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NWC TP 6776

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Phase-Shift Parameters and Small Vibrations in Resonant Optical Cavities

AD-A197 450

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SEPTEMBER 1987

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88 1 5 008

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

AD-A194 450

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b RESTRICTIVE MARKINGS:

2a SECURITY CLASSIFICATION AUTHORITY:

2b DECLASSIFICATION / DOWNGRADING SCHEDULE:

3 DISTRIBUTION / AVAILABILITY OF REPORT: A Statement; public release; distribution unlimited.

4 PERFORMING ORGANIZATION REPORT NUMBER(S): NWC TP 6776

5 MONITORING ORGANIZATION REPORT NUMBER(S):

6a NAME OF PERFORMING ORGANIZATION: Naval Weapons Center

6b OFFICE SYMBOL *(If applicable)***:**

7a NAME OF MONITORING ORGANIZATION:

6c ADDRESS *(City, State, and ZIP Code)***:** China Lake, CA 93555-6001

7b. ADDRESS *(City, State, and ZIP Code)***:**

8a NAME OF FUNDING / SPONSORING ORGANIZATION: Naval Weapons Center

8b OFFICE SYMBOL *(If applicable)***:**

9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER:

8c ADDRESS *(City, State, and ZIP Code)***:** China Lake, CA 93555-6001

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
61101E		DD	782239

11 TITLE *(Include Security Classification)*

PHASE-SHIFT PARAMETERS AND SMALL VIBRATIONS IN RESONANT OPTICAL CAVITIES (U)

12 PERSONAL AUTHOR(S): Ross, D. M., Brune, C., and Marrs, C. D.

13a TYPE OF REPORT	13b TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Final	FROM 85 May TO 85 Sep	1987, September	42

16 SUPPLEMENTARY NOTATION:

17. COSATI CODES

FIELD	GROUP	SUB-GROUP
14	02	
20	05	

18 SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*

Passive Cavity Reflectometers, Phase-Shift Method, Lasers, High-Reflectance Mirrors, Precision Characterization Techniques

19 ABSTRACT *(Continue on reverse if necessary and identify by block number)*

An exact analysis has been performed on a three-mirror passive cavity reflectometer. An expression is derived to determine the reflectance of the sample mirror including a random phase shift caused by thermal instabilities in the cavity. This exact expression is compared to a numerical analysis derived earlier for the same thermal instability. The comparison shows variations of up to several parts in 10^4 for low-loss dielectric mirrors. An error analysis is presented that quantifies the error of a three-mirror cavity.

20 DISTRIBUTION / AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS

21 ABSTRACT SECURITY CLASSIFICATION: Unclassified

22a NAME OF RESPONSIBLE INDIVIDUAL	22b TELEPHONE (Include Area Code)	22c OFFICE SYMBOL
D. M. Ross	(619) 939-3965	Code 3312

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

☆ U.S. Government Printing Office: 1986—607-044

UNCLASSIFIED

CONTENTS

INTRODUCTION

Phase-shift (References 1 through 5) and ring-down (References 4 and 5) methods of determining the reflectance of mirrors in a high Q optical cavity possess high sensitivity and precision. In maintaining this precision, the errors in the measurement must be minimized. The errors include--but are not limited to--(1) mode matching the input source to the optical cavity, (2) stable input source in power and waveform over the time required to make measurement, (3) in terms of loss, cavity mirrors equal to or better than the unknown mirror's losses, and (4) thermal stability of the cavity. Rahn (Reference 5) examined an optical cavity and included a random thermal fluctuation in the analysis to explore the range of fluctuations that can be tolerated. He was not able to evaluate two time-averaged integrals exactly but derived an expression for the loss on a numerical approach. Starting from the same assumptions as Rahn, we will derive an exact solution to a passive optical cavity, including the thermal fluctuation. Also, an error analysis of the phase-shift method is derived and examined.

THEORY

Consider an optical cavity consisting of two mirrors with their symmetry axes on a common line, as in Figure 1. Once the reflectance of one of the mirrors is known, the other can be calculated. A phase shift can be uniquely associated with the product of the cavity mirror reflectances:

$$R_t = R_1 R_2 \tag{1}$$

where R_t is the total reflectance of the cavity and R_1 and R_2 are the respective reflectance values of the cavity mirrors.

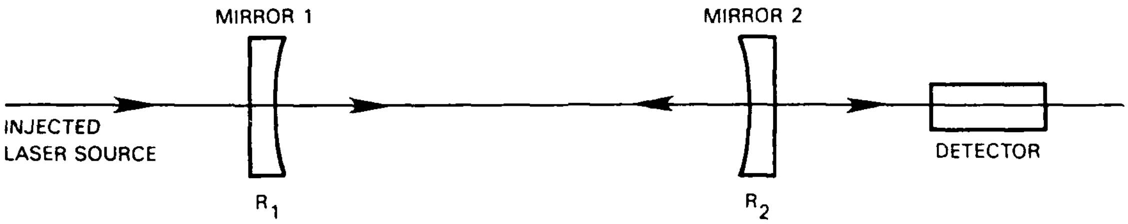

FIGURE 1. Two-Mirror Passive Cavity. Injection laser source can be short pulsed laser or modulated-CW laser source.

Another configuration--a three-mirror cavity--will be discussed in the remainder of this paper. In a three-mirror passive cavity, the optimum position of the test mirror is at the vertex of the triangle defined by the three mirrors. In the configuration of Figure 2, the total reflectance of the cavity is

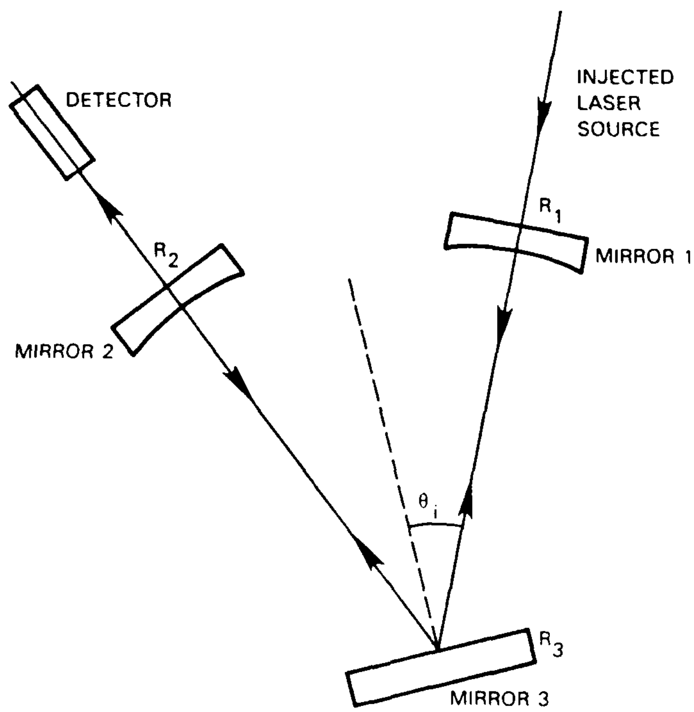

FIGURE 2. Three-Mirror Passive Cavity. Advantage over cavity in Figure 1 is in one-round trip test Mirror 3 is sampled twice, i.e., $R_T = R_1R_2R_3^2$.

$$R_t = R_1 R_2 R_3^2 \qquad (2)$$

where R_3 is the test mirror reflectance. R_3 is squared because in one round trip the test mirror is sampled twice. Thus, R_3 may be calculated if R_1, R_2, and R_t are known. To obtain high-reflectance values of R_3, high-reflectance end mirrors are required to aid in minimizing the error of the measurement. The effects of errors will be discussed later.

To model the cavity exactly, careful attention must be given to the input wave and its representation. The representation of the input laser power used by Rahn (Reference 5) will be the starting point in this analysis. This power is described by

$$I(t) = I_o \sin^2 \omega t = \frac{1}{2} I_o(1 - \cos 2\omega t) \qquad (3)$$

where I_o is the output power of the laser, f ($\omega = 2\pi f$) is the amplitude modulation frequency of this laser determined by a suitable intensity modulator, and t is time.

The passive cavity is designed to allow oscillation of the cavity modes defined by $f_o = nc/2L$, where f_o is the optical frequency of the injection source (units = sec^{-1}), n is any integer (on the order of 10^6 for optical frequencies injected into cavities with lengths from centimeters to meters), L is the length of the cavity in meters, and c is the speed of light in meters per second. If the source is a laser possessing a mode bandwidth, a change in cavity length will not quench the oscillation but will select a slightly different mode. If the length of the cavity changes beyond an integral number of half wavelengths or if only one mode is available from the laser, the oscillation may be intermittent. This will lead to errors in measurement of the cavity loss. Care in system design is necessary to prevent intermittent operation.

The beam is free to propagate through the cavity subject to losses from the mirrors. Since the modulation of the beam changes its phase with time, we can expect that as the beam traverses through the cavity in the course of one round trip, the relative phase changes by the amount (References 6 through 8)

$$\phi_r = \omega\tau = (2\pi f)(\frac{2L}{c}) = \frac{4\pi f L}{c}$$

for a single mode. The resulting intercavity intensity is therefore a summation of the intensities of all modes oscillating in the cavity. Each of these modes is subject to reductions in intensity because of the cavity losses. If Equation 3 is representative of the modulation, this summation over all modes can be expressed as (References 5 and 8)

$$I = I_o \sum_{n=o}^{\infty} R_t^n \sin^2 (\omega t + n\omega\tau) \tag{3a}$$

where $\tau = 2L/c$ and R_t is the total cavity reflectance. This summation can be evaluated by expressing the trigonometric function in its complex exponential form, so that the summation takes the form of a geometric series. In the case of Equation 3, the appropriate trigonometric power relation is applied. Details of this process are included in Appendix A. Thus,

$$I = I_o \sum_{n=o}^{\infty} R_t^n \sin^2 (\omega t + n\omega\tau) = I_o \frac{1}{2(1-R_t)}$$

$$+ \frac{1}{2} \frac{(\sin 2\omega t)(R_t \sin 2\omega\tau)-(\cos 2\omega t)(1-R_t \cos 2\omega\tau)}{1 - 2R \cos 2\omega\tau + R_t^2} \tag{3b}$$

The power and the phase angle of the transmitted beam I_t can be related since it follows a pattern similar to that of the injected beam. Thus,

$$I_T \propto \frac{I_o}{2} [1 - \cos (2\omega t + \phi)]$$

$$\propto \frac{1}{2} I_o + \frac{1}{2} I_o(\sin 2\omega t \sin \phi - \cos 2\omega t \cos \phi) \tag{4}$$

Now the time dependence can be eliminated by equating the appropriate time-related coefficients in Equations 3b and 4. Thus, equating the coefficient of sin 2ωt,

$$\sin\phi = \frac{R_t \sin 2\omega\tau}{1 - 2R_t \cos 2\omega\tau + R_t^2} \tag{5}$$

Equating the coefficient of cos 2ωt in Equations 3a and 4,

$$\cos\phi = \frac{1 - R_t \cos 2\omega\tau}{1 - 2R_t \cos 2\omega\tau + R_t^2} \tag{6}$$

So that

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{R_t \sin 2\omega\tau}{1 - R_t \cos 2\omega\tau} \tag{7}$$

This is the result if no random thermal fluctuations are included in the analysis.

To examine the effect of a random phase induced by a thermal variation in the cavity, the amplitude rather than the intensity must be used. The amplitude function is related to intensity I by

$$I = A^*A \tag{8a}$$

as is also

$$R_t = r^*r \tag{8b}$$

where A is the electric field amplitude of the laser beam, r is the reflectivity coefficient, and R_t as before is the reflectance of the cavity.

Rahn (Reference 5) suggests that a random phase shift δ could range from −π to +π; he used numerical averaging to obtain an expression for ⟨tan ϕ⟩. Increased precision may be realized if the averaging evaluation of tan ϕ can be done exactly. Starting with Equation 4 of Reference 5 for the transmitted intensity,

$$A(t) = A_0 t_1 t_2 t_3 \sum_{p=0}^{\infty} (r)^p e^{i\delta p} \sin[\omega(t - p\tau)]$$

where $r = r_1 r_2 r_3{}^2$; t_1, t_2, and t_3 are the reflection and transmission coefficients, respectively; p is the number of passes through the cavity; δ is the round-trip phase shift; and τ is the time required for a photon to make a single round trip in the cavity. Summing the infinite series gives

$$A(t) \sim \frac{\sin \omega t}{2}\left[\frac{1}{1 - re^{i(\delta + \omega\tau)}} + \frac{1}{1 - re^{i(\delta - \omega\tau)}}\right]$$

$$+ \frac{i \cos \omega t}{2}\left[\frac{1}{1 - re^{i(\delta + \omega\tau)}} \quad \frac{1}{1 - re^{i(\delta - \omega\tau)}}\right]$$

$$= \frac{\sin \omega t(1 - r_1 r_2 e^{i\delta} \cos \omega\tau) - \cos \omega t(r_1 r_2 e^{i\delta} \sin \omega\tau)}{1 - 2re^{i\delta} \cos \omega\tau + r^2 e^{2i\delta}} \qquad (9)$$

Squaring to get the transmitted power,

$$I_T = \left|A(t)\right|^2 \sim \frac{\cos 2\omega t(r \cos \omega\tau \cos \delta - \frac{1}{2} - \frac{1}{2} r^2 \cos 2\omega\tau)}{4r^2 \cos^2 \delta - 4 \cos \omega\tau(r + r_3) \cos \delta + 4r^4}$$

$$\frac{- \sin 2\omega t(r \sin \omega\tau \cos \delta - r^2 \sin \omega\tau \cos \omega\tau)}{- 2r^2 + 4r^2 \cos^2 \omega\tau + 1} \qquad (10)$$

details of which are in Appendix B. Equating trigonometric coefficients in Equations 4 and 13 for the transmitted power gives expressions for sin ϕ and cos ϕ,

$$\sin\phi = \frac{2(r^2 \sin\omega\tau \cos\omega\tau - r \sin\omega\tau \cos\delta)}{4r^2 \cos^2\delta - 4\cos\omega\tau(r+r^3)\cos\delta + 4r^4 - 2r^2 + 4r^2\cos^2\omega\tau + 1} \quad (11)$$

and

$$\cos\phi = \frac{(1 + r^2 \cos 2\omega\tau) - 2r\cos\omega\tau\cos\delta}{4r^2 \cos^2\delta - 4\cos\omega\tau(r+r^3)\cos\delta + 4r^4 - 2r^2 + 4r^2\cos^2\omega\tau + 1} \quad (12)$$

Both expressions are of the form

$$F(\delta) = \frac{a\cos\delta + b}{c\cos^2\delta - d\cos\delta + e} \quad (13)$$

where

$$c = 4r^2$$

$$d = 4r(1 + r^2)\cos\omega\tau$$

$$e = 1 + 4r^2\cos^2\omega\tau + r^4 - 2r^2$$

or

$$e = (1 - r^2)^2 + 4r^2\cos^2\omega\tau$$

For $\sin\phi$,

$$a = -2r\sin\omega\tau$$

$$b = 2r^2\sin\omega\tau\cos\omega\tau$$

For $\cos\phi$,

$$a = -2r\cos\omega\tau$$

$$b = 1 + r^2\cos 2\omega\tau$$

Equations 11 and 12 rewritten in the form of Equation 13 must be integrated between the limits 0 to 2π to obtain an average value of $\tan\phi$. Evaluation of these integrands is by partial fractions. The decomposition of the denominator to produce partial fractions is accomplished by noting that

$$c \cos^2 \delta - d \cos \delta + e = (\sqrt{c} \cos \delta - h)(\sqrt{c} \cos \delta - k)$$

$$= c \cos^2 \delta - (h + k)\sqrt{c} \cos \delta + hk$$

where h and k are coefficients that satisfy $d = (h + k)\sqrt{c}$ and $e = hk$.

These equations yield

$$k^2\sqrt{c} - kd + e\sqrt{c} = 0$$

and

$$h = \frac{d}{\sqrt{c}} - k$$

so that

$$k = \frac{1}{2\sqrt{c}}\left(d + \sqrt{d^2 - 4ec}\right) \quad , \; h = \frac{1}{2\sqrt{c}}\left(d - \sqrt{d^2 - 4ec}\right)$$

To find the partial fractions, we first write

$$\frac{a \cos \delta + b}{c \cos^2 \delta - (h+k) \sqrt{c} \cos \delta + hk} = \frac{A}{\sqrt{c} \cos \delta - h} + \frac{B}{\sqrt{c} \cos \delta - k}$$

where A and B are unknown coefficients, then

$$a \cos \delta + b = A(\sqrt{c} \cos \delta - k) + B(\sqrt{c} \cos \delta - h)$$

and evaluate at $\sqrt{c} \cos \delta = k$ and $\sqrt{c} \cos \delta = h$ separately. These yield

$$A = \frac{-(ah + b\sqrt{c})}{\sqrt{d^2 - 4ec}} \tag{14a}$$

and

$$B = \frac{ak + b\sqrt{c}}{\sqrt{d^2 - 4ec}} \tag{14b}$$

We integrate

$$\langle \sin \phi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a_1 \cos \theta + b_1) d\theta}{c \cos^2 \theta - d \cos \theta + e} = \frac{A_1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{c} \cos \theta - h}$$

$$+ \frac{B_1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{c} \cos \theta - k} = \frac{A_1}{\sqrt{h^2 - c}} + \frac{B_1}{\sqrt{k^2 - c}} \tag{15a}$$

with similar results for

$$\langle \cos \phi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a_2 \cos \theta + b_2) d\theta}{c \cos^2 \theta - d \cos \theta + e} \tag{15b}$$

so that

$$\langle \tan \phi \rangle = \frac{\langle \sin \phi \rangle}{\langle \cos \phi \rangle} = \frac{A_1\sqrt{k^2 - c} + B_1\sqrt{h^2 - c}}{A_2\sqrt{k^2 - c} + B_2\sqrt{h^2 - c}} \tag{16}$$

This becomes

$$\langle \tan \phi \rangle = \frac{r^2 \sin 2\omega\tau}{1 - r^2 \cos 2\omega\tau} = \frac{R_t \sin 2\omega\tau}{1 - R_t \cos 2\omega\tau} \tag{17}$$

the details of which appear in Appendixes C and D. Notice this is exactly equivalent to Equation 7. Thus, random phase shifts are not a major source of error over the time interval of measurement of ϕ.

An expression for R_3--the unknown mirror's reflectance--can be derived starting from Equation 17. Using a common denominator and solving for R_t gives

$$R_t = \frac{\langle \tan \phi \rangle}{\cos 2\omega\tau \langle \tan \phi \rangle + \sin 2\omega\tau} \tag{18}$$

Using Equation 2, solving for R_3, and substituting into Equation 18 for R_t gives

$$R_3 = \left[\frac{\langle \tan \phi \rangle}{R_1 R_2 (\cos 2\omega\tau \langle \tan \phi \rangle + \sin 2\omega\tau)} \right]^{1/2} \tag{19}$$

The cavity parameters $4\pi fL/c$ substitute for $\omega\tau$; thus

$$R_3 = \left[\frac{\langle \tan \phi \rangle}{R_1 R_2 \left(\langle \tan \phi \rangle \cos \frac{8\pi fL}{c} + \sin \frac{8\pi fL}{c} \right)} \right]^{1/2} \tag{20}$$

If the cavity satisfies $8\pi fL/c \ll 1$ and $R_1 \sim R_2 \sim 1$, then

$$R_3 \simeq \left(\frac{\langle \tan \phi \rangle}{\langle \tan \phi \rangle + \frac{8\pi fL}{c}} \right)^{1/2} \tag{21}$$

a good approximation for very low-loss mirrors.

A comparison of the sample reflectance R_3 derived by the numerical analysis of Rahn (Reference 5) and the exact expression, Equation 20, is given in Figure 3. The expression used by Rahn is

$$R_3 = \left(\frac{\langle \tan \theta \rangle - \frac{4\pi f L}{c}}{R_1 R_2 \langle \tan \theta \rangle} \right)^{1/2}$$

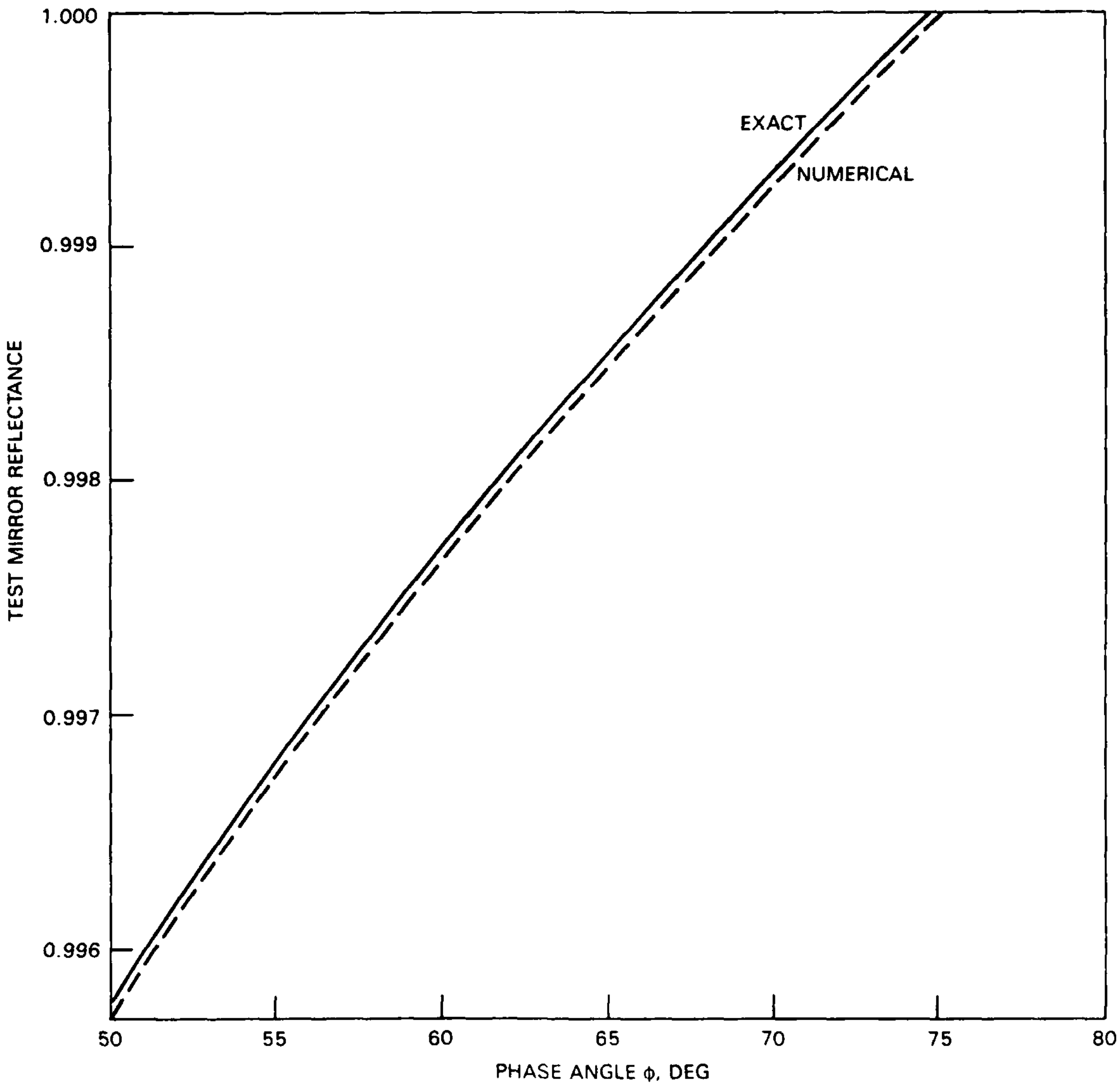

FIGURE 3. Comparison of the Numerical Analysis by Rahn (Reference 5) and the Exact Analysis for the Same Cavity Parameters, $R_1 = R_2 = 0.998$, $f = 100$ kHz, and $L = 3.864$ m. Shows sample reflectance versus measured phase angle between injected and transmitted wave.

where L is the round-trip length of the cavity. Recall that the length used in Equation 20 is defined as the length of the cavity.

Notice that the numerical analysis gives a lower sample reflectance than the exact expression, but differences are not seen in R_3 until the fourth decimal place. The numerical analysis of Rahn is adequate unless one requires higher precision, 1 part in 10^4 or better. This is the situation for characterization of state-of-the-art low-loss mirrors used in laser gyros and low-gain lasers. The high precision is also useful for detecting very small changes in the performance of an optical coating under laser loading when determining functional lifetime.

ERROR ANALYSIS FOR THREE-MIRROR CONFIGURATION

In order to evaluate the total error in the value of reflectance of a sample mirror in a passive cavity, errors must be included from four sources: errors from (1) modulation frequency; (2) cavity length; (3) cavity end-mirror; and (4) phase in the phase detector. Although several of these contribute relatively insignificantly to the overall error, the error analysis nevertheless includes them, both as reference for future systems and for the sake of completeness. Future advances in measuring systems may reduce some errors to the point that others are increasingly significant. Existing systems can have their total errors calculated without concern that error sources have become more (or less) significant.

For

$$\tan \phi = \frac{R_t \sin (2\omega\tau)}{1 - R_t \cos (2\omega\tau)}$$

$$\frac{\partial}{\partial\phi} (\tan \phi) = \frac{\partial}{\partial\phi} \frac{R_t \sin (2\omega\tau)}{1 - R_t \cos (2\omega\tau)}$$

So that

$$\sec^2(\phi) = \frac{(1-R_t \cos 2\omega\tau) \sin 2\omega\tau + R_t \sin 2\omega\tau \cos 2\omega\tau}{(1 - R \cos 2\omega\tau)^2} \frac{\partial R_t}{\partial\phi}$$

and

$$\frac{\partial R_t}{\partial \phi} = \left(\frac{1 - R_t \cos 2\omega\tau}{\cos \phi}\right)^2$$

$$\times \frac{1}{(1 - R_t \cos 2\omega\tau) \sin 2\omega\tau + R_t \sin 2\omega\tau \cos 2\omega\tau}$$

$$\frac{\partial R_t}{\partial \phi} = \left(\frac{1 - R_t \cos 2\omega\tau}{\cos \phi}\right)^2 \csc (2\omega\tau) \qquad (22)$$

Since we are concerned with the error associated with the sample mirror rather than that of the system, we must consider the previous equation $R_t = R_1R_2R_3^2$, where R_1 and R_2 are the system mirrors and R_3 is the sample, so that

$$R_3 = \frac{\sqrt{R_t}}{\sqrt{R_1R_2}}$$

and

$$\frac{\partial R_3}{\partial \phi} = \frac{1}{\sqrt{R_1R_2}} \frac{1}{2\sqrt{R_t}} \frac{\partial R_t}{\partial \phi}$$

$$= \frac{1}{2R_1R_2R_3} \frac{\partial R_t}{\partial \phi} \qquad (23)$$

The chain rule applies for the variables in the equation relating to the frequency and delay time of the system; also, so that the total error is

$$\Delta R_3 = \frac{1}{2\sqrt{R_1 R_2 R_t}} \left(\frac{\partial R_t}{\partial \phi} \Delta\phi\right)^2 + \left(\frac{\partial R_t}{\partial \omega} \Delta\omega\right)^2 + \left(\frac{\partial R_t}{\partial \tau} \Delta\tau\right)^2$$

$$+ \left(\frac{\partial R_t}{\partial R_1} \Delta R_1\right)^2 + \left(\frac{\partial R_t}{\partial R_2} \Delta R_2\right)^2$$

Evaluating the partial differentials gives the following:

$$\frac{\partial R_t}{\partial \tau} = \frac{\partial}{\partial \tau}\left[\frac{\tan\phi}{\sin 2\omega\tau + \cos 2\omega\tau \tan\phi}\right]$$

$$= \frac{-\tan\phi(2\omega\cos 2\omega\tau - 2\omega\sin 2\omega\tau\tan\phi)}{(\sin 2\omega\tau + \cos 2\omega\tau\tan\phi)^2}$$

$$= \frac{-2\omega\tan\phi(\cos 2\omega\tau - \sin 2\omega\tau\tan\phi)}{(\sin 2\omega\tau + \cos 2\omega\tau\tan\phi)^2} \qquad (24)$$

$$\frac{\partial R_t}{\partial \omega} = \frac{\partial}{\partial \omega}\left[\frac{\tan\phi}{\sin 2\omega\tau + \cos 2\omega\tau \tan\phi}\right]$$

$$= \frac{-\tan\phi(2\tau\cos 2\omega\tau - 2\tau\sin 2\omega\tau\tan\phi)}{(\sin 2\omega\tau + \cos 2\omega\tau\tan\phi)^2}$$

$$= \frac{-2\tau\tan\phi(\cos 2\omega\tau - \sin 2\omega\tau\tan\phi)}{(\sin 2\omega\tau + \cos 2\omega\tau\tan\phi)^2} \qquad (25)$$

Now

$$\left(\frac{\partial R_t}{\partial \omega} \Delta\omega\right)^2 + \left(\frac{\partial R_t}{\partial \tau} \Delta\tau\right)^2$$

$$= \frac{4\tan^2\phi(\cos 2\omega\tau - \sin 2\omega\tau\tan\phi)^2}{(\sin 2\omega\tau + \cos 2\omega\tau\tan\phi)^4}\left[(\tau\cdot\Delta\omega)^2 + (\omega\cdot\Delta\tau)^2\right]$$

$$= \frac{4R_t{}^2 \sin^2 2\omega\tau}{(1 - R_t \cos 2\omega\tau)^2} \frac{\left(\cos 2\omega\tau - \dfrac{R_t \sin^2 2\omega\tau}{1 - R_t \cos 2\omega\tau}\right)^2}{\left(\sin 2\omega\tau + \dfrac{R_t \sin 2\omega\tau \cos 2\omega\tau}{1 - R_t \cos 2\omega\tau}\right)^4}$$

$$\times [(\tau \cdot \Delta\omega)^2 + (\omega \cdot \Delta\tau)^2]$$

$$= \frac{4R_t^2 \sin^2 2\omega\tau(1 - R_t \cos 2\omega\tau)^2}{(1 - R_t \cos 2\omega\tau)^4}$$

$$\times \frac{\left[\cos 2\omega\tau - \dfrac{R_t \sin^2 2\omega\tau}{1 - R_t \cos 2\omega\tau}\right]^2}{\left[\sin 2\omega\tau + \dfrac{R_t \sin 2\omega\tau \cos 2\omega\tau}{1 - R_t \cos 2\omega\tau}\right]^4} [(\tau \cdot \Delta\omega)^2 + (\omega \cdot \Delta\tau)^2]$$

$$= \frac{4R_t^2[\sin 2\omega\tau \cos 2\omega\tau(1-R_t \cos 2\omega\tau)-R_t \sin^3 2\omega\tau]^2[(\tau \cdot \Delta\omega)^2+(\omega \cdot \Delta\tau)^2]}{[\sin 2\omega\tau(1-R \cos 2\omega\tau) + R \sin 2\omega\tau \cos 2\omega\tau]^4}$$

$$= \frac{4R_t^2 [\sin 2\omega\tau \cos 2\omega\tau - R_t \sin 2\omega\tau]^2 [(\tau \cdot \Delta\omega)^2 + (\omega \cdot \Delta\tau)^2]}{\sin^4 2\omega\tau}$$

$$= \frac{4R_t^2 (\cos 2\omega\tau - R_t)^2 [(\tau \cdot \Delta\omega)^2 + (\omega \cdot \Delta\tau^2]}{\sin^2 2\omega\tau} \qquad (26)$$

Now

$$\Delta\omega = \Delta(2\pi f) = 2\pi\Delta f \quad ; \quad \Delta\tau = \Delta(\frac{2L}{c}) = \frac{2}{c} \Delta L$$

and

$$\sec^2 \phi = 1 + \tan^2 \phi = \frac{1 - 2R_t \cos 2\omega\tau + R_t^2}{(1 - R_t \cos 2\omega\tau)^2}$$

so that Equation 22 becomes

$$\frac{\partial R_t}{\partial \phi} = \frac{(1 - R_t \cos 2\omega\tau)^2 (1 - 2R_t \cos 2\omega\tau + R_t)^2}{\sin^2 2\omega\tau(1 - R_t \cos 2\omega\tau)^2}$$

$$= \frac{(1 - 2R_t \cos 2\omega\tau + R_t)^2}{\sin^2 2\omega\tau}$$

According to Equation 2, the reflectance of the sample mirror can be expressed as

$$R_3 = \frac{\sqrt{R_t}}{\sqrt{R_1 R_2}}$$

where R_t is the reflectance of the three-mirror system and R_1 and R_2 are the end-mirror reflectances. Since they are inextricably bound to the measuring system, the end mirrors contribute to errors in the determination of the sample reflectance as follows:

$$\frac{\partial R_3}{\partial R_1} = \frac{\sqrt{R_t}}{\sqrt{R_2}} \frac{\partial}{\partial R_1} [R_1^{-1/2}] = \frac{\sqrt{R_t}}{\sqrt{R_2}} \left[- \frac{1}{2} R_1^{-3/2}\right] = - \frac{1}{2R_1} \frac{\sqrt{R_t}}{\sqrt{R_1 R_2}}$$

$$= - \frac{R_3}{2R_1} \qquad (27a)$$

and

$$\frac{\partial R_3}{\partial R_2} = - \frac{R_3}{2R_2} \qquad (27b)$$

so that the total squared error contribution from end-mirror accuracy is

$$\left(\frac{\partial R_3}{\partial R_1}\Delta R_1\right)^2 + \left(\frac{\partial R_3}{\partial R_2}\Delta R_2\right)^2 = \frac{1}{4} R_3^2 \left(\frac{\Delta R_1^2}{R_1^2} + \frac{\Delta R_2^2}{R_2^2}\right) = \frac{R_3^4}{4R_1^2 R_2^2 R_3^2}$$

$$\times \left[R_2^2 \Delta R_1^2 + R_1^2 \Delta R_2^2\right] \qquad (28)$$

and the total root-mean-square (rms) error is

$$\Delta R_3 = \frac{1}{2(R_1 R_2 R_3)} \sqrt{[(1 - 2R_t \cos 2\omega\tau + R_t^2)\csc^2(2\omega\tau)\Delta\phi]^2}$$

$$+ 4R_t^2(\cos 2\omega\tau - R_t)^2 \csc^2 \omega\tau$$

$$\times 4\left[(\pi\Delta\tau f)^2 + \left(\frac{\Delta\omega L}{c}\right)^2\right] + R_3^4(R_2^2\Delta R_1^2 + R_1^2\Delta R_2^2) \qquad (29)$$

For example, a system in which

$$f = 10^5 \pm 10^2 \text{ hertz}$$

$$L = 2 \text{ meters} \pm 0.002 \text{ meters}$$

$$R_1 = R_2 = 0.999 \pm 0.0001$$

then $\omega\tau$ equals

$$\omega\tau = \frac{4\pi f L}{c} = 0.48 \text{ degrees}$$

A sample with a nominal reflectance of 0.998 (therefore, $R_t = 0.996$) would produce an angle exiting the cavity of

$$\phi = \tan^{-1} \frac{R_t \sin 2\omega\tau}{1 - R_t \cos 2\omega\tau} = 82.706 \text{ degrees}$$

If this measurement is subject to an error of ±1 degree, then the reflectance measurement is subject to an error of

$$\Delta R_3(\phi) = \frac{1}{2R_1R_2R_3}(1 - 2R_t \cos 2\omega\tau + R_t^2) \csc (2\omega\tau)(\frac{\pi}{180})$$

$$= 1.545 \times 10^{-4}$$

The measurement due to end-mirror deviations would be

$$\Delta R_3(R_1R_2) = \frac{R_3^2}{2R_1R_2R_3}\sqrt{R_2^2\Delta R_1^2 + R_1^2\Delta R_2^2} = \frac{R_3}{2R_1R_2}\sqrt{2R_1^2\Delta R_1^2}$$

$$= \frac{R_3\Delta R_1}{\sqrt{2}\,R_1} = 7.064 \times 10^{-5}$$

Finally, the error caused by the frequency and cavity-length deviations would be equal to

$$\Delta R_3(\omega\tau) \frac{\sqrt{(\pi f \Delta\tau)^2 + (\frac{L\Delta\omega}{c})^2}}{R_1R_2R_3}\left(\frac{R_t}{1 - R_t \cos 2\omega\tau}\right) (\cos 2\omega\tau - R_t)$$

$$= \frac{\sqrt{(4.1888\times10^{-6})^2+(4.1888\times10^{-6})^2}}{0.996}(240.9)(3.85\times10^{-3})$$

$$= 5.516 \times 10^{-6}$$

so that the total rms error would be equal to

$$\Delta R_{3_{tot}} = \sqrt{(1.545 \times 10^{-4})^2 + (7.064 \times 10^{-5})^2 + (5.516 \times 10^{-6})^2}$$

$$= 1.700 \times 10^{-4}$$

and the reflectance of the sample would be expressed as $R_3 = 0.998 \pm 0.00017$. It is seen from the above that, for this system, the major sources of error in calculating the reflectance are the measurement of the phase angle of the superimposed waves and the uncertainty of end-mirror reflectances. However, the contributing sources of error are sufficiently comparable in magnitude that we suggest including them all, especially with the foreseeable modifications of greater phase-angle sensitivity and different end-mirror uncertainty values. The above analysis would not be valid for a two-mirror system, since $R_t = R_1R_2$, but those elements that are based on partial derivatives of R_t can be incorporated into a two-mirror system error analysis.

SUMMARY

An exact calculation has been performed on the influence of a random phase fluctuation induced by a thermal variation in the cavity to the performance of a passive cavity reflectometer. The analysis started with the assumptions stated by Rahn (Reference 5) in his numerical analysis of this problem. The expression that was derived is identical to the expression where no random phase fluctuations were included. Thus, thermal variation of the cavity is not a major source of error for the operation of a reflectometer. Comparison of the numerical and exact analyses show that the numerical expression for R_3 is not a good approximation where high precision--better than 1 part in 10^4--is required.

Error analysis was performed to determine the functional dependence of cavity length, modulation frequency, cavity mirror reflectance, and phase measurement errors on the precision of the reflectometer. The total error of a simple reflectometer was determined to be on the order of 2 parts in 10^4.

If high precision is required, care must be taken in the design and construction of a passive cavity reflectometer to minimize or eliminate the sources of error that are mentioned above. On the other hand, a simple reflectometer can be constructed with a precision and error of approximately 2 parts in 10^4; either analysis is adequate for the measurement of R_3.

Appendix A

DERIVATION OF THE INFINITE SUM IN EQUATION 3

Using the power relation for $\sin^2(\alpha)$,

$$I' = \sum_{n=o}^{\infty} R_t^n \sin^2(\omega t + n\omega\tau) = \frac{1}{2} \sum_{n=o}^{\infty} R_t^n [1 - \cos(2\omega t + 2n\omega\tau)]$$

Now applying the sum-and-difference formula

$$I' = \frac{1}{2} \sum_{n=o}^{\infty} R_t^n - \frac{1}{2} \sum_{n=o}^{\infty} R_t^n [\cos 2\omega t \cos 2n\omega t - \sin 2\omega t \sin 2n\omega\tau] \quad \text{(A-1)}$$

$$I' = \frac{1}{2} \left(\frac{1}{1 - R_t}\right) - \frac{1}{2} \cos 2\omega t \sum_{n=o}^{\infty} R_t^n \cos 2n\omega\tau$$

$$+ \frac{1}{2} \sin 2\omega t \sum_{n=o}^{\infty} R_t^n \sin 2n\omega\tau \quad \text{(A-2)}$$

The trigonometric identities for complex exponential yield

$$I' = \frac{1}{2(1 - R_t)} - \frac{1}{2} \cos 2\omega t \sum_{n=o}^{\infty} \frac{R_t^n}{2} (e^{i2n\omega\tau} + e^{-i2n\omega\tau})$$

$$+ \frac{1}{2} \sin 2\omega t \sum_{n=o}^{\infty} R_t^n \frac{e^{i2n\omega\tau} - e^{-i2n\omega\tau}}{2i}$$

$$= \frac{1}{2(1 - R_t)} - \frac{1}{4} \cos 2\omega t \sum_{n=o}^{\infty} R_t^n e^{i2n\omega\tau} + \sum_{n=o}^{\infty} R_t^n e^{-i2n\omega\tau}$$

$$+ \frac{1}{4i} \sin 2\omega t \sum_{n=o}^{\infty} R_t^n e^{i2n\omega\tau} - \sum_{n=o}^{\infty} R_t^n e^{-i2n\omega\tau}$$

$$= \frac{1}{2(1 - R_t)} - \frac{1}{4} \cos 2\omega t \frac{1}{1 - R_t e^{i2\omega\tau}} + \frac{1}{1 - R_t e^{-i2\omega\tau}}$$

$$+ \frac{1}{4i} \sin 2\omega t \frac{1}{1 - R_t e^{i2\omega\tau}} - \frac{1}{1 - R_t e^{-i2\omega\tau}} \qquad \text{(A-3)}$$

The complex fractions can be evaluated by expressing them with a common denominator:

$$I' = \frac{1}{2(1 - R_t)} - \frac{1}{4} \cos 2\omega t \; \frac{1 - R_t e^{-i2\omega\tau} + 1 - R_t e^{i2\omega\tau}}{1 - R_t e^{-2\omega\tau} - R_t e^{i2\omega\tau} + R_t^2}$$

$$+ \frac{1}{4i} \sin 2\omega t \; \frac{1 - R_t e^{-i2\omega\tau} - 1 + R_t e^{i2\omega\tau}}{1 - R_t e^{-i2\omega\tau} - R_t e^{i2\omega\tau} + R_t^2}$$

$$= \frac{1}{2(1 - R_t)} - \frac{1}{4} \cos 2\omega t \; \frac{2 - 2 R_t \cos 2\omega\tau}{1 - 2 R_t \cos 2\omega\tau + R_t^2}$$

$$+ \frac{1}{4i} \sin 2\omega t \; \frac{2iR_t \sin 2\omega\tau}{1 - 2R_t \cos 2\omega\tau + R_t^2}$$

$$= \frac{1}{2(1 - R_t)} - \frac{1}{2} \cos 2\omega t \; \frac{1 - R_t \cos 2\omega\tau}{1 - 2 R_t \cos 2\omega\tau + R_t^2}$$

$$+ \frac{1}{2} \sin 2\omega t \; \frac{R_t \sin 2\omega\tau}{1 - 2R_t \cos 2\omega\tau + R_t^2} \qquad \text{(A-4)}$$

Thus,

$$I' = \frac{1}{2(1 - R_t)} - \frac{1}{2} \; \frac{\cos 2\omega t(1 - R_t \cos 2\omega t) - \sin 2\omega\tau \, R_t \sin 2\omega\tau}{(1 - 2R_t \cos 2\omega\tau + R_t^2)}$$

Appendix B

DERIVATION OF A*A

Given

$$A \sim \sin \omega t \left[\frac{1 - re^{i\delta} \cos \omega\tau}{1 - 2re^{i\delta} \cos \omega\tau + r^2 e^{2i\delta}} \right]$$

$$+ \cos \omega t \left[\frac{re^{i\delta} \sin \omega\tau}{1 - 2re^{i\delta} \cos \omega\tau + r^2 e^{2i\delta}} \right]$$

$$= \frac{\sin \omega t(1 - re^{i\delta} \cos \omega\tau) + \cos \omega t(re^{i\delta} \sin \omega\tau)}{1 - 2re^{i\delta} \cos \omega\tau + r^2 e^{2i\delta}}$$

$$A^*A = \frac{\sin^2 \omega t\ (1-re^{i\delta} \cos \omega\tau)(1-re^{-i\delta} \cos \omega\tau)+\cos^2 \omega t(r^2 \sin^2 \omega\tau)}{(1 - 2re^{i\delta} \cos \omega\tau + r^2 e^{2i\delta})(1 - 2re^{-i\delta} \cos \omega\tau + r^2 e^{-2i\delta})}$$

$$+ \frac{\sin\omega t\cos\omega t \left[(1-re^{i\delta}\cos\omega\tau)re^{-i\delta}\sin\omega\tau+(1-re^{-i\delta}\cos\omega\tau)re^{i\delta}\sin\omega\tau\right]}{(1-2re^{i\delta} \cos\omega\tau+r^2 e^{2i\delta})(1-2re^{-i\delta} \cos\omega\tau+r^2 e^{-2i\delta})} \quad (B-1)$$

The denominator becomes

$$1 - 2re^{-i\delta} \cos \omega\tau + r^2 e^{-2i\delta} - 2re^{i\delta} \cos \omega\tau$$

$$+ 4r^2 \cos^2 \omega\tau - 2r^3 e^{i\delta} \cos \omega\tau$$

$$+ r^2 e^{2i\delta} - 2r^3 e^{i\delta} \cos \omega\tau + r^4$$

$$= 1 + r^4 - 4r \cos \delta \cos \omega\tau + 2r^2 \cos 2\delta + 4r^2 \cos^2 \omega\tau$$

$$- 4r^3 \cos \delta \cos w\tau$$

$$= 1 + r^4 - 4r(1 + r^2) \cos \delta \cos \omega\tau + 4r^2 \cos^2 \omega\tau$$

$$+ 2r^2 (2 \cos^2 \delta - 1)$$

$$= 4r^2 \cos^2 \delta - 4r(1 + r^2) \cos \delta \cos \omega\tau + 4r^2 \cos^2 \omega\tau$$

$$+ (1 - r^2)^2$$

Simplifying the numerator as follows,

$$\frac{1}{2} (1 - \cos 2\omega t)(1 - re^{-i\delta} \cos \omega\tau - re^{-i\delta} \cos \omega\tau + r^2 \cos^2 \omega\tau)$$

$$+ \frac{1}{2} (1 + \cos 2\omega t)(r^2 \sin^2 \omega\tau)$$

$$+ \frac{1}{2} \sin 2\omega t(re^{-i\delta} \sin \omega\tau - r^2 \sin \omega\tau \cos \omega\tau$$

$$+ re^{i\delta} \sin \omega\tau - r^2 \sin \omega\tau \cos \omega\tau)$$

$$= \frac{1}{2} (1 - 2r \cos \delta \cos \omega\tau + r^2 \cos^2 \omega\tau + r^2 \sin^2 \omega\tau)$$

$$+ \frac{1}{2} \cos 2\omega t(r^2 \sin^2 \omega\tau - 1 + 2r \cos \delta \cos \omega\tau$$

$$- r^2 \cos^2 \omega\tau)$$

$$+ \frac{1}{2} \sin 2\omega t(2r \cos \delta \sin \omega\tau - 2r^2 \sin \omega\tau \cos \omega\tau)$$

$$= \frac{1}{2} (1 - 2r \cos \delta \cos \omega\tau + r^2)$$

$$+ \frac{1}{2} \cos 2\omega t(2r \cos \delta \cos \omega\tau - 1 - 2r^2 \cos 2\omega\tau)$$

$$+ \frac{1}{2} \sin 2\omega t(2r \cos \delta \cos \omega\tau - 2r^2 \sin \omega\tau \cos \omega\tau)$$

Using double-angle identities for $\sin^2 \omega t$ and $\cos^2 \omega t$,

$A^*A \sim$

$$\frac{\frac{1}{2}\cos\omega t(2r\cos\omega\tau\cos\delta - r^2\cos 2\omega\tau - 1) - \sin 2\omega t(r\sin\omega\tau\cos\delta - r^2\sin\omega\tau\cos\omega\tau)}{4r^2\cos^2\delta - 4r(1+r^2)\cos\delta\cos\omega\tau + 4r^2\cos^2\omega\tau + (1-r^2)^2} \quad \text{(B-2)}$$

Appendix C

REDUCTION OF VARIABLES IN EQUATIONS 20 AND 21

Equation 16 is

$$\langle \tan \phi \rangle = \frac{A_1\sqrt{k^2 - c} + B_1\sqrt{h^2 - c}}{A_2\sqrt{k^2 - c} + B_2\sqrt{h^2 - c}} \qquad \text{(C-1)}$$

where

$$A_1 = \frac{-(a_1 h + b_1\sqrt{c})}{\sqrt{d^2 - 4ec}} \quad , \quad B_1 = \frac{(a_1 k + b_1\sqrt{c})}{\sqrt{d^2 - 4ec}}$$

$$A_2 = \frac{-(a_1 h + b_1\sqrt{c})}{\sqrt{d^2 - 4ec}} \quad , \quad B_2 = \frac{(a_2 k + b_2\sqrt{c})}{\sqrt{d^2 - 4ec}}$$

$$a_1 = -2r \sin \omega\tau \quad , \quad b_1 = 2r^2 \sin \omega\tau \cos \omega\tau$$

$$a_2 = -2r \cos \omega\tau \quad , \quad b_2 = 1 + r^2 \cos 2\omega\tau$$

$$c = 4r^2$$

$$h = \frac{1}{2\sqrt{c}}\left(d - \sqrt{d^2 - 4ec}\right)$$

$$k = \frac{1}{2\sqrt{c}}\left(d + \sqrt{d^2 - 4ec}\right)$$

$$d = 4r(1 + r^2) \cos \omega\tau$$

$$e = (1 - r^2)^2 + 4r^2 \cos^2 \omega\tau$$

It is noted that the expression

$$\sqrt{d^2 - 4ec}$$

is an imaginary term since

$$d^2-4ec = 16r^2(1+r^2)^2 \cos^2 \omega\tau-4[(1-r^2)^2+4r^2 \cos^2 \omega\tau](4r^2)$$

$$= -16r^2(1-r^2)^2 \sin^2 \omega\tau$$

Thus $\sqrt{d^2 - 4ec} = 4ir(1 - r^2) \sin \omega\tau$, and imaginary terms are present in Equation 16. Upon multiplying the numerator and denominator of Equation 16 by $\sqrt{h^2 - c}$,

$$\langle \tan \phi \rangle = \frac{A_1\sqrt{(k^2 - c)(h^2 - c)} + B_1(h^2 - c)}{A_2\sqrt{(k^2 - c)(h^2 - c)} + B_2(h^2 - c)} \qquad \text{(C-2)}$$

Using $hk = e$ and $(h^2 + k^2)c = d^2 - 2ec$, the argument under the radical in Equation C-2 becomes

$$(k^2-c)(h^2-c) = h^2k^2-(h^2+k^2)c+c^2$$

$$= e^2-d^2+2ec+c^2$$

$$= (e+c)^2-d^2$$

$$= [(1-r^2)^2+4r^2\cos^2\omega\tau+4r^2]^2-16r^2(1+r^2)^2\cos^2\omega\tau$$

$$= [(1+r^2)^2-4r^2\cos^2\omega\tau]^2 \qquad \text{(C-3)}$$

and $X = \sqrt{(k^2 - c)(h^2 - c)} = (1 + r^2)^2 - 4r^2 \cos^2 \omega\tau$ is entirely real. Now Equation C-2 can be written

$$\langle\tan \phi\rangle = \frac{A_1X + B_1(h^2 - c)}{A_2X + B_2(h^2 - c)} \quad , \quad X = (1 + r^2)^2 - 4r^2 \cos^2 \omega\tau$$

$$= \frac{-\left[\dfrac{a_1h + b_1\sqrt{c}}{(h - k)\sqrt{c}}\right] X + \left[\dfrac{a_1k + b_1\sqrt{c}}{(k - h)\sqrt{c}}\right](h^2 - c)}{-\left[\dfrac{a_2h + b_2\sqrt{c}}{(h - k)\sqrt{c}}\right] X + \left[\dfrac{a_2k + b_2\sqrt{c}}{(k - h)\sqrt{c}}\right](h^2 - c)}$$

$$= \frac{(a_1h + b_1\sqrt{c})\ X - (a_1k + b_1\sqrt{c})(h^2 - c)}{(a_2h + b_2\sqrt{c})\ X - (a_2k + b_2\sqrt{c})(h^2 - c)}$$

$$= \frac{(a_1h + b_1\sqrt{c})\ X - (a_1kh^2 - a_1kc + b_1\sqrt{c}h^2 - b_1c\sqrt{c})}{(a_2h + b_2\sqrt{c})\ X - (a_2kh^2 - a_2kc + b_2\sqrt{c}h^2 - b_2c\sqrt{c})} \qquad \text{(C-4)}$$

Using hk = e again,

$$\langle \tan \phi \rangle = \frac{(a_1h + b_1\sqrt{c})\ X - (a_1eh - a_1kc + b_1\sqrt{c}h^2 - b_1c\sqrt{c})}{(a_2h + b_2\sqrt{c})\ X - (a_2eh - a_2kc + b_2\sqrt{c}h^2 - b_2c\sqrt{c})}$$

$$= \frac{a_1h(X - e) + b_1\sqrt{c}(X + c) - b_1\sqrt{c}h^2 + a_1kc}{a_2h(X - e) + b_2\sqrt{c}(X + c) - b_2\sqrt{c}h^2 + a_2kc}$$

$$= \frac{a_1h(e - X) - b_1\sqrt{c}(X + c) + b_1h^2\sqrt{c} - a_1kc}{a_2h(e - X) - b_2\sqrt{c}(X + c) + b_2h^2\sqrt{c} - a_2kc} \qquad \text{(C-5)}$$

Using

$$h = \frac{1}{2\sqrt{c}}\left(d - \sqrt{d^2 - 4ec}\right)$$

$$= \frac{1}{2\sqrt{c}}(d - Di) \quad , \quad D = 4r(1 - r^2)\sin \omega\tau$$

$$h^2 = \frac{1}{4c}(d^2 - 2dDi - D^2)$$

$$k = \frac{1}{2\sqrt{c}}(d + Di)$$

and Equation C-5 can be written

$$\langle \tan\phi \rangle = \frac{\frac{a_1}{2\sqrt{c}}(d-Di)(e-X) - b_1\sqrt{c}(c+X) + \frac{b_1}{4\sqrt{c}}(d^2-2idD-D^2) - \frac{a_1\sqrt{c}}{2}(d+Di)}{\frac{a_2}{2\sqrt{c}}(d-Di)(e-X) - b_2\sqrt{c}(c+X) + \frac{b_2}{4\sqrt{c}}(d^2-2idD-D^2) - \frac{a_2\sqrt{c}}{2}(d+Di)} \quad \text{(C-6)}$$

or, multiplying numerator and denominator by $4\sqrt{c}$,

$$\langle \tan\phi \rangle = \frac{2a_1(d - Di)(e - X) - 4b_1c(c + X)}{2a_2(d - Di)(e - X) - 4b_2c(c + X)}$$

$$\frac{+ b_1(d^2 - 2idD - D^2) - 2a_1C(d + Di)}{+ b_2(d_2 - 2idD - D^2) - 2a_2C(d + Di)} \quad \text{(C-7)}$$

Separating into real and imaginary parts,

$$\langle \tan\phi \rangle = \frac{2a_1d(e-X) - 4b_1c(c + X) + b_1(d^2 - D^2) - 2a_1cd}{2a_2d(e-X) - 4b_2c(c + X) + b_2(d^2 - D^2) - 2a_2cd}$$

$$\frac{- Di[2a_1(e-X) + 2b_1d + 2a_1c]}{- Di[2a_2(e-X) + 2b_2d + 2a_2c]} \quad \text{(C-8)}$$

For

$$A = 2a_1d(e - X) - 4b_1c(c + X) + b_1(d^2 - D^2) - 2a_1cd$$

$$B = 4r(1 - r^2) \sin \omega\tau$$

$$C = 2a_2d(e - X) - 4b_2c(c + X) + b_2(d^2 - D^2) - 2a_2cd$$

$$E = 2a_2(e - X) + 2b_2d + 2a_2c$$

$$G = 2a_1(e - X) + 2b_1d + 2a_1c$$

$$\langle \tan \phi \rangle = \frac{A - BiG}{C - BiE}$$

$$= \left[\frac{A - BiG}{C - BiE}\right] \times \left[\frac{C + BiE}{C + BiE}\right]$$

$$= \frac{AC + B^2EG + Bi(AE - CG)}{C^2 + B^2E^2} \qquad \text{(C-9)}$$

Since the original integrand was real, we expect that the above expression should also be real, with the inference that $AE - CG = 0$ or

$$C = \frac{AE}{G} \qquad \text{(C-10)}$$

This is indeed the case, as shown in Appendix D. Using Equation D-1,

$$\langle \tan \phi \rangle = \frac{A(\frac{AE}{G}) + B^2EG}{(\frac{AE}{G})^2 + B^2E^2}$$

$$= \frac{A^2EG + B^2EG^3}{A^2E^2 + B^2E^2G^2} = \frac{EG(A^2 + B^2G^2)}{E^2(A^2 + B^2G^2)} = \frac{G}{E} \qquad \text{(C-11)}$$

Now,

$$\langle \tan \phi \rangle = \frac{G}{E}$$

$$= \frac{2a_1(e - X) + 2b_1 d + 2a_1 c}{2a_2(e - X) + 2b_2 d + 2a_2 c}$$

$$= \frac{a_1(e - X + c) + b_1 d}{a_2(e - X + c) + b_2 d}$$

$$= \frac{2r\sin\omega\tau(8r^2\cos^2\omega\tau) - 2r^2\sin\omega\tau\cos\omega\tau(4r\cos\omega\tau)(1+r^2)}{-2r\cos\omega\tau(8r^2\cos^2\omega\tau) + (1+r^2\cos 2\omega\tau)(4r\cos\omega\tau)(1+r^2)}$$

$$= \frac{16r^3 \sin \omega\tau \cos^2 \omega\tau - 8r^3 \sin \omega\tau \cos^2 \omega\tau(1 + r^2)}{4r \cos \omega\tau(1 + r^2)(1 + r^2 \cos 2\omega\tau) - 16r^3 \cos^3 \omega\tau}$$

$$= \frac{8r^3 \sin \omega\tau \cos^2 \omega\tau(1 - r^2)}{4r \cos \omega\tau(1 + r^2)(1 + r^2 \cos 2\omega\tau) - 16r^3 \cos^3 \omega\tau} \quad \text{(C-12)}$$

Dividing the numerator and denominator by $4r \cos \omega\tau$,

$$\langle \tan \phi \rangle = \frac{2r^2 \sin \omega\tau \cos \omega\tau(1 - r^2)}{(1 + r^2)(1 + r^2 \cos 2\omega\tau) - 4r^2 \cos^2 \omega t}$$

$$= \frac{2r^2 \sin \omega\tau \cos \omega\tau(1 - r^2)}{1 + r^2 \cos 2\omega\tau + r^2 + r^4 \cos 2\omega\tau - 4r^2 \cos^2 \omega\tau}$$

$$= \frac{2r^2 \sin \omega\tau \cos \omega\tau\ (1 - r^2)}{1 + r^2(2\cos^2 \omega\tau - 1) + r^2 + r^4 \cos 2\omega\tau - 4r^2 \cos^2 \omega\tau}$$

$$= \frac{2r^2 \sin \omega\tau \cos \omega\tau(1 - r^2)}{1 - 2r^2 \cos^2 \omega\tau + r^4(2\cos^2 \omega\tau - 1)} \qquad \text{(C-13)}$$

$$\langle \tan \phi \rangle = \frac{2r^2 \sin \omega\tau \cos \omega\tau(1 - r^2)}{1 - 2r^2 \cos^2 \omega\tau + 2r^4 \cos^2 \omega\tau - r^4}$$

$$= \frac{2r^2 \sin \omega\tau \cos \omega\tau(1 - r^2)}{(1 - r^4) - 2r^2 \cos^2 \omega\tau(1 - r^2)}$$

$$= \frac{2r^2 \sin \omega\tau \cos \omega\tau(1 - r^2)}{(1 - r^2)(1 + r^2) - (1 - r^2)(2r^2 \cos^2 \omega\tau)}$$

$$= \frac{2r^2 \sin \omega\tau \cos \omega\tau}{1 + r^2 - 2r^2 \cos^2 \omega\tau}$$

$$= \frac{r^2 \sin 2\omega\tau}{1 - r^2 \cos 2\omega\tau} \quad , \quad R_t \equiv r^2$$

$$\langle \tan \phi \rangle = \frac{R_t \sin 2\omega\tau}{1 - R_t \cos 2\omega\tau} \qquad \text{(C-14)}$$

Appendix D

PROOF OF AE - GC = 0

Starting with substitution of terms

$$AE - GC = [2a_1d(e - Xi) - 4b_1c + X) + b_1(d^2 - B^2) - 2a_1cd]$$

$$\times [2a_22a_2(e - X) + 2b_2d + 2a_2c] - [2a_1(e - X) + 2b_1d + 2a_1c]$$

$$\times [2a_2d(e - X) - 4b_2c(c + X) + b_2(d^2 - B^2) - 2a_2cd]$$

$$= 4a_1a_2(e - X)^2d + 4a_1d(e - X)(b_2d + a_2c) - 8a_2\ b_1c(c + X)$$

$$\times (e - X) - 8b_1c(c + X)(b_2d + a_2c) + 2a_2b_1(e - X)(d^2 - B^2)$$

$$+ 2b_1)d^2 - B^2)(b_2d + a_2c) - 4a_1a_2cd(e - X) - 4a_1cd$$

$$\times (b_2d + a_2c) - [4a_1a_2(e - X)^2d - 8a_1b_2c(c + X) + 2a_1b_2$$

$$\times (e - X)(d^2 - B^2) - 4a_1a_2cd(e - X) + 4a_2b_1d(e - X)$$

$$- 8b_1b_2cd(c + X) + 2b_1b_2d(d^2 - B^2) - 4a_2b_1cd^2 + 4a_1a_2cd$$

$$\times (e - X) - 8a_1b_2c^2(c + X) + 2a_1b_2c(d^2 - B^2) - 4a_1a_2c^2d]$$

$$= 4a_1d(e - X)(b_2d + a_2c) - 8a_2b_1c(c + X)(e - X) - 8b_1c$$

$$\times (c + X)(b_2d + a_2c) + 2a_2b_1)e - X)(d^2 - B^2) + 2b_1(d^2 - B^2)$$

$$\times (b_2d + a_2c) - 4a_1a_2cd(e - X) - 4a_1cd(b_2d + a_2c) + 8a_1b_2c$$

$$\times (c + X)(e - X) - 2a_1b_2(e - X)(d^2 - B^2) + 4a_1a_2cd(e - X)$$

$$- 4a_2b_1d^2(e - X) + 8b_1b_2cd(c + X) - 2b_1b_2d(d^2 - B^2)$$

$$+ 4a_2b_1cd^2 + 8a_1b_2c^2(c + X) - 2a_1b_2c(d^2 - B^2) + 4a_1a_2c^2d]$$

$$AE - GC = 4a_1b_2d^2(e - X) + 4a_1a_2cd(e - X) + 8c(c + X)(e - X)(a_1b_2$$
$$- a_2b_1) - 8b_1b_2cd(c + X) - 8a_2b_1c^2(c + X) + 2(e - X)$$
$$\times (d^2 - B^2)(a_2b_1 - a_1b_2) + 2b_1b_2d(d^2b^2) + 2a_2b_1c(d^2 - B^2)$$
$$- 4a_1a_2cd(e - X) - 4a_1b_2cd^2 - 4a_1a_2c^2d - 4a^2b_1d^2(e - X)$$
$$+ 8b_1b_2cd(c + X) - 2b_1b_2d(d^2 - B^2) + 4a_2b_1cd^2 + 8a_1b_2c^2$$
$$\times (c + X) - 2a_1b_2c(d_2 - B^2) + 4a_1a_2c^2d$$
$$= 4d^2(e - X)(a_1b_2 - a_2b_1) + 8c(c + X)(e - X)(a_1b_2 - a_2b_1)$$
$$+ 8c^2(c + X)(a_1b_2 - a_2b_1) + 2(e - X)(d^2 - B^2)(a_2b_1 - a_1b_2)$$
$$+ 2c(d^2 - B^2)(a_2b_1 - a_1b_2) + 4cd^2(a_2b_1 - a_1b_2)$$
$$= (a_1b_2 - a_2b_1)[4d^2(e - X) + 8c(c + X)(e - X) + 8c^2$$
$$\times (c + X) - 2(e - X)(d^2 - B^2) - 2c(d^2 - B^2) - 4cd^2]$$
$$= (a_1b_2 - a_2b_1)[4d^2(e - X - c) + 8c(c + X)(e - X + c)$$
$$- 2(e - X)(d^2 - B^2) - 2c(d^2 - B^2) - 4cd^2]$$
$$= (a_1b_2 - a_2b_1)[4d^2(e - X - c) + 8c(c + X)(e - X + c)$$
$$- 2(d^2 - B^2)(e - X + c)]$$
$$= (a_1b_2 - a_2b_1)[(e - X + c)(8c^2 + 8cX - 2d^2 + 2B^2)$$
$$+ 4d^2(e - X + c) - 8d^2c] = (a_1b_2 - a_2b_1)[(e - X + c)$$
$$\times (8c^2 + 8cX - 2d^2 + 2B^2 + 4d^2) - 8d^2c]$$
$$= (a_1b_2 - a_2b_1)[(e - X + c)(8c^2 + 8cX + 2d^2 + 2B^2) - 8d^2c]$$
$$= 2(a_1b_2 - a_2b_1)[(e - X + c)(4c^2 + 4cX + d^2 + B^2) - 4d^2c]$$

Using

$$c = 4r^2$$
$$X = (1 + r_2)^2 - 4r^2 \cos^2 \omega\tau$$

$$e = (1 - r^2)^2 + 4r^2 \cos^2 \omega\tau$$

then

$$\begin{aligned}
e - X + c &= (1 - r^2)^2 + 4r^2 \cos^2 \omega\tau - (1 + r^2)^2 + 4r^2 \cos^2 \omega\tau \\
&\quad + 4r^2 \cos^2 \omega\tau + 4r^2 \\
&= 1 - 2r^2 + r^4 + 8r^2 \cos^2 \omega\tau - 1 - 2r^2 - r^4 + 4r^2 \\
&= 8r^2 \cos^2 \omega\tau
\end{aligned}$$

$$\begin{aligned}
4c(c + X) &= 16r^2[4r^2 + (1 + r^2)^2 - 4r^2 \cos^2 \omega\tau] \\
&= 16r^2[4r^2 \sin^2 \omega\tau + (1 + r^2)^2]
\end{aligned}$$

$$\begin{aligned}
d^2 + B^2 &= 16r^2(1 + r^2)^2 \cos \omega\tau + 16r^2(1 - r^2)^2 \sin^2 \omega\tau \\
&= 16r^2(1 + r^2)^2 \cos^2 \omega\tau + 16r^2(1 + r^2)^2 \sin^2 \omega\tau \\
&\quad - 64r^4 \sin^2 \omega\tau \\
&= 16r^2(1 + r^2)^2 - 64r^4 \sin^2 \omega\tau
\end{aligned}$$

$$\begin{aligned}
4c(c + X) + d^2 + B^2 &= 16r^2[4r^2 \sin^2 \omega\tau] \\
&= 32r^2(1 + r^2)^2
\end{aligned}$$

$$\begin{aligned}
&(e - X + c)(4c^2 + 4cX + d^2 + B^2) - 4D^2c \\
&\quad = 8r^2 \cos^2 \omega\tau(32r^2)(1 + r^2) - 4[16r^2(1 + r^2)^2 \\
&\quad \times \cos^2 \omega\tau](4r^2) \\
&\quad = 256r^4(1 + r^2)^2 \cos^2 \omega\tau - 256r^4(1 + r^2)^2 \cos^2 \omega\tau \\
&\quad = 0
\end{aligned}$$

Therefore,

$$AE - GC = 0 \tag{D-1}$$

or

$$C = \frac{AE}{G}$$

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