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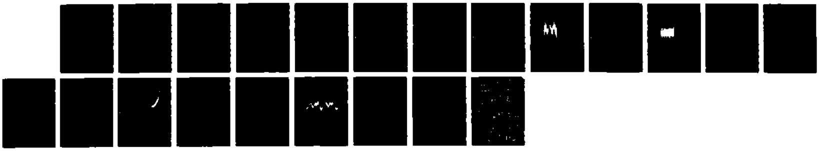

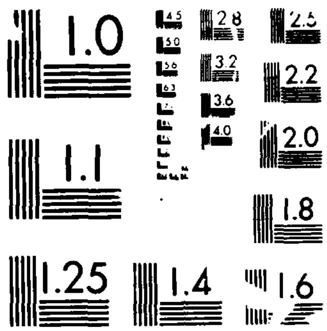
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# MASSACHUSETTS INSTITUTE OF TECHNOLOGY
# ARTIFICIAL INTELLIGENCE LABORATORY

*AI Memo 1039* *April 27, 1988*

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Gerald Jay Sussman and Jack Wisdom

**Abstract**

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**Keywords:** Dynamics, orrery, chaos, numerical methods, Pluto, Solar System.

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Jack Wisdom was also supported in part by the Planetary Geology and Geophysics Program of the National Aeronautics and Space Administration, under grant NAGW-706.

# Numerical Evidence that the Motion of Pluto is Chaotic

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April 1988

**Abstract**

The Digital Orrery has been used to perform an integration of the motion of the outer planets for 845 million years. This integration indicates that the long-term motion of the planet Pluto is chaotic. Nearby trajectories diverge exponentially with an $e$-folding time of only about 20 million years.

The determination of the stability of the Solar System is one of the oldest problems in dynamical astronomy, but despite considerable attention all attempts to prove the stability of the system have failed. Arnold has shown that a large proportion of possible solar systems are stable if the masses, and orbital eccentricities and inclinations of the planets are sufficiently small [1]. The actual Solar System, however, does not meet the stringent requirements of the proof. Certainly, the great age of the Solar System suggests a high level of stability, but the nature of the long-term motion remains undetermined. The apparent analytical complexity of the problem of determining the stability of the solar system has led us to investigate the stability by means of numerical models. We have investigated the long-term stability of the Solar System through an 845 million year numerical integration of the outer planets with the Digital Orrery [2], a special purpose computer for studying planetary motion.

Our earlier integrations [3] showed surprisingly long periods in the motion of Pluto, of order 137 million years, and longer periods or a possible secular drift in the inclination of Pluto. The motion of Pluto is involved in a surprisingly large number of resonances. The libration of the resonant argument permits the orbit of Pluto to cross the orbit of Neptune without having close encounters [4]. Williams and Benson [5] also found that the argument of perihelion was oscillating about $\pi/2$. Thus the perihelion of Pluto remains far from the line of intersection of the orbital planes of Pluto and Neptune. Williams and Benson noted that two other resonance conditions were possibly satisfied by the motion of Pluto. Using the frequencies determined in our 200 million year integrations, we find that these two resonances are satisfied to the precision of our measurements. In addition, the 137 million year components that we observed in the elements of Pluto are associated with yet another commensurability. In Hamiltonian systems resonances are associated with both enhanced stability and instability. Without the libration of the resonant argument or the argument of perihelion a Neptune crossing orbit would be highly unstable. Nevertheless, resonances are also associated with most prominent zones of instability. This is particularly true if several resonances strongly affect the motion.

The very long periods and the abundance of resonances in the motion of Pluto compelled us to pursue longer integrations. Our new integration indicates that the long-term motion of the planet Pluto, and by implication the motion of the whole Solar System, is weakly unstable. Exponential divergence of nearby trajectories indicates that the motion of Pluto is chaotic with the largest Lyapunov exponent estimated to be $10^{-7.3}\text{yr}^{-1}$.

### Our Numerical Experiment

For many years the longest direct integration of the outer planets was the one million year integration of Cohen, Hubbard, and Oesterwinter [6]. Recently several new integrations of the outer planets have been performed [7,3,8]. The longest was our set of 200 million year integrations. Our new integration is significantly longer and more accurate than all previously reported long-term integrations.

In our new integration of the motion of the outer planets the masses and initial conditions were the same as those used in our 200 million year integrations of the outer planets. The planet Pluto is taken to be a zero-mass test particle. We continue to neglect the effects of the inner planets, the mass lost by the Sun due to electromagnetic radiation and solar wind, and general relativity. The most serious limitation of our integration is our ignorance of the true masses and initial conditions. We believe that our model is close enough to the actual solar sytem that its study sheds light on the question of stability of the solar system. To draw more rigorous conclusions the sensitivity of our conclusions to the uncertainties in masses and initial conditions, and to unmodeled effects must be determined.

Our earlier integrations were limited to 100 million years forward and backward in time because of the accumulation of error, which was most seriously manifested in an accumulated longitude error of Jupiter of order 50 degrees. In our new integrations we continue to use the $12^{th}$ order Störmer predictor, but a judicious choice of stepsize has reduced the numerical errors by several orders of magnitude. In all of our integrations the error in energy of the system varies nearly linearly with time. In the regime where neither roundoff nor truncation error is dominant the slope of the energy error as a function of time depends on stepsize in a complicated way. For some stepsizes the energy error has a positive slope; for others the slope is negative. This suggests that there might be special stepsizes for which there is no linear growth of energy error. By a series of numerical experiments we indeed found that there are values of the stepsize where the slope of the linear trend of energy vanishes. The special stepsizes become better defined as the integration interval of the experiments is increased.

We chose our stepsize on the basis of a dozen 3 million year integrations, and numerous shorter integrations. For our new long integration we chose the stepsize to be 32.7 days. This seemingly innocuous change from a stepsize near 40 days dramatically reduces the slope of the energy error, by roughly three orders of magnitude. If the numerical integration were truncation-error dominated, for which the accumulated error is proportional to the $h^n$, where $h$ is the stepsize and $n$ is the order of the integrator, then this reduction of stepsize would only improve the accumulated error by about a factor of about 10.

In our new integration the relative energy error (energy minus initial energy divided by the magnitude of the initial energy) over 845 million years is $-2.6\times 10^{-10}$; the growth of the relative energy error is still very nearly linear with a slope of $-3.0 \times 10^{-19}$ per year. By comparison the rate of growth of the relative energy error in our 200 million year integrations was $1.8 \times 10^{-16}$ per year. The errors in other integrations of the outer Solar System were comparable to the errors in our 200 million year integrations. The rate of growth of energy error in the one million year integration of Cohen, Hubbard, and Oesterwinter was $2.4 \times 10^{-16}$ per year. For the 6 million year integration of Kinoshita and Nakai the relative energy error was approximately $5 \times 10^{-16}$ per year. For the LONGSTOP integration the growth of relative energy (as defined in this paper) was $-2.5 \times 10^{-16}$ per year. Thus the rate of growth of energy error in the integration reported in this paper is smaller than all previous long-term integrations of the outer planets by a factor of about 600.

We verified that this improvement in energy conservation was reflected in a corresponding improvement in position and velocity errors by integrating the outer planets forward 3 million years and then backward to recover the initial conditions, over a range of stepsizes. For the chosen stepsize of 32.7 days the error in recovering the initial positions of each of the planets is of order $10^{-5}$AU or about 1500km. Note that Jupiter has in this time travelled $2.5 \times 10^{15}$km.

The error in the longitude of Jupiter can be estimated if we assume that the energy error is mainly in the the orbit of Jupiter. The relative energy error is proportional to the relative error in orbital frequency so the error in longitude is proportional to the integral of the relative energy error: $\Delta\lambda \approx tn\Delta E(t)/E$, where $n$ is the mean motion of Jupiter and $t$ is the time of integration. Since the energy error grows linearly with time the position error grows with the square of the time. We find that the accumulated error in the longitude of Jupiter after 100 million years is only about 4 minutes of arc. This is to be compared with the 50 degree accumulated error estimated for our 200 million year integrations. The error in the longitude of Jupiter after the full 845 million years is about 5 degrees.

We have directly measured the integration error in the determination of the position of Pluto by integrating forward and backward over intervals as long as 3 million years to determine how well we can reproduce the initial conditions. Over such short intervals the roundtrip error in the position of Pluto grows as a power of the time with an exponent near 2. The error in position is approximately $1.3 \times 10^{-19} t^2$ AU (where $t$ is in years). This growth of error is almost entirely in the integration of Pluto's orbit; the roundtrip error is roughly the same when we integrate the whole system and when we integrate Pluto in the field of the Sun only. It is interesting to note that in the integrations using the 32.7 day stepsize the position errors in all the planets are comparable. Extrapolation of the roundtrip error for Pluto over the full 845 million year integration gives an error in longitude of less than 10 minutes of arc.

### Features of the orbital elements of Pluto

Our new integrations confirm and extend our previous report of extremely long periods in the orbital elements of Pluto.

For the first 450 million years of our integration we recorded the state of the system every 499983 days (about 1369 years) of simulated time. For the second 400 million years we sampled 16 times less frequently. For each planet the positions and velocities were converted to orbital elements relative to the center of mass of those bodies with smaller semimajor axes [9]. The elements were used to form the variables $h$, $k$, $p$, and $q$, defined as

$$h = e \sin \varpi \qquad k = e \cos \varpi$$
$$p = \sin(i/2) \sin \Omega \quad q = \sin(i/2) \cos \Omega$$

where $i$ is the inclination, $e$ is the eccentricity, $\Omega$ is the longitude of the ascending node, and $\varpi$ is the longitude of the perihelion. The variables $p$ and $q$ specify the orientation of the orbital plane, and the variables $h$ and $k$ specify the eccentricity and the orientation of the orbit in the orbital plane. $h$, $k$, $p$, and $q$ are natural variables for looking at the long-term behavior of planetary systems.

Figure 1 shows the variation of $h$ with time. The largest component is the 3.7 million year regression of the longitude of perihelion. The 27 million year component we previously reported is clearly visible, as is the 137 million year component. The quantity $p$ (not shown) has similar features.

Figure 2 shows the inclination of Pluto as a function of time. Besides the major 3.8 million year component we can clearly discern the 34 million year component we previously reported. Although there is no continuing secular decline in the inclination, there is a component with a period near 150 million years and evidence for a component with a period of approximately 600 million years. The existence of significant orbital variations with such long periods is quite surprising. For quasiperiodic trajectories we expect to find frequencies which are low order combinations of a few fundamental frequencies (one per degree of freedom). The natural timescale for the long

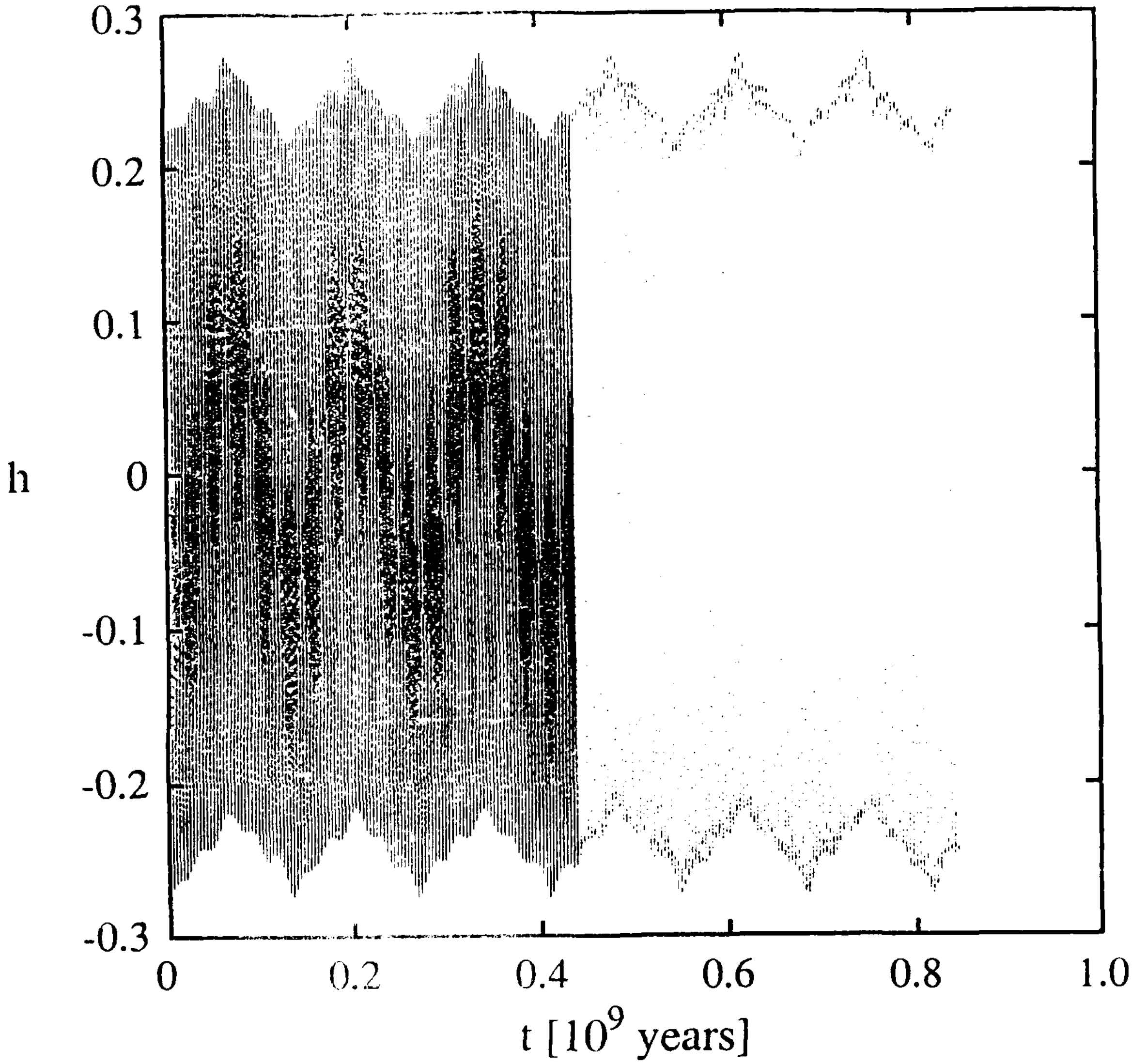


Figure 1: The orbital element $h = e \sin \varpi$ for Pluto over 845 million years. On this scale the dominant period (the 3.7 million year circulation of the longitude of perihelion) is barely resolved. The most obvious component has a period of 137 million years. The sampling interval was increased in the second half of our integration.

term evolution of a quasiperiodic planetary systems is set by the periods of the circulation of the nodes and perihelia. The presence of significant components with periods much longer than these indicates strong low-order resonances. An abundance of low-order resonances can give rise to chaotic behavior. Pluto is involved in at least five resonances, a sixth is possibly indicated by the 600 million year component that we can see in figure 2. Each time the motion of Pluto has been investigated over longer intervals longer periods have been found. This is suggestive that the motion is chaotic.

## Deterministic Chaotic Behavior

In most conservative dynamical systems Newton's equations have both regular solutions and chaotic solutions. For some initial conditions the motion is quasiperiodic; for others the motion is chaotic. Chaotic behavior is distinguished from quasiperiodic behavior by the way in which nearby trajectories diverge. [10,11] Nearby quasiperiodic trajectories diverge linearly with time on average, whereas nearby chaotic trajectories diverge exponentially with time. Quasiperiodic motion can be reduced to motion on a multidimensional torus; the frequency spectrum of quasiperiodic motion has as many independent frequencies as degrees of freedom. The frequency spectrum of chaotic motion is more complicated, usually appearing to have a broad-band component.

The Lyapunov exponents measure the average rates of exponential divergence of nearby orbits. The Lyapunov exponents are limits for large time of the quantity $\gamma = \ln(d/d_0)/(t-t_0)$, where $d$ is the distance between the trajectory and an infinitesimally nearby test trajectory, and $t$ is the time. For any particular trajectory of an $n$-dimensional system there can be $n$ distinct Lyapunov exponents, depending on the phase space direction from the reference trajectory to the nearby trajectory. In Hamiltonian systems the Lyapunov exponents are paired; for each non-negative exponent there is a non-positive exponent with equal magnitude. Thus an $m$ degree-of-freedom Hamiltonian system can have at most $m$ positive exponents. For chaotic trajectories the largest Lyapunov exponent is positive; for quasiperiodic trajectories all of the Lyapunov exponents are zero.

Lyapunov exponents can be estimated from the time evolution of the phase space distance between a reference trajectory and nearby test trajectories [10,12]. The most straightforward approach is to simply follow the

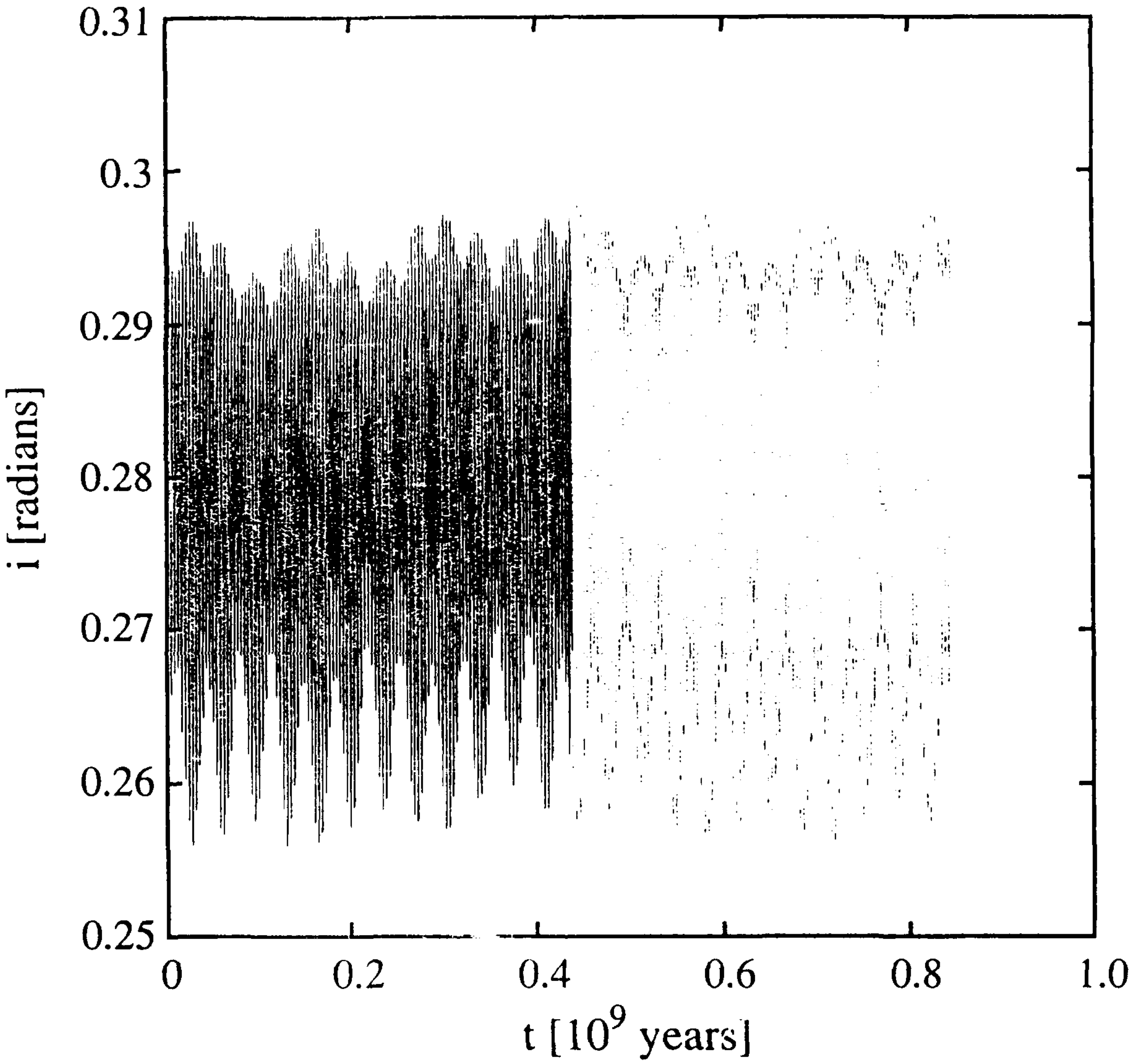


Figure 2: The inclination of Pluto over 845 million years. Besides the 34 million year component and the 150 million year component there appears to be a component with a period near 600 million years.

trajectories of a small cloud of particles started with nearby initial conditions. With a sufficiently long integration we can determine if the distances between the particles in the cloud diverge exponentially or linearly. If the divergence is exponential then for each pair of particles in the cloud we obtain an estimate of the largest Lyapunov exponent. With this method the trajectories eventually diverge so much that they no longer sample the same neighborhood of the phase space. We could fix this by periodically rescaling the cloud to be near the reference trajectory, but we can even more directly study the behavior of trajectories in the neighborhood of a reference trajectory by integrating the variational equations along with the reference trajectory. In particular, let $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ be an autonomous system of first-order ordinary differential equations and $\mathbf{y}(t)$ be the reference trajectory. We define a phase-space variational trajectory $\mathbf{y} + \delta\mathbf{y}$ and note that $\delta\mathbf{y}$ satisfies a linear system of first-order ordinary differential equations with coefficients that depend on $\mathbf{y}(t)$, $\delta\mathbf{y}' = \mathbf{J} \cdot \delta\mathbf{y}$ where the elements of the Jacobian matrix are $J_{ij} = \partial f_i / \partial y_j$.

### Lyapunov Exponent of Pluto

We estimated the largest Lyapunov exponent of Pluto by both the variational and the phase-space distance methods over the second half of our 845 million year run. Figure 3 shows the divergence of the logarithm of the phase-space distance in a representative 2-particle experiment and the growth of the logarithm of the variational phase-space distance. We measured the phase-space distance by the ordinary Euclidean norm in the 6-dimensional space with position and velocity coordinates. We measured position in AU and velocity in AU/day. Since the magnitude of the velocity in these units is small compared to the magnitude of the position, the phase space distance is effectively equivalent to the positional distance, and we refer to phase space distances in terms of AU. For both traces in this plot the average growth is linear, indicating exponential divergence of nearby trajectories with an $e$-folding time of approximately 20 million years. The shapes of these graphs are remarkably similar until the two-particle divergence grows to about an astronomical unit, verifying that the motion in the neighborhood of Pluto is properly represented. A more conservative representation of this data is to plot the logarithm of $\gamma$ versus the logarithm of time, as shown in figure 4. The leveling off of this graph indicates a positive Lyapunov exponent.

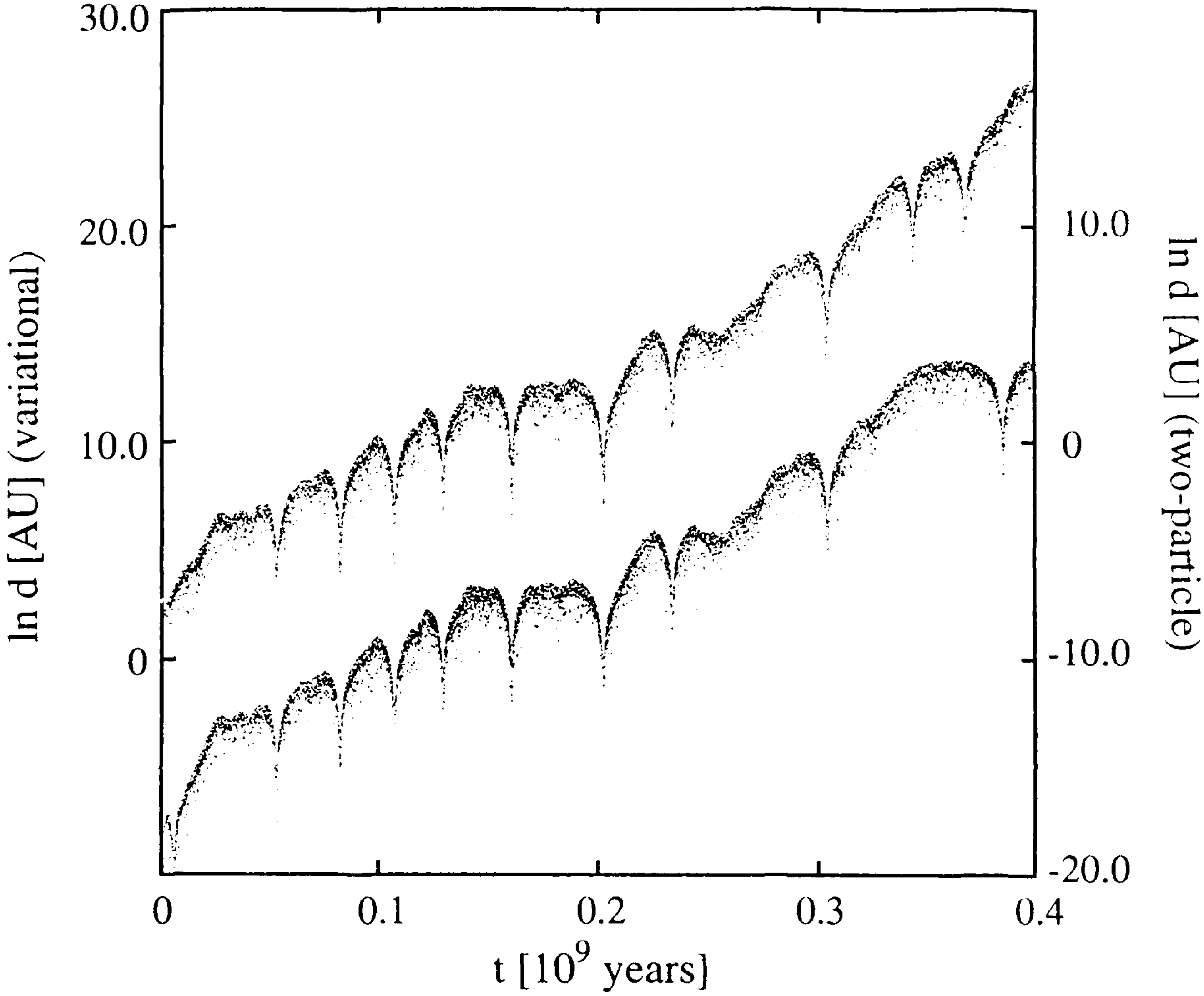


Figure 3: The exponential divergence of nearby trajectories is indicated by the average linear growth of the logarithms of the distance measures as a function of time. In the upper trace we see the growth of the variational distance around a reference trajectory. In the lower trace we see how two Plutos diverge with time. The distance saturates near 80AU when the Plutos are on opposite sides of the Sun. The variational method of studying neighboring trajectories does not have the problem of saturation. Note that the two methods are in excellent agreement until the two-trajectory method has nearly saturated.

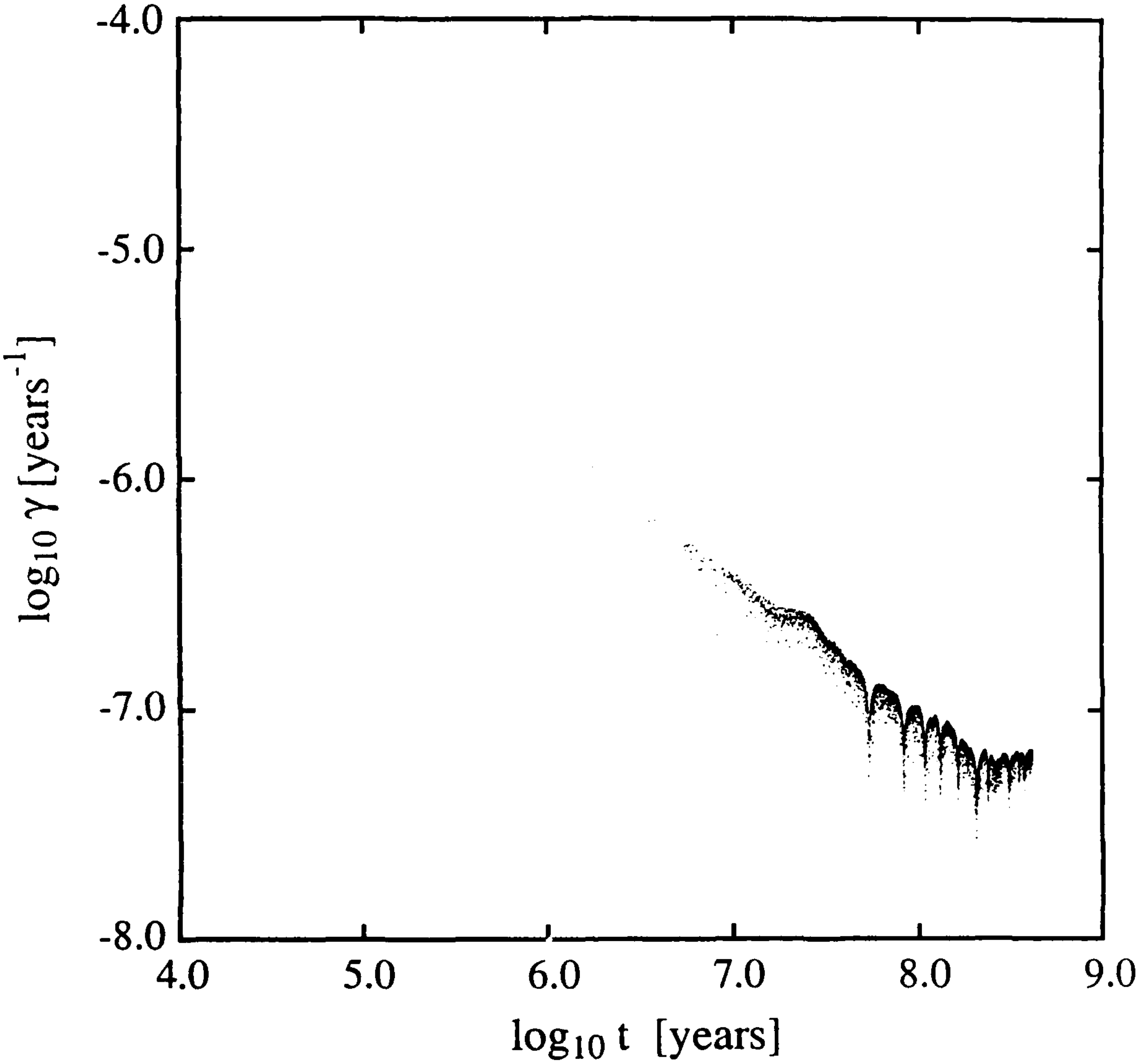


Figure 4: The conventional representation of the Lyapunov exponent calculation, the logarithm of $\gamma$ versus the logarithm of time. Convergence to a positive exponent is indicated by a leveling off; for regular trajectories this plot approaches a line with slope minus one.

To study the details of the divergence of nearby trajectories we expand the early portion of the divergence graph. Figure 5 shows the logarithms of the distances between several pairs of particles in the cloud of Plutos in our integration versus the logarithm of time. The separation starts out as a power law with an exponent near 3/2. Only after some time does the exponential take off. The power law is dominated by the exponential only after the rate of growth of the exponential exceeds the rate of growth of the power law. This suggests that the portion of the divergence of nearby trajectories that results only from the numerical error fits a 3/2 power law and that this error "seeds" the exponential divergence that is the hallmark of chaos. We tested this hypothesis by integrating a cloud of test particles with the orbital elements of Pluto in the field of the Sun alone. The divergence of these Kepler "Plutos" grows as $3.16 \times 10^{-17} t^{3/2}$ AU. This is identical to the initial divergence of the Plutos in the complete dynamical system, showing that two-body numerical error completely accounts for the initial divergence.

Only the second half of the integration was used in the computation of the Lyapunov exponents, because the measurement in the first half of our integration was contaminated by over-vigorous application of the renormalization method, and gave a Lyapunov exponent about a factor of four too large. The renormalization interval was only 275,000 years, which was far too small. The renormalization interval must be long enough that the divergence of neighboring trajectories is dominated by the exponential divergence due to the sensitive dependence on initial conditions rather than the power law divergence due to the accumulation of numerical errors. In our experiment the renormalization interval should have been greater than 30 million years. It is important to emphasize that the variational method of measuring the Lyapunov exponent has none of these problems.

## Discussion

Usually the measurement of a positive Lyapunov exponent provides a confirmation of what is already visible to the eye, chaotic trajectories usually look irregular. In this case the plots of Pluto's orbital elements do not look particularly irregular (see figures 1 and 2). The irregularity of the motion does manifest itself in the power spectra. For a quasiperiodic trajectory the power spectrum of any orbital element is composed of integral linear combinations of fundamental frequencies, where the number of fundamental

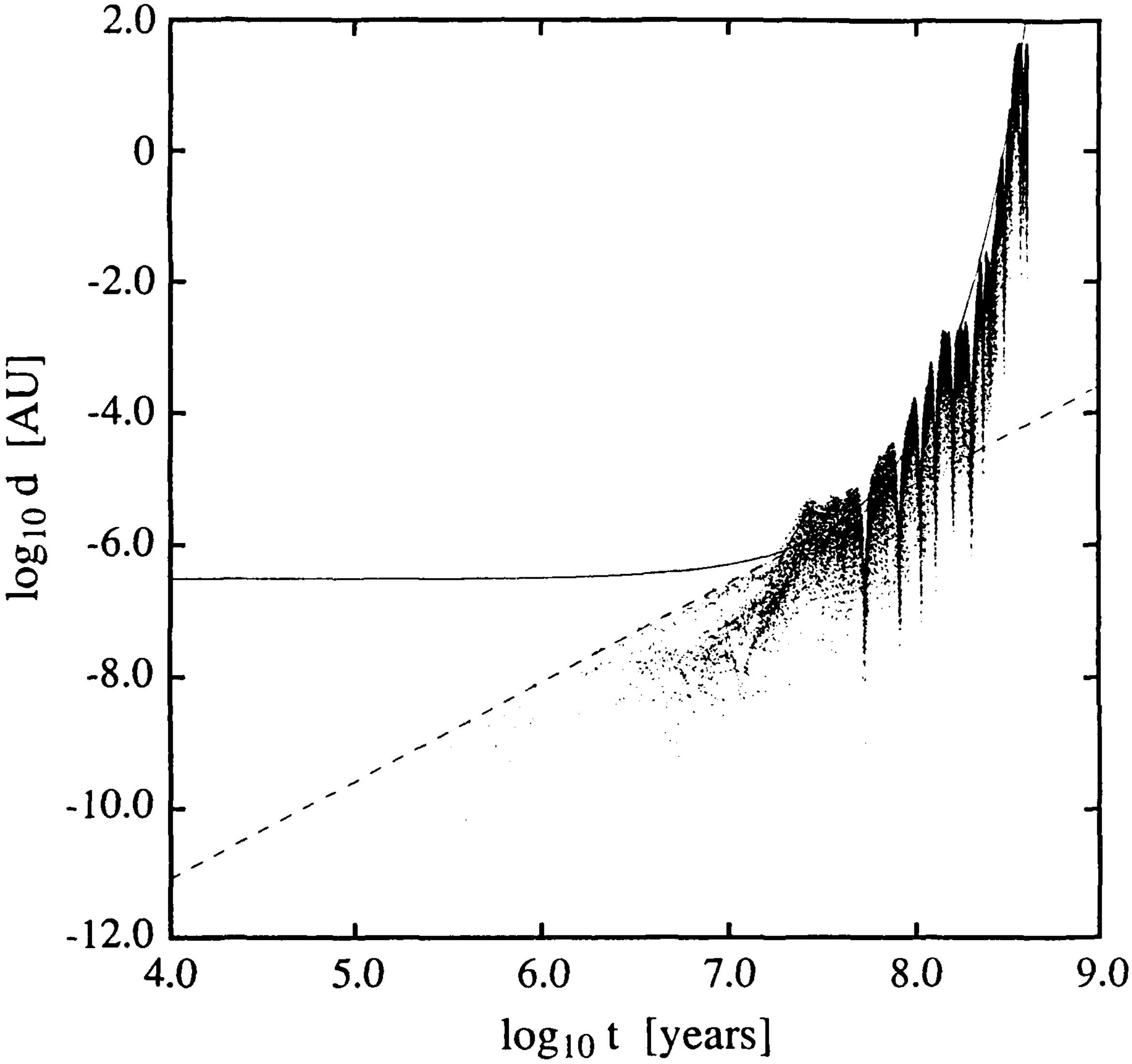


Figure 5: Common logarithm of the distance between several pairs of Plutos, in AU, versus the common logarithm of the time, in years. The initial segment of the graph closely fits a 3/2 power law (dashed line). The solid line is an exponential chosen to fit the long-time divergence of Plutos. The exponential growth takes over when its slope exceeds the slope of the power-law.

frequencies is equal to the number of degrees of freedom. The power spectrum of chaotic trajectory is more complicated, usually appearing to have some broad-band component.

A portion of the power spectrum of Neptune's $h$ is shown in figure 6. For comparison the same portion of the power spectrum of Pluto's $h$ is shown in figure 7. Hanning windows have been used to reduce spectral leakage; only the densely sampled part of the run was used in the computation of the Fourier transform. The spectrum of Neptune is quite complicated but there is no evidence that it is not a line spectrum. On the other hand the spectrum of Pluto does appear to have a broad-band component. Note that both of these spectra are computed from the same integration run, using the same numerical methods. They are subject to the same error processes, so the differences we see are dynamical in origin. The broad-band character of the Pluto spectrum is consistent with the chaotic nature of the motion as indicated by our measurement of a positive Lyapunov exponent.

The lack of obvious irregularity in the orbital elements of Pluto indicates that the portion of the chaotic zone in which Pluto is currently moving is rather small. Since the global structure of the chaotic zone is not known it is not possible for us to predict whether more irregular motions are likely. If the small chaotic zone in which Pluto is found connects to a larger chaotic region relatively sudden transitions can be made to more irregular motion. This actually occurs for the motion of asteroids near the 3/1 Kirkwood gap [13].

On the other hand, the fact that the timescale for divergence is only an order of magnitude larger than the fundamental timescales of the system indicates that the chaotic behavior is robust. It is not a narrow chaotic zone associated with a high order resonance. Even though we do not know the sensitivity of the observed chaotic behavior to the uncertainties in parameters and initial conditions, and unmodelled effects the large Lyapunov exponent suggests that the chaotic behavior is characteristic of a range of solar systems including the actual solar system.

## Conclusion

Our numerical model indicates that the motion of Pluto is chaotic. The largest Lyapunov exponent is about $10^{-7.3}\mathrm{yr}^{-1}$. Thus the $e$-folding time for the divergence of trajectories is about 20 million years. This is a remarkably short $e$-folding time, considering the age of the Solar System.

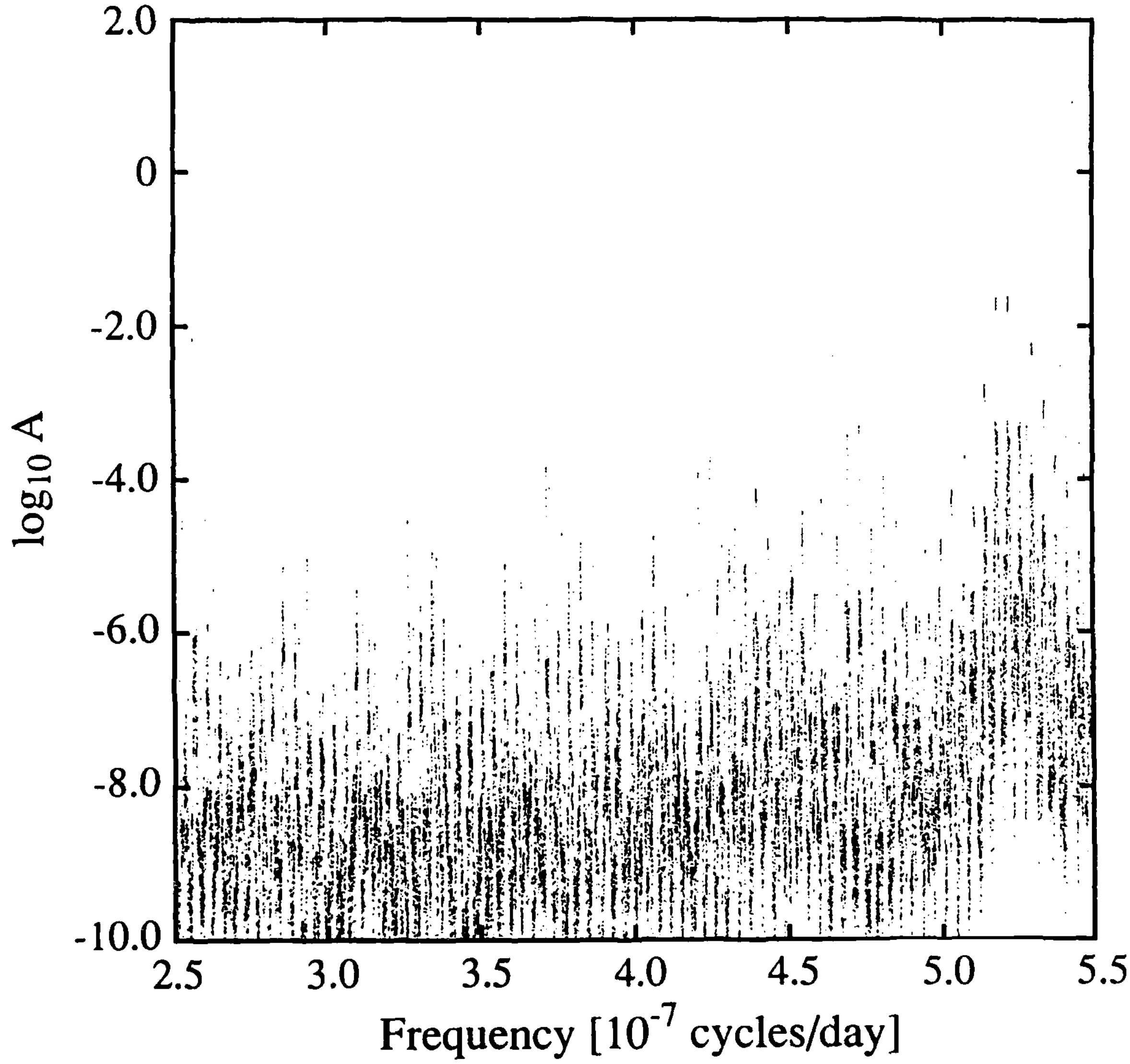


Figure 6: A portion of the power spectrum of Neptune's $h$. The spectrum is quite complicated but there is no evidence of a broadband component; the spectrum is consistent with a line spectrum.

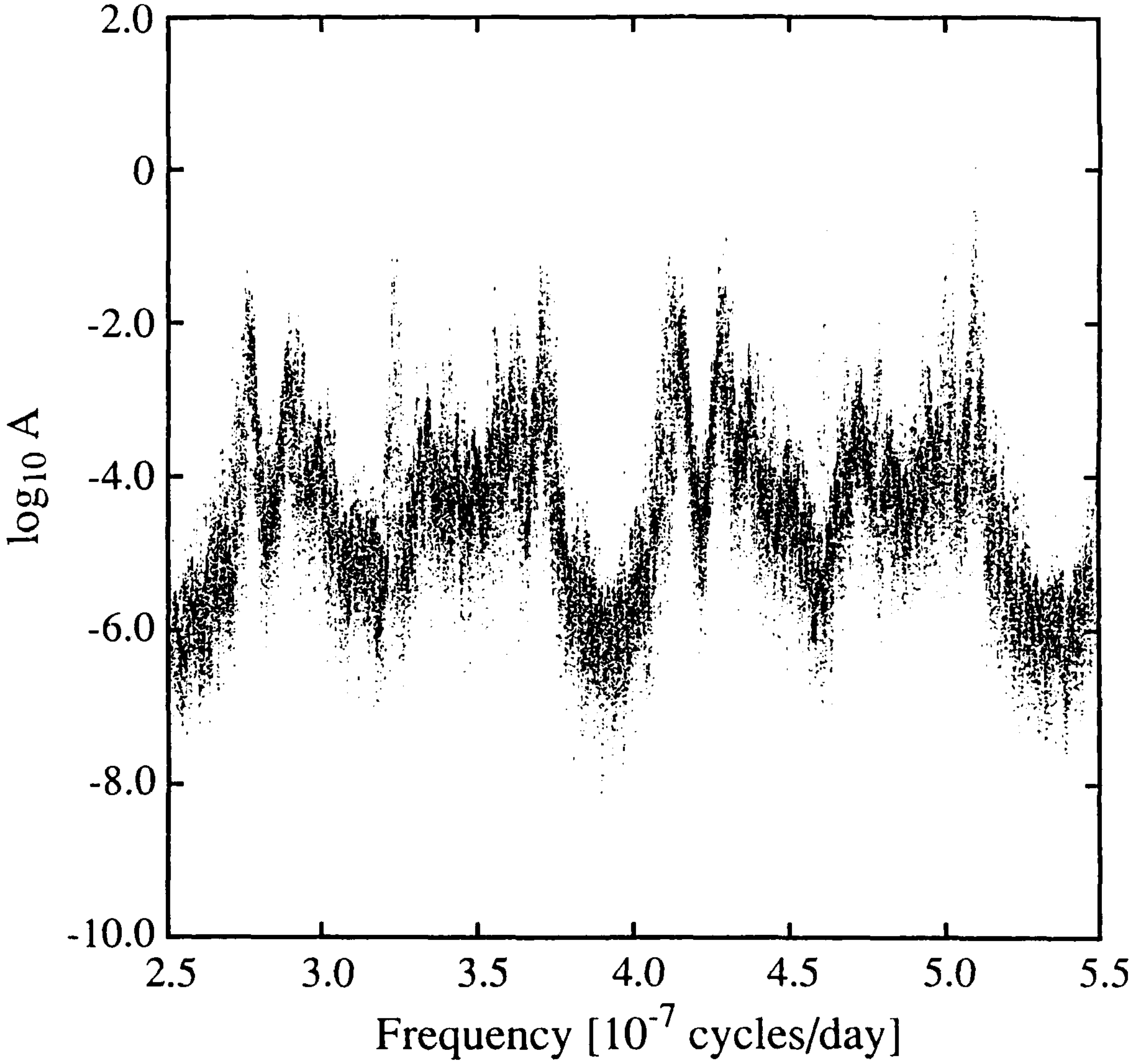


Figure 7: A portion of the power spectrum of Pluto's $h$. There appears to be a broad-band component to the spectrum. This is consistent with the chaotic character of the motion of Pluto as indicated by the positive Lyapunov exponent.

## Acknowledgements

We thank W. Kahan for suggesting that a stepsize might be chosen that essentially eliminates the linear growth in the energy error, making this paper possible. We also thank Panayotis Skordos for doing some of the experiments to localize the special stepsizes.

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The authors are listed in alphabetical order.

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