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Abel Inversion Using Transform Techniques

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Abstract - A method is presented for calculating the reconstruction of a circularly symmetric two-dimensional function from its projection, a relation known as the Abel inversion. This technique differs from techniques used previously by using integral transforms for its implementation. The frequency-domain analysis allows for experimentally obtained data, which is often noisy and off-center, to be dealt with in a systematic, rational manner. The formulation of the Abel inversion in terms of transforms, the filtering of the noise, and the estimate of the off-center shift are discussed. Sample calculations of simulated noisy data and the application of the method to an image of a laser sustained plasma are presented.

I. Introduction

Optical emission or transmission is often used to obtain experimental diagnostic measurements of axially symmetric sources such as plasmas, flames, and rocket and jet engine plumes. The line-of-sight projection of the emission or transmission coefficients onto a one-dimensional axis can be used to determine their two-dimensional radial structure. As is the case in most inverse problems, however, the conditions are ill-posed, and so considerable difficulty can arise when processing real experimentally acquired data that has inherent uncertainties.

The reconstruction of a circularly symmetric two-dimensional function from its projection onto an axis is known as Abel inversion or inverse Abel transformation of the projection. The measured intensity, $I(x)$, is given in terms of emission coefficients, $\epsilon(r)$, through the Abel transform[1]

$$I(x) = 2 \int_x^\infty \frac{r\epsilon(r)dr}{\sqrt{r^2 - x^2}} \tag{1}$$

where x is the displacement of the intensity profile and r is the radial distance in the source. The measured intensity $I(x)$ is the one-dimensional projection of the two-dimensional circularly symmetric function having $\epsilon(r)$ as a radial slice. The inversion integral, or the inverse Abel transform, is given by,

$$\epsilon(r) = \frac{-1}{\pi} \int_r^\infty \frac{(dI/dx)}{\sqrt{x^2 - r^2}} dx \tag{2}$$

In practice, the application of equation (2) is made difficult because of the singularity in the integral at the lower limit and because the derivative of the projection tends to greatly enhance the noise corrupting the data. Furthermore, the intensity is usually not available as a continuous function and so is known only at discrete sample points. Several approaches based upon geometrical techniques or numerical methods using polynomial fits have been used to perform the inversion. Nestor and Olsen[2] transformed the variables according to $r^2 = v$ and $x^2 = u$ so that the inversion integral can be approximated by a simpler sum. Bockasten[3] fitted third degree polynomials to the data points and approximated the integral by a sum. However, these methods require prior smoothing of the data and are not considered complete in themselves.

Later, least squares curve fitting methods were employed and were found to yield better results than the exact fit methods when applied to noisy data. Freeman and Kats[4] used a single polynomial curve to fit the data. Fourth order polynomials were found to give the best results among trials using up to twelfth order polynomials. Cremers and Birkebak[5] compared several inversion techniques and showed that the least squares curve fitting techniques are more favorable than the exact fit methods. Short descriptions of several other techniques and comparisons

of these various techniques can be found in reference 5. Shelby[6] divided the data into several intervals and used a least squares polynomial fit technique in each interval to smooth the scattered data. The inversion was then performed analytically and summed over the intervals to obtain emission coefficients. Maldonado *et al.*[7] expanded $\epsilon(r)$ in a series of orthogonal polynomials and derived a method to find the expansion coefficients from the intensity data.

All of these methods have drawbacks. The singularity in the lower limit of the integral causes problems for the numerical methods, but these are avoided by the analytical methods. The smoothing techniques used are essentially a kind of lowpass filtering having undetermined filter characteristics. When the inversion is performed, the spectral characteristics of the noise, of the desired signal and of the smoothing algorithm are not considered. Therefore, the possible problems of loss of information and distortion are neglected. Use of the Abel integral implies an assumption that the input data will be symmetric, but the measured intensity data often exhibit some degree of asymmetry, and the exact axis of symmetry is not usually known *a priori*. Determination of the axis of symmetry is often chosen using *ad hoc* methods. The smoothing techniques consume a large amount of computer time and the error propagation calculations are tedious[6].

We have developed a method based upon integral transforms that removes many of the difficulties and uncertainties associated with these earlier techniques and, by using the fast Fourier transform (FFT) algorithm to implement the procedure, greatly reduces the computation time. This technique is substantially different from earlier methods in its use of transform techniques and frequency domain analysis. Several principles of digital signal processing and spectral analysis are employed to solve the problems associated with the processing of actual, noise-corrupted, asymmetric data.

In Section II the Abel inversion integral is reformulated in terms of the Fourier and Hankel transforms. This *reformulation is then used to handle many of the ill-posed conditions encountered when analyzing actual noisy data, and the methods for doing this are outlined in Section III. Numerical experiments demonstrating the validity and applicability of the inversion scheme are presented in Section IV, together with an recent application of the method to an image of a laser sustained plasma.

II. Reformulation of the Abel inversion

The reformulation of the Abel inversion in terms of integral transforms can be derived beginning with equation (1). With the substitution $r = \sqrt{x^2 + y^2}$, the projection can be written

$$I(x) = \int_{-\infty}^{\infty} \epsilon(\sqrt{x^2 + y^2})dy. \tag{3}$$

The one-dimensional Fourier transform of equation (3) is

$$FT[I(x)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \epsilon(\sqrt{x^2 + y^2})exp(-j2\pi xq)dxdy. \tag{4}$$

If the variables of integration are changed from cartesian to polar coordinates, it can be shown that[8]

$$FT[I(x)] = 2\pi \int_{0}^{\infty} r\epsilon(r)J_o(2\pi rq)dr \tag{5}$$

where $J_o(.)$ denotes the zero-order Bessel function of the first kind. The right hand side of equation (5) is the zero-order Hankel transform of $\epsilon(r)$. Thus, the emission distribution of the plasma may be recovered by taking the inverse Hankel transform of the Fourier transform of the projected intensity:

$$c(r) = 2\pi \int\limits_0^\infty q J_o(2\pi r q) \int\limits_{-\infty}^\infty I(x) exp(-j2\pi x q) dx dq. \tag{6}$$

Equation (6) is mathematically equivalent to the Abel inversion integral given in equation (2), but has several advantages in its implementation with regard to actual data analysis. First, it avoids the difficulty associated with the singularity at the lower limit of integration. Second, following ing the Fourier transform of $I(x)$, filters may be applied directly in the frequency domain to reduce the noise which often corrupts the experimentally acquired data and thus smooth the signal in a known, systematic manner. As will be shown in the following section, the Fourier transform also allows the implementation of an algorithm to center the data on axis and to make it symmetric. Finally, equation (6) can be numerically evaluated using discrete fast Fourier transform algorithms available in software or on vector array processors to decrease computation time compared to the techniques used previously.

III. Application of the inversion scheme to noisy data

To apply equation (6) to actual experimentally-acquired data, it was necessary to take into account that, in general, the projected intensity measurement recorded by the detector is: a) shifted by some unknown amount from the center axis, b) corrupted with additive random noise, and c) sampled at some fixed interval into a digital signal. The resulting data was thus modeled as $d(m) = I(m + \Delta m) + n(m)$, where m is the discrete sample number, Δm is the shift from the origin, and $n(m)$ is a random process. Frequency domain techniques were employed to handle the problems associated with these effects in the inversion.

The Fourier transform of the projected intensity was approximated by the discrete Fourier transform algorithm, which could be quickly calculated using FFT array processing hardware. Following the FFT, filters could be applied to the spectrum of the data directly in the frequency domain simply by multiplication with the desired frequency response function. Because of the baseband nature of the signal of interest, a lowpass filter was suitable for removing the high frequency components of the noise while preserving the information about the signal contained in the lower frequencies. The filter design was accomplished using standard optimization programs for the desired passband-stopband specifications.

Determination of the shift of the data from the origin and the subsequent centering of the data was also performed in the frequency domain. If $D(k), G(k)$ and $N(k)$ represent the discrete Fourier transforms of the acquired data, the properly centered projected intensity, and the noise remaining on the signal after application of the filter, respectively, then they are related by

$$D(k) = G(k) exp(j2\pi k \Delta m/M) + N(k) \tag{7}$$

where M is the total number of points in the discrete Fourier transform. This may be rewritten

$$G(k) = D(k) exp(-j2\pi k \Delta m/M) - N(k) \tag{8}$$

where the noise has been assumed stationary so that multiplication of its spectrum by a phase term does not change its statistical properties, and so $N(k) exp(-j2\pi k \Delta m/M)$ can be replaced by $N(k)$ without loss of generalization.

Because the intensity is a real, even function in signal space, its Fourier transform in frequency space is also real and even. Thus, $G(k)$ in equation (8) is real and even, and any imaginary components are necessarily due to noise. This allows equation (8) to be broken up into two equations

$$G(k) = D_R(k) cos(2\pi k \Delta m/M) + D_I(k) sin(2\pi k \Delta m/M) - N_R(k) \tag{9a}$$

$$0 = D_I(k) cos(2\pi k \Delta m/M) - D_R(k) sin(2\pi k \Delta m/M) - N_I(k) \tag{9b}$$

where the subscripts R and I correspond to the real and imaginary components of each function. Equation (9b) allows the imaginary portion of the noise spectrum to be written explicitly in terms of the transform of the data, $D(k)$, and the shift of the data from the origin, Δm.

The noise, $n(m)$, was further assumed to be an uncorrelated bandlimited Gaussian random process. Its spectral components could thus be modeled as independent Gaussian random variables[9] each with probability density function

$$p_N(N_I(k)) = \frac{1}{\sqrt{2\pi}\sigma} exp[-\frac{N_I^2(k)}{2\sigma^2}]. \tag{10}$$

Substituting equation (9b) into equation (10) and taking the ensemble probability density function over all values of k as the product of the individual probability density functions yields the conditional probability density of $D(k)$ given Δm as

$$p(D(k)|\Delta m) = \frac{1}{(\sqrt{2\pi}\sigma)^M} exp(-\frac{1}{2\sigma^2}\sum_{k=0}^{M/2-1}[D_I(k)cos(2\pi k\Delta m/M) - D_R(k)sin(2\pi k\Delta m/M)]^2). \tag{11}$$

Maximizing this probability density function with respect to Δm provides a maximum likelihood estimator for the shift. The resulting transcendental equation for the estimate, $\hat{\Delta m}$, is

$$\sum_{k=0}^{M/2-1} k(D_R(k)D_I(k)cos(4\pi k\hat{\Delta m}/M) - [D_R^2(k) - D_I^2(k)]sin(4\pi k\hat{\Delta m}/M)) = 0. \tag{12}$$

Equation (12) was solved iteratively to determine an estimate for the shift. Each spectral component of the data was then multiplied by the phase term $exp(-j2\pi k\hat{\Delta m}/M)$ to shift the data back to its proper location centered about the origin. Following this multiplication, the imaginary components remaining in the spectrum of the data were by assumption due to noise, and so were set equal to zero to fully symmetrize the data and further reduce the noise. If sufficient data is available to provide ensemble averages, this noise can be used to find the statistical properties of the remaining real portion of the noise corrupting the signal, but this approach has not been taken.

The inversion scheme was completed by taking the inverse Hankel transform of the spectrum of the filtered and centered data. This integral transform was approximated by a discrete summation, and computation time was reduced by using Candel's algorithm[10]. Further reduction in computation time was achieved by a modification to this algorithm which requires only half the additions of the Candel method[11].

As a final step in the data analysis, a study of the effects of noise on the errors in the inverted signal was made. If the input noise corrupting the intensity data is assumed to be a wide-sense stationary, bandlimited random process, then the variance of the inverted output noise as a function of the radial coordinate r can be expressed

$$var[n_o(r)] = \int_0^\infty (\pi q)^2 G_n(q)[J_o^2(2\pi rq) + Y_o^2(2\pi rq)]dq \tag{13}$$

where $G_n(q)$ is the power spectral density of the input noise. For the often-assumed case of bandlimited white noise with variance $2BN_o$, and bandwidth B, the variance of the output noise has been found to be approximately equal to

$$var[n_o(r)] = \frac{N_o B^2}{2r} \tag{14}$$

for sufficiently large r. These equations served as a guide in the design of filters for preprocessing the data, and provided a benchmark for the probable errors in the inverted data.

IV. Numerical and experimental results

A number of numerical experiments have been carried out using both noise-free and noisy data to evaluate the performance of the integral transform method implemented using discrete transforms[11]. These numerical experiments using noise-free Gaussian and semi-elliptic data established the accuracy of the modified Candel method for the Hankel transform. Numerical experiments using noisy data were performed to establish the validity of the method used to symmetrize the data and to evaluate the effectiveness of the optimal filters used to smooth the data.

A noisy trial function having a known inversion was generated by sampling a Gaussian function defined as,

$$I_1(m) = exp(-\frac{m^2}{100}) \; ; \quad m = 0, 1, ..., 127. \tag{15}$$

The Abel inversion of $I_1(m)$ with an even extension in negative m axis is theoretically given by,[8]

$$\epsilon_1(l) = \frac{1}{10\sqrt{\pi}} exp(-\frac{l^2}{100}) \; ; \quad l = 0, 1, ..., 127. \tag{16}$$

A zero mean sequence of random numbers having a Gaussian density function with a variance of .001 was generated and added to the sequence of equation (16) which was then shifted by two integer sample points. Also, the sequence was zero padded so that the filtering by discrete Fourier transforms produces a linear convolution filtering. This trial sequence is shown in Figure 1.
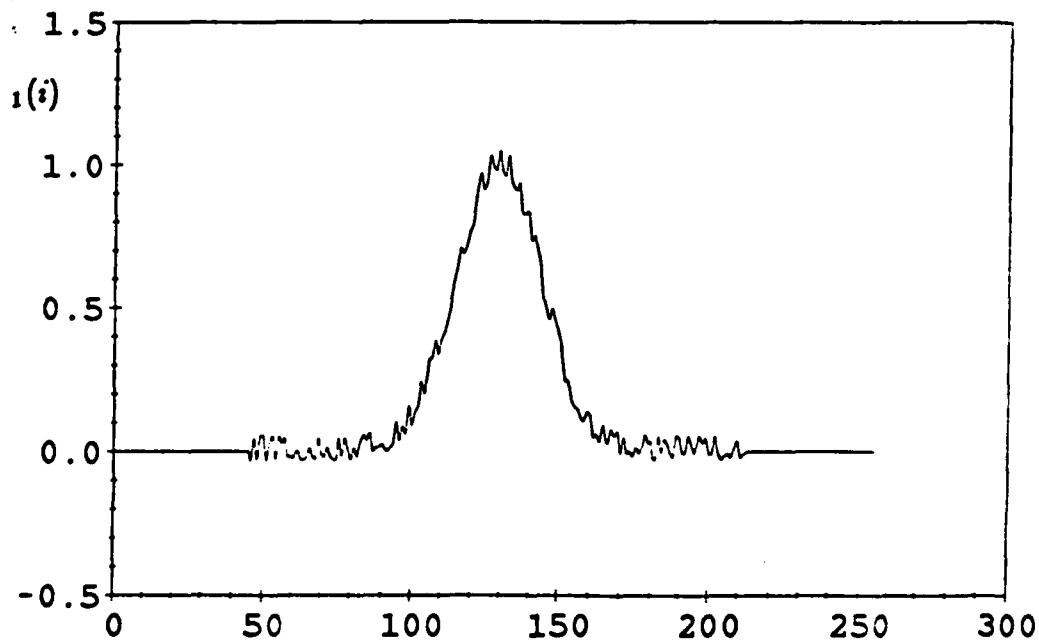

Figure 1. Generated noisy and shifted trial sequence $I_1(m)$.

The result obtained when this trial sequence was inverted without filtering or symmetrizing is shown in Figure 2. The same trial function was then inverted using the symmetrizing procedure and an optimal filter having a passband from 0.0 to 0.0525 and a stopband from 0.0725 to 0.5. The result from this inversion is shown in Figure 3. Other, more easily generated filters were found to produce similarly satisfactory inversions.

A principal advantage of this integral transform method is its computational efficiency. It has been applied to the inversions of a large number of images obtained from experiments with laser sustained plasmas[12]. Each of these digitally sampled images consisted of 512 radial scans of the plasma and each radial scan consisted of 240 sample points. The inversion of 512 sets of 240- point data consumed about eight minutes of CPU time when processed by the integral transform method, in contrast to a 20 hour CPU time required for a curve-fit inversion using the method described in reference 6. The data analysis was carried out using a Masscomp MC500 Micro Supercomputer. The symmetrizing procedure generally consumed about five to seven iterations for each set of data. Contour plots of the intensity and the emission coefficient for one of these images are given in Figure 4 and Figure 5. This particular image illustrates the performance of the method for intensity profiles which exhibit some asymmetry and which result in inversion profiles that are peaked on-axis (eg. 49 mm), flat-topped (eg. 45 mm) and having off-axis peaks (eg. 43 mm).

IV. Conclusions

A new integral transform method has been developed for the Abel inversion of measured intensity data in the presence of noise. The integral transform method has a number of advantages compared with the existing curve-fit techniques. The curve-fitting methods do not consider the spectral characteristics of the noise and the desired information, or address the problem of symmetrizing the experimental data. The Abel inversion or the inverse Abel transform is performed by the Fourier transform followed by the inverse Hankel transform. The efficiency of the well known FFT is exploited in the evaluation of both the Fourier and inverse Hankel transforms. This fast technique is made applicable to noisy and off-center experimental data by frequency domain filtering and a maximum likelihood estimator derived from the assumed noise characteristics.
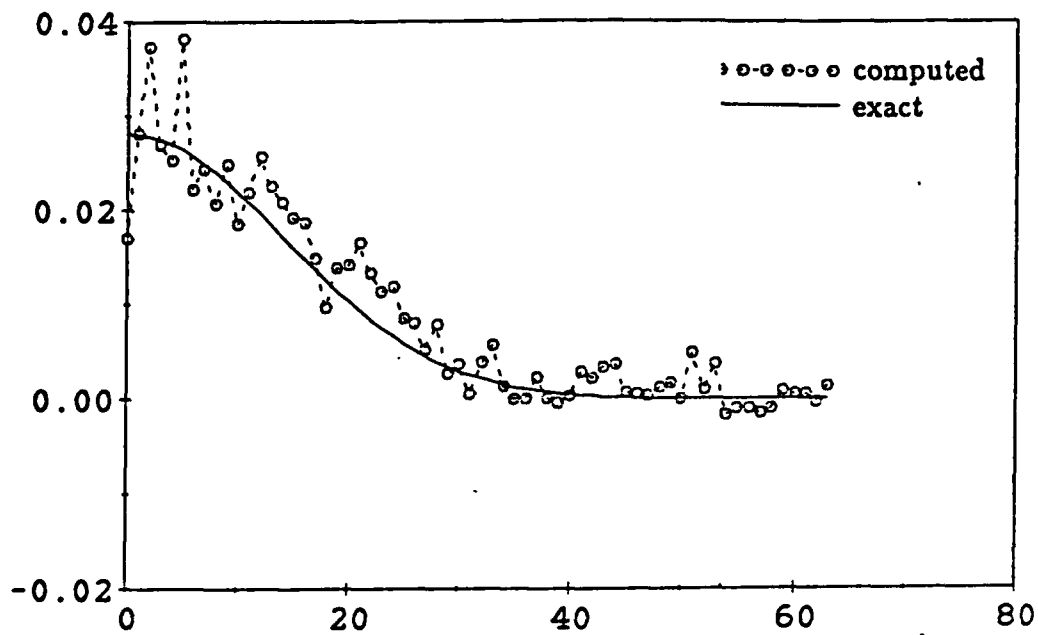
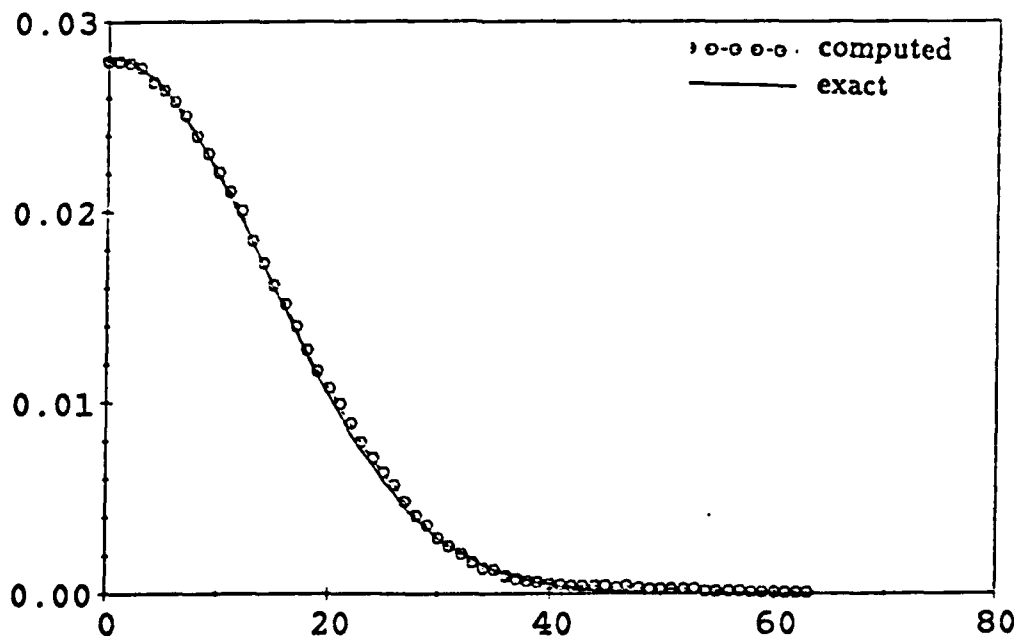Figure 2. Inversion of $I_1(m)$ without filtering or symmetrizing.

Figure 3. Inversion of $I_1(m)$ using filtering and symmetrizing technique.
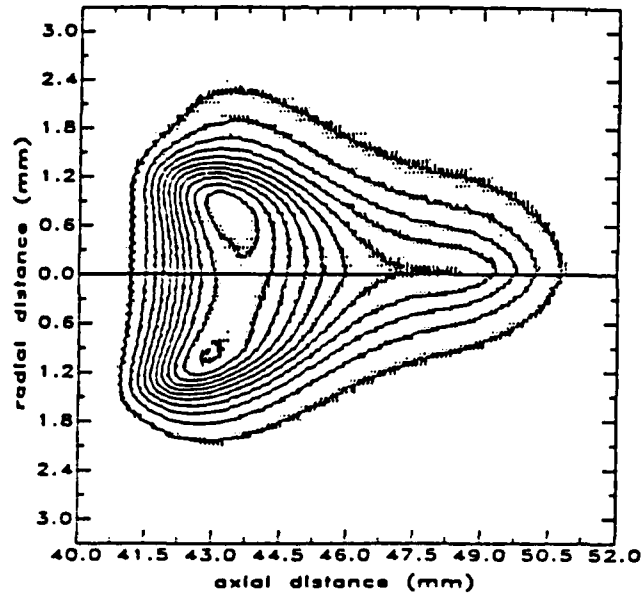
Figure 4. Contour plot of the projected intensity image data for a laser sustained plasma.
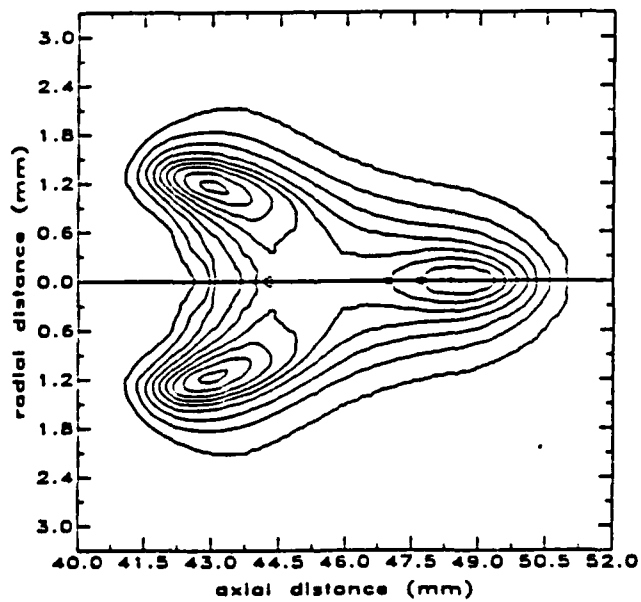

Figure 5. Contour plot of the emission coefficient image obtained by the inversion of data in Figure 5.

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