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CONTROL SYSTEMS FOR PLATFORM LANDINGS CUSHIONED BY AIR BAGS

BY
EDWARD W. ROSS

JULY 1987

FINAL REPORT
JULY 1985 TO AUGUST 1985

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This report presents an exploratory mathematical study of control systems for airdrop platform landings cushioned by airbags. The basic theory of airbags is reviewed and solutions to special cases are noted. A computer program is presented, which calculates the time-dependence of the principal variables during a landing under the action of various control systems. Two existing control systems of open-loop type are compared with a conceptual feedback (closed-loop) system for a fairly typical set of landing conditions. The feedback controller is shown to have performance much superior to the other systems. The feedback system undergoes an interesting oscillation not present in the other systems, the source of which is investigated. Recommendations for future work are included.

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PREFACE

This work is intended to explore the possibility that the performance of airbags in cushioning the landings of airdrop platforms can be improved by introducing automatic control of the vent opening. The work was done in the period May to August, 1985, under Program Element 61101A, Project No. IL161101 A91A, Task No. 07, and Work Unit Accession No. 137.

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1. INTRODUCTION

In recent years the Army has devoted considerable attention to the use of pneumatic devices (airbags) in reducing the landing shock of various air-delivery systems. A number of different designs have been investigated experimentally by Nykvist[1], Patterson[2] and others. The fundamental theory was presented first by Browning[3], then extended by Esgar and Morgan[4].

The present report describes a preliminary study of possible systems for automatic control in the course of a platform landing cushioned by a simple, cylindrical airbag. The original impetus for the study was a discussion between Dr. C.K. Lee and the author, in which Dr. Lee mentioned that small motors are now available that have response times in the millisecond range. The possibility of using such a motor to drive a device for opening and closing the vent of an airbag led to the investigation reported here.

The basic theory of airbags is reviewed in the next section and put into a dimensionless form slightly different from (but wholly equivalent to) that of Browning[3]. The resulting nonlinear system of three ordinary differential equations cannot in general be integrated exactly. However, Section 3 describes two solutions in closed form that can be obtained under certain assumptions about the vent area. Several other plausible control systems are discussed in Section 4, and Section 5 presents the results of numerical solutions, comparing the performances of these control systems. The results are discussed and conclusions presented in Sections 6 and 7.

2. BASIC AIRBAG EQUATIONS

This section is closely related to the analyses given by Browning[3] and Esgar and Morgan[4]. The context and basic assumptions are described below.

a. The load, W, is descending steadily beneath a parachute and is attached to the top of a plane platform, dropping vertically and oriented horizontally, during the entire impact process. The initial velocity of descent is V_0.

b. An airbag in the form of a cylinder with a vertical axis and horizontal end surfaces is attached to the underside of the platform. The airbag has height H and cross-sectional area A_B and initially contains air at atmospheric pressure, p_a. There is a vent to the atmosphere with area A_V.

c. At time t = 0 the lower face of the bag makes contact with the ground. Thereafter, the bag preserves always the same cross-section area, A_B, but the volume decreases as the platform descends.

d. The pressure, p, and mass-density ρ, of the air in the bag are related by the ideal gas law,

$$p/p_a = (\rho/\rho_a)^{\gamma} \tag{1}$$

where ρ_a is the density of air at atmospheric pressure and γ is the ratio of the specific heats, specifically γ = 1.4 for air.

With these assumptions the equations of motion can be written as a system of three nonlinear ordinary differential equations in

y = height of platform above ground

V = velocity (positive upward)

ρ = mass density of air in bag

namely

$$dy/dt = V \tag{2}$$

$$dV/dt = g\,[-1 + (D+R_B)/W] \tag{3}$$

$$A_b d(\rho y)/dt = -C_V A_V \rho_a q \tag{4}$$

where

g = acceleration of gravity

D = canopy drag

R_B = force transmitted to platform from airbag

q = speed of air flow from vent

C_V = vent flow coefficient

further

$$D = \tfrac{1}{2}\rho_a A_c V^2 C_D \tag{5}$$

A_c = drag area of canopy (6)

C_D = drag coefficient of canopy (7)

$$R_B = (p-p_a)A_B \tag{7}$$

$$q^2 = J_0 V^2 + S \tag{8}$$

Equation (8) is Bernoulli's law for the flow of air out the vent or orifice. J_0 is a constant that affects the definition of the flow upstream of the vent in Bernoulli's law, and S has a form which depends on the pressure in the bag. Let

$$P_c = \text{critical pressure} = p_a[1 + (\gamma-1)/2]^{\gamma/(\gamma-1)} \simeq 1.893\ p_a \tag{9}$$

Then

$$S = [2\gamma/(\gamma-1)]\ [(p/\rho) - (p_a/\rho_a)] \quad \text{if } p < p_c \tag{10}$$

$$= (\gamma p_a/\rho_a)(p/p_c)^{\gamma/(\gamma-1)} \ \text{if } p > p_c \tag{11}$$

Equation (2) is merely the definition of velocity, conservation of platform momentum is embodied in Equation (3), and Equation (4) expresses conservation of air-mass in the bag. Collectively they are, when combined with Equations (1) and (5) to (11), a system of three nonlinear, ordinary differential equations for the three functions y, V and ρ. The initial conditions are

$$\text{at } t = 0,\ y = H,\ V = -V_0 \text{ and } \rho = \rho_a.$$

The equations can be put in dimensionless form by defining

$$x_1 = y/H,\ x_2 = V/V_0,\ x_3 = \rho/\rho_a,\ t = TH/V_0$$

$$\alpha_1 = p_a A_b/W,\ \alpha_2 = gH/V_0{}^2,\ \alpha_3 = 2\gamma p_a/[(\gamma-1)\rho_a V_0{}^2]$$

$$\alpha_4 = p_c/p_a$$

$$\alpha_5 = \rho_a A_c V_0{}^2 C_D/(2W),\ \alpha_6 = J_0$$

$$\eta = p/p_a \qquad Q = q/V_0 \qquad \phi = A_V/A_B,$$

which leads to the system of equations

$$dx_1/dT = x_2 \tag{12}$$

$$dx_2/dT = \alpha_2 F_g \tag{13}$$

$$dx_3/dT = -(x_2 x_3 + QC_v \phi)/x_1 \tag{14}$$

where

$$\eta = x_3{}^{\gamma}, \quad \alpha_4 = [1 + (\gamma-1)/2]^{1-(1/\gamma)} \qquad (15),(16)$$

$$F_g = -1 + \alpha_5 x_2{}^2 + \alpha_1(\eta-1) \tag{17}$$

$$Q = [\alpha_6 x_2{}^2 + \alpha_3 \sigma]^{\frac{1}{2}} \tag{18}$$

$$\sigma = (\eta/x_3)-1 \qquad \eta \leqq \alpha_4 \tag{19}$$

$$= (\gamma-1)(\eta/\alpha_4{}^{1-(1/\gamma)} \qquad \eta \geqq \alpha_4 \tag{20}$$

and the initial conditions are

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 1 \quad \text{at} \quad T = 0 \tag{21}$$

Typical values for the physical constants in the analysis are listed in Table 1 and the corresponding dimensionless parameters α_J (J = 1,...,6) are given in Table 2.

TABLE 1

Typical Values for Physical Constants

H	initial height of platform	3 ft
V_0	initial velocity of platform	30 ft/s
A_B	cross sectional area of bag	10 ft^2
W	weight of platform and load	1000 lbs
p_a	air pressure	2117 lb/ft^2
ρ_a	mass density of air	.002 lb s^2/ft^4
g	acceleration of gravity	32.2 ft/s
A_c	drag area of canopy	1000 ft^2
A_v	cross-sectional area of vent	1 ft^2

TABLE 2

Typical Dimensionless Parameter Values

$\alpha_1 = 21.17$

$\alpha_2 = .1073$

$\alpha_3 = 8233$

$\alpha_4 = 1.893$

$\alpha_5 \simeq .9C_D$

$\alpha_6 \simeq 90\ J_0$

If the airbag has any effect, we may assume that

$$|x_1| < 1 \quad \text{and} \quad |x_2| < 1$$

throughout most of the motion because $C_D < 1$, $\alpha_5 << \alpha_1$, and it is plausible to neglect the term in (17) that contains α_5, although some inaccuracy can result near $T = 0$. A similar argument causes us to neglect the term in (18) that involves α_6, although the accuracy may be questionable near $T = 0$, especially since the value of J_0 is not well-specified beyond saying that $J_0 = O(1)$.

The remainder of this paper will be based on the equations (12) to (16), the modified equations (17) and (18),

$$F_g = -1 + \alpha_1(\eta - 1) \tag{22}$$

$$Q = (\alpha_3\sigma)^{\frac{1}{2}} \tag{23}$$

and (19) to (21).

With these approximations, the equations involve three dimensionless parameters, α_1, α_2 and α_3. We can see at a glance that model tests of the platform-airbag system will encounter scaling problems unless the model tests are carried out in a suitable, artificial atmosphere. For, if α_2 is to be the same for model and prototype, we must have $VH^{-\frac{1}{2}}$ the same, but α_3 implies that V must be the same for model and prototype if both are tested in the same medium.

The energy possessed by the platform can be found by observing that the potential energy is Wy and the kinetic energy is $WV^2/(2g)$. Then

$$\begin{aligned} \mathcal{E} &= (\text{total energy})/(\text{initial energy}) \\ &= (2\alpha_2 x_1 + x_2^{\,2})/(2\alpha_2 + 1) \end{aligned} \tag{24}$$

The purpose of the airbag is to decelerate the platform so its vertical velocity is zero when the platform strikes the ground, i.e., $x_1 = x_2 = 0$ at same time, T_f. The airbag has, therefore, to completely dissipate the initial energy.

3. SOLUTIONS FOR SPECIAL CASES

In general the system of equations is nonlinear and cannot be solved exactly in closed form. However, in certain special cases exact or approximate solutions can be found, and it is convenient to describe these here.

It is instructive first to think qualitatively about the behavior of the system. For this purpose, hodograph plots of three more or less typical cases are shown in Figure 1. All start, of course, from the initial point, $x_1 = 1$, $x_2 = -1$. The path P_0 is the trajectory followed in an ideal case where the system reaches the point $x_1 = x_2 = 0$ at time, T_f, possibly under the action of some type of control system. A case where the bag is vented too freely, and the platform crashes into the ground, i.e., $x_2 < 0$ when $x_1 = 0$, exhibits a trajectory like P_-. Contrarily, P_+ shows a path when the bag is not vented enough, and the platform bounces off the bag, i.e., $x_2 = 0$ when $x_1 > 0$.

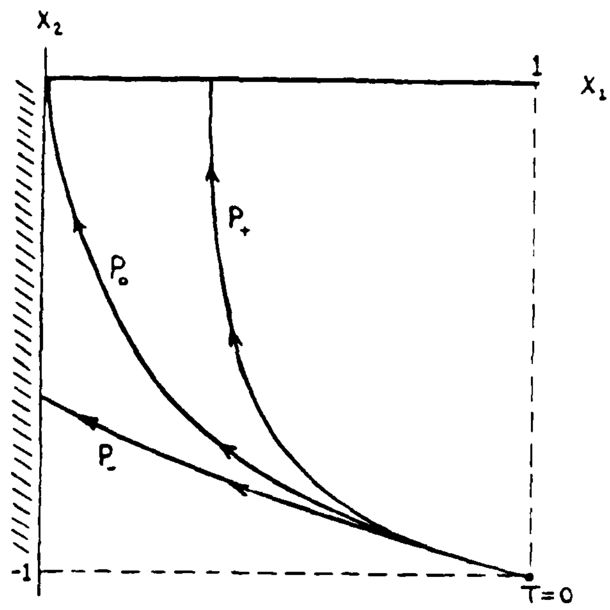

Figure 1: Hodographs for platform-landings in three typical Cases: Optimal Venting, P_0, Overventing, P_-, Underventing, P_+.

The first special case is that in which $\phi = 0$, i.e., the vent is closed. An exact solution is obtainable in implicit form in this case,

$$f(x_1) = \{1 + 2\alpha_2(1+\alpha_1)(1-x_1) + 2\alpha_1\alpha_2(\gamma-1)^{-1}(1-x_1{}^{1-\gamma})\}$$

$$x_2 = -f^{\frac{1}{2}}(x_1) \qquad x_3 = x_1{}^{-1} \qquad \eta = x_1{}^{-\gamma} \tag{29}$$

$$T(x_1) = \int_{x^1}^{1} f^{-\frac{1}{2}}(u)du$$

This solution resembles that of the trajectory P_+ in Figure 1 if it is followed long enough. The above solution loses validity when the function $f(x_1) = 0$, i.e., when $x_2 = 0$. The values of x_1 and T at which this occurs depend only on α_1, α_2 and γ, not on α_3.

An important special case is obtained if we demand that the g-force, F_g, be held constant from some time (say $T_0 > 0$) until the final time T_f. For the moment we ignore what happens for $0 \leqq T < T_0$.

If

$$x_1(T_0) = b_1 > 0, \quad x_2(T_0) = b_2 < 0, \tag{30}$$

it is easily verified that the following functions not only satisfy the three equations of motion but also the end conditions, $x_1 = x_2 = 0$ at $T = T_f$:

$$T_f = T_0 - 2b_1/b_2, \quad F_g = b_2{}^2/(2b^1\alpha_2) \tag{31}$$

$$x_1 = b_1 + b_2(T-T_0) + \tfrac{1}{2}\alpha_2 F_g(T-T_0)^2 \tag{32}$$

$$x_2 = b_2 + \alpha_2 F_g(T-T_0) \tag{33}$$

$$x_3 = \{1 + (1+F_g)/\alpha_1\}^{1/\gamma}, \quad \eta = x_3{}^\gamma \qquad (34),(35)$$

$$\phi = -x_2x_3/Q \tag{36}$$

and Q is given by (18) and (19) or (20). This solution has constant values for F_g, x_3, η and Q, quadratic time-dependence of x_1, and linear time dependence for x_2 and ϕ. Also $\phi = 0$ at $T = T_f$. In effect, Equation (36) describes the time-dependence of vent-opening that is needed in order to obtain this motion.

It is clear that, if this motion can be obtained in practice, it is an optimum solution to the problem, at least for $T_0 \leqq T \leqq T_f$. In the next section we shall discuss some aspects of this question.

4. CONTROL SYSTEMS

All of the control systems studied in this report use ϕ (the vent-area ratio) as the control variable. For simplicity in studying the action of the control, we shall also assume that $C_V = 1$ in Equation (14), a condition that causes some loss of generality if C_V depends on the state of the system but not otherwise.

The control systems that have been studied or used in the past are as follows:

(i) The constant vent opening,

$$\phi = \phi_C,$$

is probably the simplest system. It involves only a single parameter, ϕ_C.

(ii) The blow-off patch has the vent remaining closed until the pressure first attains a certain value, η_B, at which time the patch blows off (instantaneously) and the vent area jumps to its fully open value, ϕ_C, and remains at that value for the rest of the landing, i.e.,

$$\begin{aligned} &\phi = 0 && T < T_B \\ &\phi = \phi_C && T > T_B \\ &\eta(T_B) = \eta_B \end{aligned} \qquad (37)$$

Thus control depends on two parameters, η_B and ϕ_C.

Another control system, not previously examined nor used, but apparently worth investigation, is implicit in the solution of equations (30) to (36). This solution is optimal for $T_0 \leq T \leq T_f$, but the problem is in determining when to start using the control law (36) which causes the system to follow this solution. This suggests that we use a two-stage system, in which the first stage controls ϕ so that F_g is brought quickly from its initial value ($F_g = -1$) to the value given by (31). When F_g attains that value, the second stage begins, in which ϕ simply follows the control law (36). This requires that in the first stage the system must sense F_g, x_1 and x_2 and switch when (31) is satisfied, i.e.,

$$F_g = x_2{}^2/(2x_1\alpha_2) \qquad (38)$$

x_1 and x_2 can be calculated by sensing F_g and performing numerical integration of (12) and (13), so that only F_g really has to be acquired.

However, another question is whether this control system can respond quickly enough so that ϕ will be set "instantaneously" (i.e. in negligible time) to the value demanded by (36). Alternatively, can ϕ be controlled in the first stage so that it will both (a) pressurize the bag enough to cause (31) to be satisfied at some T_0 and (b) have the value demanded by (36) at that T_0?

It is not clear whether these difficulties can be surmounted in practice without making the system distastefully complicated. Also, when this system is in the second stage, the automatic control is of open loop type, i.e., it makes no use of information about F_g or any of the stated variables. The same is true of the constant vent opening. The blow-off patch is affected by the system state only to the extent of being actuated when η is large enough. To some degree, therefore, these systems all share the usual shortcomings of open loop control, in particular they may not function well if subject to unknown or random fluctuations of input.

Accordingly, an entirely different control system was investigated in this study. Since measurements of F_g are the easiest ones to obtain, the system was defined by

$$D_\phi = P_2 F_g - P_1 = P_2(F_g - r), \qquad r = P_1/P_2 \tag{39}$$

and

$$d\phi/dT = D_\phi \text{ if } D_\phi > 0 \text{ or } \phi > 0 \tag{39}$$

$$= 0 \text{ if } D_\phi \leqq 0 \text{ and } \phi = 0 \tag{39}$$

where the parameters P_1 and P_2 have to be chosen so that the system is brought to $x_1 = x_2 = 0$. This system is conceptually very simple; the vent is opened or closed at a rate proportional to $F_g - r$.

5. NUMERICAL ANALYSIS OF LANDINGS WITH CONTROLS

In order to investigate the behavior of the control systems described in Section 4, a computer program was written that carries out the numerical integration of the differential equation systems (12) to (16) and (19) to (23). This program is listed in Appendix A. It consists of a main program, MAIN, which reads the physical parameters, forms the dimensionless quantities α_1, α_2, α_3, and

$$\alpha_4 \equiv \eta_c,$$

and invokes the IMSL routine DVERK to do the numerical integration. After completing the solution, MAIN writes the results for x_1, x_2, x_3,

$$x_4 \equiv \phi$$

and F_g in file 7 and the derivatives, dx_j/dT ($j = 1,\ldots,4$), in file 8.

The routine DVERK uses a Range-Kutta integration scheme which requires that the user furnish a subroutine, FCN, for evaluating each dx_j/dT, given the values of the time and all the x_j. The control system is modelled by a set of instructions in this subroutine which define $x_4 = \phi$ or $dx_4/dT = d\phi/dT$.

We adopted the parameters given in Table 2 as a standard condition and for this condition explored the behavior of the three control systems described earlier, recalling that the objectives are to bring the system from the initial point

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 1$$

to the final point

$$x_1 = x_2 = 0$$

and to do so as smoothly as possible in the sense that the maximum of F_g during the motion should be as small as possible.

The simplest control system is that with constant vent opening,

$$x_4 \equiv \phi = \phi_c,$$

or

$$dx_4/dT = 0 \quad \text{and} \quad x_4(0) = \phi_c.$$

With this single parameter it was impossible to attain the final point $x_1 = x_2 = 0$. The "best" results (best in the sense described below) were found from a series of trials to be at $x_4 = \phi_C = 0.019$. For this case the behavior of x_1 and x_2 is shown in Figure 2 and that of F_g in Figure 3.

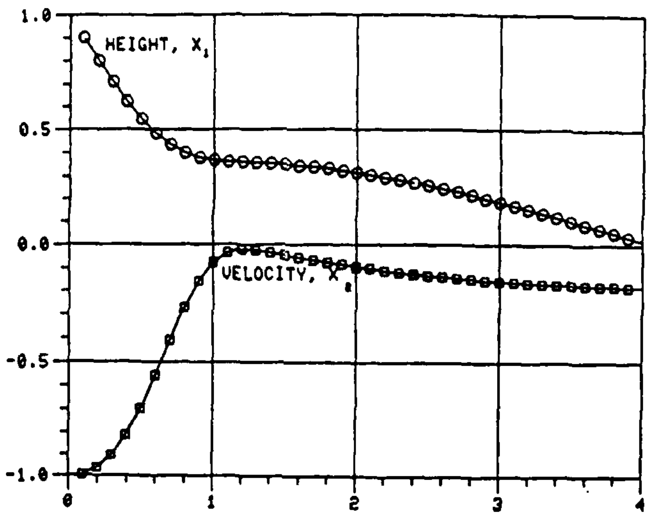

Figure 2: Dimensionless Height, X_1, and Velocity, X_2, as functions of dimensionless time, T, for the standard condition and constant vent opening, Ø = 0.019.

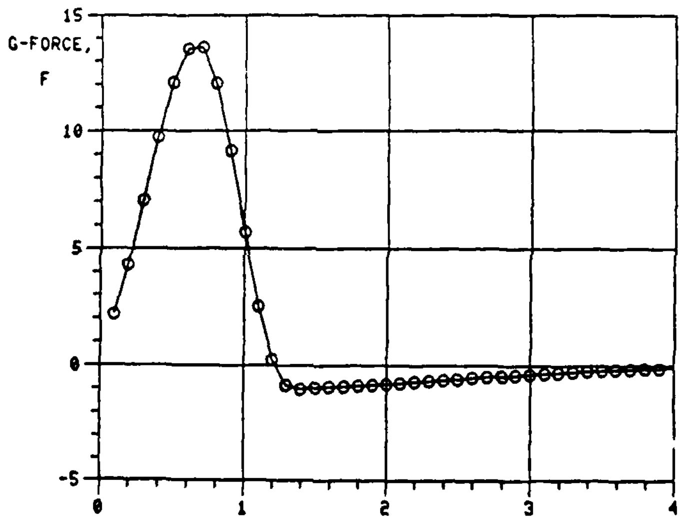

Figure 3: G-force or dimensionless acceleration, Fg, as a function of dimensionless time, T, for the standard condition and constant vent opening, Ø = 0.019.

The system has $x_2 \simeq 0$ when $1.0 < T < 1.5$, but $x_1 \simeq .36$ at that time. After almost coming to rest the system gradually resumes its descent until it strikes the ground (i.e., $x_1 = 0$) with velocity $x_2 \simeq -.18$ at time $T \simeq 4.1$. During this second phase of the landing, the pressure of the air in the bag is not great enough to equilibrate the load, so F_g is slightly negative. If $x_4 = \phi_C > 0.019$, i.e., the vent is opened wider than the optimal value, the load strikes the ground earlier and with higher velocity. For example, if $\phi_C = 0.025$, the load lands when $T = 2.17$ with $x_2 \simeq -.25$. On the other hand if $\phi_C < 0.019$, the vent is narrower than the optimal value, the load bounces off the bag and accuracy of Equations (18) to (20) is thereafter doubtful. Moreover, the maximum F_g values are higher in this case than the others. For example

$\phi_C = .015$ causes max $F_g \simeq 16$

$\phi_C = .019$ causes max $F_g \simeq 14$

$\phi_C = .025$ causes max $F_g \simeq 9$

To summarize, this simple control cannot steer this system to the origin. At best it will land this system with velocity $x_2 = -.18$ and max $F_g = 14$.

The blow-off patch control system was examined next. Control now depends on twc parameters

η_B = blow-off pressure (in atmospheres)

ϕ_C = dimensionless vent area.

A number of cases were run for various values of these two parameters. The results were qualitatively like those for the previous constant vent control, in that the system could not be steered to the point $x_1 = x_2 = 0$ by the control. Instead the system attained $x_2 = 0$ at height $\tilde{x}_1$, (i.e. it paused at height $\tilde{x}_1$) and then eventually attained $x_1 = 0$ at a velocity $\tilde{x}_2$, just as for the constant vent case. The "best" results are listed in Table 3.

TABLE 3

Optimal Results for Blow-Off Patch

η_B	ϕ_c	$\tilde{x}_1$	$\tilde{x}_2$	$\hat{F}_g$	T_f
1.3	.019	.40	-.18	14	4.4
1.5	.200	.44	-.19	14	4.6
1.8	.023	.47	-.22	16	4.4

$\tilde{x}_1$ is the value of x_1 at which $x_2 = 0$ (height of pausing).
$\tilde{x}_2$ is the value of x_2 at which $x_1 = 0$ (velocity at landing).
$\hat{F}_g$ is the maximum value of $\hat{F}_g$.

The feedback control system (39) displayed behavior quite different from the other two controls. Many different pairs of values for P_1 and P_2 were found that would steer the system to $x_1 = x_2 = 0$. For example, Table 4 shows three such sets of values and the principal properties of the trajectories that they produced. Figure 4 shows graphs of x_1 and x_2 as functions of T for the case $P_1 = 1.02$, $P_2 = .2$, and the functions x_3, F_g and ϕ are depicted in Figures 5, 6 and 7. The results were qualitatively similar for the other cases listed in Table 4.

TABLE 4

Computed Results for Feedback Control Law

P_1	P_2	$\hat{F}_g$	M	$d\phi/dT$	$\tilde{G}_f$	T_f
.50	0.1	9.3	4	-.027	4.8	1.95
1.02	0.2	8.1	6	-.024	4.9	1.93
2.08	0.4	7.4	8	-.026	5.1	1.90
4.16	0.8	6.4	9	-.027	5.2	1.90

$\hat{F}_g$ is the maximum value of F_g.
M is the number of local maxima of F_g in $0 \leqq T = T_f$
$d\phi/dT$ is the approximate slope of ϕ near $T = T_f$
$\tilde{G}_f$ is the approximate constant value of $\tilde{G}_f$ near $T = T_f$

The principal feature of these results is the oscillation in F_g, x_3 and ϕ, an oscillation which is mildly discernible also in the plot of x_2 but not of x_1. A perturbation analysis of the differential-equation system is done in Appendix B to show the origins of this behavior. However, it is clear that the F_g values obtained with this control are much lower than with either of the other controls. A practical question is whether the control system can respond quickly enough to enforce the control law during an oscillation of this type.

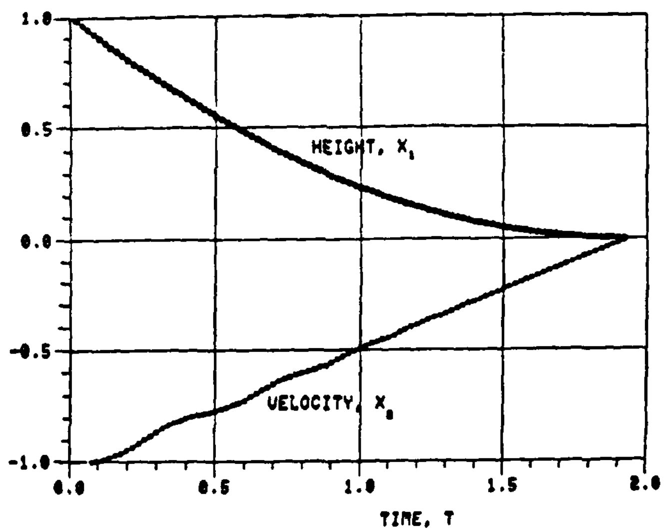

Figure 4: Dimensionless Height, X_1, and Velocity, X_2 as Functions of Dimensionless Time, T, for the Standard Condition, Feedback Control with $P_1 = 1.02$, $P_2 = 0.20$ and $T \geq 1.0$.

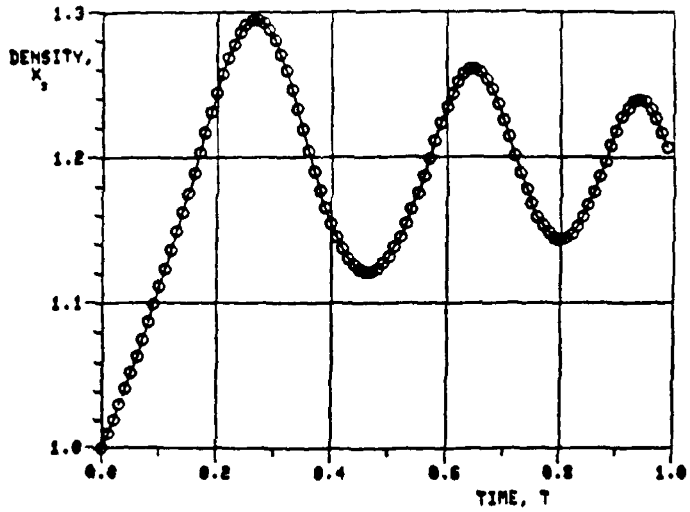

Figure 5a: Dimensionless Air Density in Bag, X_3, as a Function of Dimensionless Time, T, for the Standard Condition, Feedback Control with $P_1 = 1.02$, $P_2 = 0.20$ and $T \leq 1.0$.

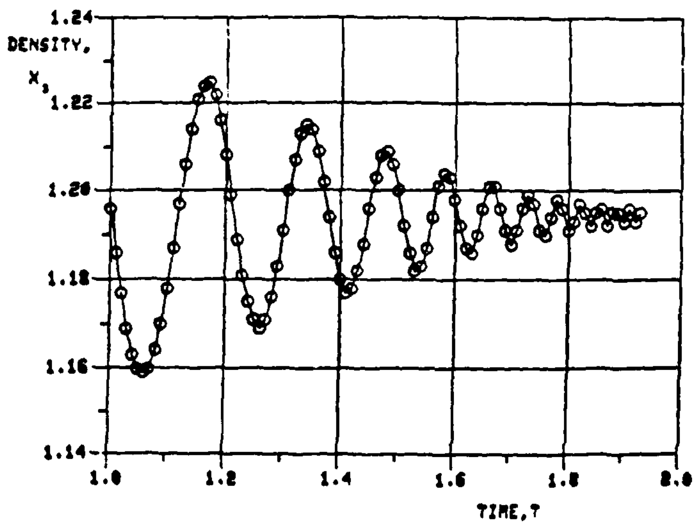

Figure 5b: Dimensionless Air Density in Bag, X_3, as a Function of Dimensionless Time, T, for the Standard Condition, Feedback Control with $P_1 = 1.02$, $P_2 = 0.20$ and $T \geq 1.0$.

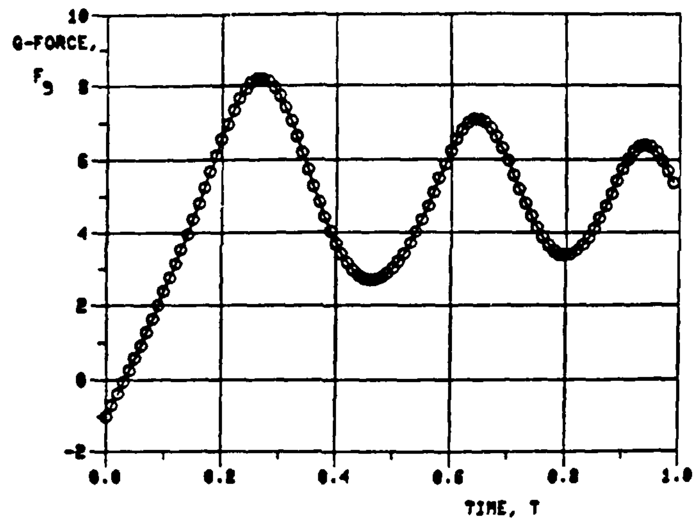

Figure 6a: G-force as a Function of Dimensionless Time, T, for the Standard Condition, Feedback Control with P_1 = 1.02, P_2 = 0.20 and T ≦ 1.0.

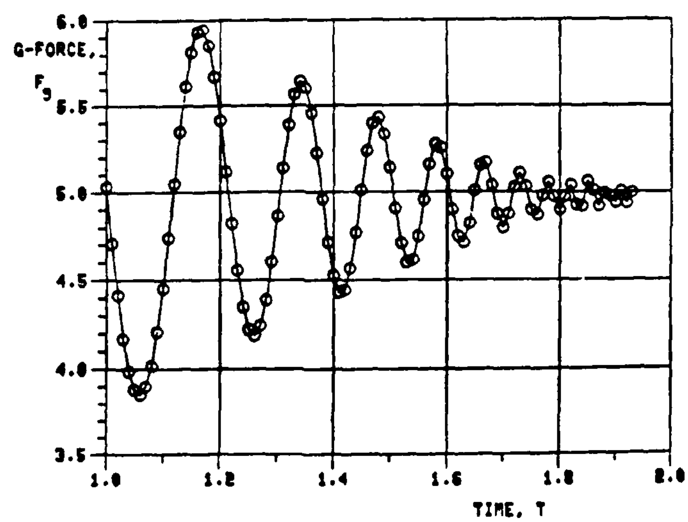

Figure 6b: G-force as a Function of Dimensionless Time, T, for the Standard Condition, Feedback Control with P_1 = 1.02, P_2 = 0.20 and T ≧ 1.0.

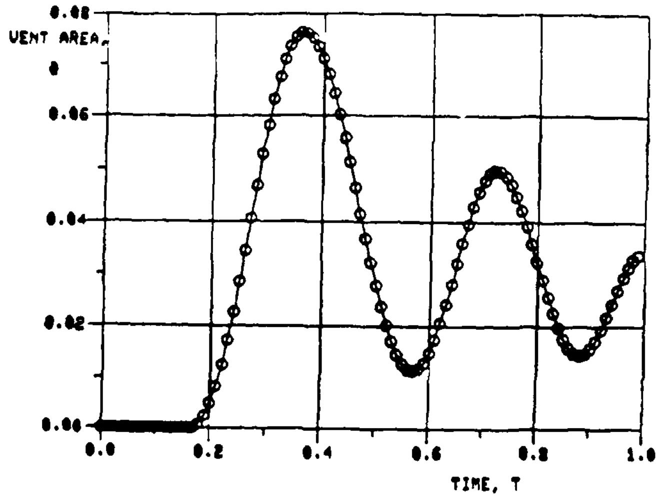

Figure 7a: Dimensionless Vent Opening, ϕ, as a Function of Dimensionless Time, T, for the Standard Condition, Feedback Control with P_1 = 1.02, P_2 = 0.20 and T $\leq$ 1.0.

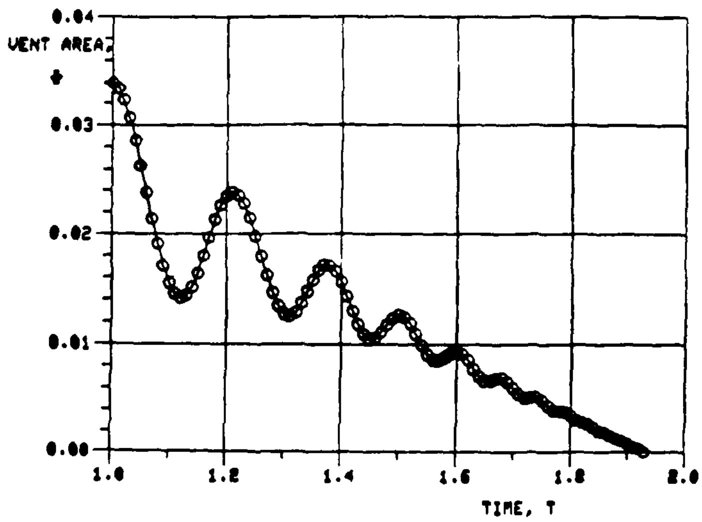

Figure 7b: Dimensionless Vent Opening, ϕ, as a Function of Dimensionless Time, T, for the Standard Condition, Feedback Control with P_1 = 1.02, P_2 = 0.20 and T $\geq$ 1.0.

6. DISCUSSION

The solution obtained in Appendix B agrees quite well with that obtained by the computer program of Appendix A. Superficially, the solutions for x_3 and F_g have the form of damped oscillations about constant values with periods that decrease as $T \to T_f$, and this is exactly what is seen in Figures 5 and 6. It is readily verified that the asymptotic values

$$x_3 \sim C_3 = 1.195$$

$$F_g \sim P_1/P_2 \approx 5$$

agree well with these Figures. Similarly, the asymptotic result

$$d\phi/dT \sim -\alpha_2 F_g{}^0 C_3/q_0 = -.0260$$

conforms closely to the estimates in Table 4, which were obtained graphically from Figure 7.

To assess the quantitative agreement, we refer to Table 5, which shows the local maxima of x_3 and Δ_3 and their times of occurrence, T_i. Two comparisons are relevant. First, if the solution (B.26) is correct,

$$\lambda\ (\ln \tau_{-1} - \ln \tau_i) \approx 2\pi$$

$$\tau_{i-1}/\tau_i = e^{2\pi/\lambda} = e^{.256} = 1.292,$$

and we see from Table 5 that successive ratios of τ_i agree very well with this estimate. Second, the maxima should satisfy

$$\ln \Delta_{3i} = A^* + (\nu - 3/2) \ln \tau_i$$

and so

$$d(\ln\Delta_{3i})/d\ln\tau_i = \nu - (3/2) \approx 1.40.$$

The points of Table 5 give a value approximately = 1.50, hence there is a small discrepancy in this comparison.

TABLE 5

Oscillation Extremes for Feedback Control Law

T_i	$\tau_i = T_f - T_i$	τ_i/τ_{i+1}	Δ_{3_i}	$\ln\Delta_3$	$\ln\tau_i$
.265	1.665	1.296	.099	-2.31	.510
.645	1.285	1.298	.065	-2.73	.251
.940	.990	1.303	.044	-3.12	-.010
1.170	.760	1.288	.030	-3.51	-.274
1.340	.590	1.297	.020	-3.91	-.528
1.475	.455	1.319			
1.585	.345	1.302			
1.665	.265	-			

Two possible sources of error in the estimate of Appendix B are these:

(i) The series expansions involved in obtaining Equations (B.10), (B.11) and (B.12) are not extremely accurate.

(ii) The estimate (B.23) is only a moderately accurate approximation to the solution of (B.22).

It is questionable whether the effort involved in improving these approximations is worth the trouble.

Several other comments can be made about this solution.

(a) The differential equation (B.25) has a singularity at $\tau = 0$, $T = T_f$ that casts doubt upon the correctness of the limiting behavior of the solution, (B.26), as $T \to T_f$.

(b) The fact that $F_g{}^0 = P_1/P_2$ furnishes a fairly accurate solution of (B.22) suggests that the zero-th order solutions are not very sensitive to the values of P_1 and P_2 separately, but only to the ratio of P_1/P_2, a conclusion supported by the numerical results in Table 4. However, the perturbation solution (B.27) depends strongly on λ, which depends on a and so is influenced by P_2 alone (see (B.20)). As P_2 increases, λ increases and the solutions oscillate more rapidly, a result that is confirmed by the values of M in Table 4.

(c) A valid question about this control system is whether the control mechanism can change the vent opening fast enough to keep up with the changes in F_g. We have

$$|d\phi/dT| \leqq |d\phi_0/dT| + |d\Delta_0/dT|$$

and

$d\phi_0/dT \simeq -.026$. Also, from (B.20)

$$|d\Delta_0/dT| = |a\Delta_3| \leqq a|\Delta_3|$$

and from the computations, Figure 5, we see that $|\Delta_3| \leqq 0.1$.

Hence

$$|d\phi/dT| \leqq .026 + 6.37 \times .01 \simeq .090.$$

Thus

$$dA_V/dT = V_0 A_B H^{-1}\, d\phi/dT \leqq 9\ ft^2/s$$

This rather crude estimate implies that the system must be able to change the vent area at a rate of .009 ft^2/ms in order to control the air bag in the manner assumed by the analysis. It is not known whether this is an attainable rate because much depends on the shape of the vent, but for any specified vent geometry (e.g., rectangular), it should be possible to decide the question.

(d) We see from Table 4 that, as P_1 and P_2 increase but remain in almost the same proportion, the maximum F_g decreases noticeably. This suggests that the most effective control is obtained with $P_1/P_2 \simeq 5$ and P_2 as large as possible. However, as mentioned under (b), the oscillation becomes more rapid as P_2 increases. When P_2 is increased, the oscillation eventually becomes so quick that the control system cannot keep up with it, and the present analysis becomes inaccurate. A more perceptive analysis, in which the effect of control system response is modelled both theoretically and numerically, would shed valuable light on the practical improvement that might be attained with a control system of this general type.

(e) The parameter, λ, that determines the frequency of the perturbation oscillation, is rather large, eg., $\lambda \simeq 24.5 >> 1$ in the example of Appendix B. We see from (B.26) that the size of λ depends on q_0 (i.e., ultimately α_3), a, and b. While $q_0 \simeq 24.7$ is fairly large, $a/b \simeq 23.8$ is almost as large, and the largeness of λ results about equally from both. In fact a is fairly large because α_1 is so, and b is rather small because α_2 is. Thus all three parameters α_1, α_2 and α_3 in the combination $\alpha_1\alpha_2^{-1}\alpha_3^{\frac{1}{2}}$ have substantial influence on λ.

Concerning the two open-loop control systems, little further comment is needed, except to remark that the numerical integration subroutine, DVERK, experienced some convergence difficulty with the discontinuity that occurs for the blow-off patch. The results for these computations are less accurate than for the other control systems although not sufficiently so to alter the main conclusions.

7. CONCLUSIONS AND RECOMMENDATIONS

The results of this preliminary study of airbag control systems are as follows:

(i) Neither of the open loop systems performed in a wholly satisfactory way for two reasons. They did not bring the system to the desired end point, and the maximum G-force was very high.

(ii) The feedback system with vent-rate proportional to the G-force was better on both counts than either open-loop system. For many pairs of values of P_1 and P_2 it brought the system to the end point and did so with a maximum G-force much smaller than either open loop control.

(iii) A number of questions remain about this particular control system. It would be desirable to eliminate the oscillation or at least reduce its frequency if that can be done without seriously increasing F_g or degrading its ability to attain the desired end point. Also, we need to clarify the behavior near $T = T_f$, and it would be desirable to conduct stability studies of this control system, i.e., how it responds to either deterministic or random errors in the inputs or environment.

(iv) The results encourage us to think that closed-loop control systems in general have much to offer in improving air-bag performance. Although the closed-loop system studied in this report is a plausible one that may be realizable in practice, there are many other possibilities. For example the control law

$$d\phi/dT = -P_1 + P_2F_g - P_3\phi^{P_4}$$

may also be realizable and, if the parameters are chosen well, have fewer undesirable side effects than the law studied here.

(v) Concerning the computer program, the values of the control parameters, P_1 and P_2, were found by manual trial-and-error in the present study. It is possible to include in the program a subroutine for nonlinear optimization, which will carry out this process "automatically". For example, the nonlinear least-squares solver NL2SOL has a number of attractive features for a study of this kind. However, even the best of these subroutines can fail if the optimal solution is not unique, so much caution is required in their use.

We recommend, therefore, that further study of both *open* and *closed-loop* control systems be undertaken, with emphasis on the latter. In particular, further study of the present control algorithm is justified, as sketched under (iii) above. Other control laws ought also to be examined. If a substantial investigation is undertaken, the computer program of Appendix A should first be enhanced by inclusion of a carefully chosen nonlinear optimization subroutine.

8. REFERENCES

1. Nykvist, William, "Balloon-skirt Airbags as Airdrop Shock Absorbers: Performance in Vertical Drops", U.S. Army Natick Research and Development Center, Natick, MA, Technical Report No. TR82/026, Dec. 1981.

2. Patterson, Timothy, "Design, Fabrication and Testing of an Airdrop Platform Utilizing Airbags as Shock Absorbers", Northeastern University, Department of Mechanical Engineering, Master Thesis, May 1985.

3. Browning, A.C., "A Theoretical Approach to Air Bag Shock Absorber Design", Royal Aircraft Establishment, Farnborough, England, Technical Note No. Mechanical Engineering 369, Feb. 1963.

4. Esgar, J.B. and Morgan, W.C., "Analytical Study of Soft Landings on Gas-filled Bags", National Aeronautics and Space Administration, Lewis Research Center, Cleveland, OH, Technical Report No. R75, 1960.

This document reports research undertaken at the U.S. Army Natick Research Development and Engineering Center and has been assigned No. NATICK/TR-88/021 in the series of reports approved for publication.

APPENDIX A: Computer Programs

The following pages contain listings with comments of the three elements needed to carry out the computations described in Section 5. These elements are presently stored in the file

BAG * FF.

on the UNIVAC 1106 at the U.S. Army Natick Research, Development and Engineering Center. The elements are

M, the main program

FNI, the subroutine, FCN, called by DVERK

D, a typical data set.

M is listed below followed by FNI and D. The data in D is that which produced the output for the constant vent-opening control system in Figures 2 and 3.

PROGRAM FOR AIRBAG CONTROL

This program calculates the motion of a platform during a landing cushioned by an airbag with an automatic control system. It is based on the system of dimensionless equations in the report "A Preliminary Study of Control Systems for Platform Landings Cushioned by Airbags" by E.W. Ross.

The main program, given below, reads in the physical quantities, converts them to dimensionless parameters and then calls the IMSL subroutine DVERK, which does numerical integration of the differential equation system, using a Runge-Kutta procedure. The subroutine DVERK call the user-supplied subroutine FCN, which calculates the derivatives, given the function values. The instructions which define the control system are embedded in this subroutine.

DEFINITIONS OF PRINCIPAL QUANTITIES:

PA,RHOA,GM	- pressure, mass-density and gamma for air
AB	- cross-sectional area of airbag
G	- acceleration of gravity
W	- weight of platform, load and bag
VO	- velocity of platform at first contact
H	- height of platform at first contact
N	- number of variables in vector X (usually 4)
TOL	- accuracy threshold for DVERK
NTS	- number of time steps in the integration
TI, TF	- initial and final time
YI	- array of initial values of the variables Y
LAM	- array of integer variables for information recording and imposing certain conditions.
CP	- array of control and information paramaters
Y	- array of main variables, as follows:
	Y(1) = dimensionless height
	Y(2) = dimensionless velocity
	Y(3) = dimensionless density
	Y(4) = dimensionless vent area (control variable)
DRV	- array of derivatives of Y
GF	- G-force (i.e., acceleration in G-units)

The calculated values of the Y's and GF are written to file no. 7 and DRV's are written to file no. 8. A typical set of input data for use with this program is in the element D in this file. The program graph in this file can be used to cause Tektronix plotting of the results stored in file 7.

```
      COMPILER (DIAG=3)
      EXTERNAL FCN
      COMMON /A/ GM,AL(4),LAM(5),GF,CP(5)DRV(4)
      REAL YI(4),Y(4),C(24),WKV(10,10)
*****  READ IN PHYSICAL AND OTHER DATA  *****
      READ (5,30) PA,RHOA,GM
      READ (5,30) AB,G,W,UO,H
      READ (5,30) N,TOL,NTS
      READ (5.30) TI,TF,(YI(J),J=1,4)
      READ (5,30) (LAM(K),K=1,5)
      READ (5,30) (CP(J),J=1,5)
*****  CALCULATE THE ARRAY AL (ALPHA) OF DIMENSIONLESS PARAMETERS
      AL(1)=PA*AB/W
      AL(2)=G*H/UO/UO
      AL(3)=2*GM*PA/(GM-1)/RHOA/UO/UO
      AL(4)=(1+GM-1)/2)**(GM/(GM-1))
      WRITE (6,40) (AL(K),K=1,4)
      IND=1
      T=TI
      DO 10  J=1,N
 10      Y(J)=YI(J)
      DT=(TF-T)/NTS
*****  MAIN LOOP FOR NUMERICAL INTERATION
      DO 20  I=1,NTS
         TS=TI=I*DT
         CALL DVERK (N,FCN,T,Y,TS,TOL,IND,C,10,WKV,IER)
         WRITE (7,40) T, (Y(J),J=1,4),GF,CP(4)
         WRITE (8,50) T, (DRV(J),J=1,4)
 20      CONTINUE
*****  END OF MAIN LOOP
 30   FORMAT ()
 40   FORMAT (7E9.4)
 50   FORMAT (F6.4,7E9.4)
      END
```

SUBROUTINE FCN

This subroutine is called by DVERK to evaluate the derivatives (DY) of the variables, given their values and those of the paramameters. The control instructions, starting at line 23, are for the particular feedback control system studied in the report, which involves the derivative of the vent area, Y(4). For a different control system, a different set of instructions may have to be written and perhaps even inserted at a different point in the program.

```
      COMPILER (DIAG=3)
      SUBROUTINE FCN (N,T,Y,DY)
      REAL Y(N),DY(N)
      COMMON /A/ GM,AL(4),LAM(5),GF,CP(5),DRV(4)
      DY (1)=Y(2)
      IF (Y(3).LT. 0.0)  WRITE (6,20) T,(Y(J),J=1,4)
      PR=Y(3)**GM
      FD=LAM(1)*Y(2)**2
      BR=AL(1)*(PR-1.)
      GF=-1.+FD+BR
      DY(2)=AL(2)*GF
      LAM(5)=0
      AA=AL(3)*(PR/Y(3)-1.)
      AP=AL(3)*(GM-1)/2
      IF (PR.GTAL(4))  AA=AP*(PR/AL(4))**(1-1/GM)
      IF (PR.GTAL(4))  LAM(5)=1
      IF (AA.LT. 5.E-8)  AA=0.
      Q=SORT(LAM(2)*Y(2)**2+AA)
      DY(3)=-(Y(2)*Y(3)+Q*Y(4))Y(1)

***** THE FOLLOWING INSTRUCTIONS EXTERT CONTROL
      D4=CP(2)*GF-CP(1)
      DY(4)=D4
      IF ((D4.LT.0.0).AND.(Y(4).LE.0.0))  DY(4)=0.0
      DO 10 J=1,4
 10      DRV(J)=DY(J)
      RETURN
 20   FORMAT (5E9.4)
      END

***** DATA SET D *****
2117.,.002,1.4
10.,32.2,1000.,30.,3.
4,1.E-4,40
0.0,4.0, 1.0,-1.0,1.0,0.019
0.0.3,0,1
0.00,0.00,0.,0.,0.
```

APPENDIX B: Perturbation Analysis

This appendix presents a perturbation solution of the motion equations with feedback control. This analysis is motivated primarily by the observed results of the computations and secondarily by the exact, optimal solution in Section 3 for F_g = constant.

The basic set of equations is (12) to (15),(19),(22),(23) with $C_V = 1$ and the control law (39). The equations are then

$$dx^1/dT = x_2 \quad ; \quad dx_2/dT = \alpha_2 F_g \qquad (B.1,2)$$

$$d(x_1 x_3) = -Q\phi \; ; \quad d\phi/dT = P_2 F_g - P_1 \qquad (B.3,4)$$

$$Q = \alpha_3^{\frac{1}{2}}(x_3^{\gamma-1}-1)^{\frac{1}{2}} \qquad (B.5)$$

$$F_g = -1 + \alpha_1(x_3^{\gamma-1}-1). \qquad (B.6)$$

We are assuming that the vent is open and the flow through it is subsonic, hence this approximate solution is not valid initially, when the vent is closed. Figures 5 and 6 show that for $T \geqq 0.3$ the variables x_3 and F_g oscillate with decreasing amplitude about constant values, and this is the behavior that we seek to explain from the above equations.

We assume that

$$x_j = x_j^0 + \Delta_j \qquad j = 1,2,3 \qquad (B.7)$$

$$\phi_j = \phi_0 + \Delta_\phi \qquad F_g = F_g^0 + \Delta_F. \qquad (B.8,9)$$

The quantities Δ_j, Δ_ϕ, Δ_F are assumed to be perturbations of the zero-th order quantities x_j^0, ϕ_0, F_g^0, respectively, with $\Delta_j << x_j^0$. Series expansions of (B.5) and (B.6) lead to

$$Q \simeq q_0(1 + q_1\Delta_3) \qquad (B.10)$$

$$q_0 = \alpha_3^{\frac{1}{2}}[(x_3^0)^{\gamma-1}-1]^{\frac{1}{2}}, \; q_1 = \tfrac{1}{2}(\gamma-1)(x_3^0)^{\gamma-2}/[(x_3^0)^{\gamma-1}-1] \qquad (B.11)$$

$$F_g^0 = \alpha_1[(x_3^0)^{\gamma-1}-1]-1 \; , \; \Delta_F = \alpha_1\gamma\,(x_3^0)^{\gamma-1}\Delta_3. \qquad (B.12)$$

Equation (B.12) shows that F_g^0 is constant if x_3^0 is so. Since we expect to obtain a solution such that x^3 and F_g oscillate about constant values, we assume

$$x^{3^0} = C_3 = \text{constant} \qquad (B.13)$$

and so

$$F_g^0 = \alpha_1(C_3^{\gamma-1}-1)-1. \qquad (B.14)$$

Then (B.1) and (B.2) imply

$$x_1^0 = b(T-T_f)^2 \ , \ x_2^0 = 2b(T-T_f) \tag{B.15}$$

$$b = \tfrac{1}{2}\alpha_2 F_g^0 \tag{B.16}$$

Equations (B.3) and (B.4) become, using (B.12)

$$d[(x_1^0 + \Delta_1)(C_3 + \Delta_3)]/dT = -q^0(1 + q_1\Delta_3)(\phi_0+\Delta_\phi)$$

$$d(\phi^0 + \Delta_\phi)/dT = P_2[F_g^0 + \alpha_1 \ \gamma \ C_3^{\gamma-1}\Delta_3]-P_1.$$

The zero-order terms lead to

$$C_3 x_2^0 = -q_0\phi_0 \tag{B.17}$$

$$d\phi_0/dT = P_2 F_g^0 - P_1 \tag{B.18}$$

and the first order terms to

$$d(x_1^0\Delta_3)/dT = -C_3 \ d\Delta_1/dT - q_0[\Delta_\phi + q_1\phi_0\Delta_3] \tag{B.19}$$

$$d\Delta_\phi/dT = a\Delta_3 \ , \ a = P_2\alpha_1\gamma \ C_3^{\gamma-1}. \tag{B.20}$$

Equations (B.17) and (B.15) imply

$$\phi^0 = -2bC_3(T-T_f)/q_0 \tag{B.21}$$

and (B.18) becomes

$$P_2 F_g^0 - P_1 + 2bC_3/q_0 = 0. \tag{B.22}$$

Since F_g^0 and q_0 all depend on C_3, this is a transcendental equation for C_3 and has to be solved by trial and error.

However, we see from (B.11) that q_0 involves the parameter α_3 which is very large for the present set of parameters, see Table 2. This implies that $q_0 >> 1$ and suggests that we attempt to solve (B.22) by neglecting the last term, which leads to

$$F_g^0 = P^1/P_2. \tag{B.23}$$

For all the values of P_1 and P_2 in Table 4, i.e. those values for which $x_1 = x_2 = 0$ at $T = T_f$, we have

$$F_g^9 \simeq 5.$$

This implies via (B.14), (B.11) and (B.16)

$$x_3^0 = C_3 = [1 + (1+F_g^0)/\alpha_1]^{1/\gamma} \approx 1.195$$

$$q_0 \approx 24.7, \quad q_1 \approx 2.43 \text{ and } b \approx .268,$$

and we can verify that the last term in (B.22) does not greatly affect the estimate (B.23).

These results are now used to solve (B.19) and (B.20). We can eliminate Δ_ϕ by differentiating (B.19) and substituting (B.20), obtaining

$$d^2(x_1^0\Delta_3)/dT^2 + q_0q_1\phi_0 d\Delta_3/dT + C_3\, d\Delta_2/dT + a\, q_0\Delta_3 + q_0q_1\Delta_3 d\phi_0/dT = 0.$$

With the aid of (B.2), (B.7) and (B.12) we find

$$C_3 d\Delta_2/dT = \mu\Delta_3, \quad \mu = \alpha_1\alpha_2\gamma C_3^{\gamma}.$$

From (B.21)

$$q_0q_1 d\phi_0/dT = -2b\nu, \quad \nu = C_3q_1.$$

Finally, if we define

$$u = x_1^0\Delta_3 \tag{B.24}$$

and use (B.15) we get

$$\frac{d^2u}{d\tau^2} - 2\,\nu\tau^{-1}\frac{du}{d\tau} + (\lambda^2+2\nu)\tau^{-2}u = 0 \tag{B.25}$$

where

$$\tau = T_f - T \geqq 0$$

$$\lambda^2 = (aq_0 + \mu)/b. \tag{B.26}$$

The numerical values of these quantities are in this case

$$\nu = 2.90, \quad \mu = 4.08$$

and, for the case where $P_2 = .2$, line 2 of Table 2,

$$a = 6.37, \quad \lambda = 24.5.$$

A general solution of this equation is

$$u = A\tau^{(\nu+\frac{1}{2})}\cos(\beta\ln\tau - S)$$

$$\beta = \{\lambda^2 + 2\nu - (1+2\nu)^2/4\}^{\frac{1}{2}} = \lambda + 0(\lambda^{-1})$$

where A and S are the arbitrary amplitude and phase.

The solution for Δ_3 is found from (B.24) and (B.15)

$$\Delta_3 = A\tau^{1[\nu-(3/2)]}\cos(\beta\ln\tau - S). \tag{B.27}$$

Also Δ_F and $d\Delta_\phi/dT$ can be found from (B.12) and (B.20).

APPENDIX C: List of Symbols

A, A^1	Arbitrary constant in perturbation solution
A_B, A_V	Cross-section areas of airbag and vent
A_C	Drag area of canopy
a	Parameter in perturbation analysis, see (B.20)
b	Parameter in perturbation analysis, see (B.16)
C_D	Parachute drag coefficient
C_V	Vent flow coefficient
C_3	Constant density value in perturbation analysis, see (B.13)
D	Canopy drag, see Equation (3)
F_g	G-force or dimensionless acceleration, see Equation (17)
g	Acceleration of gravity
H	Height of airbag
J_0	Constant in air-flow definition, see Equation (8)
M	Number of oscillations (local maxima) of F_g in Table 4
p	Pressure of air in airbag
P_1, P_2	Constants in feedback control system, see Equation (39)
p_a	Standard atmospheric pressure
p_c	Critical (sonic) pressure in vent
Q	Dimensionless air-speed in vent
q	Air-speed in vent
q_0, q_1	Constants in perturbation of Q, see (B.11)
R_B	Reaction (lift) of the airbag on platform, Equation (3)
r	P_1/P_2
S	Variable in the vent air-speed, Equation (8)
t	Time
T	Dimensionless time
T_f	Time at which platform strikes ground
T_0	Initial time for optimal solution
u	Variable in perturbation, see (B.24)
V	Platform velocity of descent
V_0	Initial platform velocity
W	Weight of platform and load
x_1, x_2, x_3	Dimensionless height, velocity and density
x_1^0, x_2^0, x_3^0	Zero-order perturbations in x_1, x_2, x_3, see (B.7)

y	Height of platform above ground during landing
$\alpha_1, \alpha_2 \ldots \alpha_6$	Dimensionless parameters
β	Constant in perturbation solution, see (B.27)
γ	Ratio of specific heats of air
$\Delta_1, \Delta_2, \Delta_3, \Delta_F, \Delta_\phi$	Perturbations of solution, see (B.8), (B.9)
η	Dimensionless pressure of air in bag
η_B	Pressure at which blow-off patch is activated
η_c	Critical, sonic pressure in vent
Θ	Energy of platform
λ	Large parameter in perturbation oscillation, see (B.26)
μ	Constant in perturbation analysis
ν	Constant in perturbation analysis
ρ	Mass density of air in bag
ρ_a	Mass density of air at standard atmospheric conditions
σ	Dimensionless variable in vent flow, see Equations (18), (19)
τ	$T_f - T$, see (B.26)
ϕ	Dimensionless vent area, control variable
ϕ_c	Constant vent opening
ϕ_0	Zero-th order perturbation in vent opening, see (B.8)