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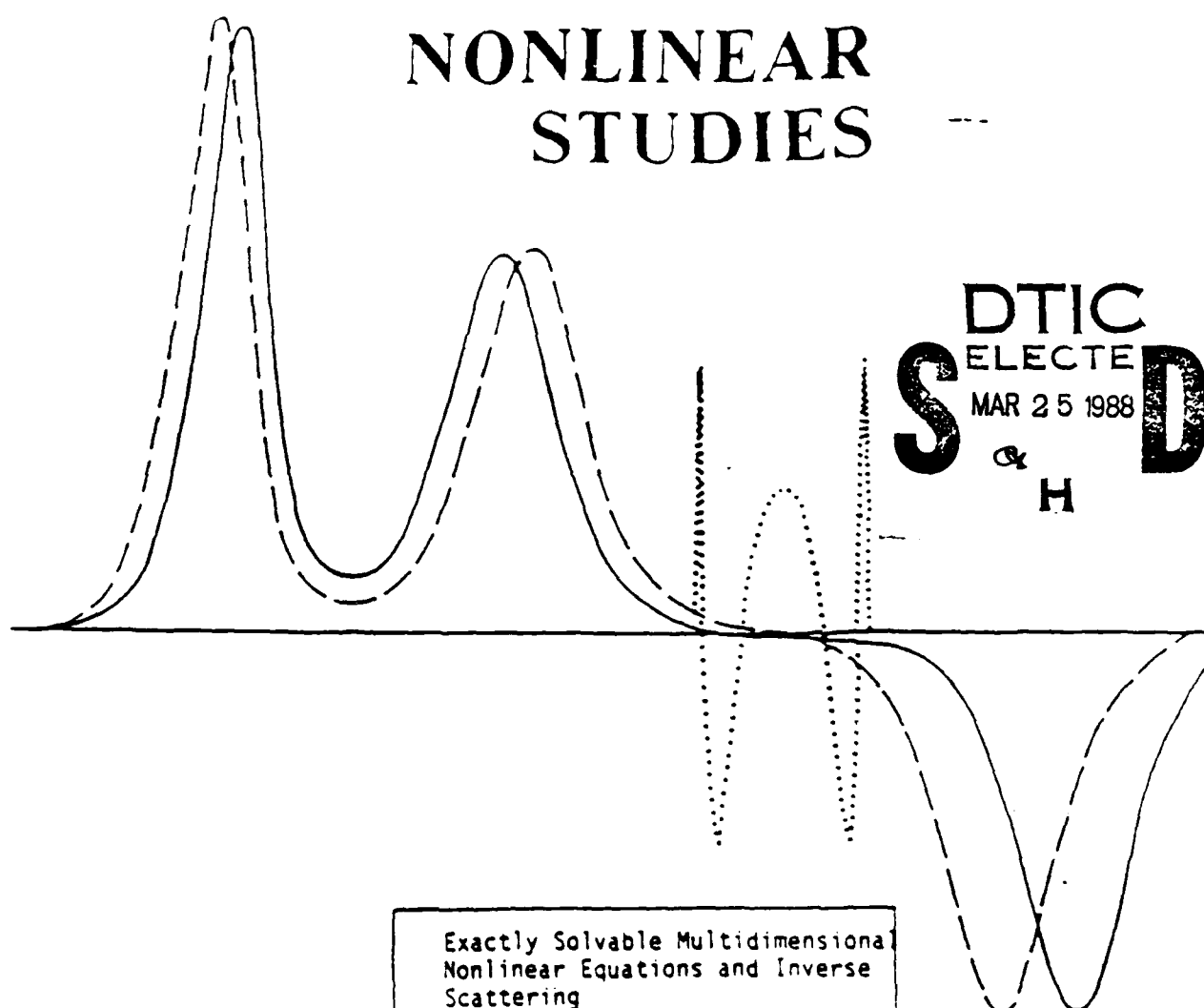
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Exactly Solvable Multidimensional
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by

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1986

N00014-86-K-0603

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December, 1986

EXACTLY SOLVABLE MULTIDIMENSIONAL NONLINEAR EQUATIONS
AND INVERSE SCATTERING

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ABSTRACT

A review of recent results associated with exactly solvable
multidimensional nonlinear systems and related questions of direct
and inverse scattering is given.

In this lecture a review of some recent results associated with exactly solvable multidimensional nonlinear systems will be given. The motivation for much of this work has come via what is commonly referred to as the Inverse Scattering Transform (I.S.T.; as a reference see, for example, [1]). IST is a method to solve certain nonlinear equations by associating them with appropriate compatible linear equations, one of which is identified as a scattering problem and the others(s) serves to fix the "time evolution" of the scattering data.

In one spatial dimension the prototype problem is the (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

The KdV equation is compatible with

$$v_{xx} + u(x,t)v = \lambda v \tag{2}$$

$$v_t = (\gamma + u_x)v - (\lambda + 2u)v_x \tag{3}$$

i.e. $v_{xxt} = v_{txx}$ implies (1). Equation (2) is the Schrödinger scattering problem, λ the eigenvalue ($\gamma =$ const. in (3)). The solution of (1) on the line: $-\infty < x < \infty$ for initial values $u(x,t=0)$ vanishing sufficiently rapidly at infinity is obtained by studying the associated direct and inverse scattering problem of (2) and using (3) to fix the time evolution of the scattering data. It turns out that the inverse problem amounts to solving a matrix Riemann-Hilbert boundary value problem (RHBVP) whose jump discontinuity depends explicitly on the scattering data. Calling $\lambda = -k^2, v(x,k) = \mu(x,k)e^{-ikx}$ the RHBVP takes the following form,

$$(\mu_+ - \mu_-)(x,t,k) = \mu_-(x,t,\sigma(k)) \, V(x,t,k) \text{ on } \Sigma$$
$$\lambda = -1, \; |k| = \infty \tag{4}$$

where

$V(x,t,k) - r(k,t) \, e^{2ikx}$, $\sigma(k) = -k$, $\Sigma = \{k : k \in \mathbb{R}\}$, and $\mu_\pm$ are the limiting boundary values as $\text{Im} k \to 0 \pm$ of meromorphic functions in the upper (+) lower (-) half plane. (4) may be converted into a linear integral equation by taking a minus projection and the potential is

reconstructed via

$$u(x,t) = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_C u(k,x,t,-k)\, V(x,t,k)\, dk \qquad (5)$$

where the contour is taken above all poles of $r(k,t)$, of which there
is at most a finite number, $k_j = i\kappa_j$, $r_j > 0$ $j = 1,---N$. The
scattering data: the reflection coefficient, $r(k,t)$ evolves simply
in time

$$r(k,t) = r(k,0)\, e^{8ik^2 t} \qquad (6)$$

The above scheme may be extended so as to solve a surprisingly
large number of interesting nonlinear evolution equations. There are
two scattering problems of particular interest in one dimension:

(i) Scalar scattering problems:

$$\frac{d^n v}{dx^n} + \sum_{j=2}^{n} u_j(x)\,\frac{d^{n-j} v}{dx^{n-j}} = \lambda v,$$

$$v(x,k),\ u_j \in \mathbb{C}$$

(ii) First order systems - generalized AKNS

$$\frac{dv}{dx} = ikJv + qv$$

$$v(x,k), q(x) \in \mathbb{C}^{N \times N},\ J = \text{diag}(J^1,---J^n)$$
$$J^i \neq J^j,\ i \neq j$$
$$q^{11} = 0.$$

Via an appropriate transformation the inverse problem associate with
(i), (ii) can be expressed as a matrix RHBVP of the form (4). The
potentials u_j,q can be shown to satisfy nonlinear evlution equations
by appending to (i), (ii) suitable linear time evolution equations.
One then finds that the scattering data $V(x,t,k)$ evolves simply in
time. Well known solvable nonlinear equations include the Boussinesq,
modified KdV, sine-Gordon, nonlinear Schrodinger, and three wave in-
teraction equations. The reader may wish to consult for example[2a-e]
for a detailed discussion of some of this material.

It is most significant that these concepts can be generalized to
2 spatial plus one time dimension.   Here the prototype euqation is
the Kadomtsev-Petviashvili (K-P) equation:

$$(u_t + 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} \qquad (7)$$

which is the compatibility equation between the following linear prob-
lems:

$$v_y + v_{xx} + u(x,y,t)v = 0 \qquad (8)$$

$$v_t + 4v_{xxx} + 6uv_x + 3(u_x - \sigma \int_{-\infty}^{x} u_y dx')v + \gamma v = 0 \qquad (9)$$

(γ = const.). We shall consider the question of solving (7) for
$u(x,y,0)$ decaying sufficiently rapidly in the plane $r^2 = x^2 + y^2 \to \infty$.
Physically speaking, both cases $\sigma^2 = -1$ (KPI) $\sigma^2 = +1$ (KPII) are of
interest. Whereas KPI can be related to a RHBVP of a certain type
(nonlocal; see ref.[3]) KPII turns out to require new ideas. Letting

$$v = \mu(x,y,k)e^{ikx + k^2 y/\sigma}$$

$\sigma = \sigma_R + i\sigma_I, \; \sigma_R \neq 0$. Then there exist functions μ bounded for all
x,y satisfying $\mu \to 1$ as $|k| \to \infty$. However such a function turns out
to be nowhere analytic in k, rather it depends nontrivially on both
the real and imaginery parts of $k = (k_R + ik_I)$. $\mu = \mu(x,y,k_R,k_I)$.

In fact μ satisfies a generalization of a RHBVP - namely a
$\bar{\partial}$ (DBAR) problem where μ satisfies,

$$\frac{\partial \mu}{\partial k} = \mu(x,y,\xi_0,k_I) V(x,y,k_R,k_I) \qquad (10)$$

where $\frac{\partial}{\partial k} = \frac{1}{2}(\frac{\partial}{\partial k_R} + i \frac{\partial}{\partial k_I})$ and V has the structure

$$V(x,y,k_R,k_I) = \frac{\text{sgn}(k_0)e^{i\beta(x,y,k_R,k_I,\xi_0)}}{\pi \sigma R!} T(k_R,k_I)$$

$$\beta(x,y,k_R,k_I,\xi_0) = (x + 2y \frac{k_I}{\sigma_R})(\xi_0 - k_R) = -2(x + 2y \frac{k_I}{\sigma_R})k_0$$

$$\xi_0 = -k_R - \frac{2\sigma_I}{\sigma_R}k_I, \; k_0 = k_R + \frac{\sigma_I}{\sigma_R}k_I \qquad (11)$$

(11) may be converted into a linear integral equation by employing
the generalized Cauchy formula. $T(k_R,k_I)$ is viewed as the ("nonphysi-
cal" data, i.e. inverse scattering data: i.e. inverse data) and the
potential is reconstructed via

$$u(x,y) = \frac{2i}{\pi} \frac{\partial}{\partial x} \int\int u(x,y,\xi_0,k_I)V(x,y,k_R,k_I)dk_R dk_I.$$ (12)

For K-P the evolution of the data obeys $(\gamma = 4ik^3$ in (9))

$$\frac{\partial T}{\partial t} = (8ik_0)(6kk_0 - 4k_0^2 - 3k^2)T$$ (13)

where $k_0 = k_R - \frac{ik_I}{R}$, $k = k_R + ik_I$.

Similar ideas apply to higher order scalar problems

(iii) $\qquad \frac{\partial v}{\partial y} + \frac{\partial^n v}{\partial x^n} + \sum_{j=2}^{n} u_j(x)\frac{\partial^{n-j}v}{\partial x^{n-j}} = 0$

where: $v, u_j \in \mathbb{C}$ and to first order systems

(iv) $\qquad \frac{\partial v}{\partial y} + J\frac{\partial v}{\partial x} + q(x,y)v = 0$

where: $v, q \in \mathbb{C}^{N\times N}$, $J=\text{diag}(J^1,\dots,J^N)$, $J^i \neq J^j$, $i \neq j$ with $q^{ii} = 0$.
Interested readers many consult reference[4a,b] for associated details.

The notion of $\bar{\partial}$ extends to higher dimensional scattering and inverse scattering problems. However as we shall mention, despite the fact that the inverse scattering problem is essentially tractable there does not appear to be any local nonlinear evolution equations in dimensions greater than $2 + 1$ associated with multidimensional generalizations of (iii) or (iv).

Our prototype scattering problem will be

$$\sigma v_y + \Delta v + u(x,y)v = 0$$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R}.$$ (14)

Letting

$$v = \mu(x,y,k)e^{ik\cdot x + k^2 y/\sigma}$$

$$k = k_R + ik_I, \quad k \in \mathbb{C}^n$$

$$k\cdot x = \sum_{1}^{n} k_j x_j, \quad \sigma = \sigma_R + i\sigma_I.$$

Then there exist functions μ bounded for all x, y satisfying $\mu \to 1$ as $|k_j| \to \infty$, $j = 1,\dots,n$. When $\sigma_R \neq 0$ μ turns out to be nonanalytic in each of the variables k, i.e. $\mu = \mu(x,y,k_{R_1},\dots,k_{R_n}, k_{I_1},\dots,k_{I_n})$ and

satisfies a $\bar{\partial}$ problem linear in ... in each of the variables k_j, i.e. ... satisfies an equation of the form,

$$\frac{\partial \omega}{\partial \bar{k}_j} = \bar{T}_j(..); \qquad j = 1,\ldots,n \qquad (15)$$

where $\bar{T}_j(u)$ is an appropriate linear integral operator which depends only on one scalar scattering function T : $\bar{T}_j = \bar{T}_j[T]$, $T=T(k_R,k_I,\xi)$ ξ being $(n-1)$ integration parameters in the nonlocal operator $\bar{T}_j$. The inverse problem is redundant, i.e. we are given $T(k_R,k_I,\xi)(3n-1)$ parameters) and we must reconstruct a local potential $u(x,y)(n+1$ parameters). A serious issue is how to characterize admissible inverse data T, i.e. data that really arises from a local potential (small generic changes in $T(k_R,k_I,\xi)$ cannot be expected to arise from a local potential $u(x,y)$). Insight into this question is obtained by requiring $\partial^2 u/\partial \bar{k}_i \partial \bar{k}_j = \partial^2 u/\partial \bar{k}_j \partial \bar{k}_i$ ($i \neq j$). The form of this constraint is given by

$$\mathcal{L}_{ij}(T) = \bar{N}_{ij}[T] \qquad (16)$$

where $\mathcal{L}_{ij}$ is a linear operator and $\bar{N}_{ij}$ a nonlinear (quadratic) non-local operator. Details can be found in[5a,b]. Equation (16) can be integrated and this integrated version may be used to reconstruct $u(x,y)$ as well as give a characterization for admissible scattering data: $T(k_R,k_I,\xi)$. However (16) also indicates why simple local nonlinear evolution equations have not been associated with equation (8). Namely in the previous lower dimensional (2+1 and 1+1) problems the time evolution of the scattering data obeyed a particularly simple equation, (e.g. $\frac{\partial T}{\partial t} = \omega(k_R,k_I)T$. However in this case such a simple flow will not be maintained - due to the nonlinear constraint (16).

These ideas can be generalized to first order systems:

$$(v) \qquad \frac{\partial v}{\partial y} + c \sum_{j=1}^{n} J_j \frac{\partial v}{\partial x_j} = qv$$

$v, q \in \mathbb{C}^{N \times N}$, $J_j = \text{diag}(J_j^1,\ldots,J_j^N)$

$J_j^k \neq J_j^\ell$, $k \neq \ell$.

with similar results obtained[6a,b]. Again the scattering data satis-
fies a nonlinear constraint. In general, there is no compatible local
nonlinear evolution equation associated with (v). However when cer-
tain restrictions are put on J_j then the constraint equation becomes
linear and the so-called N wave interaction equations are compatible
with the system (v). Nachman and Ablowitz[6a] showed that at most, the
system would be $3+1$ dimensional, and Fokas[6b] showed that indeed the
system is reducible to $2+1$ dimensions by a transformation of independ-
ent variables (characteristic variables).

Beals and Coifman have given an alternative but similar formula-
tion[7a,b] in the scalar case.

There is an n-dimensional problem which also fits within
the framework of IST: The so-called generalized wave and generalized
sine-Gordon equation (GWE and GSGE). These equations arise in the
context of differential geometry and serve to extend the classical re-
sults of Bäcklund for the sine-Gordon equation to n-dimensions[8].
The n-dimensional Bäcklund tranformation is given by:

$$dX + XA^tX = A - XB, \qquad (17)$$

where

$$dX = \sum_{j=1}^{n} \frac{\partial X}{\partial x_j} dx_j,$$

$$A_{ij} = \theta_i(z) a_{ij} dx_j,$$

$$B_{ij} = \frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i} dx_j - \frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j} dx_i, \quad 1 \leq i, j \leq n, \quad (18)$$

and $a = (a_{ij}) \in R^{n \times n}$. Equations (17-18) reduce to the Bäcklund trans-
formation for the generalized sine-Gordon equation (GSGE) when

$$\theta_i(z) = (z^2 + (2\delta_{i1} - 1))/2z, \qquad (19)$$

and for the generalized wave equation (GWE) when

$$\theta_i(z) = -(1-z^2)/2z \equiv \lambda(z). \qquad (20)$$

The compatibility condition required for the existence of solu-
tions to these Backlund transformations results in a system of second-.

order partial differential equations for an orthogonal $n \times n$ matrix $a = \{a_{ij}\}$ in (17) which is a function of n independent variables $a = a(x_1, x_2, \ldots, x_n)$. The equation has the form

$$\frac{\partial}{\partial x_i} \left(\frac{1}{a_{1i}} \frac{\partial a_{1i}}{\partial x_i} \right) + \frac{\partial}{\partial x_j} \left(\frac{1}{a_{1k}} \frac{\partial a_{1i}}{\partial x_j} \right)$$

$$+ \sum_{k \neq i,j} \frac{1}{a_{1k}^2} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1j}}{\partial x_k} = \kappa a_{1i} a_{1j}, \quad i \neq j,$$

$$\frac{\partial}{\partial x_k} \left(\frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j} \right) = \frac{1}{a_{1k} a_{1j}} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1k}}{\partial x_j}, \quad i, j, k \text{ distinct,}$$

$$\frac{\partial a_{1k}}{\partial x_k} = \frac{\partial a_{ji}}{\partial a_{1i}} \frac{\partial a_{1k}}{\partial x_i}, \quad i \neq k, \quad (21)$$

where $\epsilon = 1$ for the GSGE and $\epsilon = 0$ for the GWE.

We observe that when $n = 2$ and $\epsilon = 1$ (GSGE), the orthogonal matrix $a = \{a_{ij}\}$ given by

$$a = \begin{pmatrix} \cos \frac{1}{2} u & \sin \frac{1}{2} u \\ -\sin \frac{1}{2} u & \cos \frac{1}{2} u \end{pmatrix} \quad (22)$$

for the function $u = u(x,t)$ reduces the GSGE to the classical sine-Gordon equation ($\kappa = -1$),

$$u_{tt} - u_{xx} - \kappa \sin u = 0. \quad (23)$$

On the other hand when $n = 2$ and $\kappa = 0$, then with (22) the GWE reduces to the wave equation (23). When $n \geq 3$ the generalization of the wave equations discussed here is nonlinear.

The Bäcklund transformations (17) described above are in fact matrix Riccati equations. Linearizations of such a system can be performed in a striaghtforward manner (see for example [9]). Introducing the transformation

$$x = UV^{-1}, \quad (24)$$

where U, V and $n \times n$ matrix functions of $x_1 \ldots x_n$, the following linear

system is deduced:

$$\begin{pmatrix} dU \\ dV \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{25}$$

with the components of A, B given by (18). Compatibility ensures that the orthogonal matrix $a = \{a_{ij}\}$ satisfies the GSGE with (19) and GwE with (20). Alternatively, if we call

$$\begin{pmatrix} U \\ V \end{pmatrix} = \psi .$$

the following linear system of $2n$ o.d.e.'s are obtained:

$$\frac{\partial \psi}{\partial x_j} = \lambda \tilde{A}_j \psi + C_j \psi. \tag{26}$$

where $\tilde{A}_j$, C_j are $2n \times 2n$ matrices with the block structure

$$\tilde{A}_j = \begin{pmatrix} 0 & \tilde{a}_j \\ \tilde{a}_j^t & 0 \end{pmatrix} , \quad C_j = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix} . \tag{27}$$

Here $\tilde{a}_j$, $\tilde{\gamma}_j$ are $n \times n$ matrices having the following structure:

$$\tilde{a}_j = (\frac{\delta}{\lambda} - 1)e_i a_j + a_j ,$$

$$a_j = a e_j \tag{28}$$

where $e_j = \{e_j\}_{ik}$ is the unit matrix

$$\{e_j\}_{ik} = \begin{cases} 1 & i = k = j, \\ 0 & \text{otherwise}, \end{cases} \tag{29}$$

and in component form γ_j takes the form

$$(\gamma_j)_{k\ell} = (1-\delta_{kj})\frac{1}{a_{\ell k}} \frac{\partial a_{\ell j}}{\partial x_k} \delta_{\ell j} - (1-\delta_{\ell j})\frac{1}{a_{\mu}} \frac{\partial a_{\ell j}}{\partial x_\ell} \delta_{kj} . \tag{30}$$

In (28) a is the orthogonal matrix $R^n \to SO(n)$ associated with the GwE when $\delta = \lambda$ and with the GSGE when $\delta = \frac{1}{2}(z + 1/z)$, $\lambda = \frac{1}{2}(z - 1/z)$, and γ_j is the matrix (30): $R_n \to M_n(R)$, $\gamma_j + \gamma_j^t = 0$. Equations (21) arise as the compatibility condition associated with (26). More explicitly, for the GwE the scattering problem takes the form $[\psi = \psi(x,\lambda)]$

$$\frac{\partial \psi}{\partial x_j} = \lambda A_j \psi + C_j \psi \tag{31}$$

with

$$A_j = \begin{pmatrix} 0 & a_j \\ a_j^t & 0 \end{pmatrix}.$$ (32)

and C_j given by (27,30).

For the GSGE the scattering problem for $\psi = \psi(x,z)$ takes the form

$$\frac{\partial \psi}{\partial x_j} = \delta(z) \begin{pmatrix} 0 & e_1 a_j \\ a_j^t e_1 & 0 \end{pmatrix} \psi$$

$$+ \lambda(z) \begin{pmatrix} 0 & (1-e_1)a_j \\ a_j^t(1-e_1) & 0 \end{pmatrix} \psi + C_j \psi$$ (33)

$\delta(z)$, $\lambda(z)$, C_j given above, or equivalently

$$\frac{\partial \psi}{\partial x_j} = \frac{z}{2} A_j \psi + \frac{2}{z} B_j \psi + C_j \psi.$$ (34)

where

$$B_j = \begin{pmatrix} 0 & ua_j \\ a_j^t u & 0 \end{pmatrix}, \quad u = \text{diag}(+1, -1, \ldots, -1).$$ (35)

In[8] it is shown how these linear problems may be viewed as a direct and inverse scattering problem for the GWE and GSGE. Namely the direct and inverse problem may be solved for matrix potentials, depending on the orthogonal matrix a, tending to the identity sufficiently fast in certain "generic" directions. It should be noted that solving the n-dimensional GWE and GSGE reduces to the study of the scattering and inverse scattering associated with a coupled system of n one-dimensional o.d.e.'s. This is in marked contrast to other attempts described earlier to isolate solvable (local) multidimensional nonlinear evolution equation which are compatibility conditions of two Lax-type operators, e.g.,

$$L\psi = \lambda\psi$$ (36)

$$\psi_t = M\psi$$ (37)

where L is a partial differential operator with the variable t entering only parametrically. Although as we have seen nonlinear evolution

equations in three independent variables can be associated with such Lax pairs (e.g. the K-P, Davey-Stewartson, three wave interaction equations, etc.) little progress via this route has been made in more than three dimensions. As discussed earlier one has to overcome a serious constraint inherent in the scattering/inverse scattering theory for higher dimensional partial differential operators in order to be able to isolate associated solvable nonlinear equations, i.e. the scattering data generally satisfies a nonlinear equation (e.g. (16)). The analysis associated with the GWE and GSGE avoids these difficulties since the GWE and GSGE problems are simply a compatible set of nonlinear one-dimensional o.d.e.'s. The results in [8] demonstrate that the initial value problem is posed with given data along lines and not on $(n-1)$ dimensional manifolds.

Similar ideas apply to certain n-dimensional extensions of the so-called anti-self-dual Yang-Mills equations (SDYM). In[9] it is shown that these multi-dimensional nonlinear equations are associated with compatible two-dimensional linear systems. Broad classes of solutions may be calculated by the $\bar{\partial}$ method. Since the overall compatible linear systems are coupled two-dimensional equations, the scattering data does not satisfy the nonlinear constraint discussed earlier.

Finally we remark that there is a class of nonlocal equations which can be reduced to exactly solvable equations. In the context of multidimensional nonlinear equations perhaps the most interesting example is

$$(u_t + u_{xxx} + 2(uH_z u)_x)_x = -3c^2 u_{yy}, \qquad (38)$$

where

$$(H_z u)(x,y,z,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x,y,\xi,t)}{\xi - z} \, d\xi \qquad (39)$$

and $\int_{-\infty}^{\infty}$ denotes the Cauchy principal value integral. (38) is reduced to the K-P equation

$$(w_t + w_{xxx} - 1(w^2)_x)_x = - 3c^2 w_{yy} \qquad (40)$$

via the transformation

$$w = u + iH_z u. \qquad (41)$$

Details and other examples are given in [10].

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