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*B-Spline-Bezier Representation  
of  
Tau-Splines*

Dieter Lasser

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# B-Spline-Bezier Representation of Tau-Splines

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**Abstract:** We present a B-spline-Bezier representation of  $\tau$  splines, curvature and torsion continuous quintics which have been introduced in CAGD by *Hagen* in 1985. Explicit formulas are given for the conversion of the B-spline-Bezier representation to the  $\tau$  spline representation and vice versa, and conditions and certain ranges of tension values are derived which insure the positivity of the design parameters.

## 0. Introduction

In 1974 *Nielson* [*Nielson 74*] gave a piecewise polynomial alternative to splines under tension [*Schweikert 66*], [*Cline 74*], the so-called  $\nu$  spline. The  $\nu$  splines are curvature continuous interpolating cubics and they are the solution of the minimization of

$$\int_{t_0}^{t_N} \|X''(t)\|^2 dt + \sum_{I=0}^N v_I \|X'(t_I)\|^2, \quad v_I \geq 0 \quad (1)$$

over the space

$$H^2 = \{X: X'' \in L^2[t_0, t_N], X' \text{ absolutely continuous on } [t_0, t_N]\}$$

subject to interpolation conditions and certain end conditions. [*Nielson 74, 86*]

In [*Boehm 85*] a B-spline-Bezier representation of  $\nu$  splines was given, the  $\gamma$  spline, and *Boehm* also pointed out the close relation to *Barsky's*  $\beta$  splines. [*Barsky 81*] Some details on this relation can be found in [*Fritsch 86*] especially conversion equations between  $\beta$  splines and  $\nu$  splines.

In 1985 Hagen [Hagen 85] generalized Nielson's approach by using

$$\int_{t_0}^{t_N} \|\mathbf{X}^{(K)}(t)\|^2 dt + \sum_{l=0}^N \sum_{L=1}^{K-1} v_{l,L} \|\mathbf{X}^{(L)}(t_l)\|^2 dt, \quad v_{l,L} \geq 0 \quad (2)$$

( $K \geq 2$ ), rather than (1), minimizing now with respect to

$$H^K = \{ \mathbf{X}: \mathbf{X}^{(K)} \in L^2[t_0, t_N], \mathbf{X}^{(K-1)} \text{ absolutely continuous on } [t_0, t_N] \}$$

and satisfying interpolation and generalized end conditions. Hagen's concept of 'geometric spline curves' includes for  $K = 2$  Nielson's  $v$  splines and yields for  $K = 3$  to curvature and torsion continuous quintics, the  $\tau$  splines.

The aim of this paper is to give a B-spline-Bezier representation of  $\tau$  splines. We also derive the conversion equations between the Bezier and the originally given Hermite representation in [Hagen 85]. We discuss positivity conditions on the design parameters of the Bezier representation and value ranges for the point weights, the  $v_i$ 's.

Because we like to find a B-spline-Bezier representation of  $\tau$  splines, we first introduce the Bezier representation of segmented curves. In section II we give a short discussion of Nielson's  $v$  splines while Hagen's  $\tau$  splines are discussed in section III.

## I. Bezier representation of segmented curves

Let  $\mathbf{X}(t)$  be a planar or spatial parametrized curve defined with respect to a partition of the domain space by 'knots'

$$t_0 < t_1 < \dots < t_N.$$

The parameter space segmentation induces a curve segmentation in Segments  $\mathbf{X}_I: [t_I, t_{I+1}] \rightarrow \mathbb{R}^d$  ( $d = 2, 3$ ). A local parameter  $u \in [0, 1]$  can be introduced such that

$$I = 0, \dots, N-1 \quad \mathbf{X}(t) = \mathbf{X}_I(u) \quad \text{for } t \in [t_I, t_{I+1}]$$

by the linear interpolation of  $t_I$  and  $t_{I+1}$ :

$$t = (1-u)t_I + ut_{I+1}, \quad \text{where } u \in [0, 1].$$

The derivatives have to be calculated now by the chain rule, i.e.

$$\mathbf{X}^{(r)}(t) \equiv \frac{d^r}{dt^r} \mathbf{X}(t) = \frac{d^r u}{dt^r} \frac{d^r}{du^r} \mathbf{X}_I(u) = \frac{1}{\Delta_I^r} \frac{d^r}{du^r} \mathbf{X}_I(u)$$

where  $\Delta_I \equiv t_{I+1} - t_I$ .

Now the segments might be given in Bezier representation, that means

$$\mathbf{X}_I(u) = \sum_{k=0}^n \mathbf{b}_{nI+k} B_k^n(u) \quad (3)$$

where  $\mathbf{b}_{nI+k} \in \mathbb{R}^d$  ( $d = 2, 3$ ),  $u \in [0, 1]$  and

$$B_k^n(u) = \binom{n}{k} u^k (1-u)^{n-k}$$

are the (ordinary) **Bernstein polynomials** of degree  $n$  in  $u$ . The coefficients  $\mathbf{b}_j$  are called **Bezier points**. They form in their natural ordering given by their subscripts the vertices of the so called **Bezier polygon**.

The derivatives of  $X_I(u)$  with respect to  $u$  are given by

$$\frac{d^r}{du^r} X_I(u) = \frac{n!}{(n-r)!} \sum_{k=0}^{n-r} \Delta^r b_{nI+k} B_k^{n-r}(u)$$

where

$$\Delta^r b_\alpha \equiv \Delta^{r-1}(b_\alpha + 1 - b_\alpha)$$

so that in  $X(t_i) = X_{I-1}(1) = X_I(0)$ , the common boundary point of  $X_{I-1}(u)$  and of  $X_I(u)$ , we have for the left sided derivatives of  $X(t)$

$$X^{(r)}(t_i^-) \equiv \lim_{t \rightarrow t_i^-} X^{(r)}(t) = \frac{1}{\Delta_{I-1}^r} \frac{n!}{(n-r)!} \Delta^r b_{nI-r}$$

and for the right sided derivatives

$$X^{(r)}(t_i^+) \equiv \lim_{t \rightarrow t_i^+} X^{(r)}(t) = \frac{1}{\Delta_I^r} \frac{n!}{(n-r)!} \Delta^r b_{nI}$$

## II. Nu-splines

Nielson's  $\nu$  splines are solutions of the minimization of (1) over the space  $H^2$  subject to the interpolation conditions

$$I = 0, \dots, N \quad X(t_I) = X_I$$

and one of the following end conditions

- i.)  $X'(t_0) = X'_0, \quad X'(t_N) = X'_N,$
- ii.)  $X''(t_0^+) = \nu_0 X'(t_0), \quad X''(t_N^-) = \nu_N X'(t_N),$
- iii.)  $X(t_0) = X(t_N), \quad X'(t_0) = X'(t_N), \quad X''(t_0^+) - X''(t_N^-) = (\nu_0 + \nu_N) X'(t_0).$

$\nu$  splines fulfill at any knot  $t_I$  ( $I = 1, \dots, N-1$ ) the continuity conditions

$$X(t_I^+) = X(t_I^-) \quad (4)$$

$$X'(t_I^+) = X'(t_I^-) \quad (5)$$

$$X''(t_I^+) = X''(t_I^-) + \nu_I X'(t_I^-) \quad (6)$$

what can be written in matrix form as

$$X_I^+ = A_I X_I^-$$

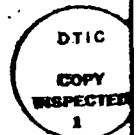
The  $(r+1)^2$  matrix  $A_I$  is called **connection matrix** and the 'vector'  $X_I^\pm$  with  $r+1$  elements of  $\mathbb{R}^d$  ( $d \geq r$ , here is  $r=2$ ) is sometimes called the **r-jet** of  $X$ .

Because the curvature of a planar resp. spatial curve is given by

$$\kappa = \frac{|X', X''|}{\|X'\|^3} \quad \text{resp.} \quad \kappa = \frac{\|X' \times X''\|}{\|X'\|^3} \quad (7)$$

we see that a  $\nu$  spline is curvature continuous.

B-Spline-Bezier Representation of Tau-Splines



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A Bezier representation of  $v$  splines can be derived by inserting (3) for  $n = 3$  into (4) to (6). We get as continuity conditions of the Bezier representation (Figure 1)

$$(1 + q_I) b_I = q_I b_{3I-1} + b_{3I+1}$$

$$(1 + \gamma_I q_I) b_{3I-1} = \gamma_I q_I b_{3I-2} + s_I$$

$$(\gamma_I + q_I) b_{3I+1} = q_I s_I + \gamma_I b_{3I+2}$$

where

$$q_I = \frac{\Delta_I}{\Delta_{I-1}},$$

and

$$\gamma_I = \frac{1}{1 + \frac{1}{1 + q_I} \frac{\Delta_I}{2} v_I} \quad (8)$$

$I = 1, \dots, N - 1$ . (8) allows the evaluation of the  $\gamma_I$ 's of the Bezier representation of a  $v$  spline, i.e. the evaluation of the  $\gamma_I$ 's for given  $v_I$  values. On the other side, the corresponding  $v$  spline to a given curvature continuous cubic Bezier spline, a so-called  $\gamma$  spline, has  $v_I$  values given by

$$I = 1, \dots, n - 1 \quad v_I = \frac{2}{\Delta_I} (1 + q_I) \left( \frac{1}{\gamma_I} - 1 \right). \quad (9)$$

(9) was first given by Boehm in [Boehm 85]. He also presented a B-spline representation for curvature continuous cubics, and pointed out the relation to Barsky's uniformly-shaped  $\beta$  splines. [Barsky 81] The connection to general  $\beta$  splines, sometimes called explicit, discrete or discretely-shaped  $\beta$  splines [Hoellig 86], [Bartels et al. 87], was also pointed out by Nielson and given by Fritsch (see [Fritsch 86]).<sup>1</sup> In fact, looking at the connection matrices of  $\gamma$  splines,  $v$  splines and (general)  $\beta$  splines [Dyn et al. 85] we see that  $\beta$ ,  $\gamma$  and  $v$  splines are nothing else than different representations of curvature continuous cubics.

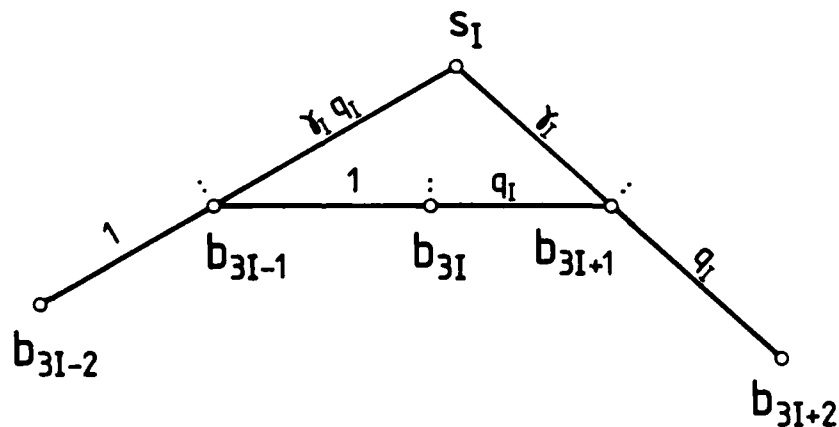


Figure 1. Construction of the Bezier polygon for  $v$  splines

Dyn and Micchelli [Dyn et al. 85] have shown that the existence of non-negative local support basis functions which sum to one follows - for geometrically continuous spline curves, i.e. spline curves with continuous differential geometric invariants like curvature, torsion, etc. - from the total positivity of the connection matrix. For  $v$  splines the total positivity of  $A_I$  has the meaning  $v_I \geq 0$  and that is exactly the range for the tension parameters  $v_I$  covered by the minimum norm

<sup>1</sup> The first local basis for  $GC^2$  splines was developed by Nielson and Lewis in 1975. [Lewis 75] A local basis for uniformly-shaped  $\beta$  splines was given in [Barsky 81] and for discretely-shaped  $\beta$  splines in [Bartels et al. 84], see also [Cohen 87].

characterization of  $v$  splines. [Nielson 74, 86] For  $v_I \geq 0$  we obtain from (9) the  $\gamma_I$  range:  $\gamma_I \leq 1$  (remark:  $v_I = 0 \Leftrightarrow \gamma_I = 1$  is the usual  $C^2$  cubic spline). But working with  $\gamma$  splines we know that  $\gamma_I > 1$  are possible as well. Indeed, if we request **positive design parameters**, i.e.  $\gamma_I > 0$ , so that such important properties like the convex hull and the variation diminishing property are given, (8) yields to

$$I = 1, \dots, N-1 \quad v_I > -\frac{2}{\Delta_I} (1 + q_I). \quad (10)$$

Hence not only positive tension values but  $v_I$  values in the range given by (10) guarantee positive design parameters (remark:  $v_I < 0$  yields to  $\gamma_I > 1$ ) and therefore properties like the two mentioned above. This result goes conform with work done by Barsky [Barsky 84] who extended the theory of  $v$  splines by identifying certain ranges for the  $v_I$ 's that guarantee a **unique solution of the interpolation problem**. In the special case of a uniform, an equidistant parametrization, i.e.  $q_I = 1$ , Barsky gives the range  $v_I > -4$  which is also given by (10).

### III. Tau-splines

Hagens's  $\tau$  splines are solutions of the minimization of (2) over the space  $H^K$  for  $K = 3$  subject to the interpolation conditions

$$I = 0, \dots, N \quad \mathbf{X}(t_I) = \mathbf{X}_I$$

and one of the following end conditions ( $L = K, \dots, 2(K-1)$ )

$$\text{i.)} \quad \mathbf{X}^{(2K-1-L)}(t_0) = \mathbf{X}_0^{(2K-1-L)}, \quad \mathbf{X}^{(2K-1-L)}(t_N) = \mathbf{X}_N^{(2K-1-L)},$$

$$\text{ii.)} \quad \mathbf{X}^{(L)}(t_0^+) = v_{0,2K-1-L} \mathbf{X}^{(2K-1-L)}(t_0), \quad \mathbf{X}^{(L)}(t_N^-) = v_{N,2K-1-L} \mathbf{X}^{(2K-1-L)}(t_N),$$

$$\text{iii.)} \quad \mathbf{X}(t_0) = \mathbf{X}(t_N), \quad \mathbf{X}^{(2K-1-L)}(t_0) = \mathbf{X}^{(2K-1-L)}(t_N),$$

$$\mathbf{X}^{(L)}(t_0^+) - \mathbf{X}^{(L)}(t_N^-) = (v_{0,2K-1-L} - v_{N,2K-1-L}) \mathbf{X}^{(2K-1-L)}(t_0).$$

$\tau$  splines fulfill at any knot  $t_I$  ( $I = 1, \dots, N-1$ ) the continuity conditions

$$\mathbf{X}(t_I^+) = \mathbf{X}(t_I^-) \quad (11)$$

$$\mathbf{X}'(t_I^+) = \mathbf{X}'(t_I^-) \quad (12)$$

$$\mathbf{X}''(t_I^+) = \mathbf{X}''(t_I^-) \quad (13)$$

$$\mathbf{X}'''(t_I^+) = \mathbf{X}'''(t_I^-) + v_{I,2} \mathbf{X}'(t_I^-) \quad (14)$$

$$\mathbf{X}'''(t_I^+) = \mathbf{X}'''(t_I^-) - v_{I,1} \mathbf{X}'(t_I^-) \quad (15)$$

Because the curvature of a spatial curve is given by (7) and the torsion by

$$\tau = \frac{|\mathbf{X}', \mathbf{X}'', \mathbf{X}'''|}{\|\mathbf{X}' \times \mathbf{X}''\|^2}$$

we see that a  $\tau$  spline is curvature and torsion continuous.



The connection matrix of a general torsion continuous contact of two curve segments of  $\mathbb{R}^d$  ( $d \geq 3$ ) of degree  $n$  ( $n \geq 4$ ) - we speak about **geometric  $C^3$  continuity**, briefly  **$GC^3$  continuity** - is given by [Dyn et al. 85], [Lasser, Eck 88]:

$$A_{GC^3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_{11} & 0 & 0 \\ 0 & \omega_{12} & \omega_{11}^2 & 0 \\ 0 & \omega_{23} & \omega_{13} & \omega_{11}^3 \end{bmatrix}$$

Comparing  $A_{GC^3}$  with (11) to (14), we see that  $\tau$  splines do not take advantage of all shape parameters offered by the concept of torsion continuity. They rather form a subset of the set of  $GC^3$  continuous curves.

Furthermore we like to mention that  $\tau$  splines and **visual  $C^3$  continuous curves**, briefly  **$VC^3$  continuous curves**, i.e. curves having **contact of order  $r$**  [Geise 62] with  $r = 3$ , are spanning two 'almost totally separated' subsets of the set of  $GC^3$  continuous curves. The connection matrix of two curve segments having contact of order 3 is given by [Dyn et al. 85], [Lasser 88]:

$$A_{VC^3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & v_1 & 0 & 0 \\ 0 & v_2 & v_1^2 & 0 \\ 0 & v_3 & 3v_1v_2 & v_1^3 \end{bmatrix}$$

and therefore, comparing  $A_{VC^3}$  with  $A_{GC^3}$ ,  $VC^3$  continuous curves form a subset of the set of  $GC^3$  continuous curves, and, comparing  $A_{VC^3}$  with (11) to (14), we see that the only curves being  $\tau$  splines as well as  $VC^3$  continuous curves are the usual  $C^3$  continuous curves.

To find a Bezier representation of  $\tau$  splines we insert (3) for  $n = 5$  into (11) to (15) and get as continuity conditions of the Bezier representation (Figure 2)

$$(1 + q_I) \mathbf{b}_I = q_I \mathbf{b}_{5I-1} + \mathbf{b}_{5I+1} \quad (16)$$

$$(1 + \gamma_I q_I) \mathbf{b}_{5I-1} = \gamma_I q_I \mathbf{b}_{5I-2} + \mathbf{s}_I \quad (17.1)$$

$$(\gamma_I + q_I) \mathbf{b}_{5I+1} = q_I \mathbf{s}_I + \gamma_I \mathbf{b}_{5I+2} \quad (17.2)$$

$$(1 + \delta_I q_I) \mathbf{b}_{5I-2} = \delta_I q_I \mathbf{b}_{5I-3} + \mathbf{e}_I^- \quad (18.1)$$

$$(\delta_I + \varepsilon_I q_I) \mathbf{s}_I = \varepsilon_I q_I \mathbf{e}_I^- + \delta_I \mathbf{e}_I^+ \quad (18.2)$$

$$(\varepsilon_I + q_I) \mathbf{b}_{5I+2} = q_I \mathbf{e}_I^+ + \varepsilon_I \mathbf{b}_{5I+3} \quad (18.3)$$

$$(1 + \rho_I q_I) \mathbf{b}_{5I-3} = \rho_I q_I \mathbf{b}_{5I-4} + \mathbf{f}_I^- \quad (19.1)$$

$$(\rho_I + \sigma_I q_I) \mathbf{e}_I^- = \sigma_I q_I \mathbf{f}_I^- + \rho_I \mathbf{t}_I \quad (19.2)$$

$$(\sigma_I + \tau_I q_I) \mathbf{e}_I^+ = \tau_I q_I \mathbf{t}_I + \sigma_I \mathbf{f}_I^+ \quad (19.3)$$

$$(\tau_I + q_I) \mathbf{b}_{5I+3} = q_I \mathbf{f}_I^+ + \tau_I \mathbf{b}_{5I+4} \quad (19.4)$$

where

$$q_I = \frac{\Delta_I}{\Delta_{I-1}}$$

and

$$r_I = 1$$

and

$$\delta_I = \frac{1}{1 + \frac{1}{(1+q_I)^2} \frac{\Delta_I}{3} v_{I,2}} \quad (20)$$

$$\varepsilon_I = \frac{1}{1 + \frac{q_I}{(1+q_I)^2} \frac{\Delta_I}{3} v_{I,2}} \quad (21)$$

and

$$\rho_I = \frac{1}{3q_I - R_I} \quad (22)$$

$$\sigma_I = q_I^2 \frac{1}{T_I - 3 + (T_I - S_I) \frac{q_I}{\varepsilon_I}} \quad (23)$$

$$\tau_I = q_I \frac{1}{3 - T_I} \quad (24)$$

with

$$R_I = \frac{3q_I - 1 + \frac{1}{(1+q_I)^3} \frac{\Delta_I^3}{24} v_{I,1}}{1 + \frac{1}{(1+q_I)^3} \frac{\Delta_I}{3} v_{I,2}}$$

$$S_I = \frac{2(1-q_I) + \frac{1-q_I}{(1+q_I)^3} \frac{\Delta_I^3}{24} v_{I,1}}{1 + \frac{2q_I}{(1+q_I)^3} \frac{\Delta_I}{3} v_{I,2}}$$

$$T_I = \frac{3 - q_I + \frac{q_I}{(1+q_I)^3} \frac{\Delta_I^3}{24} v_{I,1}}{1 + \frac{q_I^2}{(1+q_I)^3} \frac{\Delta_I}{3} v_{I,2}}$$

(20) to (24) allows the evaluation of the design parameters of the Bezier representation of a  $\tau$  spline and by this the Bezier representation is given. The design parameters are of course not independent of each other any longer. This is obvious because two shape parameters are given by (11) to (15) but five are given by (16) to (19). The dependences are

$$\delta_I = \frac{q_I \varepsilon_I}{1 - (1 - q_I) \varepsilon_I} \quad \text{resp.} \quad \varepsilon_I = \frac{\delta_I}{q_I + (1 - q_I) \delta_I} \quad (25)$$

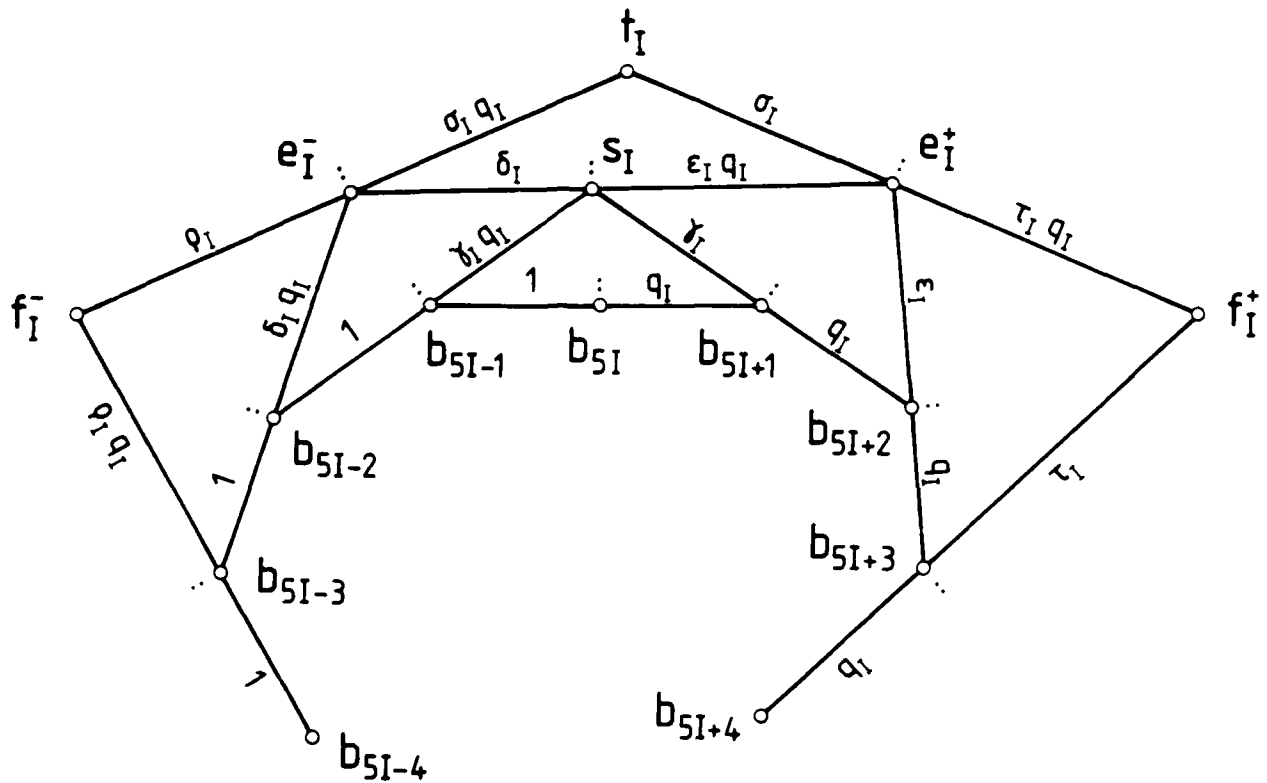


Figure 2. Construction of the Bezier polygon for a  $\tau$  spline

and

$$\rho_I = \frac{1}{\delta_I} \frac{(1 + q_I \delta_I)(\epsilon_I + \delta_I)\sigma_I}{3(1 + q_I)\sigma_I - (\delta_I + q_I \epsilon_I)} \quad \text{resp.} \quad \sigma_I = \frac{(\delta_I + q_I \epsilon_I)\delta_I \rho_I}{3(1 + q_I)\delta_I \rho_I - (1 + q_I \delta_I)(\epsilon_I + \delta_I)} \quad (26)$$

$$\sigma_I = \frac{(\delta_I + q_I \epsilon_I)\epsilon_I \tau_I}{3(1 + q_I)\epsilon_I \tau_I - (\epsilon_I + q_I)(\epsilon_I + \delta_I)} \quad \text{resp.} \quad \tau_I = \frac{1}{\epsilon_I} \frac{(\epsilon_I + q_I)(\epsilon_I + \delta_I)\sigma_I}{3(1 + q_I)\sigma_I - (\delta_I + q_I \epsilon_I)} \quad (27)$$

$$\tau_I = \frac{(\epsilon_I + q_I)\delta_I \rho_I}{((1 + q_I \delta_I)\epsilon_I)} \quad \text{resp.} \quad \rho_I = \frac{(1 + q_I \delta_I)\epsilon_I \tau_I}{(\epsilon_I + q_I)\delta_I} \quad (28)$$

That means if an interpolating  $\tau$  spline is given, i.e.  $v_{I,1}$  and  $v_{I,2}$  values, then the design parameters of the Bezier representation of the  $\tau$  spline have to be determined in such a way, that they fulfill the equations (25) to (28). And the equations (25) to (28) are valid if the design parameters are determined by (20) to (24). On the other side, the design parameters of a torsion continuous quintic Bezier spline curve have to fulfill the conditions (25) to (28) to make the quintic Bezier spline being an interpolating  $\tau$  spline in Bezier representation. That means either  $\delta_I$  or  $\epsilon_I$  can be chosen as independent design parameter and, let's say in case of  $\delta_I$ ,  $\epsilon_I$  has to be determined by (25) as  $\epsilon_I = \epsilon_I(\delta_I)$  and in addition either  $\rho_I$  or  $\sigma_I$  or  $\tau_I$  can be chosen as independent design parameter and, e.g. in case of  $\rho_I$ ,  $\sigma_I$  and  $\tau_I$  have to be determined by (26) and (28) as  $\sigma_I = \sigma_I(\rho_I)$  and  $\tau_I = \tau_I(\rho_I)$ . If the design parameters of the Bezier representation are determined in this way, then the Bezier spline is equivalent to an interpolating  $\tau$  spline having the property of minimizing (2) and we can calculate the  $v_{I,1}$  and  $v_{I,2}$  that means the point weights defining the jumps of the third and fourth derivatives in the knots of the  $\tau$  spline by

$$v_{I,2} = \frac{3}{\Delta_I} (1 + q_I)^2 \left( \frac{1}{\delta_I} - 1 \right) \quad \text{resp.} \quad v_{I,2} = \frac{3}{\Delta_I} \frac{(1 + q_I)^2}{q_I} \left( \frac{1}{\epsilon_I} - 1 \right)$$

and

$$v_{l,1} = \frac{24}{\Delta_l^3} (1+q_l)^2 \left[ (1+q_l)(1-3q_l) - \left(\frac{1}{\delta_l} + q_l\right) \left(\frac{1}{\rho_l} - 3q_l\right) \right]$$

resp.

$$v_{l,1} = \frac{24}{\Delta_l^3} \frac{(1+q_l)^2}{q_l} \left[ \left(\frac{1}{\varepsilon_l} + \frac{q_l}{\delta_l}\right) \left(\frac{q_l}{\sigma_l} + q_l - 2\right) + \frac{2}{\varepsilon_l} (1 - q_l^2) \right] \frac{\varepsilon_l \delta_l}{\varepsilon_l + \delta_l}$$

resp.

$$v_{l,1} = \frac{24}{\Delta_l^3} \frac{(1+q_l)^2}{q_l} \left[ (1+q_l)(q_l - 3) - \left(1 + \frac{q_l}{\varepsilon_l}\right) \left(\frac{q_l}{\tau_l} - 3\right) \right]$$

and so on.

The  $\tau$  splines form a subset of the set of  $GC^3$  continuous quintics. The **B-spline-Bezier representation of Tau splines** is therefore identical with the B-spline-Bezier representation of  $GC^3$  continuous quintics which was given in [Eck 87], [Lasser,Eck 88], with the restriction that for a  $\tau$  spline the design parameters  $\gamma_l, \delta_l, \varepsilon_l, \rho_l, \sigma_l$  and  $\tau_l$  can not be chosen independently of each other as in case of the  $GC^3$  continuous quintic spline curve. We rather have to set  $\gamma_l = 1$  and have to choose  $\delta_l, \varepsilon_l, \rho_l, \sigma_l$  and  $\tau_l$  according to the dependences given by (25) to (28).

Let's discuss now the **positivity of the design parameters** for  $\tau$  splines. The dependences (25) to (28) imply the following:

Because of (25), positivity of  $\delta_l = \delta_l(\varepsilon_l)$  needs

$$\begin{aligned} 0 < \varepsilon < \frac{1}{1 - q_l} & \quad \text{in case of } q_l < 1 \Leftrightarrow \Delta_l < \Delta_{l-1} \\ \varepsilon > 0 & \quad \text{in case of } q_l \geq 1 \Leftrightarrow \Delta_l \geq \Delta_{l-1} \end{aligned}$$

On the other side, positivity of  $\varepsilon_l = \varepsilon_l(\delta_l)$  needs

$$\begin{aligned} \delta > 0 & \quad \text{in case of } q_l \leq 1 \Leftrightarrow \Delta_l \leq \Delta_{l-1} \\ 0 < \delta < \frac{-q_l}{1 - q_l} & \quad \text{in case of } q_l > 1 \Leftrightarrow \Delta_l > \Delta_{l-1} \end{aligned}$$

Because of (27) and (28), positivity of  $\rho_l = \rho_l(\tau_l)$  and of  $\sigma_l = \sigma_l(\tau_l)$  needs

$$\tau > \frac{(\varepsilon_l + q_l)(\varepsilon_l + \delta_l)}{3(1 + q_l)\varepsilon_l}$$

On the other side, because of (26) and (27), positivity of  $\rho_l = \rho_l(\sigma_l)$  and of  $\tau_l = \tau_l(\sigma_l)$  needs

$$\sigma > \frac{\delta_l + q_l \varepsilon_l}{3(1 + q_l)}$$

and because of (26) and (28), positivity of  $\sigma_l = \sigma_l(\rho_l)$  and of  $\tau_l = \tau_l(\rho_l)$  needs

$$\rho > \frac{(1 + q_l)\delta_l(\varepsilon_l + \delta_l)}{3(1 + q_l)\delta_l}$$

The minimum norm characterization of  $\tau$  splines works for non-negative tension parameter  $v_{I,1}$  and  $v_{I,2}$  only. As for  $v$  splines we can extend the theory of  $\tau$  splines by requesting the **positivity of the design parameters**.

Because of (20) and (21), for  $\varepsilon_I > 0$  and  $\delta_I > 0$  the tension parameter  $v_{I,2}$  has to be within the range

$$v_{I,2} > \max \left\{ -(1+q_I)^2 \frac{3}{\Delta_I}, -\frac{(1+q_I)^2}{q_I} \frac{3}{\Delta_I} \right\} \quad (29)$$

that means

$$v_{I,2} > -\frac{(1+q_I)^2}{q_I} \frac{3}{\Delta_I} \quad \text{if } q_I > 1 \Leftrightarrow \Delta_I > \Delta_{I-1}$$

$$v_{I,2} > -\frac{12}{\Delta_I} \quad \text{if } q_I = 1 \Leftrightarrow \Delta_I = \Delta_{I-1}$$

$$v_{I,2} > -(1+q_I)^2 \frac{3}{\Delta_I} \quad \text{if } q_I < 1 \Leftrightarrow \Delta_I < \Delta_{I-1}$$

Thus not only non-negative but also certain negative  $v_{I,2}$  values are allowed.

Because of (22) to (24), for  $\rho_I > 0$ ,  $\sigma_I > 0$  and  $\tau_I > 0$  the tension parameter  $v_{I,1}$  has to be within the ranges given by (30).

$$v_{I,1} < \frac{24}{\Delta_I^3} [(1+q_I)^3 + q_I \Delta_I v_{I,2}] \quad (30.1)$$

$$v_{I,1} > -\frac{24}{\Delta_I^3} \frac{(1+q_I)^3 + 2(1-q_I+q_I^2) \frac{\Delta_I}{3} v_{I,2}}{2 + \frac{1}{1+q_I} \frac{\Delta_I}{3} v_{I,2}} \quad (30.2)$$

First  $v_{I,2}$  has to be chosen such that  $\varepsilon_I > 0$  and  $\delta_I > 0$  is fulfilled, i.e.  $v_{I,2}$  has to be chosen within the range given by (29), then  $v_{I,1}$  can be chosen within the range given above.

For  $q_I = 1$  (30) yields for example to

$$-4 \frac{24}{\Delta_I^3} < v_{I,1} < \frac{24}{\Delta_I^3} (8 + \Delta_I v_{I,2})$$

Barsky [Barsky 84] extended the theory of  $v$  splines by identifying certain ranges for the  $v_i$ 's that guarantee a **unique solution of the interpolation problem**. This idea allows especially the consideration of different end conditions. The same can be done for  $\tau$  splines, and is indeed the topic of actual research.

Furthermore the idea of [Salkauskas 75] and [Foley 86,87] of introducing interval weights can be picked up to create **interval weighted geometric spline curves**<sup>2</sup> minimizing

$$\sum_{I=0}^N p_I \int_{t_{I-1}}^{t_I} \|\mathbf{X}^{(K)}(t)\|^2 dt + \sum_{I=0}^N \sum_{L=1}^{K-1} v_{I,L} \|\mathbf{X}^{(L)}(t_I)\|^2 dt .$$

For  $K = 3$  we get interval weighted  $\tau$  splines. Actually we are working on this too.

<sup>2</sup> These interval weighted geometric spline curves are in general not curvature, torsion, etc. continuous.

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