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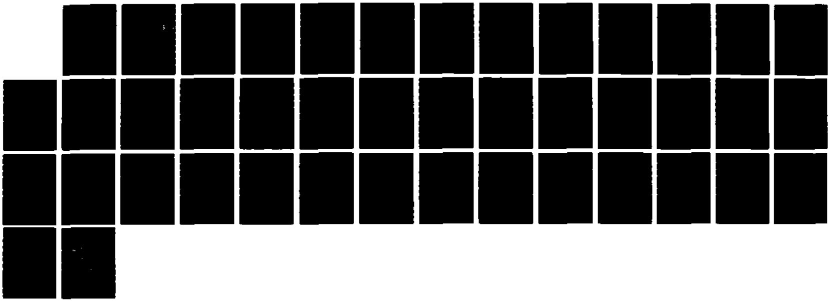

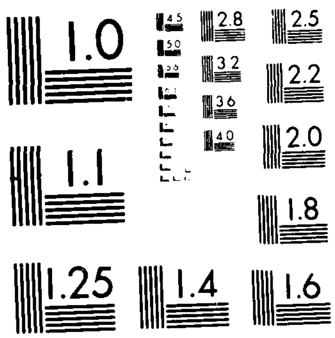
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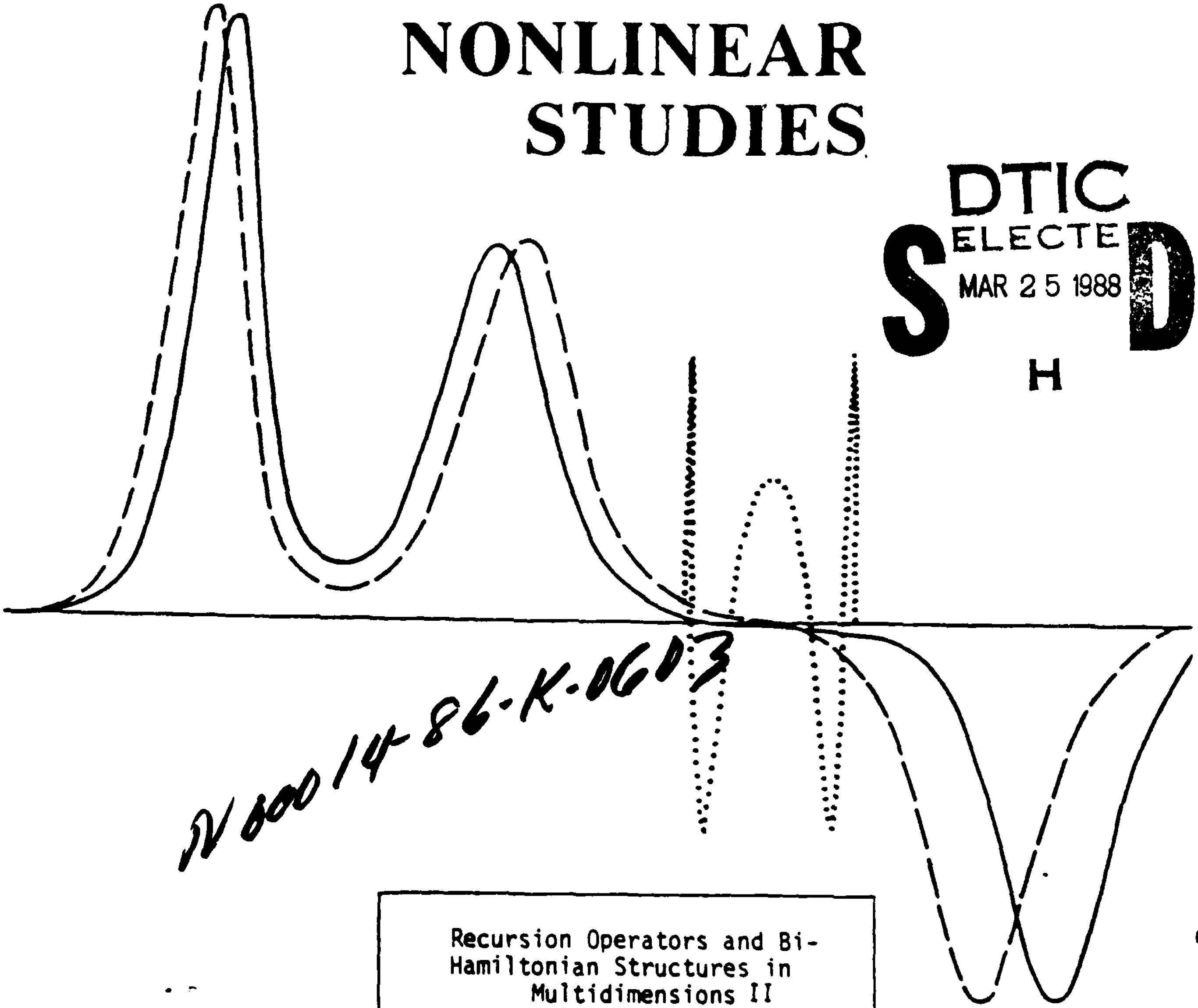

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Recursion Operators and Bi-Hamiltonian Structures in Multidimensions II

by

A. S. Fokas and P. M. Santini+

July 1986



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# RECURSION OPERATORS AND BI-HAMILTONIAN STRUCTURES IN MULTIDIMENSIONS II

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## ABSTRACT

We analyze further the algebraic properties of bi-Hamiltonian systems in two spatial and one temporal dimensions. By utilizing the Lie algebra of certain basic (starting) symmetry operators we show that these equations possess infinitely many time dependent symmetries and constants of motion. The master symmetries $\tau$ for these equations are simply derived within our formalism. Furthermore, certain new functions $T_{12}$ are introduced, which algorithmically imply recursion operators $\Phi_{12}$. Finally the theory presented here and in a previous paper is both motivated and verified by regarding multidimensional equations as certain sngular limits of equations in one spatial dimension.

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Recursion Operators and Bi-Hamiltonian

Structures in Multidimensions II

by

A.S. Fokas and P.M. Santini

## I. INTRODUCTION

This paper investigates certain algebraic aspects of exactly solvable evolution equations in 2+1 (i.e. in two spatial and in one temporal dimensions). It is a continuation of [1], although it can be read independently.

We consider evolution equations in the form

$$q_t = K(q), \tag{1.1}$$

where $q(x,y,t)$ is an element of a suitable space S of functions vanishing rapidly for large x,y. Let K be a differentiable map on this space and assume that it does not depend explicitly on x,y,t. If equation (1.1) is integrable then it belongs to some hierarchy (generated by a recursion operator $\Phi_{12}$), hence in association with (1.1) we shall study $q_t = K^{(n)}(q)$. Fundamental in our theory is to write these equations in the form

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} \Phi_{12}^n \hat{K}_{12}^0 \cdot 1 \doteq \int_{\mathbb{R}} dy_2 \delta_{12} K_{12}^{(n)} = K_{11}^{(n)}, \tag{1.2}$$

where $\delta_{12} = \delta(y_1 - y_2)$ denotes the Dirac delta function $q_i \doteq q(x,y_i,t)$, $i = 1,2$, $K_{12}^{(n)}(q_1,q_2)$ belong to a suitably extended space $\tilde{S}$, $\Phi_{12}$, $\hat{K}_{12}^0$ are operator valued functions in $\tilde{S}$. If q is a matrix function then 1 is replaced by the identity matrix.

Throughout this paper m and n are non-negative integers.

The following results were obtained in [1]:

i) There is an algorithmic approach for obtaining the recursion operator $\Phi_{12}$ from the associated isospectral eigenvalue problem. ii) This operator is hereditary. iii) Each member of the hierarchy $(\Phi_{12}^m \hat{K}_{12}^0 \cdot 1)_{11} \doteq \int_{\mathbb{R}} dy_2 \delta_{12} \Phi_{12}^m \hat{K}_{12}^0 \cdot 1$, where $K_{12}^0 \cdot 1$ is a starting symmetry, is a symmetry (1.2). For example the Kadomtsev-Petviashvili (KP) equation and the Davey-Stewartson (DS) equation admit two such hierarchies of commuting symmetries. iv) If the hereditary operator admits a factorization in terms of two Hamiltonian operators, then hierarchies of commuting symmetries give rise to hierarchies of constants of motion in involution with respect to two different Poisson brackets. For example, the KP and the DS equations admit two such hierarchies of conserved quantities.

The above results extend the theory of [2] - [4] to equations in 2+1. Novel aspects of the theory in 2+1 include: i) The role of the Frechét derivative is now played by a certain directional derivative. If subscripts f and d denote these derivatives then there is a simple relationship between directional and total Frechét derivatives:

$$K_{12_d}[\delta_{12} F_{12}] = K_{12_f}[F] \doteq K_{12_{q_1}}[F_{11}] + K_{12_{q_2}}[F_{22}], \qquad (1.3a)$$

where $K_{12}$ is an arbitrary function in $\tilde{S}$, and $K_{12_{q_i}}$ denotes the Frechét derivative of $K_{12}$ with respect to $q_i$, i.e.

$$K_{12_{q_i}}[F_{ii}] \doteq \frac{\partial}{\partial \varepsilon} K_{12}(q_i + \varepsilon F_{ii}, q_j) \Big|_{\varepsilon=0}, \quad i,j = 1,2, \quad i \neq j. \qquad (1.3b)$$

Operators on which directional derivatives are defined are called admissible [1] (applications of the d-derivative in explicit examples can be found in Appendix A, see also Appendix C of [1]).

ii) The starting symmetry $k_{12}^0$ can be written as $\hat{K}_{12}^0 \cdot 1$, where $\hat{K}_{12}^0$ is an admissible operator. Essential to our theory is that the operators $\hat{K}_{12}^0$, acting on suitable functions $H_{12}$, form a Lie algebra.

1. For the equations associated with the KP equation,

$$\Phi_{12} = D^2+q_{12}^+ + Dq_{12}^+D^{-1}+q_{12}^-D^{-1}q_{12}^-D^{-1}, \quad q_{12}^\pm \doteq q_1 \pm q_2+\alpha(D_1 \mp D_2), \tag{1.4}$$

where $D_i \doteq \frac{\partial}{\partial y_i}$. The starting operators $\hat{K}_{12}^0$ are given by

$$\hat{N}_{12} \doteq q_{12}^-, \quad \hat{M}_{12} \doteq Dq_{12}^+ + q_{12}^-D^{-1}q_{12}^-, \tag{1.5}$$

and $H_{12}$ is an arbitrary function independent of x, i.e.

$$H_{12} = H_{12}(y_1,y_2). \tag{1.6}$$

The Lie algebra of $\hat{K}_{12}^0$ is given by

$$[\hat{N}_{12}H_{12}^{(1)}, \hat{N}_{12}H_{12}^{(2)}]_d = -\hat{N}_{12}H_{12}^{(3)}, \; [\hat{N}_{12}H_{12}^{(1)}, \hat{M}_{12}H_{12}^{(2)}]_d = -\hat{M}_{12}H_{12}^{(3)},$$

$$[\hat{M}_{12}H_{12}^{(1)}, \hat{M}_{12}H_{12}^{(2)}]_d = -\Phi_{12}\hat{N}_{12}H_{12}^{(3)}, \tag{1.7}$$

where

$$[K_{12}^{(1)},K_{12}^{(2)}]_d = K_{12_d}^{(1)}[K_{12}^{(2)}] - K_{12_d}^{(2)}[K_{12}^{(1)}], \tag{1.8}$$

$$H_{12}^{(3)} \doteq [H_{12}^{(1)},H_{12}^{(2)}]_I \doteq \int_{\mathbb{R}} dy_3(H_{13}^{(1)}H_{32}^{(2)} - H_{13}^{(2)}H_{32}^{(1)}). \tag{1.9}$$

2. For the equations associated with the DS equation

$$\Phi_{12} = \sigma(P_{12} - Q_{12}^+P_{12}^{-1}Q_{12}^+), \quad Q_{12}^\pm F_{12} \doteq Q_1F_{12} \pm F_{12}Q_2,$$
$$P_{12}F_{12} \doteq F_{12_x} - JF_{12_{y_1}} - F_{12_{y_2}}J, \tag{1.10}$$

where $J = \alpha\sigma$, $\sigma$=diag(1,-1), Q is a 2 x 2 off-diagonal matrix containing the potentials $q_1(x,y)$, $q_2(x,y)$ and $\Phi_{12}$ is defined on off-diagonal matrices. The starting operators $\hat{K}_{12}^0$ are given by:

$$\hat{N}_{12} \doteq Q_{12}^{-}, \quad \hat{M}_{12} \doteq Q_{12}^{-}\sigma, \tag{1.11}$$

and $H_{12}$ is an arbitrary matrix function satisfying the following properties:

$$H_{12} \text{ diagonal matrix}, \; P_{12}H_{12} = 0. \tag{1.12}$$

Also

$$\begin{gathered}[\hat{N}_{12}H_{12}^{(1)}, \hat{N}_{12}H_{12}^{(2)}]_d = -\hat{N}_{12}H_{12}^{(3)}, \quad [\hat{N}_{12}H_{12}^{(1)}, \hat{M}_{12}H_{12}^{(2)}]_d = -\hat{M}_{12}H_{12}^{(3)}, \\ [\hat{M}_{12}H_{12}^{(1)}, \hat{M}_{12}H_{12}^{(2)}]_d = -\hat{N}_{12}H_{12}^{(3)}.\end{gathered} \tag{1.13}$$

iii) The recursion operator $\Phi_{12}$ is admissible and enjoys a simple commutation operator relation with $h_{12} = h(y_1 - y_2)$:

$$[\Phi_{12}, h_{12}] = -\beta h'_{12}, \quad h'_{12} \doteq \frac{\partial h_{12}}{\partial y_1}, \tag{1.14}$$

*which implies that* $\delta_{12}K_{12}^{(n)} = \delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0} \cdot 1 = \sum_{\ell=0}^{n} \beta^{\ell}\binom{n}{\ell}\Phi_{12}^{n-\ell}\delta_{12}^{\ell}\hat{K}_{12}^{0} \cdot 1$, where $\delta_{12}^{\ell} \doteq \partial^{\ell}\delta_{12}/\partial y_1^{\ell}$.

The starting operator $\hat{K}_{12}^{0}$ is also admissible and its commutator relation with $h_{12}$ implies that $\delta_{12}K_{12}^{(n)}$ can be written in the form

$$\delta_{12}K_{12}^{(n)} = \delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0} \cdot 1 = \sum_{\ell=1}^{n} b_{n,\ell}\Phi_{12}^{n-\ell}\hat{K}_{12}^{0} \cdot \delta_{12}^{\ell} \tag{1.15}$$

for suitable constants $b_{n,\ell}$.

1. For the two classes of evolution equations associated with the KP equation we have that

$$\beta = -4\alpha, \; [\hat{N}_{12}, h_{12}] = 0, \quad [\hat{M}_{12}, h_{12}] = -\tilde{\beta}Dh'_{12}, \; \tilde{\beta} = \bar{\beta}/2, \tag{1.16}$$

and

$$b_{n,\ell} = \begin{cases} \beta^{\ell}\binom{n}{\ell}, & \text{for } \hat{K}_{12}^{0} = \hat{N}_{12} \\ \sum_{s=0}^{\ell} \beta^{\ell-s}\tilde{\beta}^{s}\binom{n-s}{\ell-s}, & \text{for } \hat{K}_{12}^{0} = \hat{M}_{12}. \end{cases} \tag{1.17}$$

2. For the two classes of evolution equations associated with the DS

equation we have that

$$\beta = 2\alpha \quad , [\hat{N}_{12},h_{12}] = [\hat{M}_{12},h_{12}] = 0 \tag{1.18}$$

and

$$b_{n,\ell} = \beta^{\ell}\binom{n}{\ell} . \tag{1.19}$$

In [1] we assume knowledge of the underlying isospectral problem. This problem implies: a) a hereditary operator $\Phi_{12}$; b) suitable starting operators, say $\hat{M}_{12}$ and $\hat{N}_{12}$, and functions $H_{12}$; c) two skew symmetric operators such that $\Phi_{12} = \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$. Furthermore, it can be shown that $\Phi_{12}$ is a strong symmetry for the starting symmetries. One then needs to: a) Find $\beta$ and $b_{n,\ell}$ appearing in equations (1.14), (1.15). b) Compute the Lie algebras of $\hat{M}_{12}$, $\hat{N}_{12}$ on function $H_{12}$ (i.e. obtain equations analogous to (1.7), (1.13). c) Verify that the starting symmetries correspond to extended gradients, i.e. verify that $((\Theta_{12}^{(1)})^{-1}\hat{K}_{12}^{0}\cdot H_{12})_d$, $\hat{K}_{12}^{0} = \hat{M}_{12}$ or $\hat{N}_{12}$, is symmetric with respect to the bilinear form

$$\langle g_{12},f_{12}\rangle \doteq \int_{\mathbb{R}^3} dx\, dy_1 dy_2 \quad \text{trace} \quad g_{21}f_{12}. \tag{1.20}$$

d) Verify that $\Theta_{12}^{(1)}$, $\Theta_{12}^{(2)}$ are compatible Hamiltonian operators.

In this paper the following results are presented.

i) In §2 we investigate further the Lie algebra of the starting symmetries $\hat{K}_{12}^{0}H_{12}$. In [1] we only used a subclass of solutions of (1.6) and (1.12), given by $H_{12}=h_{12}=h(y_1-y_2)$ and $H_{12}=h_{12}(aI + b\sigma)$, a, b, constants, respectively. This gave rise to time-independent commuting symmetries. We now choose $H_{12}$ to be a more general solution of the above equations; this gives rise to time dependent symmetries.

ii) In §3, using the Lie algebra of $\hat{K}_{12}^{0},H_{12}$ and an isomorphism between Lie and Poisson brackets we prove directly that $\Phi_{12}^{n}\hat{K}_{12}^{0}H_{12}$ correspond to conserved quantities. This derivation, which capitalizes on the arbitrariness of $H_{12}$, has the advantage that does not use the bi-Hamiltonian factorization of $\Phi_{12}$. In other words, for the theory developed in this paper one needs only to verify a)-c) above.

We recall that Fuchssteiner and one of the authors (ASF) introduced an alternative way for generating symmetries, the so called master-symmetry approach. A master-symmetry is a function $\tau$ which has the property that its Lie commutator with a symmetry is also a symmetry. The $\tau$ functions for the Benjamin-Ono and the KP equations were given in [5] and [6-7] respectively. Several a ors (E.g. [8]-[12]) have noticed that master-symmetries also exist for equations in I as well as for finite dimensional systems [13]. Let $\tau$ and T denote mastery-symmetries for equations in 2+1 and 1+1 respectively. If $\Phi$ is the recursion operator and $\Sigma = tK + T_0$ is the scaling symmetry of an equation in 1+1, $q_t = K$, then $T = \Phi T_0$ is a master symmetry. However, there exists a fundamental difference between $\tau$ and T. The function $\Theta^{-1}T$ ($\Theta$ is a Hamiltonian operator) is not a gradient function; this can be used to constructively obtain $\Phi$ from T. But $\Theta^{-1}\tau$ is a gradient and hence the above construction of $\Phi$ from $\tau$ fails.

In this paper we show that $\tau$ is not the proper analogue of T. Let us consider the KP for concreteness. As it was mentioned earlier, $\Phi_{12}^n \hat{K}_{12}\cdot 1$ generate time-independent symmetries; it will be shown here that $\Phi_{12}^n \hat{K}_{12}(y_1+y_2)^m$ generates time dependent symmetries. It turns out that $\tau = (\Phi_{12}^2 \hat{K}_{12}(y_1 + y_2))_{1\bar{1}}$ (see §IID). But $\Theta_{12}^{-1}\Phi_{12}^n \hat{K}_{12}H_{12}$ is an extended gradient for all $H_{12}$, hence $\Theta^{-1}\tau$ is a gradient function. In §4 we show that the proper analogue of T for the KP is $T_{12} \doteq \Phi_{12}^2 \delta_{12}$ (it corresponds to $\Phi^2\cdot 1$ for the KdV). Actually, $\Theta_{12}^{-1}T_{12}$ is not an extended gradient and it can be used to constructively obtain $\Phi_{12}$.

In §5 we show that exactly solvable 2+1 dimensional equations are exact reductions of nonlocal evolution equations generated via nonlocal isospectral eigenvalue problems. This result both motivates the basic ideas and concepts introduced in [1] and in this paper, as well as verifies several results presented in the above papers.

## II. A LIE-ALGEBRA FOR EQUATIONS IN 2+1

In developing a theory for time-dependent symmetries in 2+1 it is useful first to: i) characterize the commutator properties of these symmetries, ii) study the action of $\Phi$ on the Lie commutator $[a,b]_L$, where

$$[a,b]_L \doteq a_L[b] - b_L[a], \tag{2.1}$$

and $a_L$ denotes an appropriate derivative. This derivative is linear and satisfies the Leibnitz rule. For equations in 1+1 one only needs $[a,b]_f$, while for equations in 2+1 one also needs $[a_{12},b_{12}]_d$ (see (1.3)).

**Lemma 2.1**

$\sigma^{(r)}$ is a time dependent symmetry of order r of the equation $q_t = K$, i.e.

$$\frac{\partial\sigma^{(r)}}{\partial t} + [\sigma^{(r)},K]_L = 0, \tag{2.2}$$

iff

$$\sigma^{(r)} = \sum_{j=0}^{r} t^j \Sigma^{(j)}, \quad \Sigma^{(j)} \doteq -\frac{1}{j}[\Sigma^{(j-1)},K]_L, \quad j=1,\ldots r, \quad [K,\Sigma^{(r)}]_L = 0. \tag{2.3}$$

The above result follows from the definition of a symmetry and the assumption that $\Sigma^{(j)}$ is time independent. It implies that constructing a symmetry of order $\ell$ is equivalent to finding a function $\Sigma^{(0)}$ with the property that its $(\ell+1)^{st}$ commutator with K is zero.

The action of a hereditary operator $\Phi$ on a Lie commutator is given by:

Theorem 2.1

Let

$$S \neq \Phi_L[K] + [\Phi, K_L] . \tag{2.4}$$

Then

a$_1$) $$\Phi^n[K_1,K_2]_L = [K_1,\Phi^n K_2]_L + (\sum_{r=1}^{n} \Phi^{n-r} S_1 \Phi^{r-1})K_2. \tag{2.5}$$

If $\Phi$ is hereditary, i.e. if

$$\Phi_L[\Phi v]w - \Phi\Phi_L[v]w \text{ is symmetric w.r.t. } v,w \tag{2.6}$$

then the following are true

a$_2$) $$\Phi_L[\Phi^n K] + [\Phi,(\Phi^n K)_L] = \Phi^n S, \tag{2.7}$$

a$_3$) $$\Phi^{n+m}[K_1,K_2]_L = [\Phi^n K_1,\Phi^m K_2]_L + \Phi^n(\sum_{r=1}^{m} \Phi^{m-r} S_1 \Phi^{r-1})K_2 - \Phi^m(\sum_{r=1}^{n} \Phi^{n-r} S_2 \Phi^{r-1})K_1 . \tag{2.8}$$

(m,n are non-negative integers).

Proof.

To prove (2.5) use induction: $(2.5)_0$ is an identity. Applying $\Phi$ on $(2.5)_n$ we obtain

$$\Phi^{n+1}[K_1,K_2]_L = \Phi[K_1,\Phi^n K_2]_L + \Phi(\sum_{r=1}^{n} \Phi^{n-r} S_1 \Phi^{r-1})K_2.$$

Equation $(2.5)_{n+1}$ follows from the above and the following identity

$$\Phi[K_1, M]_L = [K_1,\Phi M]_L + S_1 M.$$

Equation (2.7) also follows from induction. To prove (2.8) first note that

(2.5) implies

$$\phi^m[K_1,K_2]_L - (\sum_{r=1}^{m}\phi^{m-r}S_1\phi^{r-1})K_2 = [K_1,\phi^m K_2]_L. \tag{2.9}$$

Equation (2.5) also implies

$$\phi^n[K_1,\tilde{K}_2]_L = [\phi^n K_1,\tilde{K}_2]_L - (\sum_{r=1}^{n}\phi^{n-r}\tilde{S}_2\phi^{r-1})K_1.$$

Let $\tilde{K}_2 = \phi^m K_2$, then (2.6) implies $\tilde{S}_2 = \phi^m S_2$, and the above equation becomes

$$\phi^n[K_1,\phi^m K_2]_L = [\phi^n K_1,\phi^m K_2]_L - (\sum_{r=1}^{n}\phi^{n-r}\phi^m S_2\phi^{r-1})K_1.$$

Applying $\phi^n$ on (2.9) and using the above we obtain (2.8).

Corollary 2.1

Let the hereditary operator $\phi$ be a strong symmetry for both $K_1$ and $K_2$, i.e. $S_1 = S_2 = 0$. Then

$$\phi^{n+m}[K_1,K_2]_L = [\phi^n K_1,\phi^m K_2]_L. \tag{2.10}$$

In the rest of this section we characterize extended symmetries $\sigma_{12}$. The following theorem, proven in [1], maps extended symmetries $\sigma_{12}$ to symmetries $\sigma_{11}$.

Theorem 2.2

Assume that the commutor of $\phi_{12}$ with $h_{12}$ is given by (1.14) and that the starting operator $\hat{K}^0_{12}$ are such that (1.15) is valid. If $\sigma_{12}$ is an extended symmetry of (1.2), i.e. if

$$\frac{\partial\sigma_{12}}{\partial t} + [\sigma_{12},\delta_{12}\phi^n_{12}\hat{K}^0_{12}\cdot 1]_d = 0, \tag{2.11}$$

then $\sigma_{11}$ is a symmetry of (1.2), i.e.

$$\frac{\partial\sigma_{11}}{\partial t} + [\sigma_{11},K^{(n)}_{11}]_f = 0. \tag{2.12}$$

In the above

$$[\sigma_{11},K_{11}^{(n)}]_f = \sigma_{11_{q_1}}[K_{11}^{(n)}] - K_{11_{q_1}}^{(n)}[\sigma_{11}], \tag{2.13}$$

and

$$[\sigma_{12},\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1]_d = \sum_{\ell=0}^{n} b_{n,\ell}[\sigma_{12},\Phi_{12}^{n-\ell}\hat{K}_{12}^{0}\delta_{12}^{\ell}]_d. \tag{2.14}$$

It is necessary to rewrite $\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1$ in the form appearing in (2.14) since the directional derivative is defined only for functions of the form $\hat{L}_{12}H_{12}$, where $\hat{L}_{12}$ is an admissible operator.

Using Lemma 2.1, Corollary 2.1 and the Lie algebra of $\hat{K}_{12}^{0}H_{12}$ (with appropriate $H_{12}$) we obtain extended symmetries, which then via Theorem 2.2 give rise to symmetries.

<u>Proposition 2.1</u>

Assume that the hereditary operator $\Phi_{12}$ is a strong symmetry for the admissit starting operators $\hat{M}_{12},\hat{N}_{12}$, and that (1.14), (1.15) hold. Further assume that $\hat{M}_{12}$, $\hat{N}_{12}$ form a Lie algebra (analogous to (1.7), (1.13)). Consider the following hierarch

$$q_{1_t} = \int_{\mathbf{R}} dy_2\delta_{12}\Phi_{12}^{n}\hat{N}_{12}\cdot 1 = \int_{\mathbf{R}} dy_2\delta_{12}N_{12}^{(n)} = N_{11}^{(n)}, \tag{2.15a}$$

$$q_{1_t} = \int_{\mathbf{R}} dy_2\delta_{12}\Phi_{12}^{n}\hat{M}_{12}\cdot 1 = \int_{\mathbf{R}} dy_2\delta_{12}M_{12}^{(n)} = M_{11}^{(n)}, \tag{2.15b}$$

Then:

a) $(\Phi_{12}^{m}\hat{M}_{12}\cdot 1)_{11}$, $(\Phi_{12}^{m}\hat{N}_{12}\cdot 1)_{11}$, are symmetries of equations (2.15).

b) Appropriate linear combinations of $\{\Phi_{12}^{m}\hat{M}_{12}H_{12}^{(r)}\}_{11}$, $\{\Phi_{12}^{m}\hat{N}_{12}H_{12}^{(r)}\}_{11}$ for suitable functions $H_{12}^{(r)}$ generate time dependent symmetries for equations (2.15).

Rather than proving the above proposition in general, we use for concreteness, the Lie algebra (1.6) to sketch how the above results can be derived. Details are given in II.A, II.B.

Let

$$\hat{N}_{12}^{(n)} \doteq \Phi^n \hat{N}_{12}, \quad \hat{M}_{12}^{(n)} \doteq \Phi^n \hat{M}_{12} . \tag{2.16}$$

Then, using corollary 2.1, equations (1.7) imply

$$[\hat{N}_{12}^{(m)} H_{12}^{(1)}, \hat{N}_{12}^{(n-\ell)} H_{12}^{(2)}]_d = -\hat{N}^{(m+n-\ell)} H_{12}^{(3)}, \quad [\hat{N}_{12}^{(m)} H_{12}^{(1)}, \hat{M}_{12}^{(n-\ell)} H_{12}^{(2)}]_d = -\hat{M}_{12}^{(m+n-\ell)} H_{12}^{(3)} ,$$

(2.17)

$$[\hat{M}_{12}^{(m)} H_{12}^{(1)}, \hat{N}_{12}^{(n-\ell)} H_{12}^{(2)}]_d = -\hat{M}_{12}^{(m+n-\ell)} H_{12}^{(3)}, \quad [\hat{M}_{12}^{(m)} H_{12}^{(1)}, \hat{M}_{12}^{(n-\ell)} H_{12}^{(2)}]_d = -\hat{N}_{12}^{(m+n+1-\ell)} H_{12}^{(3)} .$$

Part a) of the proposition is a direct consequence of equations (2.17) and (2.14). For example

$$[\hat{N}_{12}^{(m)} \cdot 1, \delta_{12} \hat{N}_{12}^{(n)} \cdot 1]_d = -\sum_{\ell=0}^{n} b_{n,\ell} \hat{N}^{(m+n-\ell)} \cdot \tilde{H}_{12}^{(\ell)} = 0,$$

since $\tilde{H}_{12}^{(\ell)} = [1, \delta_{12}^{\ell}]_I = 0$; thus $\hat{N}_{12}^{(m)} \cdot 1$ are extended symmetries of (2.15a).

Consider part b) of Proposition 2.1. Let us first consider symmetries of order one in t. Then

$$\tilde{N}_{12}^{(m)}(y_1 + y_2) - t2\beta\binom{n}{1}\hat{N}_{12}^{(m+n-1)} \cdot 1$$

$$\tilde{M}_{12}^{(m)}(y_1 + y_2) - t2\beta\binom{n}{1}\hat{M}_{12}^{(m+n-1)} \cdot 1 \tag{2.18}$$

are first order time dependent extended symmetries of (2.15a). Similarly

$$\hat{N}_{12}^{(m)} \cdot (y_1 + y_2) - t2\, b_{n,1}\, \hat{M}_{12}^{(n+m-1)} \cdot 1, \tag{2.19a}$$

$$\hat{M}_{12}^{(m)} \cdot (y_1 + y_2) - t2(b_{n,1})\hat{N}^{(m+n)} \cdot 1, \tag{2.19b}$$

are extended symmetries of (2.15b) with $b_{n,1} = (-4\alpha) \sum_{s=0}^{1} 2^{-s}\binom{n-s}{\ell-s}$.

To derive the above we use Lemma 2.1 and equations (2.17). For example, to derive (2.18) we look for a function $\Sigma_{12}^{(0)}$ such that its commutator with $\delta_{12}\hat{N}_{12}^{(n)} \cdot 1$, commutes with $\delta_{12}\hat{N}_{12}^{(n)} \cdot 1$ . Clearly $\Sigma_{12}^{(0)} = \hat{N}_{12}^{(m)}(y_1 + y_2)$ or

$\hat{M}_{12}^{(m)}(y_1 + y_2)$. For, (2.17a) implies

$$[\hat{N}_{12}^{(m)}(y_1 + y_2),\ \delta_{12}\hat{N}_{12}^{(n)} \cdot 1]_d = 2\beta\binom{n}{1}\hat{N}^{(m+n-1)} \cdot 1,$$

since, $\tilde{H}_{12}^{(\ell)} = [y_1+y_2,\ \delta_{12}^{\ell}]_I = -2\delta_{1,\ell}$, where $\delta_{1,\ell} = 0$ if $\ell \neq 1$ or 1 if $\ell = 1$.

In a similar manner

$$\hat{N}_{12}^{(m)}(y_1 + y_2)^2 - t4\beta\binom{n}{1}\hat{N}_{12}^{(m+n-1)}(y_1+y_2) - t^2 4\beta^2\binom{n}{1}^2\hat{N}_{12}^{(m+2n-2)} \cdot 1$$

$$\hat{M}_{12}^{(m)}(y_1 + y_2)^2 - t4\beta\binom{n}{1}\hat{M}_{12}^{(m+n-1)}(y_1+y_2) + t^2 4\beta^2\binom{n}{1}^2\hat{M}_{12}^{(m+2n-2)} \cdot 1 \qquad (2.20)$$

are second order time dependent extended symmetries of (2.15b). Similarly

$$\hat{N}^{(m)} \cdot (y_1 + y_2)^2 - t4b_{n,1}\hat{M}^{(m+n-1)} \cdot (y_1+y_2) + t^2 4b_{n,1}^2\hat{N}^{(m+2n-1)} \cdot 1, \qquad (2.21a)$$

$$\hat{M}^{(m)} \cdot (y_1 + y_2)^2 - t4b_{n,1}\hat{N}^{(m+n)} \cdot (y_1+y_2) + t^2 4b_{n,1}^2\hat{M}^{(m+2n-1)} \cdot 1,$$

$$b_{n,1} = (-4\alpha)(n + \tfrac{1}{2}), \qquad (2.21b)$$

are extended symmetries of (2.15b). Indeed

$$[\hat{N}_{12}^{(m)}(y_1 + y_2)^2,\ \delta_{12}\hat{N}_{12}^{(n)} \cdot 1]_d = 4\beta\binom{n}{1}\hat{N}_{12}^{(m+n-1)}(y_1+y_2),$$

since, $[(y_1+y_2)^2, \delta_{12}^{\ell}] = -4(y_1 + y_2)\delta_{1,\ell}$. Also

$$[\hat{N}_{12}^{(m+n-1)}(y_1+y_2),\ \delta_{12}\hat{N}_{12}^{(n)} \cdot 1] = 2\beta\binom{n}{1}\hat{N}_{12}^{(m+2n-2)} \cdot 1 \ .$$

The extension of the above results to any order in time is straightforward: To generate $\sigma_{12}^{(r)}$ consider $\Sigma_{12}^{(0)} = \hat{N}_{12}^{(m)}(y_1+y_2)^r$ or $\hat{M}_{12}^{(m)}(y_1+y_2)^r$. The commutator of $(y_1+y_2)^r$ with $\delta_{12}^{\ell}$ produces $(y_1 + y_2)^{r-1}$. Thus the $r^{th}$ commutator of $(y_1+y_2)^r$ with $\delta_{12}^{\ell}$ produces 1 which commutes with $\delta_{12}^{(\ell)}$; hence Lemma 2.1 guarantees the existence of an $r^{th}$ order symmetry.

## II.A. Time Dependent Symmetries for the Equations Associated with the KP Equation.

Following the construction and the argument sketched above, extended *symmetries of order r in time*

$$\sigma_{12}^{(r)} = \sum_{j=0}^{r} t^j \Sigma_{12}^{(j)} \tag{2.22}$$

are generated through Proposition 2.1, starting with $\Sigma_{12}^{(0)} = \hat{N}_{12}^{(m)} \cdot H_{12}^{(r)}$ or $\hat{M}_{12}^{(m)} \cdot H_{12}^{(r)}$, where $H_{12}^{(r)}$, is defined by

$$H_{12}^{(r)} \doteq (y_1 + y_2)^r; \tag{2.23}$$

more generally, any homogeneous polynomial of degree r in $y_1$ and $y_2$ could be used as well (note $H_{12}^r$ solves (1.6)). Using

$$[H_{12}^{(r)}, \delta_{12}^s]_I = -(1 - (-1)^s)\theta(r-s)\frac{r!}{(r-s)!} H_{12}^{(r-s)}, \tag{2.24}$$

$$\theta(a) = \begin{cases} 1, & a \geq 0, \\ 0, & a < 0, \end{cases} \tag{2.25}$$

we can show that

i) The class of evolution equations (2.15a) with $\hat{N}_{12} = q_{12}^-$ admits t-dependent symmetries of order r given by

$$\Sigma_{12}^{(0)} = \hat{N}_{12}^{(m)} \cdot H_{12}^{(r)} \tag{2.26a}$$

$$\Sigma_{12}^{(j)} = \Sigma \nu(r,j,s) \hat{N}_{12}^{(m+jn-\sum_{\ell=1}^{j} 2s_\ell + 1)} . H_{12}^{(r-\sum_{\ell=1}^{j} 2s_\ell + 1)}, \tag{2.26b}$$

and by

$$\Sigma_{12}^{(0)} = \hat{M}_{12}^{(m)} \cdot H_{12}^{(r)} \tag{2.27a}$$

$$\Sigma_{12}^{(j)} = \Sigma \nu(r,j,s) \hat{M}^{(m+jn-\sum_{\ell=1}^{j} 2s_\ell + 1)} . H_{12}^{(r-\sum_{\ell=1}^{j} 2s_\ell + 1)}, \tag{2.27b}$$

where the summation $\Sigma$ is from $s_{--}$ $s_{-----}s$, zero to P .

where $j \geq 1$, $P_n = (n-1)/2$ if n is odd and $(n-2)/2$ if n is even. Also

$$\nu(r,j,s) \doteq \frac{(-2)^j}{j!}\left(\prod_{\mu=1}^{j}\theta(r-\sum_{\ell=1}^{j}2s+1)\right)\left(\prod_{\ell=1}^{j}b_{n,2s_\ell+1}\right)\frac{r!}{(r-\sum_{\ell=1}^{j}2s_\ell+1)!}, \quad (2.28)$$

and $b_{n,\ell} = (-4\alpha)^\ell\binom{n}{\ell}$.

ii) The KP class (2.15b) with $\hat{M}_{12} = Dq^+_{12} + q^-_{12}D^{-1}q^-_{12}$ admits t-dependent symmetries of order r given by

$$\Sigma^{(0)}_{12} = \hat{N}^{(m)}_{12} \cdot H^{(r)}_{12}, \quad (2.29a)$$

$$\Sigma^{(2j)}_{12} = \Sigma\nu(r,2j,s)\hat{N}_{12}^{(m+2jn+j-\sum_{\ell=1}^{2j}2s_\ell+1)} \cdot H_{12}^{(r-\sum_{\ell=1}^{2j}2s_\ell+1)}, \quad (2.29b)$$

$$\Sigma^{(2j-1)}_{12} = \Sigma\nu(r,2j-1,s)\hat{M}_{12}^{(m+(2j-1)n+j-1-\sum_{\ell=1}^{2j-1}2s_\ell+1)} \cdot H_{12}^{(r-\sum_{\ell=1}^{2j-1}2s_\ell+1)} \quad (2.29c)$$

and by

$$\Sigma^{(0)}_{12} = \hat{M}^{(m)}_{12} \cdot H^{(r)}_{12}, \quad (2.30a)$$

$$\Sigma^{(2j)}_{12} = \Sigma\nu(r,2j,s)\hat{M}_{12}^{(m+2jn+j-\sum_{\ell=1}^{2j}2s_\ell+1)} \cdot H_{12}^{(r-\sum_{\ell=1}^{2j}2s_\ell+1)}, \quad (2.30b)$$

$$\Sigma^{(2j-1)}_{12} = \Sigma\nu(r,2j-1,s)\hat{N}_{12}^{(m+(2j-1)n+j-\sum_{\ell=1}^{2j-1}2s_\ell+1)} \cdot H_{12}^{(r-\sum_{\ell=1}^{2j-1}2s_\ell+1)} \quad (2.30c)$$

with $j \geq 1$ and $b_{n,\ell} = \sum_{s=0}^{\ell} \beta^{\ell-s}\beta^s\binom{n-s}{\ell-s} = (-4\alpha)^\ell\sum_{s=0}^{\ell}2^{-s}\binom{n-s}{\ell-s}$.

## II.B. Time Dependent Symmetries for the Equations Associated with the Davey-Stewartson Equation.

The construction of t-dependent symmetries for the equations associated with the DS equation is similar. Extended symmetries of order r in time are generated through Lemma 2.1, starting with $\Sigma_{12}^{(0)} = \hat{N}^{(m)} H_{12}^{(r)}$ or $\hat{M}_{12}^{(m)} H_{12}^{(r)}$, where the solution $H_{12}^{(r)}$ is defined by,

$$H_{12}^{(r)} \doteq \mathrm{diag}(\xi_{+12}^{r}, \xi_{-12}^{r}), \quad \xi_{\pm 12} \doteq y_1 + y_2 \pm 2\alpha x \qquad (2.41)$$

$H_{12}^{(r)}$ satisfies the same formula (2.24), obviously replacing $[H_{12}^{(r)}, \delta_{12}^{s}]_I$ by $[H_{12}^{(r)}, \delta_{12}^{s} I]_I$. Then, using Corollary 2.1 and equations (1.13), one can show that

i) The class of evolution equations (2.15a) with $\hat{N}_{12} = Q_{12}^{-}$ admits t-dependent symmetries of order r given by equations (2.26) and (2.27), where $b_{n,\ell} = \beta^{\ell}\binom{n}{\ell} = (2\alpha)^{\ell}\binom{n}{\ell}$ and $j \geq 1$.

ii) The class of evolution equations (2.15b) with $\hat{M}_{12} = Q_{12}^{-}\sigma$ admits t-dependent symmetries of order r given by equations (2.29-30), replacing: $\hat{N}^{(\cdot)} \to \hat{N}^{(\cdot-j)}$ in equation (2.29b), $\hat{M}^{(\cdot)} \to \hat{M}^{(\cdot-j+1)}$ in equation (2.29c), $\hat{M}^{(\cdot)} \to \hat{M}^{(\cdot-j)}$ in equation (2.30b), $\hat{N}^{(\cdot)} \to \hat{N}^{(\cdot-j)}$ in equation (2.30c) and using $b_{n,\ell} = (2\alpha)^{\ell}\binom{n}{\ell}$.

## II.C. Connection with Known Results.

Before the discovery [14] of the recursion operator of the KP equation, a different approach, the so-called master-symmetries approach, was used to generate an infinite sequence of commuting symmetries [6], as well as t-dependent symmetries [7], [11], of the KP equation (see also [18], [19]).

The existence of a hereditary operator in 2+1 dimensions, together with the Lie algebra of the starting symmetries allows a simple and elegant characterization of the 2+1 dimensional (gradient) master-symmetries introduced in the above papers. Here we briefly consider the KP example.

In Proposition 2.1 and in §II.B. we have shown that the functions

$$\tau_{12}^{(m,r)} \doteq \Phi_{12}^{m} K_{12}^{0} H_{12}^{(r)}, \tag{2.32}$$

(where $H_{12}^{(r)}$ is defined in (2.23), but it could be any homogeneous polynomial of degree r in $y_1$, $y_2$, and $\hat{K}_{12}^{0}$ is $\hat{N}_{12}$ or $\hat{M}_{12}$) have the property that their $(r+1)^{st}$ commutator with $\delta_{12}K_{12}^{(n)}$ is zero, namely

$$\underbrace{[\cdot\ \cdot\ \cdot}_{r+1\ \text{times}}[\tau_{12}^{(m,r)},\ \delta_{12}K_{12}^{(n)}]_d \underbrace{\cdot\ \cdot\ \cdot\ ]_d}_{r+1\ \text{times}} = 0. \tag{2.33}$$

Then Theorem 4.1 of [1] implies that

$$\underbrace{[\cdot\ \cdot\ \cdot}_{r+1\ \text{times}}[\tau_{11}^{(m,r)},\ K_{11}^{(n)}]_f \underbrace{\cdot\ \cdot\ \cdot\ ]_f}_{r+1\ \text{times}} = 0, \tag{2.34}$$

*namely* $\tau_{11}^{(m,r)}$ *are the so-called master-symmetries of degree r of KP* [11]. Equation (2.33) essentially follows from the fact that a single commutator of $\tau_{12}^{(m,r)}$ with $\delta_{12}K_{12}^{(n)}$ generates a linear combination of lower degree master-symmetries; in fact, choosing for concreteness $\tau_{12}^{(m,r)}=\Phi_{12}^{m}\hat{N}_{12}(y_1 + y_2)^r$ and $K_{12}^{(n)} = M_{12}^{(n)}$, we have

$$[\tau_{12}^{(m,r)},\delta_{12}M_{12}^{(n)}]_d = -\sum_{\ell=0}^{n} b_{n,\ell}\hat{M}_{12}^{(m+n)}[(y_1+y_2)^r,\ \delta_{12}^{\ell}]_I =$$

$$= \sum_{\ell=1}^{n} \theta(r-\ell)\frac{r!}{(r-\ell)!}\, b_{n,\ell}\, \tau_{12}^{(m+n,r-\ell)}, \tag{2.35}$$

which implies

$$[\tau_{11}^{(m,r)},M_{11}^{(n)}]_f = \sum_{\ell=1}^{n} \theta(r-\ell)\frac{r!}{(r-\ell)!}\, b_{n,\ell}\, \tau_{11}^{(m+n,r-\ell)}. \tag{2.36}$$

For r = 1 equation (2.36) becomes

$$[\tau_{11}^{(m,1)},M_{11}^{(n)}]_f = b_{n,1}M_{11}^{(m+n)}; \tag{2.37}$$

master-symmetries of degree 1 generate equations which belong to the given hierarchy

## III. LIE AND POISSON BRACKETS FOR EQUATIONS IN 2+1

In this section we first derive an isomorphism between Lie and Poisson brackets. Then, using this isomorphism and the Lie algebra of the operators $\hat{K}^0_{12}$, we prove that $\Theta^{-1}_{12}\hat{K}^0_{12}H_{12}$ are extended gradients. This implies that all extended symmetries of the previous section give rise to conserved quantities.

Theorem 3.1

Let $[a,b]_L = a_L[b] - b_L[a]$ be a Lie commutator and $\langle f,g\rangle$ be an appropriate symmetric bi-linear form. Let grad I be the gradient of a functional I, defined by $I_L[v] = \langle \text{grad } I,v\rangle$; then $\gamma$ is a gradient function iff $\gamma_L = \gamma^*_L$, where $M^*$ denotes the adjoint of the operator M with respect to the above bi-linear form, i.e. $\langle M^*f,g\rangle = \langle f,Mg\rangle$. Then if the operator $\Theta$ is a Hamiltonian operator, i.e. if

$$\Theta^* = -\Theta, \quad \langle a,\Theta_L[\Theta b]c\rangle + \text{cyclic permut} = 0, \tag{3.1}$$

then

$$[\Theta f,\Theta g]_L = \Theta\,\text{grad}\langle f,\Theta g\rangle + \Theta\{(f_L-f^*_L)[\Theta g] - (g_L-g^*_L)[\Theta f]\}. \tag{3.2}$$

Proof.

$$\text{grad}\langle f,\Theta g\rangle[v] = \langle f_L[v],\Theta g\rangle + \langle f,\Theta_L[v]g\rangle + \langle f,\Theta g_L[v]\rangle$$

$$= \langle f^*_L[\Theta g] - g^*_L[\Theta f]\rangle$$

$$= \langle f^*_L[\Theta g] + M^*_g f - g^*_L[\Theta g],\ v\rangle,$$

where $\langle f,\Theta_L[v]g\rangle = \langle f,M_g[v]\rangle$ and $M_g$ denotes a linear operator depending on g. Hence

$$[\Theta f,\Theta g]_L - \Theta\text{grad}\langle f,\Theta g\rangle = \Theta_L[\Theta g]f + \Theta f_L[\Theta g] - \Theta_L[\Theta f]g - \Theta g_L[\Theta f]$$

$$-\Theta f^*_L[\Theta g] + \Theta g^*_L[\Theta g] - \Theta M^*_g f$$

$$= \Theta_L[\Theta g]f - \Theta_L[\Theta f]g - \Theta M^*_g f + \Theta\{(f_L-f^*_L)[\Theta g] - (g_L-g^*_L)[\Theta f]\}.$$

But the sum of the first three terms of the above equals zero because of (3.1). Hence (3.2) follows.

In the above $\partial_L$ denotes an appropriate directional derivative. For equations in 1+1:

$$[a,b]_L = [a,b]_f, \quad \langle f,g\rangle = \int_{\mathbf{R}} dx \text{ trace } gf. \tag{3.3}$$

For equation in 2+1

$$[a_{12},b_{12}]_L = [a_{12},b_{12}]_d, \quad \langle f_{11},g_{11}\rangle = \int_{\mathbf{R}^2} dx\, dy \text{ trace } g_{11}f_{11},$$

$$\langle f_{12},g_{12}\rangle = \int_{\mathbf{R}^2} dx\, dy_1 dy_2 \text{ trace } g_{21}f_{12} \tag{3.4}$$

(if f and g are scalars, then delete trace), where $[\ \ ]_f$, $[\ \ ]_d$ are defined in (2.13), (2.4). Furthermore the following double representation of the functional I

$$I = \int_{\mathbf{R}^2} dx\, dy_1 \text{trace } \rho_{11} = \int_{\mathbf{R}^3} dx\, dy_1 dy_2 \delta_{12} \text{trace } \rho_{12} \tag{3.5}$$

allows to define the extended gradient $\text{grad}_{12} I$ and the gradient grad I of the functional I by

$$I_d[v_{12}] = \int_{\mathbf{R}^3} dx\, dy_1 dy_2 \delta_{12} \text{trace } \rho_{12}[v_{12}] \doteq \langle \text{grad}_{12} I, v_{12}\rangle, \tag{3.6a}$$

$$I_f[v_{11}] = \int_{\mathbf{R}^2} dx\, dy_1 \text{trace } \rho_{11_f}[v_{11}] \doteq \langle \text{grad } I, v_{11}\rangle. \tag{3.6b}$$

The following theorem, proven in [1], maps extended gradients $\gamma_{12}$ to gradients $\gamma_{11}$:

<u>Theorem 3.2.</u>

a) $\gamma_{12}$ and $\gamma_{11}$ are extended gradients and grdients respectively iff $\gamma^*_{12_d} = \gamma_{12_d}$ and $\gamma^*_{11_f} = \gamma_{11_f}$, with respect to the bilinear forms (3.4c) and (3.4b) respectively.

b) If $\gamma_{12}$ is an extended gradient, then $\gamma_{11}$ is a gradient corresponding to the same potential, namely if $\gamma_{12} = \text{grad}_{12} I$, then $\gamma_{11} = \text{grad } I$.

Proposition 3.1

Assume that the hereditary operator $\Phi_{12}$ is a strong symmetry for the starting symmetries $\hat{M}_{12}H_{12}$ and $\hat{N}_{12}H_{12}$. Further assume that $\hat{M}_{12}$, $\hat{N}_{12}$ form a Lie algebra (analogous to (1.7) and (1.13)) and that $\Theta_{12}$ is a Hamiltonian operator whose inverse exists. Then

$$\Theta_{12}^{-1}\Phi_{12}^{m}\hat{K}_{12}^{0}H_{12},\quad \hat{K}_{12}^{0} = \hat{M}_{12} \text{ or } \hat{N}_{12} \tag{3.7}$$

are extended gradients, provided that $\Theta_{12}^{-1}\hat{K}_{12}^{0}H_{12}$ are extended gradient.

Proof

For concreteness we proof the above proposition for the recursion operator and starting symmetries associated with the two dimensional Schrödinger and 2 x 2 AKNS problems.

IIIA. Conserved Quantities for Equations Related to KP Equations

Corollary 3.1

Let

$$\hat{N}_{12} \doteq q_{12}^{-},\ \hat{M}_{12} \doteq Dq_{12}^{+} + q_{12}^{-}D^{-1}q_{12}^{-},\ H_{12} \doteq H(y_1,y_2),\ \hat{M}_{12}^{(n)} \doteq \Phi_{12}^{n}\hat{M}_{12},$$
$$\hat{N}_{12}^{(n)} = \Phi_{12}^{n}\hat{N}_{12},\quad \Theta_{12} = D, \tag{3.8}$$

where $\Phi_{12}$ is the recursion operator associated with the KP and is defined by (1.4). Then

$$\begin{aligned} D^{-1}\hat{M}^{(n+1)}H_{12}^{(3)} &= \text{grad}\langle \hat{M}^{(n)}H_{12}^{(1)},\ D^{-1}\hat{N}^{(1)}H_{12}^{(2)}\rangle, \\ D^{-1}\hat{N}^{(n+1)}H_{12}^{(3)} &= \text{grad}\langle \hat{M}^{(n)}H_{12}^{(1)},\ D^{-1}\hat{M}H_{12}^{(2)}\rangle. \end{aligned} \tag{3.9}$$

Proof

We first note that the assumptions of Proposition 3.1 are fulfilled. Namely $\Phi_{12}$ is hereditary and is a strong symmetry of $\hat{M}_{12}H_{12}$, $\hat{N}H_{12}$ (see Lemma 4.2 and Appendix C.1a of [1]). The operator $D^{-1}$ is obviously a Hamiltonian

Since $D^{-1}\hat{M}_{12}H_{12}$ is an extended gradient, Theorem 3.1 and (1.7c) imply that $D^{-1}\hat{N}_{12}^{(1)}H_{12}$ is an extended gradient. Then Theorem 3.1 and $[\hat{M}^{(n)}H_{12}^{(1)},\hat{N}^{(1)}H_{12}^{(2)}]_d = -\hat{M}^{(n+1)}H_{12}^{(3)}$ imply by induction (3.9a). Finally Theorem 3.1 and $[\hat{M}^{(n)}H_{12}^{(1)},\ \hat{M}H_{12}^{(2)}]_d = -\hat{N}^{(n+1)}H_{12}^{(3)}$ imply by induction (3.9b).

A consequence of the above result is that all symmetries derived in §II.B. give rise to consderved quantities. For example, the following t-dependent extended symmetries (see (2.19b) and (2.21a))

$$\sigma_{12}^{(1)} = \hat{M}_{12}^{(m)}(y_1 + y_2) + t12\alpha\hat{N}_{12}^{(m+1)} \cdot 1,$$

$$\sigma_{12}^{(2)} = \hat{N}_{12}^{(m)}(y_1 + y_2)^2 + t24\alpha\hat{M}_{12}^{(m)}\cdot(y_1+y_2) + t^2 144\alpha^2\hat{N}_{12}^{(m+1)} \cdot 1,$$

of the KP equation $q_{1_t} = M_{11}^{(1)} = 2(q_{1_{xxx}} + 6q_1q_{1_x} + 3\alpha^2 D^{-1}q_{1_{y_1y_1}})$ correspond to extended gradient functions $D^{-1}\sigma_{12}^{(1)}$ and $D^{-1}\sigma_{12}^{(2)}$; then they give rise to the following t-dependent conserved quantities (see equations (4.15))

$$I^{(1)} = \int_{\mathbb{R}^2} dx\, dy_1\left(\frac{1}{2(2m+3)}(D^{-1}\hat{M}_{12}^{(m+1)}(y_1 + y_2))_{11} + \frac{t_3}{m+2}(D^{-1}\hat{N}_{12}^{(m+2)} \cdot 1)_{11}\right),$$

$$I^{(2)} = \int_{\mathbb{R}^2} dx\, dy_1\left(\frac{1}{4(m+1)}(D^{-1}\hat{N}^{(m+1)}(y_1+y_2)^2)_{11} + \frac{t12\alpha}{2m+3}(D^{-1}\hat{M}_{12}^{(m+1)}(y_1+y_2))_{11} + \right.$$

$$\left. + \frac{t^2 36\alpha^2}{m+2}(D^{-1}\hat{N}_{12}^{(m+2)} \cdot 1)_{11}\right).$$

## III.B. Conserved Quantities for Equations Related to DS Equations.

Corollary 3.2

Let

$$\hat{M}_{12} \doteq Q_{12}\sigma,\ \hat{N}_{12} \doteq Q_{12}^{-},$$

$H_{12}$ diagonal and such that $P_{12}H_{12}=0$, $\hat{M}_{12}^{(n)} \doteq \Phi_{12}^{n}\hat{M}_{12}, \hat{N}_{12}^{(n)}=\Phi_{12}^{n}\hat{N}_{12}, \theta_{12} = \sigma,$

(3.10)

where $\Phi_{12}$ is the recursion operator associated with the DS equation and is defined by (1.9). Then

$$\sigma\hat{M}^{(n+1)}H_{12}^{(3)} = \text{grad}\langle\hat{M}^{(n)}H_{12}^{(1)},\ \sigma\hat{N}^{(1)}H_{12}^{(2)}\rangle, \tag{3.11}$$

$$\sigma\hat{N}^{(n)}H^{(3)} = \text{grad}\langle\hat{M}^{(n)}H^{(1)},\ \sigma\hat{M}H^{(1)}\rangle$$

**Proof**

The assumptions of Proposition 3.1 are again fulfilled (see Lemma 4.2 and Appendix C.2a of [1]). The operator $\sigma$ is obviously Hamiltonian in a space of off-diagonal matrices. Furthermore, $\sigma\hat{M}_{12}H_{12}$, $\sigma\Phi_{12}\hat{N}_{12}H_{12}$ are extended gradients (see Appendix A).

Since the above are gradients, $[\hat{M}^{(n)}H_{12}^{(1)}, \hat{N}^{(1)}H_{12}^{(2)}]_d = -\hat{M}^{(n+1)}H_{12}^{(3)}$ implies (3. ). Then $[\hat{M}^{(n)}H_{12}^{(1)}, \hat{M}H_{12}^{(2)}] = -\hat{N}^{(n)}H_{12}^{(3)}$ implies (3.11).

The above implies that the symmetries derived in §II.C. give rise to conserved quantities. For example, the $1^{st}$ and $2^{nd}$ order t-dependent symmetries

$$\sigma_{12}^{(1)} = \hat{M}_{12}^{(m)}H_{12}^{(1)} - 8\alpha t\ \hat{N}_{12}^{(m)} \cdot I$$

$$\sigma_{12}^{(2)} = \hat{N}_{12}^{(m)}H_{12}^{(2)} - t16\alpha\hat{M}_{12}^{(m)}H_{12}^{(1)} + t^2 64\alpha^2\hat{N}_{12}^{(m+2)} \cdot I$$

of the DS euqation $Q_{1_t} = M_{11}^{(2)} = -[2\sigma(Q_{1_{xx}} + \alpha^2 Q_{1_{y_1y_1}}) - Q_1A_1 + A_1Q_1]$, $(D-JD_1)A_1 = -2(D+JD_1)\sigma Q_1^2$, obtained from equations (2.29-30), correspond to the extended gradients $\sigma\sigma_{12}^{(1)}$, $\sigma\sigma_{12}^{(2)}$; then they give rise to the following t-dependent conserved quantities (see equations (4.24)):

$$I^{(1)} = \int_{\mathbf{R}^2} dx\ dy\ \text{trace}\,\sigma[Q_1, \frac{1}{2(m+1)}(D^{-1}\hat{M}_{12}^{(m+1)}H_{12}^{(1)})_{11} - \frac{t4\alpha}{m+1}\hat{N}_{12}^{(m+1)} \cdot I],$$

$$I^{(2)} = \int_{\mathbf{R}^2} dx\ dy_1 \text{trace}\,\sigma[Q_1, \frac{1}{2(m+1)}(D^{-1}\hat{N}_{12}^{(m+1)}H_{12}^{(2)})_{11} - \frac{t8\alpha}{m+1}(D^{-1}\hat{M}_{12}^{(m+1)}H_{12}^{(1)})_{11} + \frac{t^2 32\alpha^2}{m+3}(D^{-1}\hat{N}_{12}^{(m+3)} \cdot I)_{11}].$$

## IV. ON A NON-GRADIENT MASTER-SYMMETRY.

In this section we make extensive use of the isomorphism between Lie and Poisson brackets. Hence it is useful to investigate the properties of

$$\Theta(g_L - g_L^*) = T_L + \Theta T_L^* \Theta^{-1}; \quad T \triangleq \Theta g,\ \Theta_L = 0. \tag{4.1}$$

Lemma 4.1: Let

$$S = \Phi_L[T] + [\Phi, T_L], \tag{4.2}$$

with its adjoint

$$S^* = \Phi_L^*[T] + [T_L^*, \Phi^*]. \tag{4.3}$$

a) If $\Phi$ is hereditary then

$$\Phi_L^*[\Phi^n T] + (\Phi^n T)_L^* \Phi^* - \Phi^* (\Phi^n T)_L^* = S^* \Phi^{*n}. \tag{4.4}$$

b) If $\Phi$ is factorizable in terms of compatible Hamiltonian operators, i.e. if $\Phi = \Omega\Theta^{-1}$, where $\Omega + \nu\Theta$ is a Hamiltonian operator, $\Theta$ is invertible and $\nu$ is an arbitrary constant, then

$$(\Phi T)_L + \Theta(\Phi T)_L^* \Theta^{-1} = \Phi(T_L + \Theta T_L^* \Theta^{-1}) + \Theta S^* \Theta^{-1}, \tag{4.5}$$

where we have assumed for simplicity that $\Theta_L = 0$.

c) $$(\Phi^n T)_L + \Theta(\Phi^n T)_L^* \Theta^{-1} = \Phi^n (T_L + \Theta T_L^* \Theta^{-1}) + \sum_{r=1}^{n} \Phi^{r-1} \Theta \Phi^{*n-r} S^* \Theta^{-1}. \tag{4.6}$$

Proof

Equation (4.4) is the adjoint of (2.7) for K = T. Equation (4.5) is derived in Appendix B, and (4.6) follows from (4.5) by induction.

Theorem 4.1

*Assume that* $\Phi$ *is factorizable in terms of* compatible Hamiltonian operators and that $\Theta_L = 0$. Further assume that $\Theta^{-1}\Phi^n M$ is a gradient function and that $\Phi$ is a strong symmetry for M. Then

$$\Phi^m \sum_{r=1}^{n} \Phi^{n-r} S \Phi^{r-1} M = \Theta \operatorname{grad}\langle \Theta^{-1}\Phi^n M, \Phi^m T\rangle - \sum_{r=1}^{m} \Phi^{r-1} \Theta \Phi^{*m-r} \tilde{S}^* \Theta^{-1} \Theta^n M$$
$$-\Phi^m (T_L + \Theta T_L^* \Theta^{-1}) \Phi^n M - \Phi^{n+m}[M, T]_L. \tag{4.7}$$

Proof

Using the fact that $\Theta^{-1}\Phi^n M$ is a gradient, equation (3.2) becomes

$$[\Phi^n M, \Phi^m T]_L = \Theta \operatorname{grad}\langle \Theta^{-1}\Phi^n M, \Phi^m T\rangle - \{(\Theta^m T)_L + \Theta(\Phi^m T)_L^* \Theta^{-1}\}\Phi^n M. \tag{4.8}$$

Since $M$ is a strong symmetry of $\Phi$, Theorem 2.1 implies

$$[\Phi^n M, \Phi^m T]_L = \Phi^{n+m}[M,T]_L + \Phi^m\left(\sum_{r=1}^{n} \Phi^{n-r} S \Phi^{r-1}\right)M. \tag{4.9}$$

Using the above and (4.6) in (4.8) we obtain (4.7).

Equations (4.6) and (4.9) are useful in finding non-gradient master-symmetries for equations in 2+1. Furthermore, Theorem 4.1 is useful for deriving the potentials of various gradients. Formulae (4.6), (4.9) and (4.7) take a particularly simple form if the function $T_{12}$ is such that

i) $$S_{12} = S_{12}^* = c1, \tag{4.10a}$$

where 1 is the identity operator and c is an arbitrary constant, and

ii) $$T_{12_d} + \Theta_{12} T_{12_d}^* \Theta_{12}^{-1} = 0. \tag{4.10b}$$

In the following two examples the non-gradient master-symmetries are generated through functions $T_{12}$ that satisfy equations (4.10).

## IVA. Equations Associated with the KP Equations

Corollary 4.1

a) $\Phi_{12}^2 \delta_{12}$ is a non-gradient master-symmetry for the KP and the equations related to KP:

$$[\Phi_{12}^n \hat{K}_{12}^0 H_{12}, \Phi_{12}^2 \delta_{12}]_d = b_n \Phi_{12}^{n+1} \hat{K}_{12}^0 H_{12}, \tag{4.11}$$

$$8\Phi_{12} = (\Phi_{12}^2 \delta_{12})_d + \Theta_{12}(\Phi_{12}^2 \delta_{12})_d^* \Theta_{12}^{-1}, \tag{4.12}$$

where $b_n$ and $H_{12}$ are given by

$$b_n = 4n,\ H_{12} = H(y_1, y_2) \text{ arbitrary}, \quad \text{if } \hat{K}_{12}^0 = \hat{N}_{12} \tag{4.13a}$$

and by

$$b_n = 2(2n+1),\quad H_{12} = (y_1+y_2)^r, \quad r = 0,1, \quad \text{if } \hat{K}_{12}^0 = \hat{M}_{12}. \tag{4.13b}$$

b) Let

$$\hat{\Upsilon}^{(n)}_{12} \doteq \Phi^{*n}_{12}\hat{\Upsilon}^{0}_{12}, \quad \hat{\Upsilon}^{0}_{12} \doteq \Theta^{-1}_{12}\hat{K}^{0}_{12}. \tag{4.14}$$

Then

$$\hat{\Upsilon}^{(n)}_{12}H_{12} = \text{grad}_{12}I_n, \tag{4.15a}$$

$$I_n \doteq \frac{1}{b_{n+1}} \langle\hat{\Upsilon}^{(n+1)}_{12}H_{12},\delta_{12}\rangle = \frac{1}{b_{n+1}} \int_{\mathbf{R}^3} dx\, dy_1 dy_2\, \delta_{12}\hat{\Upsilon}^{(n+1)}_{12}H_{12} =$$

$$= \frac{1}{b_{n+1}} \int_{\mathbf{R}^2} dx\, dy_1 (\hat{\Upsilon}^{(n+1)}_{12}H_{12})_{11}, \tag{4.15b}$$

where $b_n$ and $H_{12}$ are given in (4.13).

<u>Proof</u>.

If

$$T_{12} = \delta_{12}, \tag{4.16}$$

equation (4.10b) is trivially satisfied and equation (4.10a) holds for $c = 4$, since $\Phi_{12_d}[\delta_{12}] = \Phi^{*}_{12_d}[\delta_{12}] = 4$. Equation (4.12) is a simple consequence of (4.6) for $n = 2$; using the following results

$$\Phi^{n}_{12}[\hat{N}_{12}H_{12},\delta_{12}]_d = 0, \tag{4.17a}$$

$$\Phi^{n}_{12}[\hat{M}_{12}(y_1+y_2)^r,\delta_{12}]_d = 2\Phi^{n-1}_{12}\hat{M}_{12}(y_1+y_2)^r, \quad r = 0,1, \tag{4.17b}$$

(see Appendix A) in equations (4.9) and (4.7) (with $M = \hat{K}^{0}_{12}H_{12}$ and $H_{12}$ as in (4.13)), we obtain

$$[\Phi^{n}_{12}\hat{K}^{0}_{12}H_{12},\Phi^{m}_{12}\delta_{12}]_d = b_n\Phi^{n+m-1}_{12}\hat{K}^{0}_{12}H_{12} \tag{4.18}$$

(that reduces to (4.11) for m=2), and

$$b_n\Phi^{n+m-1}_{12}\hat{K}^{0}_{12}H_{12} = \Theta_{12}\text{grad}_{12}\langle\hat{\Upsilon}^{(n)}_{12}H_{12},\Phi^{m}_{12}\delta_{12}\rangle, \tag{4.19}$$

where we have used $\Phi^{n}_{12}\Theta_{12} = \Theta_{12}\Phi^{*n}_{12}$. Equation (4.19) reduces to (4.15) if one uses the definition of $\langle f_{12},g_{12}\rangle$ given by (1.20) and (3.4c).

Remark 4.1

i) $T \doteq \phi^2 1$ is a non-gradient master-symmetry for the KdV equation. Given $T$ one recovers $\phi$ from $T_f + \Theta T_f^* \Theta^{-1}$. Equation (4.12) is the two-dimensional analogue of this well known formula [8]-[10].

ii) Theorem 3.2 implies that equations (4.15) with m=1, $H_{12}=1$ reduce to the following formula [6]

$$\gamma_{11}^{(n)} = \frac{1}{b_{n+1}} \text{ grad} \int_{R^2} dx\, dy_1 \gamma_{11}^{(n+1)}. \tag{4.20}$$

An analogous formula, for the KdV equation is well known

$$\gamma^{(n)} = \frac{1}{2(2n+3)} \text{grad} \int_{R^2} dx \gamma^{(n+1)}.$$

iii) We observe that equation (4.18) for $H_{12} = 1$ cannot be projected into equation (2.37).

## IVB. Equations Associated with the DS Equation

Corollary 4.2.

a) $\Phi_{12}^2 T_{12}$, $T_{12} \doteq \frac{x}{2} \sigma Q_{12}^{+} \delta_{12} I$, $I = \mathrm{diag}(1,1)$, is a non-gradient master-symmetry for the DS and the equations related to DS:

$$[\Phi_{12}^n \hat{K}_{12}^0 H_{12}, \Phi_{12}^2 T_{12}]_d = n\Phi_{12}^{n+1} \hat{K}_{12}^0 H_{12}, \tag{4.21}$$

$$2\Phi_{12} = (\Phi_{12}^2 T_{12})_d + \Theta_{12} (\Phi_{12}^2 T_{12})_d^* \Theta_{12}^{-1}, \ \Theta_{12} = \sigma, \tag{4.22}$$

where $\hat{K}_{12}^0 H_{12}$ is defined in (1.11-12).

b) Let

$$\hat{\gamma}_{12}^{(n)} = \Phi_{12}^{*n} \hat{\gamma}_{12}^0, \ \hat{\gamma}_{12}^0 = \Theta_{12}^{-1} \hat{K}_{12}^0, \ \Theta_{12} = \sigma. \tag{4.23}$$

Then

$$\hat{\gamma}_{12}^{(n)} H_{12} = \mathrm{grad}_{12} I_n, \tag{4.24a}$$

$$I_n \doteq \frac{1}{n+1} \langle \hat{\gamma}_{12}^{(n+1)} H_{12}, T_{12} \rangle = -\frac{1}{2(n+1)} \int_{\mathbf{R}^3} dx\, dy_1 dy_2 \,\mathrm{trace}\, \delta_{12} Q_{12}^{+} \sigma \hat{\gamma}_{12}^{(n+1)} H_{12} =$$

$$= \frac{1}{2(n+1)} \int_{\mathbf{R}^2} dx\, dy_1 \,\mathrm{trace}\, \sigma[Q_1, (\hat{\gamma}_{12}^{(n+1)} H_{12})_{11}]. \tag{4.24b}$$

Proof:

If

$$T_{12} \doteq \frac{x}{2}\sigma Q^{+}_{12}\delta_{12}I, \tag{4.25}$$

equation (4.10b) is satisfied and equation (4.10a) holds for $c = 1$ (see Appendix A). Then the derivation of equations (4.21), (4.22) and (4.24) is analogous to the one of Corollary 4.1 (see Appendix A).

## V. 2+1 DIMENSIONAL EQUATIONS AS REDUCTIONS OF NON-LOCAL SYSTEMS.

In [1] and [14] the classes of evolution equations

$$q_{1_t} = \int_{\mathbf{R}} dy_2 \delta_{12} \Phi^{n}_{12} \hat{K}^{0}_{12} \cdot 1, \tag{5.1}$$

where $\Phi_{12}$ and $\hat{K}^{0}_{12}$ are defined in (1.4-5), were algorithmically derived from the spectral problem

$$w_{xx} + q(x,y)w + \alpha w_y = 0. \tag{5.2}$$

In this section we show that equations (5.1) are exact reductions of equations non-local in $y$, generated by the following non-local analogue of (5.2)

$$w_{xx} + \tilde{q}w + \alpha w_y = \lambda w, \tag{5.3}$$

where

$$(\tilde{q}f)(x,y) \doteq \int_{\mathbf{R}} dy_2 \; q(x,y,y_2)f(x,y_2). \tag{5.4}$$

Hereafter the symbols $\tilde{u}$ and $u_{12}$ indicate the integral operator defined by

$$(\tilde{u}f)(x,y) \doteq \int_{\mathbf{R}} dy_2 u(x,y,y_2)f(x,y_2) \tag{5.5}$$

and its kernel $u_{12} \doteq u(x,y_1,y_2)$, respectively.

The algorithmic derivation of the classes of evolution equations associated with (5.3) is standard; its main steps are:

i) Compatibility.

A compatibility between equation (5.3), written in the more convenient form $\binom{w}{w_x}_x = \begin{pmatrix} 0, & 1 \\ \lambda-\tilde{q}-D_y, & 0 \end{pmatrix}\binom{w}{w_x}$, and the linear evolution equation $\binom{w}{w_x}_t = \tilde{V}\binom{w}{w_x}$, yields the following operator equation

$$\tilde{q}_t = \tilde{c}_{xxx} + [\tilde{q} + \alpha D_y, \tilde{c}]^+_x + [\tilde{q} + \alpha D_y, \tilde{c}_x]^+ + [\tilde{q} + \alpha D_y, D^{-1}[\tilde{q} +$$

$$+ \alpha D_y, \tilde{c}]] + \lambda(-4\tilde{c}_x + \tilde{A}_0 - \tilde{E}_0) + \tilde{E}_0(\tilde{q} + \alpha D_y) - (\tilde{q} + \alpha D_y)\tilde{A}_0, \quad (5.6)$$

where the scalar integral operator $2\tilde{c}$ is the 1,2 component of the 2x2 matrix integra operator $\tilde{V}$, $\tilde{A}_{0_x} = \tilde{E}_{0_x} = 0$ and [ , ] and $[\ ,\ ]^+$ are the usual commmutator and anti-commutator.

ii) Equation for the kernel.

The operator equation (5.6), together with the definition (5.5), implies the following equation for the kernels $q_{12}$, $c_{12}$, $A_{12}$ and $E_{12}$:

$$q_{12_t} = D\tilde{\psi}_{12}c_{12} + \frac{1}{2}(\tilde{q}^+_{12} - \tilde{q}^-_{12})A_{12} - \frac{1}{2}(\tilde{q}^+_{12} + \tilde{q}^-_{12})E_{12} + \lambda(-4c_{12_x} + A_{12} - E_{12}), \quad (5.7)$$

where

$$\tilde{\psi}_{12} \doteq D^2 + \tilde{q}^+_{12} + D^{-1}\tilde{q}^+_{12}D + D^{-1}\tilde{q}^-_{12}D^{-1}\tilde{q}^-_{12}, \quad (5.8a)$$

$$\tilde{q}^{\pm}_{12}f_{12} = \int_{\mathbb{R}}(q_{13}f_{32} \pm f_{13}g_{32})dy_3 + \alpha(D_1 \mp D_2)f_{12} . \quad (5.8b)$$

iii) Expansion in powers of $\lambda$.

Let us first assume that

$$c_{12} = \sum_{j=0}^{n}\lambda^j c^{(j)}_{12}, \; E_{12} = A_{12} = 0, \quad (5.9)$$

equating the coefficients of $\lambda^j (0 \leq j \leq n)$ to zero we obtain:
$c^{(n)}_{12} = H^{(n)}_{12}$; $c^{(j-1)}_{12} = \frac{1}{4}\tilde{\psi}_{12}c^{(j)}_{12} + H^{(j-1)}_{12} (1 \leq j \leq n)$; $q_{12_t} = D\tilde{\psi}_{12}c^{(0)}_{12}$; where $H^{(j)}_{12} = H^{(j)}(y_1,y_2)$.

Then $C^{(0)}_{12} = \sum_{s=0}^{n} 4^{s-n}\tilde{\Psi}^{n-s}_{12}H^{(n-s)}_{12}$ and

$$q_{12_t} = \sum_{s=0}^{n} 4^{s-n}D\tilde{\Psi}^{n-s+1}_{12}H^{(n-s)}_{12} = \sum_{s=0}^{n} 4^{s-n}\tilde{\Phi}^{n-s+1}_{12}\cdot H^{(n-s)}_{12}, \qquad (5.10)$$

where

$$\tilde{\Phi}_{12} \doteq D\tilde{\Psi}_{12}D^{-1} = D^2 + \tilde{q}^{+}_{12} + D\tilde{q}^{+}_{12}D^{-1} + \tilde{q}^{-}_{12}D^{-1}\tilde{q}^{-}_{12}D^{-1}. \qquad (5.11)$$

If we assume that

$$C_{12} = \sum_{j=0}^{n}\lambda^j C^{(j)}_{12},\ E_{12} = A_{12} = -4\sum_{j=0}^{n+1}\lambda^j\bar{H}^{(j)}_{12},\ \bar{H}^{(j)}_{12} = \bar{H}^{(j)}(y_1,y_2),$$

then $C^{(n)}_{12} = D^{-1}\tilde{q}^{-}_{12}\cdot\bar{H}^{(n+1)}_{12}+H^{(n)}_{12}$; $C^{(j-1)}_{12} = \frac{1}{4}\tilde{\Psi}_{12}C^{(j)}_{12} + D^{-1}\tilde{q}_{12}\cdot H^{(j)}_{12}+H^{(j-1)}_{12}\ (1 \le j \le n)$;

$q_{12_t} = D\tilde{\Psi}_{12}C^{(0)}_{12}+4\tilde{q}^{-}_{12}\bar{H}^{(0)}_{12}$, where $H^{(j)}_{12} = H^{(j)}(y_1,y_2)$. The choice $H^{(j)}_{12} = 0$ for $0 \le j \le n$ yields $C^{(0)}_{12} = \sum_{s=0}^{n} 4^{s-n}\tilde{\Psi}^{n-s}_{12}D^{-1}\tilde{q}^{-}_{12}\cdot\bar{H}^{(n-s+1)}_{12}$ and

$$q_{12_t} = \sum_{s=0}^{n+1} 4^{s-n}D\tilde{\Psi}^{n-s+1}_{12}D^{-1}\tilde{q}^{-}_{12}\cdot\bar{H}^{(n-s+1)}_{12} = \sum_{s=0}^{n+1} 4^{s-n}\tilde{\Phi}^{n-s+1}_{12}\tilde{q}^{-}_{12}\cdot\bar{H}^{(n-s+1)}_{12}. \qquad (5.12)$$

Thus the isospectral problem (5.3) generates the classes of evolution equations (5.10) and (5.12)

It turns out that the transformation $q_{12} \to \delta_{12}q_1$, $q_1 = q(x,y_1)$, is an exact re tion of equations (5.10-11) if, at the same time, $4^{s-n}H^{(n-s)}_{12}$, $4^{s-n}\bar{H}^{(n+1-s)}_{12}$ $\to \beta^s\binom{n+1}{s}\delta^s_{12}$. In this case $\tilde{q}^{\pm}_{12} \to \tilde{q}^{\pm}_{12}$, $\tilde{\Phi}_{12} \to \Phi_{12}$ and

$$\delta_{12}q_{1_t} = \sum_{\ell=0}^{n+1}\beta^\ell\binom{n+1}{\ell}\Phi^{n+1-\ell}_{12}D\cdot\delta^\ell_{12} = \delta_{12}\Phi^{n+1}_{12}D\cdot 1 = \delta_{12}\Phi^{n}_{12}\hat{M}_{12}\cdot 1, \qquad (5.13a)$$

$$\delta_{12}q_{1_t} = \sum_{\ell=0}^{n+1}\beta^\ell\binom{n+1}{\ell}\Phi^{n-\ell+1}_{12}q^{-}_{12}\cdot\delta^\ell_{12} = \delta_{12}\Phi^{n}_{12}q^{-}_{12}\cdot 1 = \delta_{12}\Phi^{n}_{12}\hat{N}_{12}\cdot 1. \qquad (5.13b)$$

Proceeding exactly in the same way it is possible to show that the nonlocal eigenvalue problem

$$W_x = JW_y + \tilde{Q}W + \lambda JW, \qquad (5.14)$$

generates the following class of evolution equations

$$Q_{12_t} = \sum_{\ell=0}^{n} a_{n,\ell}\tilde{\Phi}_{12}^{n-\ell}\tilde{Q}_{12}^{-} \cdot H_{12}^{(\ell)}, \quad Q_{12} \doteq Q(x,y_1,y_2), \tag{5.15}$$

where

$$\tilde{\Phi}_{12}F_{12} \doteq \sigma(P_{12} - \tilde{Q}_{12}^{+}P_{12}^{-1}\tilde{Q}_{12}^{+})F_{12}, \quad F_{12} \doteq F(x,y_1,y_2) \text{ off-diagonal} \tag{5.16a}$$

$$\tilde{Q}_{12}^{\pm}F_{12} \doteq \int_{\mathbb{R}} dy_3(Q_{13}F_{32} \pm F_{13}Q_{32}), \tag{5.16b}$$

$\sigma = \text{diag}(1,-1)$ and $H_{12}^{(j)}$ is defined by

$$P_{12}H_{12}^{(j)} = 0, \quad H_{12}^{(j)} \text{ diagonal}. \tag{5.16c}$$

Also in this case the transformation $Q_{12} \to \delta_{12}Q_1$ is a reduction of (5.15) if $a_{n,\ell} \to \beta^{\ell}\binom{n}{\ell}$ $(\beta = 2\alpha)$ and $H_{12}^{(\ell)} \to \delta_{12}^{\ell}I$ or $\delta_{12}^{\ell}\sigma$. In fact, $\tilde{Q}_{12}^{\pm} \to Q_{12}^{\pm}$, $\tilde{\Phi}_{12} \to \Phi_{12}$. Thus one obtains the following classes of equations

$$\delta_{12}Q_{11_t} = \sum_{\ell=0}^{n} \beta^{\ell}\binom{n}{\ell}\Phi_{12}^{n-\ell}Q_{12}^{-}\delta_{12}^{\ell}I = \delta_{12}\Phi_{12}^{n}Q_{12}^{-}I \tag{5.17a}$$

or

$$\delta_{12}Q_{11_t} = \sum_{\ell=0}^{n} \beta^{\ell}\binom{n}{\ell}\Phi_{12}Q_{12}^{-}\delta_{12}^{\ell}\sigma = \delta_{12}\Phi_{12}^{n}Q_{12}^{-}\sigma, \tag{5.17b}$$

associated with the eigenvalue problem

$$W_x = JW_y + WQ + \lambda JW.$$

The above results clearly imply that all the notions introduced in [1] to characterize the algebraic properties of equations in 2+1 dimensions can be justified and interpreted in terms of the algebraic structure of the corresponding non-local versions. For example:

i) The above derivations both motivate and explain the derivation of the recursion operators introduced in [1] and [14]. In particular the crucial role played by the integral representation of differential operators is clarified.

ii) The directional derivative introduced in [1], which is the main tool needed to investigate the algebraic properties of equations in 2+1 dimensions, can be derived from the usual Frechét derivative in the space of non-local operators. For example, the Frechét derivative of $\bar{q}^{\pm}_{12}g_{12}$ in a direction $f_{12}$ is

$$\bar{q}^{\pm\prime}_{12}[f_{12}]g_{12} = f^{\pm}_{12}g_{12}, \tag{5.18a}$$

$$f^{\pm}_{12}g_{12} \doteq \int_{R} dy_3(f_{13}g_{32} \pm g_{13}f_{32}), \tag{5.18b}$$

which is exactly the direction derivative $q^{\pm}_{12_d}[f_{12}]g_{12}$ introduced in [1].

iii) The definition of an admissible function and of its derivative follows from the fact that reduced functions admit a double representation; for example (5.13b) implies

$$\sum_{\ell=0}^{n} \beta^{\ell}\binom{n}{\ell}\phi^{n-\ell}_{12}q^{-}_{12}\delta^{\ell}_{12} = \delta_{12}\phi^{n}_{12}q^{-}_{12}\cdot 1. \tag{5.19}$$

But the directional derivative is defined <u>only</u> on the admissible representation given by the left hand side of (5.19), which is the form of the function before the reduction: $\sum_{\ell=0}^{n} a_{n,\ell}\bar{\phi}^{n-\ell}_{12}\bar{q}^{-}_{12}H^{(\ell)}_{12}$.

In Appendix A we investigate (equations (A.3)) the algebra of the nonlocal operators $a^{\pm}_{12}$ defined in (5.18b). Here we remark that this algebra can also be interpreted as an algebra of matrices in which $\pm$ indicates the oeprations of anticommutator and commutator respectively, namely $a^{\pm}b = ab \pm ba$. (See also Appendix C of [1].) This is not a coincidence and the following important observations, here illustrated on the recursion operator $\Phi_{12}$ of the KP class, can be made.

i) Integral operators:

$$q^{\pm}_{12}f_{12} = \int_{R} dy_3(q_{13}f_{32} \pm f_{13}q_{32}). \tag{5.20a}$$

$$q_{12} = \delta_{12} q_1 + \alpha \delta'_{12}, \tag{5.20b}$$

is equivalent to the introduction of the integral operator $\tilde{q}^{\pm}_{12}$. Then $\Phi_{12}$ becomes the nonlocal recursion operator $\tilde{\Phi}_{12}$, defined in (5.11) and associated with the nonlocal eigenvalue problem (5.3).

ii) Matrix operators:

$$q^{\pm}f \doteq qf \pm fq; \quad q,f \text{ matrices}, \tag{5.21}$$

reduces $\Phi_{12}$ to the well-known matric recursion operator

$$\Phi \doteq D^2 + q^+ + Dq^+D^{-1} + q^-D^{-1}q^-D^{-1}, \tag{5.22}$$

associated with the N x N matrix Schroedinger eigenvalue problem in one dimension [15].

The directional derivative $q^{\pm}_{12_d}[f_{12}]g_{12}$ of $q^{\pm}_{12}$:

$$q^{\pm}_{12_d}[f_{12}]g_{12} = f^{\pm}_{12} g_{12}, \tag{5.23}$$

i) is exactly the usual Fréchet derivative $\tilde{q}^{\pm\prime}_{12}[f_{12}]g_{12}$ of $\tilde{q}^{\pm}_{12}$.

ii) Corresponds to the usual Fréchet derivative $q^{\pm}[f]g$ of $q^{\pm}$:

$$q^{\pm\prime}[f]g = f^{\pm}g = fg \pm gf. \tag{5.24}$$

Since the $\pm$ operators in (5.20a), (5.8b), (5.21) and (5.18b) satisfy the same algebraic identities (A.3), then important algebraic properties of the recursion operator $\Phi_{12}$ of the KP equation (like hereditariness) are equivalent to the corresponding properties of the nonlocal recursion operator $\tilde{\Phi}_{12}$ (5.11) and, even more remarkable, of the matrix recursion operator $\Phi_{12}$ (5.22).

In order to make this connection with the matrix formalism more clear, we observe that the nonlocal problem (5.3) can be obtained taking the $N \to \infty$ limit of the N x N matrix one dimensional Schroedinger problem

$$\underline{W}_{xx} + \underline{\underline{q}}\, \underline{W} = \lambda \underline{W}, \tag{5.25}$$

where the coefficients of the matrix $\underline{\underline{q}}$ are chosen in the form

$$(\underset{\approx}{q})_{ij} = q_{ij}(x,t) + a(\delta_{i,j+1} - \delta_{i,j-1}), \tag{5.26}$$

with the obvious prescriptions

$$q_{ij}(x,t) \underset{N\to\infty}{\to} q(x,t,y_1,y_2);\ a(\delta_{i,j+1} - \delta_{i,j-1}) \underset{N\to\infty}{\to} \alpha\frac{\partial}{\partial y_1}\ . \tag{5.27}$$

The connection between equations in 2+1 and N x N matrix equations in 1+1 was first used by P. Caudrey. He introduced in [16] a NxN spectral problem (similar to (5.25)) which reduces to (5.2) in the limit $N\to\infty$. Then he showed that the NxN Riemann-Hilbert formalism associated with it becomes, in the limit $N\to\infty$, the nonlocal Riemann-Hilbert and the $\bar{\partial}$ formalisms of (5.2) [17].

The connection established in this section between the spectral problems (5.25), (5.3) and (5.2) implies that the well established theory of recursion operators and their connection to the bi-Hamiltonian formalism in 1+1 dimensions, once applied to the matrix problem (5.25), gives rise, in the limit $N\to\infty$, to the corresponding theory developed in [1] and in this paper for 2+1 dimensional systems.

It is remarkable that both algebraic properties and methods of solution for integrable systems in 2+1 dimensions can be justified and obtained from the corresponding properties of 1+1 dimensional systems.

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## APPENDIX A.

In this Appendix we present some of the explicit calculations necessary to apply the results presented in this paper to the classes of evolution equations associated with the KP and the DS equations. In order to make this paper self-contained, we first present some results contained in Appendices B, C of [1].

The directional derivatives of the basic operators $q^{\pm}_{12}$ and $Q^{\pm}_{12}$, defined in (1.4b) and (1.10b) respectively, are

$$q^{\pm}_{12_d}[f_{12}]g_{12} = f^{\pm}_{12}g_{12}, \quad f_{12},\ g_{12} \text{ scalars}, \tag{A.1a}$$

$$Q^{\pm}_{12_d}[f_{12}]g_{12} = f^{\pm}_{12}g_{12}, \quad f_{12} \text{ off-diagonal matrix}, \tag{A.1b}$$

where $f^{\pm}_{12}$ are defined by

$$f^{\pm}_{12}g_{12} \triangleq \int_{\mathbb{R}} dy_3(f_{13}g_{32} \pm g_{13}f_{32}). \tag{A.2}$$

The integral operators $f^{\pm}_{12}$ have the following algebraic properties

$$a^{\pm}_{12}b_{12} = \pm b^{\pm}_{12}a_{12}, \tag{A.3a}$$

$$(a^{\pm}_{12}b^{\pm}_{12} - b^{\pm}_{12}a^{\pm}_{12})c_{12} = (a^{-}_{12}b_{12})^{-}c_{12} = -c^{-}_{12}a^{-}_{12}b_{12}, \tag{A.3b}$$

$$(a^{+}_{12}b^{-}_{12} \mp b^{\mp}_{12}a^{\pm}_{12})c_{12} = (a^{\mp}_{12}b_{12})^{\pm}c_{12} = \pm c^{\pm}_{12}a^{\mp}_{12}b_{12}, \tag{A.3c}$$

$$a^{\pm *}_{12} = \pm a^{\pm}_{12}. \tag{A.3d}$$

Moreover the integral representations

$$q^{\pm}_{12}f_{12} = \int_{\mathbb{R}} dy_3(q_{13}f_{32} \pm f_{13}q_{32}),\ q_{12} = \delta_{12}q_1 + \alpha\delta'_{12},$$

$$Q^{\pm}_{12}f_{12} = \int_{\mathbb{R}} dy_3(Q_{13}f_{32} \pm f_{13}Q_{32}), \quad Q_{12} = \delta_{12}Q_1,$$

imply that the operators $q^{\pm}_{12}$ and $Q^{\pm}_{12}$ satisfy equations (A.3) as well. Equations (A.3) are conveniently used to show that:

a) The recursion operators $\Phi_{12}$ (1.4) and (1.10) are strong symmetries of the starting symmetries $\hat{K}^{0}_{12}H_{12}$ (1.5-6) and (1.11-12) respectively. For example, if $\hat{K}^{0}_{12} = Q^{-}_{12}$ and $H_{12}$ is given by (1.12)

$$\Phi_{12_d}[Q^{-}_{12}H_{12}]f_{12} - (Q^{-}_{12}H_{12})_d[\Phi_{12}f_{12}] + \Phi_{12}(Q^{-}_{12}H_{12})_d[f_{12}] =$$

$$= -\sigma[(Q^{-}_{12}H_{12})^{+}P^{-1}_{12}Q^{+}_{12} + Q^{+}_{12}P^{-1}_{12}(Q^{-}_{12}H_{12})^{+}]f_{12} - (\sigma(P_{12}-Q^{+}_{12}P^{-1}_{12}Q^{+}_{12})f_{12})^{-}H_{12} +$$

$$+ \sigma(P_{12} - Q^{+}_{12}P^{-1}_{12}Q^{+}_{12})f^{-}_{12}H_{12} = 0,$$ since the terms without $Q^{\pm}_{12}$ give $-\sigma(P_{12}f_{12})^{-}H_{12} + \sigma P_{12}f^{-}_{12}H_{12} = 0$, and the terms with $Q^{\pm}_{12}$ give $-\sigma(Q^{-}_{12}H_{12})^{+}P^{-1}_{12}Q^{+}_{12}f_{12}$

$$-\sigma Q^{+}_{12}P^{-1}_{12}(Q^{-}_{12}H_{12})^{+}f_{12} + (\sigma Q^{+}_{12}P^{-1}_{12}Q^{+}_{12}f_{12})^{-}H_{12} - \sigma Q^{+}_{12}P^{-1}_{12}Q^{+}_{12}f^{-}_{12}H_{12} =$$

$$= -\sigma(((Q^{-}_{12}H_{12})^{+} + H^{-}_{12}Q^{+}_{12})P^{-1}_{12}Q^{+}_{12}f_{12} + Q^{+}_{12}P^{-1}_{12}(f^{+}_{12}Q^{-}_{12}H_{12} + Q^{+}_{12}f^{-}_{12}H_{12})) =$$

$$= -\sigma Q^{+}_{12}P^{-1}_{12}(H^{-}_{12}Q^{+}_{12}f_{12} + f^{+}_{12}Q^{-}_{12}H_{12} + Q^{+}_{12}f^{-}_{12}H_{12}) = 0.$$

b) The Lie algebra of the starting symmetries is given by equations (1.7) and (1.13). For example

i) if $\hat{K}^{0}_{12}H_{12}$ are given by (1.5-6):

$$[\hat{N}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d = ((Dq^{+}_{12} + q^{-}_{12}D^{-1}q^{-}_{12})H^{(2)}_{12})^{-}H^{(1)}_{12} - D(q^{-}_{12}H^{(1)}_{12})^{+}H^{(3)}_{12} -$$

$$-(q^{-}_{12}H^{(1)}_{12})^{-}D^{-1}q^{-}_{12}H^{(2)}_{12} - q^{-}_{12}D^{-1}(q^{-}_{12}H^{(1)}_{12})^{-}H^{(2)}_{12} = -Dq^{+}_{12}(H^{(1)}_{12})^{-}H^{(2)}_{12} +$$

$$+q^{-}_{12}D^{-1}(-(H^{(1)}_{12})^{-}q^{-}_{12}H^{(2)}_{12} + (H^{(2)}_{12})^{-}q^{-}_{12}H^{(1)}_{12}) = -\hat{M}_{12}H^{(3)}_{12};$$

ii) if $\hat{K}^{0}_{12}H_{12}$ are given by (1.11-12)

$$[\hat{N}_{12}H^{(1)}_{12}, \hat{M}_{12}H^{(2)}_{12}]_d = (Q^{-}_{12}\sigma H^{(2)}_{12})^{-}H^{(1)}_{12} - (Q^{-}_{12}H^{(1)}_{12})^{-}\sigma H^{(2)}_{12} =$$

$$= -(H^{(1)}_{12})^{-}Q^{-}_{12}\sigma H^{(2)}_{12} + (\sigma(H^{(2)}_{12}))^{-}Q^{-}_{12}H^{(1)}_{12} = -\hat{M}_{12}H^{(3)}_{12} .$$

c) The functions $T_{12}$ given by (4.16) and (4.25) satisfy equations (4.10);

for examples

i) if $T_{12} = \delta_{12}$, then

$$S_{12}f_{12} = \Phi_{12_d}[\delta_{12}]f_{12} = (2\delta^+_{12} + \delta^-_{12}D^{-1}q^-_{12} + q^-_{12}D^{-1}\delta^-_{12})f_{12} = 4f_{12},$$

since $\delta_{12_d} = 0$ and $\delta^+_{12}f_{12} = 2f_{12}$, $\delta^-_{12}f_{12} = 0$.

ii) If $T_{12} = \frac{x}{2}\sigma Q^+_{12}\delta_{12}I$, then equations (4.10) are satisfied using the following results:

$$T_{12_d}[f_{12}] = \frac{x}{2}\sigma f^+_{12}\delta_{12}I = x\sigma f_{12},$$

$$T^{\pm}_{12}f_{12} = x(\sigma Q_1 f_{12} \pm f_{12}\sigma Q_2) = x\sigma\begin{cases} Q^{\mp}_{12}f_{12}, & f_{12} \text{ off-diagonal} \\ Q^{\pm}_{12}f_{12}, & f_{12} \text{ diagonal} \end{cases}.$$

*For instance:*

$$S_{12}f_{12} = -\sigma(T^+_{12}P^{-1}_{12}Q^+_{12} + Q^+_{12}P^{-1}_{12}T^+_{12})f_{12} + \sigma(P_{12}-Q^+_{12}P^{-1}_{12}Q^+_{12})x\sigma f_{12} -$$

$$- x(P_{12} - Q^+_{12}P^{-1}_{12}Q^+_{12})f_{12} = f_{12}.$$

d) $\Phi^n_{12}[\hat{K}^0_{12}H_{12}, T_{12}] = 0$, if $\hat{K}^0_{12}H_{12}$ and $T_{12}$ are given by (1.11-12) and (4.25) respectively, or by $q^-_{12}H_{12}$, $H_{12} = H(y_1,y_2)$, and $\delta_{12}$. For example

i) $\Phi^n_{12}[q^-_{12}H_{12},\delta_{12}]_d = \Phi^n_{12}\delta^-_{12} \cdot H_{12} = 0.$

ii) $\Phi^n_{12}[Q^-_{12}H_{12},T_{12}]_d = \Phi^n_{12}(T^-_{12}H_{12} - T_{12_d}[Q^-_{12}H_{12}]) =$

$$= \Phi^n_{12}(x\sigma Q^-_{12} - x\sigma Q^-_{12}) \cdot H_{12} = 0.$$

e) Equation (4.17b) holds. It follows from $\hat{M}_{12_d}[\delta_{12}] =$ $D\delta^+_{12} + \delta^-_{12}D^{-1}q^-_{12} + q^-_{12}D^{-1}\delta^-_{12} = 2D$, which implies

$$\Phi_{12}^{n}[\hat{M}_{12}\cdot H_{12},\delta_{12}] = 2\Phi_{12}^{n}D\cdot H_{12}. \qquad \text{(A.4)}$$

Different choices of $H_{12} = H(y_1,y_2)$ give different results. As it was shown in Appendix B of [1]

$$\Phi_{12}^{n+1}D\cdot H_{12} = \sum_{\ell=0}^{n}(2\alpha)^{\ell}\Phi_{12}^{n-\ell}\cdot\hat{M}_{12}\cdot H_{12}^{(\ell)},\ H_{12}^{(\ell)} \doteq \partial^{\ell}\frac{H(y_1-y_2)}{\partial y_1^{\ell}}; \qquad \text{(A.5)}$$

an analogous, although more tedious derivation, gives

$$\Phi_{12}^{n+1}D\cdot H_{12} = \Phi_{12}^{n}\hat{M}_{12}\cdot H_{12} + \sum_{\ell=1}^{\nu_n}a_{\ell}(2\alpha)^{2\ell}\Phi_{12}^{n-2\ell}\hat{M}_{12}\cdot H_{12}^{(2\ell)}, \qquad \text{(A.6a)}$$

$$H_{12}^{(\ell)} \doteq \frac{\partial^{\ell}H(y_1+y_2)}{\partial y_1^{\ell}},\quad a_{\ell} \doteq C_{\ell}^{(\ell)},\quad \nu_n \doteq \begin{cases}(n-1)/2, & n\ \text{odd}\\ n/2, & n\ \text{even}\end{cases} \qquad \text{(A.6b)}$$

and the coefficients $C_{\ell}^{(\ell)}$ are obtained through the following recursive construction:

$$C_{\ell}^{(m)} = C_{\ell}^{(m-1)} + 2C_{\ell-1}^{(m-1)} + C_{\ell-2}^{(m-1)}, \qquad \text{(A.7)}$$
$$C_{0}^{(0)} = 1,$$

where $C_{b}^{(a)} = 0$ if $b < 0$ and $b > a$. Equations (A.4) and (A.6) imply equation (4.17b).

f) $\Theta_{12}^{-1}\Phi_{12}^{n}\hat{K}_{12}^{0}H_{12}$ are extended gradients; for example if

i) $\hat{K}_{12}^{0} = \hat{N}_{12} \doteq q_{12}^{-}$, $H_{12} = H(y_1,y_2)$, $\Theta_{12} = D$ and $n = 0$:

$$\langle f_{12},(D^{-1}\hat{N}_{12}H_{12})_d[g_{12}]\rangle = \langle f_{12}, D^{-1}g_{12}^{-}H_{12}\rangle = \langle D^{-1}f_{12},H_{12}^{-}g_{12}\rangle =$$
$$= -\langle H_{12}^{-}D^{-1}f_{12},g_{12}\rangle = \langle D^{-1}f_{12}^{-}H_{12},g_{12}\rangle.$$

ii) $\hat{K}_{12}^{0} = \hat{M}_{12} \doteq Dq_{12}^{+} + q_{12}^{-}D^{-1}q_{12}^{-}$, $H_{12} = H(y_1,y_2)$, $\Theta_{12} = D$ and $n = 0$:

$$\langle f_{12},(D^{-1}\hat{M}_{12}H_{12})_d[g_{12}]\rangle = \langle f_{12},g_{12}^{+}H_{12} + D^{-1}g_{12}^{-}D^{-1}q_{12}^{-}H_{12} + D^{-1}q_{12}^{-}D^{-1}g_{12}^{-}H_{12}\rangle =$$
$$= \langle f_{12},(H_{12}^{+}-D^{-1}((D^{-1}q_{12}^{-}H_{12})^{-} + q_{12}^{-}D^{-1}H_{12}^{-}))g_{12}\rangle = \langle (H_{12}^{+}-((D^{-1}q_{12}^{-}H_{12})^{-} +$$
$$+ H_{12}^{-}D^{-1}q_{12}^{-})D^{-1})f_{12},g_{12}\rangle = \langle (H_{12}^{+}-D^{-1}((D^{-1}q_{12}^{-}H_{12})^{-} + q_{12}^{-}H_{12}^{-}D^{-1}))f_{12},g_{12}\rangle.$$

iii) $\hat{K}^{0}_{12} = \hat{M}_{12} \doteq Q^{-}_{12}\sigma$, $H_{12}$ defined in (1.12) and $n = 0$:

$$<f_{12}, (\sigma\hat{M}_{12}H_{12})_d g_{12}> = <f_{12}, -H^{+}_{12}g_{12}> = <-H^{+}_{12}f_{12}, g_{12}> .$$

iv) $\hat{K}^{0}_{12} = \hat{N}_{12} \doteq Q^{-}_{12}$, $H_{12}$ defined in (1.12) and $n=1$:

$$<f_{12},(\sigma\hat{N}^{(1)}_{12}H_{12})_d g_{12}> = <f_{12},(-(P^{-1}_{12}Q^{+}_{12}Q^{-}_{12}H_{12})^{+} - P_{12}H^{-}_{12} +$$

$$+ Q^{+}_{12}H^{-}_{12}P^{-1}_{12}Q^{+}_{12})g_{12} = <(-(P^{-1}_{12}Q^{+}_{12}Q^{-}_{12}H_{12})^{+} - P_{12}H^{-}_{12} + Q^{+}H^{-}_{12}P^{-1}_{12}Q^{+}_{12})f_{12}, g_{12}.>$$

g) Equation (4.24b) holds, since

$$<\hat{Y}^{(n+1)}_{12}H_{12}, \ x\sigma Q^{+}_{12}\delta_{12}I> = -<xQ^{+}_{12}\sigma\hat{Y}^{(n+1)}_{12}H_{12}, \delta_{12}I> =$$

$$= \int_{\mathbb{R}^3} dx\, dy_1 dy_2 \delta_{12} \text{trace } Q^{+}_{12}\sigma\hat{Y}^{(n+1)}_{12}H_{12}.$$

## APPENDIX B.

IN this Appendix we show that if $\Phi$ is factorizable in terms of compatible Hamiltonian operators $\Omega$ and $\Theta$ in the form $\Phi = \Omega\Theta^{-1}$, and if $\Theta$ is invertible and $\Theta_L = 0$, then equation (4.5) holds.

We first show that

$$(\Phi T)^{*}_{L} = \mathcal{L}^{*}_{T} + T^{*}_{L}\Phi^{*}, \quad \mathcal{L}_{T}b \doteq \Phi_L[b]T, \tag{B.1}$$

$$\Phi_L[v]T + \Theta\mathcal{L}^{*}_{T}\Theta^{-1}v = \Phi_L[T]b. \tag{B.2}$$

(B.1) simply follows from the definition of the adjoint:

$$<(\Phi T)^{*}_{L} a,b> = <a,\Phi_L[b]T + \Phi T_L[b]> = <(\mathcal{L}^{*}_{T} + T^{*}_{L}\Phi^{*})a,b>,$$

while (B.2) requires the use of all the hypothesis of this Lemma.:

$$<\Phi_L[v]T + \Theta\mathcal{L}^{*}_{T}\Theta^{-1}v,\alpha> = <\Omega_L[\Theta(\Theta^{-1}v)]\Theta^{-1}T,\alpha> + <\Omega_L[\Theta\alpha]\Theta^{-1}v, \ \Theta^{-1}T> =$$

$$\langle \alpha, \Omega_L[T]\Theta^{-1}v \rangle = \langle \alpha, \Phi_L[T]v \rangle .$$

Then, using (B.1-2) and (4.4) for $n = 0$, we obtain equation (4.5):

$$((\Phi T)_L + \Theta(\Phi T)^*_L\Theta^{-1})v = \Phi(T_L[v] + \Theta T^*_L\Theta^{-1}v) + \Phi_L[v]T +$$

$$+ \Theta L^*_T\Theta^{-1}v + \Theta(T^*_L\Phi^* - \Phi^*T^*_L)\Theta^{-1}v =$$

$$= \Phi(T_L[v] + \Theta T^*_L\Theta^{-1}v) + \Theta S^*\Theta^{-1}v.$$


REFERENCES

[1] P.M. Santini and A.S. Fokas, Recursion Operators and Bi-Hamiltonian Structures in MUltidimension I, preprint, INS #65, Clarkson University, July 1986.

F. Magri, J. Math. Phys. 19, 1156 (1978); Nonlinear Evolution Equations and Dynamical Systems, edited by M. Boiti, F. Pempinelli and G. Soliani, Lecture Notes in Physics, Vol. 120 (Springer, New York, 1980), p. 233.

[3] I.M. Gel'fand and I.Y. Dorfman, Fuct. Anal. Appl. 13, 13 (1979); 14, 71 (1980).

[4] A.S. Fokas and B. Fuchssteiner, Lett. Nuovo Cimento 28, 299 (1980); B. Fuchssteiner and A.S. Fokas, Physica 4D, 47 (1981).

[5] A.S. Fokas and B. Fuchssteiner, Phys. Lett. A86, 341 (1981).

[6] W. Oevel and B. Fuchssteiner, Phys. Lett. A88, 323 (1982).

[7] H.H. Chen, Y.C. Lee and J.E. Lin, Physica 9D, 493 (1983).

[8] H.H. Chen, J.E. Lin and Y.C. Lee, On the Direct Construction of the Inverse Scattering Operators of Nonlnear Hamiltonian Systems (preprint) 1986.

[9] W. Oevel, A geometrical Approach to Integrable Systems Admitting Scaling Symmetries, (preprint) University of Paderborn, January 1986.

[10] F. Magri, (private communication).

[11] B. Fuchssteiner, Prog. Theor. Phys. 70, 150 (1983).

[12] I.Y. Dorfman, Deformations of the Hamiltonian Structures and Integrable Systems (preprint).

[13] W. Oevel, Mastersymmetries for Finite Dimensional INtegral Systems: The Calogero-Moser System.

[14] A.S. Fokas and P.M. Santini, Stud. Appl. Math. 75, 179 (1986).

[15] F. Calogero and A. Degasperis, Nuovo Cimento, 39B, 1 (1977).

[16] P. Caudrey, Discrete and Periodic Spectral Transforms Related to the Kadomtsev-Petviashvili Equation, Preprint U.M.I.S.T.,

[17] A.S. Fokas and M.J. Ablowitz, Stud. Appl. Math. 69, 211 91983); M.J. Ablowitz, D. BarYaacov and A.S. Fokas, Stud. Appl. Math. 69, 135 (1983).

[18] K.M. Case, J. Math. Phys. 26, 1201 (1985).

[19] D. David,N. Kamran, D. Levi and P. Winternitz, Symmetry Reduction for the KP Equation Using a Loop Algebra, (preprint), Université De Montréal (1985).

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