:

$$\deg(\mathfrak{A}_{g\times h}) \geq \sup\{\{\deg(\mathcal{C}^{\xi_i})\}_{i\in I},\{\deg(\mathcal{C}^{\eta_j})_{j\in J}\}\} \geq \underset{\sim}{0}'' \qquad \square$$

From Arrow and Debreu [1954] Walrasian models of general equilibrium are strategically equivalent to N-person non-cooperative games in the sense of Nash [1950]. Obviously, any two recursive presentations of such structures would bi-equivalent in terms of the Turing degree of complexity of realizing their tasks.

Definition 2.10: An N-person non-cooperative game in the sense of Nash is a structure

$$\mathcal{L}_{a} = \langle N, \{S_j\}_{j\varepsilon N}, \{\phi_j\}_{j\varepsilon N}\rangle$$

where N is a finite nonempty set of players, and each S_j is an $r - 1$ dimensional simplex for all $j \varepsilon N$, and where the functions $\{\phi_j\}_{j\varepsilon N}$ are such that

$$\phi_j : \prod_{j=1}^{n} S_j \to \mathbb{R} .$$

Suitable recursive presentations for structures of the form can be constructed by first obtaining a recursive metric space for $\mathbb{R}^{N_r}$ and then taking the functions $\{\phi_j\}_{j\varepsilon N}$ to be uniformly recursive in j on $\sum_1^0$-indexed enumerations in the product space

$$R(S_{j_1}) \times \cdots \times R(S_{j_N})$$

where $R(S_j)$ is the image of S_j in the recursive metric space for $\mathbb{R}^{N_r}$.

Theorem 2.5: Let $M(\mathbb{R}^N r)$ be a recursive metric space and let $\mathcal{L}_g$ be a recursively presented N-person non-cooperative game such that $g: N \times \omega \to \omega$ is a uniform $\sum_1^0$-index for sequences of recursive subsets $\{\mathbb{F}^j_{R_t}\}_{t<\omega} \subseteq R(S_j)$ and such that $\{\mathbb{F}^j_{R_t}\}_{t<\omega}$ is full for all $j \in N$. If $\mathcal{L}_g$ is nontrivial, then $\underset{\sim}{0}'' \leq \deg(\mathcal{L}_g)$.

<u>Proof</u>: This theorem follows from Theorem 4.4 and the fact that $\mathcal{L}_g$ is strategically equivalent to some recursively presented Walrasian model of general equilibrium.

The reader is advised to consult and Arrow and Debreu [1954] for the details of the strategic equivalence of maximizing behavior for the participants in Walrasian game-theoretic structures and Nash N-person noncooperative game-theoretic structures. The structures that generalize the Nash N-person noncooperative game-theoretic structures to Walrasian models and provide strategic equivalence in this setting are the <u>abstract economies</u> of the form:

$$\mathcal{L}_V = \langle V, U_1, \ldots, U_V, f_1, \ldots, f_V, A_1(\bar{a}_1), \ldots, A_V(\bar{a}_V)\rangle$$

where the U_1 are subsets of $\mathbb{R}^n$ $(n \geq 1)$ and an equilibrium point for the game associated with $\mathcal{L}(V)$ is a point $a^* = \langle a_1^*, \ldots, a_V^*\rangle \in \prod_{i=1}^{v} U_i \subseteq \mathbb{R}^{nv}$ such that (i) $a_i^* \in A_i(\bar{a}_i^*)$ for all $i = 1, \ldots, v$ and (ii) $f_i(\bar{a}_i^*, a_i^*) = \max_{a_i \in A_i(\bar{a}_i^*)} f_i(\bar{a}_i^*, a_i)$ for all $i = 1, \ldots, v$. The notation $\bar{a}_i^*$ denotes the deleted vector $\bar{a}_i^* = \langle a_i^*, \ldots, a_{i-1}^*, a_{i+1}^*, \ldots, a_v^*\rangle$. Given strategic equivalence for structures of the form $\mathcal{L}(V)$ and Walrasian models, the theorem is obvious when $\mathcal{L}(V)$ is given a suitable recursive presentation since the class of structures of the form $\mathcal{L}(V)$ contains the normal-form Nash N-person noncooperative structures $\mathcal{L} = \langle N, \{S_j\}_{j\in N}, \{\phi_j\}_{j\in N}\rangle$ as a subclass.

3. A General Impossibility Result

In this section, we show that the impossibility result of Kramer [1974] is a special instance of Theorem 2.2.

The following definitions will be required:

Definition 3.1: By an alphabet one means an ordered finite set of primitive symbols, denoted as K.

Definition 3.2: By a string is meant a finite line or sequence of elements in K and we say that the string is in K.

Definition 3.3: For a set $\{x_j\}_{j=1}^{n} \subset K$, let $(x_1,\ldots,x_n)$ be the string formed by $\{x_j\}_{j=1}^{n}$. Then the length of the string is n.

Definition 3.4: Let x and y be strings in K then the string xy is in K and is termed the concatenation of x and y, which is defined by $(x_1,\ldots,x_n,y_1,\ldots,y_m)$ and is of length $n + m$.

By an elementary formal system over an alphabet K, (ξ), is meant the following set of items (1)-(5).

(1) The alphabet K.

(2) An alphabet of symbols, V, the variables.

(3) An alphabet of symbols, P, the predicates; each of finite degree.

(4) A pair of symbols, $(\rightarrow, ,)$ called implication and punctuation.

(5) A finite sequence, $A_1, \ldots, A_n$ of wffs. termed the axioms of (ξ). A wff. of (ξ) is an expression of the form $P \ t_i, \ldots, t_m$ for $t_1, \ldots, t_m$ terms, or $F_1 \rightarrow F_2 ' \ldots \rightarrow F_n$ where each F_j has the form $Pt_1, \ldots, t_m$.

We let $\tilde{K}$ denote the set of all finite strings in K and let the term attributes denote a set in $\tilde{K}$ or a member of P.

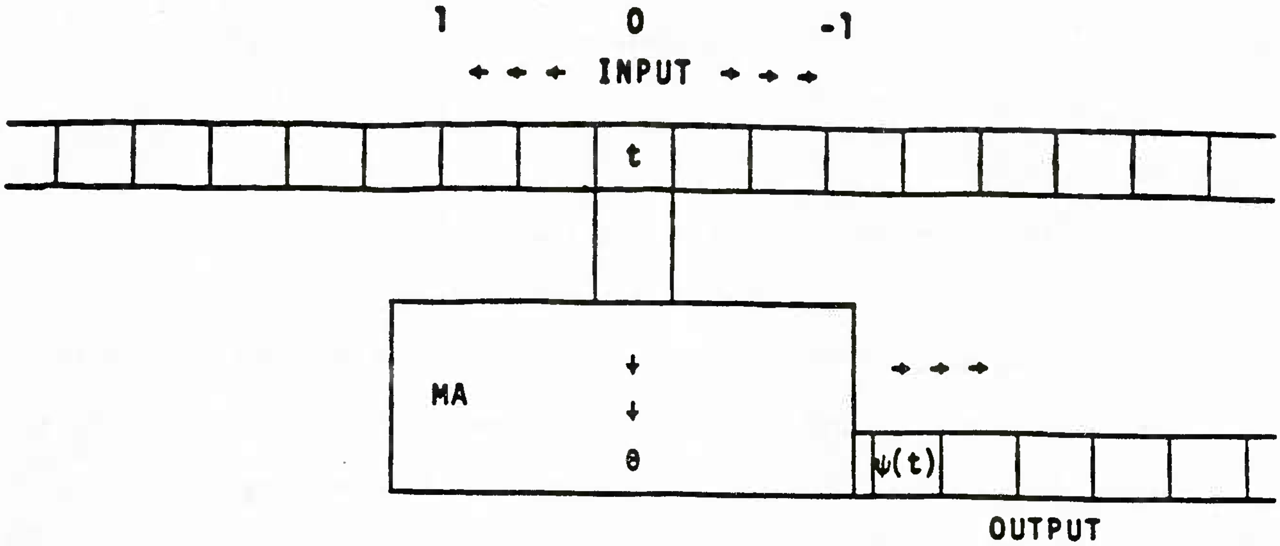
Turning now the framework of Kramer [1974], we now consider whether or not a rational choice function, in the sense of Richter, admits of a constructive representation within a formal system that we shall interpret by finite automata.

Consider a computing device that has k components, each of which can obtain m states, where m and k are finite integers. Let the description of the machine's computing process be given by the following:

$$MA = \langle s, t, \theta(s), \psi(t), \ m = \{1, 0, -1\} \rangle$$

The machine can be viewed as a kind of scanning device that looks at symbols on a tape and then signifies an output. The description, MA, for Mealy automaton (cf. Starke [1972] Ch. I), provides the following kind of rule: If in state s, and if the input symbol t is scanned, then go to the state $\theta(s)$, and signify the output $\psi(t)$, and then move the input tape either right one space, -1; left one space, 1; or leave the tape where it is, 0.

Diagramatically, we can visualize MA as:



Let there be two finite languages (not necessarily distinct) L_I and L_O corresponding to an input language, and an output language, respectively, L_I contains distinguished elements $\{\Delta, \#\}$ that are used to indicate when distinguished segments of the input tape are begun and terminated. In the language L_O there is a symbol Λ to indicate the null output. In the manner of the above definitions, we may construct on each language the elementary formal system $\xi(L_I)$ and $\xi(L_O)$. One can then view the input tape as comprised of strings of wff.s in $\xi(L_I)$, while the output tape can be viewed as strings of wff.s in $\xi(L_O)$. It is then permissible to view the automaton MA as a composite formal system with components $\xi(L_I)$ and $\xi(L_O)$ and ψ a rule of derivation.

Consider now the quotient space $X/\sim$, of indifference classes of a set of infinite alternatives upon which a complete pre-order $\succsim$ is defined. Select then from $2^{X/\sim}$ the class of all finite sets, and call it IF. Let us make the assumption that $X/\sim$ is denumerably infinite. If we assume that any finite set in IF has a well-formed representation as a string in $\xi(L_I)$, then from the denumerability of IF, we may form the input tape T(IF) that encodes the members of IF as wff.s in $\xi(L_I)$.

Denote the totality of states for machine MA as S. Then $\|S\| = k^m$ and is finite, for k the number of components in MA. For a given string x on T(IF), the function:

$$\tau_x = S \times \{-1,1\} \to [S \times \{-1,1\}] \quad \{0\}$$

specifies the transition rule of the machine MA with respect to x. For the pair, (s,-1), the machine is in state s, at the right most symbol of x. If the machine runs the string x to the right and goes to state $\theta(s) = s'$, then set the value of the function $\tau_x(s,-1) = (\theta(s),-1)$.

One sees readily that the total number of such functions for strings on T(IF) is $Q = (2k^m + 1)^{2k^m} = \|R(\tau)\|^{\|D(\tau)\|}$, for $R(\tau) =$ the range of τ and $D(\tau) =$ the domain of τ. Then, the sets $\{T_{\tau_{x_i}}\}_{i=1}^{Q}$ give a partition of T(IF) such that if $x, y \in T_{\tau_{x_i}}$, then $\tau_x = \tau_y$, i.e., each $T_{\tau_{x_i}}$ is an equivalence class of strings that generate the same transition function, τ_{x_i}.

Before proceeding, we need to formalize precisely what is means for a Mealy automaton to realize a function.

<u>Definition 3.5</u>: Allow $f\colon I \to B$ to be a partial function defined on aribitrary sets I and B. A Mealy automaton given as $MA = \langle s,t,\theta(s),\phi(t),\ m = \{1,0,-1\}\rangle$ is said <u>to realize f on I</u> if and only if:

(a) for every $i \in D(f) \subseteq I$ there is a unique wff. in $\xi(L_I)$ that formally represents i.

(b) for every $b \in R(f) \subseteq B$ for which there is an $i \in D(f) \subseteq I$ such that $f(i) = b$, there is a unique wff. in $\xi(L_0)$ that formally represents b.

(c) if t is the wff. in $\xi(L_i)$ that formally represents an $i \in D(f) \subseteq I$, then $\phi(t)$ is the unique wff. in $\xi(L_0)$ that formally represents that $b \in R(f) \subseteq B$ for which $f(i) = b$.

The following is Kramer's impossibility result.

<u>Theorem 3.1</u>: Let $X/\sim$ be denumerably infinite and let $\succsim$ be a complete pre-order on X that rationalizes the choice function $\mathcal{C}(A) = \{x \in A\colon \forall\, y \in A[x \succsim y]\}$ for every $A \in \mathbb{F}$. Then no Mealy automaton of the form: $MA = \langle S,\theta(s),\phi(t),\ m = \{1,0,-1\}\rangle$ can realize $\mathcal{C}$ on $\mathbb{F}$.

<u>Proof</u>: Suppose MA realized the choice function $\mathcal{C}$ when $A \in F$, then A would be encoded by a string $x_1 \ldots x_n$ in $\xi(L_I)$ and appear on $T(\mathbb{F})$, the input tape. Then we require that $\phi(x_1 \ldots x_n\}$ be a

wff. in $\xi(L_0)$. Clearly, the nature $\gtrsim$ requires that if $x_1 \neq x_2$ in $\xi(L_I)$ then $\phi(x_1) \neq \phi(x_2)$ in $\xi(L_0)$.

It can be demonstrated that for an input tape segment $x_1 \ldots x_n$ such that $x_i \in T_{\tau_{x_i}}$, if the machine accepts the segment scanning to the right and printing $\phi(x_1 \ldots x_n)$, then

$$Ln(\phi(x_1 \ldots x_n)) = \sum_{j=1}^{n} f_j(x_j)$$

where $Ln(\phi)$ is the length of the output string $\phi(x_1 \ldots x_n)$ and each of the functions are such that $f_j\colon T_{x_j} \neq \omega$. (cf. Kramer [1974] p. 48-49) Since the cardinality of inputs on $T(\mathbb{F})$ at least one member of $\{T_{\tau_{x_i}}\}_{i=1}^{Q}$, $T_{\tau_{x_0}}$, must contain infinitely many substrings, representing distinct members of $\mathbb{F} \subseteq 2^{X/\sim}$. The set $P = \{x_{01}, x_{02}, x_{03}, \ldots, x_{0n}, \ldots\}$ can then be formed in terms of distinguished singletons, one each from the members of $T_{\tau_{x_0}}$.

One sees readily that all sets of the form $\{\alpha_i\} \cup \{\alpha_j\}$ or $\{\alpha_j\} \cup \{\alpha_i\}$ must yield output strings of identical length, when represented as $x_{0i}x_{0j}$ or $x_{0j}x_{0i}$ in $\xi(L_I)$, i.e. $Ln(\phi(x_{0i}, x_{0j})) = Ln(\phi(x_{0j}, x_{0i}))$. Clearly, for distinct $\alpha_i, \alpha_j \in X/\sim$, either $\alpha_j \gtrsim \alpha_i$ or $\alpha_i \gtrsim \alpha_j$ 1/, and if $A = \{\alpha_i\} \cup \{\alpha_i\}$. then, we have $\mathcal{C}(A) = \alpha_i$. Then it follows that $LN(\phi(x_{0i}x_{0j})) = Ln(\phi(x_{0j}, x_{0i})) = W$, where $W = Ln(\phi(x_{0i}))$ or $Ln(\phi(x_{0j}))$ depending on whether $\alpha_i \gtrsim \alpha_j$ or $\alpha_j \gtrsim \alpha_i$ respectively.

However, the output alphabet is finite for $\xi(L_0)$, say of cardinality n. Then the number of distinct strings in $\xi(L_0)$ of length W is bounded sharply by $(n)^W$.

Then for q sufficiently large, say $q > (n)^L$, if i^*, $j^* > q$, then if x_{0i*} and x_{0j*} represent α_{i*} and α_{j*} respectively, since both i^* and j^* are in excess of q, $Ln(\phi(z_{0i*})) = Ln(\phi(x_{0j*}))$ must imply that $\phi(x_{0i*}) = \phi(x_{0j*})$, and therefore if MA were to realize $\mathcal{C}(A)$ for $A = \{\alpha_{j*}\}$ or $\{\alpha_{i*}\}$, $\mathcal{C}(\{\alpha_{j*}\}) = \mathcal{C}(\{\alpha_{i*}\})$. But, as $x_{0j*} \neq x_{0i*}$ only if $\alpha_{j*} \neq \alpha_{i*}$, by the preference structure, no common choice is possible. □

We will now obtain Theorem 3.1 as a corollary to Theorem 2.2.

Theorem 3.2: Theorem 2.2 implies Theorem 3.1.

Proof: We give proof in stages.

First, by means of an admissible coding, the alphabet and the wff.s of an elementary formal system (ξ) can be arithmetized by means of a Gödel numbering[2/]. By an admissible coding is simply meant as an injection α: $\tilde{K} \to \omega$, and any element of $\tilde{K}$ is said to be coded by α. Under the simplifying assumption that for MA the input and output languages are identical, and thus that $(\xi) = \xi(L_I) = \xi(L_0)$. Inputs t, and output $\phi(t)$ of the Mealy automaton become strings of elements of ω by way of $\alpha(t)$ and $\alpha(\phi(t))$; where if t (or $\phi(t)$) is a string of (ξ) of the form $x_1 \ldots x_n$, then $\alpha(t)$ (or $\alpha(\phi(t))$ is a string of the form $\alpha(x_1)\ldots\alpha(x_n)$ comprised of elements of ω. Taking now the strings of (ξ) as wff.s the occurrences of "wff. in $\xi(L_I)$" or "wff. in $\xi(L_0)$", may be replaced by "a set of elements in ω" by means of the coding α. The strings of ω generated thus by α may further be reduced to elements of ω by the mapping β: $\alpha(\tilde{K}) \to \omega$ where $\prod_{j=0}^{n-1} P_j^{\alpha(x_{j+1})}$ for t of the form $x_1 \ldots x_n$, and $\{P_0,\ldots,P_{n-1}\}$ the set of n initial primes.

<u>Second</u>, if we are given a partial function f: $\tilde{K} \to \tilde{K}$, by means of the mapping $\gamma = \beta o \alpha$, we code f by means of a function $\hat{f}$: $\omega \to \omega$ such that $\hat{f}$ maps the code of elements in $\tilde{K} \cap D(f)$ to the code of elements in $R(f) = \tilde{K}$ under γ. In explicit terms, we may take $\hat{f}$ to be given by

$$\hat{f} = \gamma o f o \gamma^{-1} .$$

<u>Third</u>, the definitions of effectively computable function derived from Church's Thesis and the equivalences for partial functions with domain in $\tilde{K}$ and range $\tilde{K}$ by way of stipulating that the partial function $\hat{f}$: $\tilde{K} \to \tilde{K}$ is effectively computable if and only if the function $\hat{f}$: $\omega \to \omega$ is effectively comptable in terms of one of the equivalences. In particular, f: $\tilde{K} \to \tilde{K}$ is recursive if and only if $\hat{f}:\mathbb{N} \to \mathbb{N}$ is recursive. Similarly, it follows that a set $A \subseteq \tilde{K}$ is a recursive attribute if and only if the characteristic function of $\alpha(A)$, $\chi_{\alpha(A)}$ is a recursive function. Since, therefore, any string in

K is finite, the wff.s of (ξ) are recursive attributes from the fact that any finite set of elements of ω is recursive.

Finally, the need is to show that a choice function that is rationalized by a complete pre-order defined on $X/\sim$ is realized on $\mathbb{F}$ by a Mealy automaton only if the numerical representation of the choice function under the codings induced by α and γ is recursively realizable. Assume then that $\mathcal{C}$ on $\mathbb{F}$ is realized by some Mealy automaton. Then from the fact that each distinct finite set of $X/\sim$ is given a representation in (ξ) as a unique wff., and if t is a wff. of (ξ) then $\alpha(t)$ is a recursive subset of ω, and distinct enumeration of the members of $\mathbb{F}$ by way of their representations in (ξ) becomes an enumeration of recursive subsets of ω under the coding α. If we next allow $\alpha(X/\sim)$ and $\alpha(\mathbb{F})$ to denote the image in ω under α of wff.s in (ξ) representing elements of $X/\sim$ and $\mathbb{F}$, $\langle\alpha(X/\sim), \alpha(\mathbb{F})\rangle$ is a recursive sub-space of alternatives, for suitable representations of X[3/], of a recursive metric space. Further, if t is a wff. of (ξ) that represents an element of $\mathbb{F}$, then $\psi(t)$ is the unique wff. of (ξ) that represents the image of that element under the choice function $\mathcal{C}$. Then, since for any $A \in \mathbb{F}$, $\mathcal{C}(A) \subseteq A$, it follows that $\alpha(\psi(t))$ must also be recursive from the fact that any $A \in \mathbb{F}$ is finite and $\psi(t)$ is a wff. of a subset of A. Then it follows that if $F_j \in \mathbb{F}$ for all $j \in \omega$, for the following enumerated pairs under the coding α, we obtain the fact that

$$\{\langle\alpha(F_j),\ \alpha(\mathcal{C}(F_j))\rangle\}_{j\varepsilon\mathbb{N}} \subseteq \alpha(\mathbb{F}) \times \alpha(\mathbb{F})$$

and the rest of the argument is clear from the following.

Proposition 3.3: Let $X/\sim$ be denumerably infinite and let $\gtrsim$ be a complete pre-order on X that rationalizes the choice function $\mathcal{C}(A) = \{x \ \varepsilon\ A\colon\ \forall y \ \varepsilon\ A[x \gtrsim y]\}$ for every $A \ \varepsilon\ \mathbb{F}$. Then if $\mathcal{C}$ is realized by a Mealy automaton of the form $MA = \langle S,\theta(s),\phi(t),$ $m = \{1,0,-1\}\rangle$, there exists a Turing machine T_M such that under the coding $\gamma = \beta o \alpha$, T_M computes the values of $\gamma(\mathcal{C}(A))$ from $\gamma(A)$ for all $A \ \varepsilon\ \mathbb{F}$.

Proof: Since the mapping α and β are both injective, if MA realizes the choice function $\mathcal{C}$ on $\mathbb{F}$ then under the coding $\gamma = \beta o \alpha$, MA also realizes the function $\hat{\mathcal{C}}\colon\ \omega \to \omega$ on $\gamma(\mathbb{F}) = \{n \ \varepsilon\ \omega\colon\ \exists\ A \ \varepsilon\ \mathbb{F}\ [\gamma(A) = n]\}$, where $\hat{\mathcal{C}} = \gamma o \mathcal{C} \gamma^{-1}$. This is merely the result of renaming the wff.s in (ξ) that represent the member of $\mathbb{F}$ and the members of $\mathcal{C}(\mathbb{F}) = \{\mathcal{C}(A)\colon\ \exists\ A \ \varepsilon\ \mathbb{F}\}$ by way of arithmetization.

Next, we represent elements of ω on the encoded input tape of MA by finite strings of 1's so that if $x \ \varepsilon\ \omega$, $\overline{x}$ is a finite sequence of 1's representing x on the input tape of MA. Observe that if MA realizes $\hat{\mathcal{C}}$ on $\gamma(\mathbb{F})$, then by Theorem 1 of Appendix 2 of Ritchie [1963], for every $A \ \varepsilon\ \mathbb{F}$, there exists a finite set of instantaneous descriptions of MA, $\{D_1,\ldots,D_n\}$ where for $j = 1,\ldots,n$ each $D_j = (\overline{\quad x},s,p)$, for $s \ \varepsilon\ S$ and p the number of the square of the input tape that corresponds to the rightmost 1 of

$\overline{x}$ for $x \in \omega$, and such that $D_1 = (\overline{\gamma(A)}, s_0, m)$ and $D_n = (\overline{\hat{\mathcal{C}}(A)}, f, \ell)$ where s_0, $f \in S$ denote the initial and final states of MA respectively, and $m, \ell \in \omega$ such that $\ell = m + Ln(\overline{\hat{\mathcal{C}}(A)}) \leq \|S\| + Ln(\overline{\gamma(A)})$, for $LN(\overline{\hat{\mathcal{C}}(A)})$ and $Ln(\overline{\gamma(A)})$ the respective lengths of $\overline{\hat{\mathcal{C}}(A)}$ and $\gamma(A)$. Furthermore, for $j > 1$, each instantaneous description D_j is obtained from its predecessor by means of the following scheme:

$D_j \to D_{j+1}$ if and only if either

(a) $D_j = (\overline{x}, s, p)$ and $D_{j+1} = (\overline{y}, s', p+1)$

if $1 \leq j < n$ and $m \leq p < \ell - 1$,

or

(b) $D_j = (\overline{x}, s, \ell-1)$ and $D_{j+1} = (\overline{\hat{\mathcal{C}}(A)}, f, \ell)$

if $j = n - 1$

The proposition now follows by setting values of a Turing machine $T_M : S \times \{0,1\} \to \{0,1\} \times \{L,R\} \times S$, the set of states of MA, in correspondence to the successive derivations of the instantaneous descriptions $\{D_1, \ldots, D_n\}$. $\square$

By the proposition, if a Mealy automaton realized the choice function $\mathcal{C}$ on $\mathbb{F}$, then under the coding γ, $\langle \gamma(F_j), \gamma(\mathcal{C}(F_j)) \rangle\}_{j \in \omega}$ would have to be a subset of $\omega \times \omega$ of degree less than $\underset{\sim}{0}'$ by way of the equivalence between functions computable by

Turing machines and recursive functions. However, by Theorem 2.2 this cannot occur, and the proof is finished. □

APPENDIX I

In this appendix we provide a brief description of Turing machines, and demonstrate that, within the framework of the descriptions, finite automata can be viewed as a special instance of Turing machines.

(A) Turing Machines

Definition 1: By an alphabet A, we will mean a finite set of elements called symbols which includes a distinguished symbol B, termed the blank symbol.

Definition 2: A Turing machine Z over the alphabet A is a quadruple (S,m,s_0,f) when S is a finite set, s_0 and f are elements of S, and $m: A \times (S - \{f\} \rightarrow A \times S \times \{1,-1,0\}$. The set S is called the set of states of Z, s_0 the initial state, f the final state, and m the transition function.

Definition 3: For a given Turing machine $Z = (S,m,s_0,f)$ over an alphabet A, an instantaneous description of Z is a triplet (t,s,p) for t a finite sequence of elements of A; p positive integer not greater than the length of t, and s as an element of S. t is called the tape in Z, p the number of the scanned square, and s the state of Z.

Definition 4: For a given Turing machine $A = (s,m,s_0,f)$ over an alphabet A, the yield operation $\rightarrow$, on instantaneous descriptions of Z is defined as follows $X(Z) \rightarrow Y(Z)$ if and only if at least one of the following obtains where a_i and b_i are in A for all positive integers i:

1. $X = (a_1 \ldots a_n, s, p)$ and $Y = (b_1 \ldots b_n, s', p')$ with $a_j = b_j$ for all $j \neq p$, $m(a_p, s) = (b_p, s', p', -p)$ and either $p < n$ or $p = p' = n$.

2. $X = (a_1 \ldots a_{n-1} a_n^p, s, n)$ and $Y = (a_1 \ldots a_{n-1} b_p^\beta, s', n+1)$ where $m(a_n, s) = (b_n, s', 1)$.

3. $X = (a_1 \ldots a_{n-1} a_n, s, n)$ and $Y = (a_1 \ldots a_{n-1} b_n, s', n-1)$ where $m(a_n, s) = (b_n, s, -1)$ and $b_n \neq \beta$.

4. $X = (a_1 \ldots a_{n-1} a_n, s, n)$ and $Y = (a_1 \ldots a_{n-1}, s', n-1)$ where $m(a_n, s) = (\beta, s', -1)$.

Definition 5: A computation by a Turing machine Z over an alphabet A is a finite sequence $X_1, \ldots, X_q$ of instantaneous descriptions of A such that for all $i = 1, \ldots, q-1$, $X_i(Z) \rightarrow X_{i+1}(Z)$ and for a finite sequence of elements t of A and some integer p, $X_q = (t, f, p)$. We then say that X_1 begins the computation and that X_q is the resultant of X_1.

Definition 6: Given a subset D of $\mathbb{F} = \bigcup_{j \in \mathbb{N}} A^j$, then $\mathbb{F}$ the set of all finite sequences of elements of the alphabet A, the function $\Phi = D \rightarrow \mathbb{F}$ is said to be computed by the Turing machine Z over the alphabet A if the following conditions hold. For each $t \in D$ there is a computation by Z beginning with $(t, s_0, 1)$ such that the resultant of $(t, s_0, 1)$ is $(\Phi(t), f, p)$ for some integer p.

<u>Definition 7</u>: Let $\mathbb{F}_0$ denote $\bigcup_{j \varepsilon \mathbb{N}} \{0,1\}^j$, the collection of all finite sequences of elements from the two element set $\{0,1\}$. The Turing machine over the alphabet $\{0,1,\beta\}$ is said to compute the function f from n-tuples of non-negative integers to non-negative integers if it computes the function $\tilde{f}: D_0 \to \mathbb{F}_0$ when

1. D_0 is the set of strings of the form:

$$\bar{n}_1 \beta \bar{n}_2 \beta \ldots \beta \bar{n}_n \quad \text{for} \quad (n_1, \ldots, n_n) \; \varepsilon \; \mathrm{Dom}(f).$$

2. $\tilde{f}(\bar{n}_1 \beta \ldots \beta \bar{n}_n)$ is defined as $\overline{f(n_1, \ldots, n_n)}$ when $\bar{n}$ for $n \; \varepsilon \; \mathbb{F}$ denotes the binary encoding of the natural number n.

(B) Finite Automata

<u>Definition 8</u>: For a given Turing machine over an alphabet A, the $\mathbb{F}$-yield operation $\underset{\mathbb{F}}{\to}$ between instantaneous descriptions of Z is defined as follows: $X(Z) \underset{\mathbb{F}}{\to} Y(Z)$ if and only if at least one of the following conditions obtains where a_i and b_i are in A for a positive integer i.

1. $X = (a_n a_{n-1} \ldots a_1 a_0, s, p)$ and $Y = (b_n b_{n-1} \ldots b_1 b_0, s', p+1)$ for $p < n$, $b_j = a_j$ for all $j \neq p$, and also that $m(a_p, s) = (b_p, f, -1)$.

2. $X = (a_n a_{n-1} \ldots a_1 a_0, s, p)$ and $Y = (b_n b_{n-1} \ldots b_1 a_0, f, n+1)$ with $m(a_n, s) = (b_n, f, -1)$.

3. $X\ (a_n a_{n=1} \ldots a_1 a_0, s, n)$ and $Y = (\beta b_n a_{n-1} \ldots a_1 a_0, s', n+1)$ with $m(a_n, s) = (b_n, s', -1)$ and $s' \neq f$.

Definition 9: An $\mathbb{F}$ computation by a Turing machine Z over the alphabet A is a finite sequence $X_1 \ldots X_z$ of instantaneous descriptions of Z such that for all $i = 1, \ldots, q-1$ $X(Z) \underset{\mathbb{F}}{\rightarrow} Y(Z)$ and $X = (t, s_0, 0)$ and $S_q = (t', f, \ell(t'))$ where t and t' and finite sequences of elements in A and $\ell(t')$ denotes the length of the sequence t'. We say that X_1 begins the computation and that X_q is the resultant of X_1.

Definition 10: For a fixed subset D of $\mathbb{F} = \bigcup_{j \in \mathbb{N}} A^j$, the function Φ: $D \rightarrow \mathbb{F}$ is said to be computed by the Turing machine Z viewed as a finite automaton over A if the following conditions hold. For each $t \in D$ there is an $\mathbb{F}$-computation by Z beginning with $(t, s_0, 0)$ and the resultant of $(t, s_0, 0)$ is $(\Phi(t), f, \Phi(t))$.

A computation by a finite state machine always begins on the rightmost square of the input tape and proceeds by moving one square to the left at each stage of its computation. The tape of the finite state machine can be extended indefinitely, however unlike the Turing machine, finite state machines cannot add or take away blank squares. The ability to "print" and "erase" is the major distinction between the two forms as can be seen from a comparison of the conditions that define their respective computation processes.

A further difference between the two forms of Turing machine is that the class of functions computable by a finite state machine is restricted by the length of input tapes. The result below taken from the paper by Ritchie demonstrates this restriction, which was actually employed in the result of Kramer [1974].

Theorem (Ritchie op. cit. p. 164): If Φ is a function computed by a finite automaton Z with K non-final states then, for each argument t is the domain of Φ, the length $\Phi(t)$ is at most $K+\ell(t)$ where $\ell(t)$ is the length of the tape t.

Proof" Since A is finite, after Z has read all of t it proceeds to move left reading blank squares. However, if it, (Z), enters the same state twice it cycles and will then fall into an infinite loop. Since K is the number of distinct non-final states of A and t is in the domain of Φ, Z must enter the final state f within K steps after reading the last symbol of t. Therefore, the length of $\Phi(t)$ is at most K plus the length of t. Q.E.D.

A discussion of further limitations of finite state machines can be found in the article by C.C. Elgot, "Decision Problems of Finite Automata Design and Related Arithmetics," Transactions AMS, 98, [1961] pp. 21-51.

It should further be observed that the comparative strength in computing capability obtained by Turing machines, relative to that of finite automata, serves to distinguish our approach from the works of Futio and Gottinger, which, like Kramer's approach, consider the item of rationality in decision making for social decision rules as representable by finite state machines. The issue of complexity in their approach is defined in terms of Krohn-Rhodes decomposition theory. Within our framework of Turing computability, obtianed by means of Church's Thesis, complexity takes the form of degrees of unsolvability.

APPENDIX II

We present in this appendix a brief summary of some of the important structural features of the recursive metric space $M(\mathbb{R}) = \langle(\mathbb{N}(R){\sim_{\mathbb{N}(R)}}), \mathcal{D}_R\rangle$. The terminology is that of Moschovakis [1965], wherein can be found proofs of the propositions.

Definition 1: A sequence $\{\alpha_j\}_{j\varepsilon\mathbb{N}}$ for each $\alpha_j \varepsilon M(\mathbb{R})$ is said to be recursive if there is a general recursive function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $j \varepsilon \mathbb{N}$, $f(j) \varepsilon \mathbb{N}(\mathbb{R})$ and $\alpha_j[f(j)]$, where $[f(j)]$ is an equivalence class under $\sim_{\mathbb{N}(R)}$. The Gödel number of f, $n(f)$ is said to index the sequence.

Definition 2: A sequence $\{\alpha_j\}_{j\varepsilon\mathbb{N}}$ for each $\alpha_j \varepsilon M(\mathbb{R})$ is said to be recursively Cauchy if there is a general recursive function $g\colon \mathbb{N} \to \mathbb{N}$ such that for all j, $K \varepsilon \mathbb{N}$ $\mathcal{D}_R(\alpha_{g(j)}, \alpha_{g(j)+K}) < 2^{-j}$. The function g is called a Cauchy criterion for the sequence $\{\alpha_j\}_{j\varepsilon\mathbb{R}}$ and the Gödel number of g, $n(g)$ is termed a criterion index for the sequence.

A typical property of $\mathbb{R}$ that one would wish $M(\mathbb{R})$ to preserve in recursive analogue is that it is complete. We state the fact that $M(\mathbb{R})$ has such a property in terms of the following.

Property A: A recusive metric space is said to have Property A is there is a partial recursive function $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ called a convergence function, such that if $n(F)$ is an index of a recursive sequence with a criterion index $n(g)$, and if there is an α such that

$\alpha = \lim_{j\to\infty} \alpha(j)$, then $h(n(f),n(g))$ is well defined as an element of the notation for the metric space and $\alpha = [h(n(f),n(g))]$.

Definition 3: If a recursive metric space satisfies Property A and if every recursively Cauchy sequence has a limit, it is said to be recursively complete.

Proposition 1: The recursive metric space $M(\mathbb{R})$ is recursively complete.

Another feature of $\mathbb{R}$ that one would desire $M(\mathbb{R})$ to possess is that $\mathbb{R}$ is separable. That $M(\mathbb{R})$ is in fact separable can be verified immediately by the constructions of $QM(\mathbb{R})$ from R-indices of the rational numbers which by Proposition 3 are recursive real numbers. Further, since the rationals can be made isomorphic to $\mathbb{N}$, they form a recursively enumerable subset of $M(\mathbb{R})$.

Definition 4: A recursive metric space is recursively separable if there is a recursively enumerable subset of the space that is dense.

Proposition 2: $M(\mathbb{R})$ is recursively separable.

Definition 5: A listable predicate of n-tuples of R-indices in $M(\mathbb{R})$ is a predicate $P:(\mathbb{N}(R))^n \to \{1,0\}$ for which there is a partial recursive function $f: \mathbb{N}^n \to \{1,0\}$ for which it is true that

$$f(n_1,\ldots,n_n) = 1 \text{ if and only if } P(n(f_1),\ldots,n(f_n)) = 1.$$

Proposition 3: For a fixed $\alpha_0 \in M(\mathbb{R})$ and for any $K \in \mathbb{N}$, the open sphere, $S(\alpha_0,K)$ with center α_0 and radius 2^{-K} defined as

$$S(\alpha_0,K) = \{\beta \in M(\mathbb{R}) : \mathcal{D}_R(\alpha_0,\beta) < 2^{-K}\}$$

is a listable subset of $M(\mathbb{R})$.

We next obtain by way of Proposition 3 the fact that $M(\mathbb{R})$ is connected in the natural topology on $M(\mathbb{R})$ induced by the metric $\mathcal{D}_R$ with the spheres $S(\alpha_0,K)$ as a basis.

Proposition 4: $M(\mathbb{R})$ is connected in the natural topology.

Proof: The open sets in the natural topology on $M(\mathbb{R})$ are taken as the recursive union of spheres, i.e., an open set has the form:

$$O = \bigcup_{j,f(j)} S([f(j)], \quad (j))$$

for $f\colon \mathbb{R} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ partial recursive. The functions f and g are said to index O.

By Theorem 2 of Moschovakis [1965], one observes that if O is an open set in a recursive metric space satisfying Property A, then its complement is recursively closed, i.e., contains the recursive limits of its recursive sequences. In particular, this is true of $M(\mathbb{R})$ since it has Property A. To see that $M(\mathbb{R})$ is connected, we show that no proper subset of $M(\mathbb{R})$ is both recursively open and recursively closed.

Let $\mathbb{F}$ be a proper recursively closed subset of $M(\mathbb{R})$. We show that $\mathbb{F}$ is not recursively open. Choose $\alpha_x \notin \mathbb{F}$ and $\alpha_y \notin \mathbb{F}$ which can be done since $\emptyset \neq \mathbb{F} \neq M(\mathbb{R})$. Assume that $\alpha_x > \alpha_y$ in the order on $M(\mathbb{R})$ induced by the order on $\mathbb{R}$ and define $\alpha_z = \sup\{\alpha_t \in \mathbb{F}: \alpha_t < \alpha_x\}$. Then it is true that $\alpha_x \geqslant \alpha_z \geqslant \alpha_y$. Then $\alpha_z \in \mathbb{F}$, because $\mathbb{F}$ is recursively closed and for any $S(\alpha_z,K)$, $(S(\alpha_z,K) \cap \mathbb{F}) \neq \emptyset$. But if $\alpha_z \in \mathbb{F}$, then $\alpha_x > \alpha_z$, but then any $S((\alpha_x - \alpha_z)/\alpha_2,K)$ with K sufficiently large is such that it is true that $(S((\alpha_x = \alpha_z)/\alpha_2,K) \cap \mathbb{F}) = \emptyset$ and since the choice of α_x is arbitrary, α_z cannot be interior to $\mathbb{F}$ and so $\mathbb{F}$ cannot be recursively open. Q.E.D.

Finally we state two results that refer to the fact that $M(\mathbb{R})$ is a Baire space in the sense that it is not the recursive union of recursively closed, nowhere dense sets of which in the classical setting yields that every denumerable subset of a perfect metric space is of the first category; a metric space being perfect if it has no isolated point which is true of $M(\mathbb{R})$.

<u>Proposition 5</u>: Every recursively enumerable subset of a perfect recursive metric space, and therefore of $M(\mathbb{R})$, is of the first category.

<u>Proposition 6</u>: The complement of a recursively enumerable subset of a recursively separable, recursively complete, perfect recursive metric space, and therefore of $M(\mathbb{R})$, is recursively dense.

From Proposition 4 one sees that the recursive closure of the subspace $QM(\mathbb{R})$ is in fact $M(\mathbb{R})$ from Proposition 2.

FOOTNOTES

1/ Of course, by $\alpha_j \gtrsim \alpha_i$ we mean that for some x, $y \in X$ s.t. $x \in \alpha_j$ and $y \in \alpha_i \gtrsim y$.

2/ Two of the most frequently employed Gödel numberings are the lexicographic and dyadic. (cf. Rogers [1967])

3/ I.e. let X be the set of "rationals" in $[0,1]^n$ form some $n > 0$, under a suitable coding into $M(\mathbb{R}^n)$.

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