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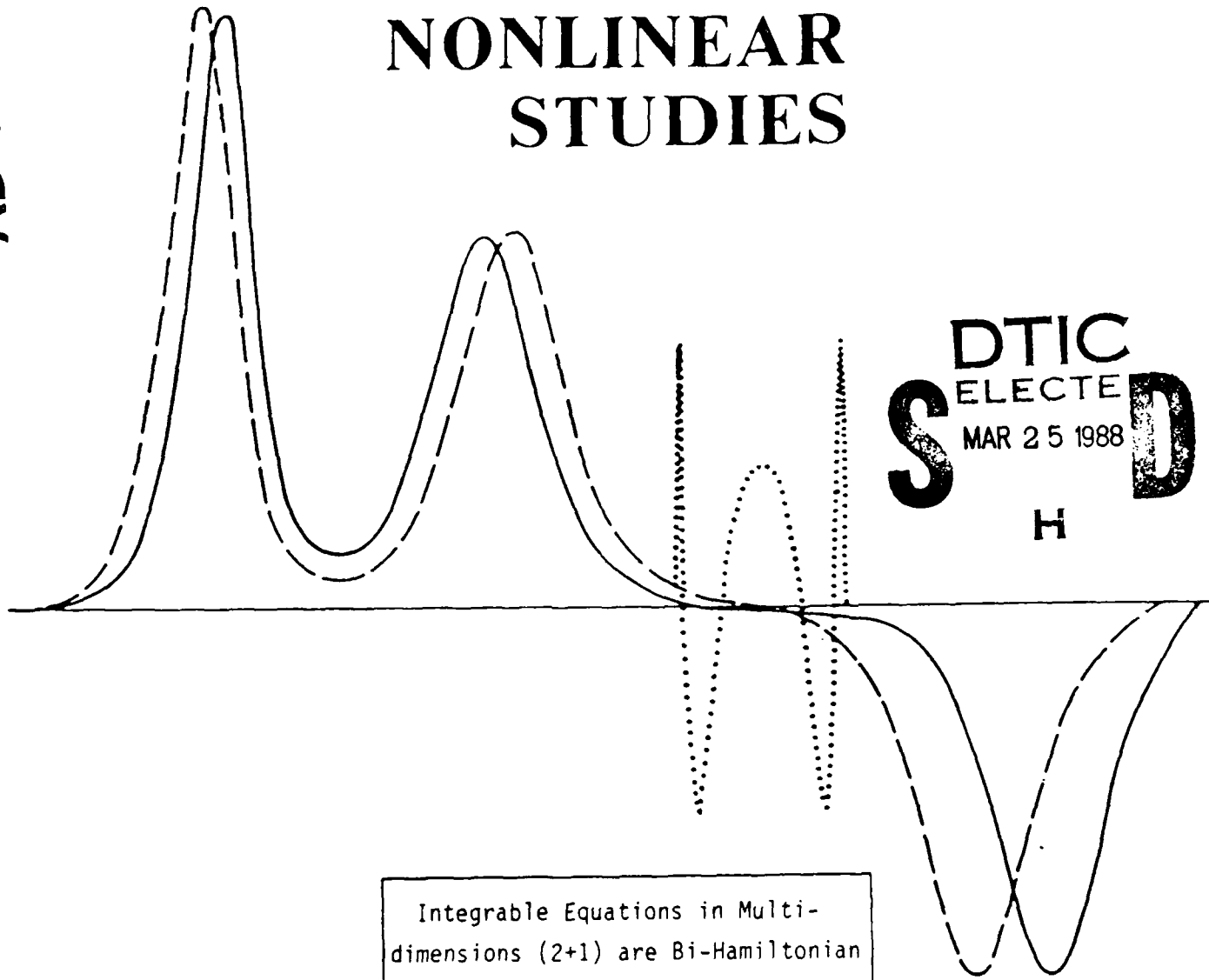
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Integrable Equations in Multidimensions (2+1)

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1.   INTRODUCTION

Ablowitz, Kaup, Newell and Segur [1], following ideas of Lax [2] were the first to solve in the concrete case of the Dirac problem the following question: Given a linear eigenvalue problem find all nonlinear equations that are related to it. They found that associated with a given eigenvalue problem there exists a hierarchy of infinitely many equations. This hierarchy is generated by a certain linear operator. This operator is the squared eigenfunction operator of the underlying linear eigenvalue problem. The operator generating the KdV hierarchy (i.e. the squared eigenfunction operator of the Schrödinger eigenvalue problem) was found by Lenard. For other eigenvalue problems see [3]-[10].

Olver [11] established the group theoretical origin of the above hierarchy: Finding the hierarchy associated with a given equation is equivalent to finding the non-Lie point symmetries of the given equation. He thus interpreted the squared eigenfunction operator as an operator

mapping symmetries onto symmetries; this lead to a simple mathematical
characterization of the recursion operator ϕ. Olver,
was thus the first to establish that certain integrable nonlinear equa-
tions possess infinitely many symmetries. This motivates the following
question: Is there an algorithmic way for generating equations possess-
ing infinitely many symmetries? Fuchssteiner [12] discovered such a
way: If an operator ϕ has a certain mathematical property called
hereditary then the equations $u_t = \phi^n u_x$, n integer,
possess infinitely many symmetries. From the above discussion it
follows that both linear eigenvalue problems and hereditary operators
yield hierarchies of equations possessing infinitely many symmetries.
Actually Anderson and the author [13], following ideas of Fuch-
ssteiner, have shown that eigenvlaue problems algorithmically imply
hereditary operators.

Equations solvable by the Inverse Scattering Transform are Hamil-
tonian systems. Magri, in a pioneering paper [14], realized that
integrable Hamiltonian systems have additional structure: They are bi-
Hamiltonian systems. Actually the underlying hereditary operator can
be factorized in terms of the two associated Hamiltonian operators.
The theory of factorizable hereditary operators has been further developed
by Fuchssteiner and the author [15] and by Gel'fand and Dorfman [16].

The understanding of the central role played by factorizable
hereditary operators for equations in 1+1, motivated a search for here-
ditary operators for equations in 2+1. However, in this direction
several negative results have appeared in the literature. For example,
Zakharov and Konopelchenko [17], in an interesting paper proved that
recursion operators (of a certain type naturally motivated from the re-
sults in 1+1) did not exist in multidimensions. A similar result has

been proven for the Benjamin-Ono (BO) equation [18]. It should be noted
that the BO equation has more similarities [19] with the Kadomtsev-
Petviashvili (KP) equation than with the KdV equation. Fuchssteiner and
the author [18] after failing to find a recursion operator for the BO
introduced the concept of the master-symmetries τ. Subsequently Oevel and
Fuchssteiner [20] found a master-symmetry for the KP equation. The τ
theory for equations in 2+1 has been developed by Dorfman [21] and Fuch-
ssteiner [22]. However, the τ is not related to the underlying iso-
spectral problem and also can not be used to construct a second Hamil-
tonian operator. This is a serious drawback: several prominent invest-
igators, for example Gel'fand [23] have considered the existence
of a bi-Hamiltonian formulation as fundamental to integrability. With-
out finding a recursion operator Φ, one cannot find the second
Hamiltonian operator. Several investigators have noticed that master-
symmetries also exist for equations in 1+1. The theory for the master-
symmetries T in 1+1 was developed by Oevel [24] (see also [25])
and is more satisfactory than the theory in 2+1: If one assumes that an
equation is invariant under scaling then there exist a one to one con-
structive relationship between T and the recursion operator Φ.

Recently P.M. Santini and the author [26]-[28] have found the
recursion operator and the bi-Hamiltonian formulation of a large class
of equations in 2+1. They have also established the general theory
associated with factorizable recursion operators in multidimensions.
Furthermore, both gradient and non-gradient (the 2+1 analogue of T)
master-symmetries are simply derived and their general theory is developed.

2.   MASTER SYMMETRIES

In this section we review certain aspects of non-gradient master-symmetries in 1+1 and gradient mastersymmetries in 2+1.

Definition 2.1.

A function τ is a master-symmetry of the equation $q_t = K$ iff the map

$$[\cdot,\tau]_L \quad \text{where} \quad [a,b]_L \doteq a'[b] - b'[a] \tag{2.1}$$

maps symmetries onto symmetries (prime denotes Fréchet derivative).

The first example of a master-symmetry was given for the Benjamin-Ono equation

$$q_t = Hq_{xx} + 2qq_x, \quad (Hf)(x) \doteq \frac{1}{\pi} \int_R \frac{d\xi f(\xi)}{\xi - x}. \tag{4.2}$$

It was shown in [18] that if $\tau \doteq x(Hq_{xx} + 2qq_x) + q^2 + \frac{3}{2}Hq_x$ and σ_n is a symmetry then $\sigma_{n+1} \doteq [\sigma_n, \tau]$ is also a symmetry. It was further shown in [18] that $D^{-1}\tau$ is a gradient function $(\tau'D + D\cdot\tau'^* = 0)$.

Master-symmetries are intimately related to time-dependent non-Lie-point symmetries [25]. Indeed, the first non-Lie-point time-dependent symmetry is a natural candidate for a master-symmetry: Consider the evolution equation $q_t = K^{(1)}$ and let $K^{(2)}$, $K^{(3)}$,... denote its time-independent non-Lie-point symmetries. Let

$$\sigma^{(2)} = tK^{(2)} + \tau \tag{2.3}$$

be a time-dependent non-Lie-point symmetry. Then

$$K^{(2)} + [tK^{(2)} + \tau, K^{(1)}]_L = 0, \quad \text{or} \quad K^{(2)} = [K^{(1)}, \tau]_L$$

and τ is a candidate for a master-symmetry.

2.1. Master-symmetries for equations in 1+1.

Lemma 2.1.

Let

$$S_i \doteq \Phi'[K_i] + [\Phi, K_i'], \qquad i = 1,2. \qquad (2.4)$$

If Φ is hereditary, i.e. if

$\Phi'[\Phi v]w - \Phi\Phi'[v]w$ is symmetric w.r.t. v, w,   then

$$\Phi^{n+m}[K_1,K_2]_L = [\Phi^n K_1, \Phi^m K_2]_L + \Phi^n(\sum_{r=1}^{m} \Phi^{m-r}S_1\Phi^{r-1})K_2 - \Phi^m(\sum_{r=1}^{n} \Phi^{n-r}S_2\Phi^{r-1})K_1, \qquad (2.5)$$

m,n are non-negative integers.

Proof.

See Theorem 2.1 of [28].

Corollary 2.1.

Assume that τ_0 is a scaling of both K and of the hereditary operator Φ, i.e.

$$[K,\tau_0] = \alpha K, \qquad \Phi'[\tau_0] + [\Phi,\tau_0'] = \beta\Phi. \qquad (2.6)$$

Then

(i) $$(\alpha + n\beta)\Phi^{n+1}K = [\Phi^n K, \Phi\tau_0]_L, \qquad (2.7)$$

   i.e. $\Phi\tau_0$ is a master-symmetry for $q_t = K$.

(ii) $$(\alpha + n\beta)\Phi^{n+m}K = [\Phi^n K, \Phi^m\tau_0]_L, \qquad (2.8)$$

   i.e. $\Phi^m\tau_0$ is a master-symmetry of order m for $q_t = K$.

(iii) $\Sigma \doteq (\alpha + n\beta)t\Phi^{n+1}K + \Phi\tau_0$ is a symmetry of $q_t = \Phi^n K$.

Proof.

(i) Apply Theorem 2.1 with

$$K_1 = K, \quad K_2 = \tau_0, \quad [K_1, K_2] = \alpha K, \quad L_1 = 0, \quad L_2 = \beta \Phi.$$

(ii) Similar to (i).

(iii) Use the definition of a symmetry.

In the above we derive T from Φ. Now we obtain Φ from T.

Lemma 2.2.

Let Φ be a hereditary operator such that $\Phi\Theta = \Theta\Phi^+$, where Θ is a constant, invertible, skew-symmetric operator. Then

$$(\Theta T)' + \Theta(\Phi T)'^+ \Theta^{-1} = \Phi(T' + \Theta(T')^+ \Theta^{-1}) + \Theta S^+ \Theta^{-1}, \tag{2.9}$$

where

$$S^+ \equiv \Phi'^+[T] + [T'^+, \Phi].$$

Proof.

See [28].

Theorem 2.1.

(i) If the hereditary oeprator Φ admits the scaling τ_0 then $\Phi\tau_0$ is a master-symmetry for the hierarchy generated by Φ.

(ii) Assume that the hereditary operator Φ admits the scaling τ_0 and that it also satisfies $\Phi\Theta = \Theta\Phi^+$, where Θ is a constant, invertible, skew-symmetric operator which also admits the scaling τ_0. Then

$$\Phi = (\Phi\tau_0)' + \Theta(\Phi\tau_0)'^+ \Theta^{-1}. \tag{2.10}$$

Proof.

(i) If Φ admits a scaling and K is generated from Φ then K also admits a scaling. Hence Corollary 2.1 implies (i) above.

(ii) Since ϕ admits a scaling, ϕ^+ also admits a scaling, hence S^+ is proportional to ϕ^+, thus $\Theta S^+ \Theta^{-1}$ is proportional to ϕ. Furthermore, since Θ admits the scaling τ_0, $\tau_0'\Theta + \Theta\tau_0'^+ = \alpha\Theta$, thus $\tau_0' + \Theta(\tau_0')^+\Theta^{-1}$ equals a constant. Hence (2.9) implies (2.10).

EXAMPLE

1. $\phi = D + q + q_x D^{-1}$ is the hereditary operator associated with Burgers equation. It admits the scaling $q \to \alpha q$, $x \to \alpha^{-1}x$, i.e. $\tau_0 = q + xq_x$. Thus $x(q_{xx} + 2qq_x) + q^2$ is a master-symmetry of Burgers equation.

2. $\phi = D^2 + 4q + 2q_x D^{-1}$ admits the scaling $q \to \alpha q$, $x \to \alpha^{-2}x$, i.e. $\tau_0 = q + 2xq_x$. Thus $T = \phi\tau_0$ is a master-symmetry of the KdV.

3. If $\tau_0 = q + 2xq_x$, then $\tau_0' + D(\tau_0')^+D^{-1} = -3$. Hence if T is the master-symmetry of KdV,

$$\phi = T' + D(T')^+D^{-1}$$

is the recursion operator of the KdV.

2.2. Gradient master-symmetries for equations in 2+1.

A straightforward generalization of Theorem 2.1 to equations 2+1 fails: i) ϕ could not be found, ii) the known master-symmetries τ were gradient functions, hence $\tau' + \phi(\tau')^+\phi^{-1} = 0$. It will be shown

in §3 that for equations in 2+1: i) Suitable generalizations of ϕ, de-
noted by ϕ_{12} can be found, ii) there exist non-gradient master-symmetries
T_{12} (for example for the KP $T_{12} = \phi_{12}^2 \delta(y_1-y_2)$, where δ denotes the Dirac
delta function). Hence a generalization of theorem 2.1 to equations in
2+1 is given in §3.

One can still develop a theory for master-symmetries without
using the connection with the recursion operator ϕ: see [21], [22].

3.   SYMMETRIES FOR EQUATIONS IN 2+1.

In this section we review the theory recently developed by Paolo
Santini and the author. We use the KP as an illustrative example and
quote the basic theorems when needed. We hope that this form of present-
ation will aid the non-expert reader to become familiar with the notions
and methods developed in [26]-[28]. We advise the non-expert reader to
read [15] before reading this paper since many of the results presented
here are two dimensional generalizations of results given in [15].

3.1. Derivation of Recursion Operators.

Given an isospectral eigenvalue problem there exists a simple
algorithmic way of obtaining a recursion operator. This approach in-
volves three steps: compatibility, an integral representation of a certain
differential operator, and an expansion in terms of delta functions.
Let us consider the eigenvalue equation

$$w_{xx} + q(x,y)w + \alpha w_y = 0, \qquad \alpha \text{ is a constant} \qquad (3.1)$$

and for convenience of notation we suppress the t-dependence. Using
vector notation, (3.1) yields

$$W \doteq \begin{pmatrix} w \\ w_x \end{pmatrix}, \qquad W_x = \begin{pmatrix} 0 & 1 \\ -\dot{q} & 0 \end{pmatrix} W, \qquad \dot{q} \doteq q + \alpha D_y; \; D_y = \frac{\partial}{\partial y} . \qquad (3.2)$$

1. Compatibility

Associated with $W_x = UW$ we look for compatible flows $W_t = VW$ where

$$V = \begin{pmatrix} A & 2C \\ B & E \end{pmatrix}, \quad A,B,C,E \text{ polynomials in } D_y.$$

Compatibility implies the <u>operator</u> equation

$$U_t = V_x - [U,V],$$

or

$$\begin{pmatrix} 0 & 0 \\ -\hat{q}_t & 0 \end{pmatrix} = \begin{pmatrix} A_x & 2C_x \\ B_x & E_x \end{pmatrix} - \left[\begin{pmatrix} 0 & 1 \\ -\hat{q} & 0 \end{pmatrix}, \begin{pmatrix} A & 2C \\ B & E \end{pmatrix} \right].$$

Solving in the above equation for A,B,E in terms of C we obtain the following operator equation:

$$q_t = C_{xxx} + [\hat{q},C]_x^+ + [\hat{q},C_x]^+ + [\hat{q},D^{-1}[\hat{q},C]] + A_0\hat{q}, -\hat{q}A_0, \qquad (3.3)$$

where

$[\ ,\ ]$ is a commutator, $[\ ,\ ]^+$ is an anticommutator, A_0

is an operator such that $A_{0_x} = 0$ and $(D^{-1}f)(x,y) = \int_{-\infty}^{x} f(\xi,y)d\xi$.

In what follows we take $A_0 = 0$ (the general case is considered in [27]).

2. An Integral Representation.

The crucial step is to use an integral representation for the differential operator C:

$$(Cf)(x,y_1) = \int_R dy_2 T(x,y_1,y_2)f(x,y_2). \qquad (3.4)$$

Let

$$q_i \doteq q(x,y_i), \quad D_i \doteq D_{y_i}, \quad i = 1,2, \quad T_{12} \doteq T(x,y_1,y_2). \quad (3.5)$$

Equation (3.4) implies similar integral representations for all quantities appearing in the RHS of (3.3). For example

$$(\hat{q}_1 C)f = \int_R dy_2 \{q_1 T_{12} + \alpha(D_1 + D_2)T_{12}\}f_2.$$

For

$$(q_1 C)f = \int_R dy_2(q_1 T_{12})f, \quad D_1(Cf) = (D_1 C)f + Cf_{y_1} = \int_R dy_2 T_{12_{y_1}} f_2,$$

$$Cf_{y_1} = \int_R dy_2 T_{12} f_{2_{y_2}} = -\int_R dy_2 T_{12_{y_2}} f_2.$$

Thus $(D_1 C)f = \int_R dy_2(T_{12_{y_1}} + T_{12_{y_2}})f_2.$

Similarly

$$(\hat{q}_1 C \pm C\hat{q}_1)f = \int_R dy_2(q_{12}^{\pm} T_{12})f_2,$$

where the operators $q_{12}^{\pm}$ are defined by

$$q_{12}^{\pm} \doteq q_1 \pm q_2 + \alpha(D_1 \mp D_2). \quad (3.6)$$

Using the above integral representations in (3.3) we obtain

$$\delta_{12} q_{1_t} = T_{12_{xxx}} + (q_{12}^+ T_{12})_x + q_{12}^+ T_{12_x} + q_{12}^- D^{-1} q_{12}^- T_{12}, \quad \delta_{12} \doteq \delta(y_1 - y_2)$$

or

$$\delta_{12} q_{1_t} = D\gamma_{12} T_{12}, \quad \gamma_{12} \doteq D^2 + q_{12}^+ + D^{-1} q_{12}^+ D + D^{-1} q_{12}^- D^{-1} q_{12}^-. \quad (3.7)$$

Let us introduce the operator ϕ_{12} via

$$D\Psi_{12} = \phi_{12}D, \quad \phi_{12} \doteq D^2 + q_{12}^+ + Dq_{12}^+ D^{-1} + q_{12}^- D^{-1}q_{12}^- D^{-1}. \tag{3.8}$$

Thus

$$\delta_{12}q_{1_t} = D\Psi_{12}T_{12} = \phi_{12}DT_{12}. \tag{3.9}$$

3.   Expansions in terms of delta functions.

We expand T_{12} in the form

$$T_{12} = \sum_{j=0}^{n} \delta_{12}^j T_{12}^{(j)}; \qquad \cdot \qquad \delta_{12}^j \doteq \frac{d^j}{dy_1^j} \delta(y_1 - y_2). \tag{3.10}$$

It turns out that Ψ_{12} admits a simple commutator relationship with respect to $h_{12} = h(y_1 - y_2)$. Actually the following <u>operator</u> equation is valid

$$[\Psi, h_{12}] = 4\alpha h_{12}'; \quad h_{12}' \doteq \frac{d}{dy_1} h_{12}. \tag{3.11}$$

Hence equation (3.7) yields

$$\delta_{12}q_{1_t} = \sum_{j=0}^{n} D\Psi_{12}\delta_{12}^j T_{12}^{(j)} = \sum_{j=0}^{n} \delta_{12}^j D\Psi_{12} T_{12}^{(j)} + 4\alpha \sum_{j=1}^{n+1} \delta_{12}^j DT_{12}^{(j-1)}.$$

Thus

$$T_{12_x}^{(n)} = 0, \quad T_{12}^{(j-1)} = -\frac{1}{4\alpha} \Psi_{12}T_{12}^{(j)}, \quad \delta_{12}q_{1_t} = \delta_{12}D\Psi_{12}T_{12}^{(0)}. \tag{3.12}$$

Letting $T_{12}^{(n)} = 1$ we have the following proposition:

Proposition 3.1.

The isospectral equation

$$w_{xx} + \hat{q}w = 0, \quad \hat{q} \doteq q + \alpha D_y; \quad \alpha \text{ constant} \tag{3.13}$$

is associated with the equations

$$q_{1_t} = \beta_n \int_R dy_2 \delta_{12} D \Psi_{12}^{n+1} \cdot 1 = \beta_n \int_R dy_2 \delta_{12} \Phi_{12}^n (\Phi_{12} D) \cdot 1, \quad \beta_n \text{ constant} \tag{3.14}$$

where

$$\Psi_{12} \doteq D^2 + q_{12}^+ + D^{-1} q_{12}^+ D + D^{-1} q_{12}^- D^{-1} q_{12}^-, \quad \Phi_{12} \doteq D^2 + q_{12}^+ + D q_{12}^+ D^{-1} + q_{12}^- D^{-1} q_{12}^- D^{-1},$$
$$\tag{3.15}$$

and Ψ_{12}, Φ_{12} and related via $D\Psi_{12} = \Phi_{12}D$. The operators $q_{12}^{\pm}$ are defined by

$$q_{12}^{\pm} \doteq \hat{q}_1 \doteq \hat{q}_1^{\star}, \quad \hat{q}_1 \doteq q_1 + \alpha D_{y_1}, \quad \hat{q}_1^{\star} \doteq q_2 - \alpha D_{y_2}. \tag{3.16}$$

(The notation $\hat{q}_1^{\star}$ is justified, since $\hat{q}_1^{\star}$ is indeed the adjoint of $\hat{q}_1$, see §3.2).

EXAMPLE.

1.   Equation (3.14) with $n = 0$ and $\beta_0 = 1/2$ implies $q_{1_t} = q_{1_x}$.

2.   Equaton (3.14) with $n = 1$ and $\beta_1 = 1/2$ implies the KP equation

$$q_{1_t} = q_{1_{xxx}} + 6q_1 q_{1_x} + 3\alpha^2 D^{-1} q_{1_{y_1 y_1}}. \tag{3.17}$$

Remark 3.1.

(i)   The operators Φ_{12} and Ψ_{12} with $y_2 = y_1$ and $\alpha = 0$ reduce to Φ and Φ^+ respectively, where Φ is the recursion operator of the KdV.

(ii) The starting symmetry $(\phi_{12}D) \cdot 1$ is given by $q_{1_x} + q_{2_x} +$

$(q_1-q_2)D^{-1}(q_1-q_2) + \alpha D^{-1}(q_{1y_1} - q_{2y_2})$. Thus it reduces to q_{1_x},

the starting symmetry of the KdV, when $y_2 = y_1$.

3.2. A New Directional Derivative and a New Bilinear Form.

Recall that ϕ generates symmetries and ϕ^+ generates conserved

covariants. Similarly, it will turn out, that ϕ_{12} and ϕ_{12}^* generate

extended symmetries and extended conserved covariants respectively. To

define these extended notions we need to introduce a new bilinear form

and a new directional derivative:

(i) A new bilinear form.

$$<g_{12},f_{12}> \doteq \int_{R^3} dx \, dy_1 dy_2 \text{trace } g_{21}f_{12}, \qquad (3.18)$$

where f_{12} and g_{12} are matrix valued functions of x, y_1, y_2, and obviously

the trace is dropped if f_{12}, g_{12} are scalars. In association with the

above form we define L_{12}^* to be the adjoint of L_{12} iff

$$<L_{12}^*g_{12},f_{12}> = <g_{12},L_{12}f_{12}>. \qquad (3.19)$$

We recall that the usual bilinear form and the usual adjoint are defined

by

$$(g,f) \doteq \int_{R^2} dx \, dy \text{ trace } gf, \quad (L^+g,f) = (g,Lf), \qquad (3.20)$$

where f,g are matrix valued functions of x,y.

EXAMPLE.

1.   The adjoint of $\dot{q}_1$ is given by $\dot{q}_1^* = q_2 - iD_2$

2. $\quad (q_{12}^+)^* = q_{12}^+, \quad (q_{12}^-)^* = -q_{12}^-$ (3.21)

3. $\quad \phi_{12}^* = \psi_{12}$.

Note that the fastest way to compute the adjoint of an operator L_{12} is to evaluate the adjoint as usually and then interchange $1 \longleftrightarrow 2$.

Let I be a functional given by

$$ I = \int_{R^2} dx\, dy_1 \text{ trace } \rho_{11} = \int_{R^3} dx\, dy_1 dy_2 \delta_{12} \text{trace } \rho_{12}. \quad (3.22)$$

The __extended gradient__ of this functional is defined by

$$ \langle \text{grad}_{12} I, \cdot \rangle \doteqdot I_d[\cdot] = \int_{R^3} dx\, dy_1 dy_2 \delta_{12} \rho_{12_d}[\cdot], \quad (3.23)$$

where subscript d denotes a suitable directional derivative.
It is easily seen that a function γ_{12} is an extended gradient function, i.e. it has a potential I, iff

$$ \gamma_{12_d} = \gamma_{12_d}^* . \quad (3.24)$$

Also

$$ (\text{grad } I, \cdot) \doteqdot I_f[\cdot] = \int_{R^2} dx\, dy\, \rho_f[\cdot], \quad (3.25)$$

and γ is a gradient function iff $\gamma_f = \gamma_f^+$.

(ii) __A new directional derivative.__

Recall the crucial integral representation

$$ (\hat{q}_1 f)(x,y_1) = \int_R dy_3 q(x,y_1,y_3) f(x,y_3) . $$

Allowing f also to depend on y_2 we obtain

$$ \hat{q}_1 f_{12} = \int_R dy_3 q_{13} f_{32} . $$

The above mapping between an operator and its kernel induces a mapping between derivatives: Let subscript d denote the new directional derivative. Then

$$\hat{q}_{1_d}[\sigma_{12}]f_{12} = \int_R dy_3 \sigma_{13}f_{32}.$$

The integral representation for $\hat{q}_1$ also induces, via (3.18) an integral representation for the adjoint of $\hat{q}_1$:

$$\langle g_{21}, \hat{q}_1 f_{12} \rangle = \int_{R^3} dy_1 dy_2 dx \; g_{21} \int_R dy_3' q_{13'} f_{3'2} = \int_{R^4} dy_3' dy_2 dy_1 dx \; g_{23'} q_{3'1} f_{12}$$

$$= \int_{R^3} dy_1 dy_2 dx \; G_{21} f_{12}, \quad \text{where we have used } 3' \longleftrightarrow 1, \text{ and}$$

$$G_{21} = \int_R dy_3' g_{23'} q_{3'1}, \quad \text{thus} \quad G_{12} = \int_R dy_3' g_{13'} q_{3'2}. \quad \text{Thus } \hat{q}_1^* f_{12} = \int_R dy_3 q_{32} f_{13}.$$

Furthermore, the $\hat{q}_1^*$ mapping induces a mapping between derivatives. Thus

$$\hat{q}_1 f_{12} \doteq (q_1 + \alpha D_1) f_{12} = \int_R dy_3 q_{13} f_{32}, \quad \hat{q}_1^* f_{12} \doteq (q_2 - \alpha D_2) f_{12} = \int_R dy_3 q_{32} f_{13}$$

$$\tag{3.26}$$

$$\hat{q}_{1_d}[\sigma_{12}]f_{12} = \int_R dy_3 \sigma_{13} f_{32}, \qquad \hat{q}_{1_d}^*[\sigma_{12}]f_{12} = \int_R dy_3 \sigma_{32} f_{13}. \tag{3.27}$$

The above derivatives with respect to $\hat{q}_1$ and $\hat{q}_1^*$ imply the following derivatives with respect to q_{12}^+, q_{12}^-:

$$q_{12_d}^{\pm}[\sigma_{12}]f_{12} \doteq \int_R dy_3 (\sigma_{13} f_{32} \pm \sigma_{32} f_{13}). \tag{3.28}$$

Furthermore, using the chain rule and (3.28), if an operator K_{12} depends only on q_{12}^+, q_{12}^- its directional derivative $L_{12_d}[\sigma_{12}]$ is well defined.

This derivative is linear, and satisfies the Leibnitz rule. Also, using (3.28) it follows that the directional derivative in the direction of δ_{12} reduces to the usual total Fréchet derivative:

$$K_{12_d}[\delta_{12}F_{12}] = K_{12_f}[F] \doteq K_{12_{q_1}}[F_{11}] + K_{12_{q_2}}[F_{22}], \qquad (3.29)$$

where the subscript f stands for a Fréchet derivative and

$$K_{12_{q_i}}[F_{ii}] = \frac{\partial}{\partial \epsilon} K_{12}(q_i + F_{ii}, q_j)\Big|_{\epsilon=0}, \quad i,j = 1,2, \qquad (3.30)$$
$$i \neq j.$$

Operators which depend only on $q_{12}^{\pm}$ are called admissible. Similarly, a function K_{12} is called admissible if it can be written in the form $K_{12} = \hat{K}_{12}H_{12}$, where $\hat{K}_{12}$ is an admissible operator and H_{12} is an appropriate function (for the KP, $H_{12} = H(y_1, y_2)$).

EXAMPLE.

The function $\hat{M}_{12}\delta_{12} \doteq Dq_{12}^+\delta_{12} + q_{12}^-D^{-1}q_{12}^-\delta_{12}$ is an admissible function since the operator $\hat{M}_{12}$ depends only on $q_{12}^{\pm}$, and $\delta_{12} = \delta(y_1 - y_2)$. It is easy to compute its directional derivative:

$$(\hat{M}_{12}\delta_{12})_d[\sigma_{12}] = D\sigma_{12}^+\delta_{12} + \sigma_{12}^-D^{-1}q_{12}^-\delta_{12} + q_{12}^-D^{-1}\sigma_{12}^-\delta_{12},$$

where $\sigma_{12}^{\pm}F_{12} \doteq \int_R dy_3(\sigma_{13}f_{32} \pm \sigma_{32}f_{13})$. Hence $(\hat{M}_{12}\delta_{12})_d[\sigma_{12}] = 2D\sigma_{12}$.

3.3. Isospectral problems yield hereditary operators.

Using the same methods as in 1+1, it can be shown that if the extended gradient $(G_\lambda)_{12}$ of the eigenvalue λ of an isospectral problem satisfies

$$\Psi_{12}(G_\lambda)_{12} = \mu(\lambda)(G_\lambda)_{12}, \tag{3.31}$$

then $\phi_{12} \doteq \Psi_{12}^*$ is a hereditary operator. (One must again assume comleteness, a proof of which should follow a two dimensional version of the method developed in [6]).

EXAMPLE.

Consider the isospectral problem

$$V_{1_{xx}} + (\hat{q}_1 - \lambda)V_1 = 0. \tag{3.32}$$

Taking the directional derivative of the above it follows that

$$(D^2 + \hat{q}_1 - \lambda)V_{1_d}[f_{12}] + (\hat{q}_{1_d}[f_{12}] - \lambda_d[f_{12}])V_1 = 0.$$

Multiplying the above by V_1^+, where V_1^+ satisfies the adjoint of (3.32), integrating with respect to $dx\, dy_1$, and assuming $\int_{R^2} dx\, dy_1 V_1 V_1^+ = 1$, we obtain

$$\lambda_d[f_{12}] = \langle \text{grad}_{12}\lambda, f_{12} \rangle = \int_{R^2} dx\, dy_1 V_1^+ \hat{q}_{1_d}[f_{12}]V_1.$$

Using (3.26) to evaluate $\hat{q}_{1_d}[f_{12}]$ it follows that

$$(G_\lambda)_{12} \doteq \text{grad}_{12}\lambda = V_1 V_2^+. \tag{3.33}$$

It is easy to show that $\ddagger_{12}^*$ as defined by (3.7) satisfies

$$\ddagger_{12}^* V_1 V_2^+ = 4\lambda V_1 V_2^+. \tag{3.34}$$

Hence ϕ_{12} is a hereditary operator.

Remark 3.2.

Konopelchenko and Dubrovsky [29] were the first to establish the
importance of working with $V(x,y_1)V^+(x,y_2)$, as opposed to $V(x,y_1)V^+(x,y_2)$.
They also found a linear equation satisfied by $V_1V_2^+$. However, they
failed to recognize that this equation could actually yield the recur-
sion operator of the entire associated hierarchy of nonlinear equations.
Indeed, they used the above equation to obtain "local" recursion oper-
ators. Thus the question of studying the remarkably rich structure of
these recursion operators in particular its connection to symmetries,
conservation laws, and bi-Hamiltonian operators were not even posed.

3.4   Starting Symmetries.

The theory of symmetries for equations in 1+1 is based on the
existence of "starting" symmetries K^0, which via ϕ generate infinitely
many symmetries. For example, for the KdV $K^0 = q_x$. For equations in
2+1 we find that the starting symmetries K_{12}^0 has the following important
properties: (i) Can be written in the form $\hat{K}_{12}^0 H_{12}$, where $\hat{K}_{12}^0$ is an ad-
missible operator and H_{12} is an appropriate function. (ii) The start-
ing operators $\hat{K}_{12}^0$ have simple commutator properties with respect to
$h_{12} = h(y_1-y_2)$. (iii) The Lie algebra of the starting operator $\hat{K}_{12}^0$
acting on functions H_{12} is closed. (iv) Using (ii) and the fact that
ϕ_{12} also admits a simple commutator relationship with h_{12}, it can be
shown that $\delta_{12}\phi_{12}^n\hat{K}_{12}^0 \cdot 1 = \sum_{\ell=0}^{n} b_{n,\ell}\phi_{12}^{n-\ell}\hat{K}_{12}^0 \cdot \delta_{12}^\ell$, where $b_{n,\ell}$ are
appropriate constants; hence $\delta_{12}\phi_{12}^n\hat{K}_{12}^0 \cdot 1$ are admissible functions.
It is thus clear that in 1+1 one considers the Lie algebra of functions
K^0, while in 2+1 one considers the Lie algebra of operators $\hat{K}_{12}^0$. This

richer algebraic structure of equations in 2+1 can be exploited in a
variety of ways. For example different choices of H_{12} yield both time-
independent and time-dependent symmetries. Furthermore, all these
symmetries correspond to gradient functions.

We now discuss (i)-(iv) above for the concrete case of the KP:
It should be first noted that given an operator Φ_{12} there exists an
algorithmic way of finding its starting symmetries: One looks for
operators $\hat{S}_{12}$ such that $\hat{S}_{12}H_{12} = 0$ but $\Phi_{12}\hat{S}_{12}H_{12} = \hat{K}^0_{12}H_{12} \neq 0$. It can
be shown that if a starting symmetry is constructed in the above way
and Φ_{12} is hereditary then Φ_{12} is a strong symmetry for this starting
symmetry.

(i)   For the KP there exist two starting symmetries:

$$\hat{M} \doteq Dq^+_{12} + q^-_{12}D^{-1}q^-_{12}, \quad \hat{N} \doteq q^-_{12}, \quad H_{12} \doteq H(y_1, y_2) \tag{3.35}$$

corresponding to $\hat{S}_{12} = D$ and $\hat{S}_{12} = D(q^-_{12})^{-1}D$ respectively.

(ii)   The following operator equations are valid:

$$[\hat{M}_{12}, h_{12}] = 2aDh'_{12}, \qquad [\hat{N}_{12}, h_{12}] = 0. \tag{3.36}$$

(iii) The Lie algebra of $\hat{M}_{12}$, $\hat{N}_{12}$ is given by

$$[\hat{N}_{12}H^{(1)}_{12}, \hat{N}_{12}H^{(2)}_{12}]_d = -\hat{N}_{12}H^{(3)}_{12}, \quad [\hat{N}_{12}H^{(1)}_{12}, \hat{M}_{12}H^{(2)}_{12}]_d = -\hat{M}_{12}H^{(3)}_{12},$$

$$\tag{3.37}$$

$$[\hat{M}_{12}H^{(1)}_{12}, \hat{M}_{12}H^{(2)}_{12}]_d = -\Phi_{12}\hat{N}_{12}H^{(3)}_{12},$$

where

$$[K^{(1)}_{12}, K^{(2)}_{12}]_d \doteq K^{(1)}_{12_d}[K^{(2)}_{12}] - K^{(2)}_{12_d}[K^{(1)}_{12}], \tag{3.38}$$

$$H_{12}^{(3)} \doteq [H_{12}^{(1)}, H_{12}^{(2)}]_I \doteq \int_R dy_3 (H_{13}^{(1)} H_{32}^{(2)} - H_{13}^{(2)} H_{32}^{(1)}). \qquad (3.39)$$

Let us derive (3.32a):

$$q_{12}^-[q_{12}^- H_{12}^{(2)}] H_{12}^{(1)} = \int_{R^2} dy_3 dy_3' \{ (q_{13}, H_{3'3}^{(2)} - q_{3'3} H_{13}^{(2)'}) H_{32}^{(1)} -$$

$$- H_{13}^{(1)} (q_{33}, H_{3'2}^{(2)} - q_{3'2} H_{33'}^{(2)}) \}$$

$$= \int_{R^2} dy_3 dy_3' \cdot \{ H_{32}^{(1)} [q_1 - q_3 - \alpha(D_1 + D_3)] H_{13}^{(2)} - H_{13}^{(1)} [q_3 - q_2 - \alpha(D_3 + D_2)] H_{32}^{(2)} \}.$$

Hence

$$[q_{12}^- H_{12}^{(1)}, q_{12}^- H_{12}^{(2)}]_d = -\{q_1 - q_2 + \alpha(D_1 + D_2)\} [H_{12}^{(1)}, H_{12}^{(2)}]_I.$$

Remark 3.3.

The bracket (3.39) can also be traced back to the integral representation of $\hat{q}_1$ (see [27]).

(iv) Equations (3.36) and the operator equation (see (3.11))

$$[\phi_{12}, h_{12}] = 4\alpha h_{12}' \qquad (3.40)$$

imply

$$\phi_{12}^{'} \phi_{12}^n \dot{N}_{12} \cdot 1 = \sum_{i=1}^{n} (-4\alpha)^i \binom{n}{i} \phi_{12}^{n-i} \dot{N}_{12} \phi_{12}^i , \qquad (3.41)$$

$$\phi_{12}' \phi_{12}^n \dot{M}_{12} \cdot 1 = \sum_{i=1}^{n} b_{n,i} \phi_{12}^{n-i} \dot{M}_{12} \phi_{12}^i , \quad b_{n,i} \doteq (-4\alpha)^i \sum_{j=0}^{i} 2^{-j} \binom{n-j}{i-j}. \qquad (3.42)$$

Let us indicate how the above equations can be derived: Introducing an operator D, which commutes with all admissible operators ϕ_{12} and which

has the property that

$$D \cdot h_{12} = h'_{12} \, ,$$

it follows that

$$\delta_{12}\phi_{12}^{n}\hat{N}_{12} \cdot 1 = (\phi_{12}-4\alpha D)^{n}\delta_{12}\hat{N}_{12} \cdot 1 = (\phi_{12}-4\alpha D)^{n}\hat{N}_{12} \cdot \delta_{12} =$$

$$= \sum_{\ell=1}^{n} (-4\alpha)^{\ell}\binom{n}{\ell}\phi_{12}^{n-\ell}\hat{N}_{12} \cdot \delta_{12}^{\ell} \, .$$

To derive equation (3.42) note that

$$\delta_{12}\phi_{12}^{n}\hat{M}_{12} \cdot 1 = (\phi_{12}-4\alpha D)^{n}\delta_{12}\hat{M}_{12} \cdot 1 = (\phi_{12}-4\alpha D)^{n}(\hat{M}\cdot\delta_{12}-2\alpha D\cdot\delta'_{12})$$

$$= \sum_{\ell=0}^{n} (-4\alpha)^{\ell}\binom{n}{\ell}\phi_{12}^{n-\ell}\hat{M} \cdot \delta_{12}^{\ell} - 2\alpha \sum_{\ell=0}^{n} (-4\alpha)^{\ell}\binom{n}{\ell}\phi_{12}^{n-\ell}D \cdot \delta_{12}^{\ell+1}. \quad (3.43)$$

The next step is to express $\phi_{12}^{j}D$ in terms of $\phi_{12}^{j'}\hat{M}_{12}$, where j, j' are integers. This can be achieved as follows: It can be shown that $\phi_{12}^{n+1}D \cdot 1 = \phi_{12}^{n}\hat{M} \cdot 1$. This equation implies

$$\phi_{12}^{n+1}D \cdot h_{12} = \sum_{j=0}^{n} (2\alpha)^{j}\phi_{12}^{n-j}\hat{M}_{12} \cdot h_{12}^{j}; \quad h_{12}^{j} \doteq \frac{d^{j}}{dy_{1}^{j}} h_{12}. \quad (3.44)$$

For example, multiplying $\phi_{12}D \cdot 1 = \hat{M}_{12} \cdot 1$ by h_{12}, it follows that $(\phi_{12}-4\alpha D)h_{12}D \cdot 1 = (\hat{M}_{12}-2\alpha D)\cdot h$, or $\phi_{12}D \cdot h_{12} = \hat{M}_{12} \cdot h_{12}$. Similarly $\phi_{12}^{2}D \cdot 1 = \phi_{12}\hat{M}_{12} \cdot 1$ implies $\phi_{12}^{2}D \cdot h_{12} = \phi_{12}\hat{M}_{12} \cdot h_{12} + 2\alpha\hat{M}_{12} \cdot h_{12}$, etc. Using (3.44) into (3.43) yields (3.42).

3.5. Basic Notions and Results

We consider exactly solvable evolution equations in the form
$q_t = K(q)$, $q(x,y,t)$, on a normed space M of vector-value functions on
R; K is a suitable C^∞ vector field on M. We assume that the space of
smooth vector fields on M is some space S of C^∞ functions on the
plane vanishing rapidly as $x, y \to \pm \infty$. The above equation is a member
of a hierarchy generated by ϕ_{12}, hence more generally we shall study
$q_t = K^{(n)}(q)$. Fundamental in our theory is to write these equations
in the form

$$q_{1_t} = \int_R dy_2 \delta_{12} \phi_{12}^n K_{12}^0 \cdot 1 \doteq \int_R dy_2 \delta_{12} K_{12}^{(n)} = K_{11}^{(n)}, \qquad (3.45)$$

(in the matrix case, 1 is replaced by the identity matrix I), where
$K_{12}^{(n)}(q_1,q_2)$ belongs to a suitably extended space $\bar{S}$, and $\bar{S}^*$ denotes the
dual of $\bar{S}$. In the extended spaces $\bar{S}$ and $\bar{S}^*$ we define the new direction-
al derivative (3.28) and the new bilinear form (3.18); the notions of the
adjoint and of a gradient are well defined with respect to (3.18) (see
(3.19), (3.23), (3.24)). In analogy with definition 2.1 we have:

Definition 3.1.

(i)   A function $\sigma_{12} \in \bar{S}$ is called an extended symmetry of

$$q_{1_t} = \int_R dy_2 \delta_{12} K_{12} = K_{11} \qquad (3.46)$$

iff

$$\frac{\partial \sigma_{12}}{\partial t} + \sigma_{12_f}[K] - (\delta_{12} K_{12})_d[\sigma_{12}] = 0. \qquad (3.47)$$

(ii) A function $\gamma_{12} \in \bar{S}^*$ is called an extended conserved gradient

(i.e. it is the extended gradient of a conserved functional I)

of (3.46) iff

$$\frac{\partial \gamma_{12}}{\partial t} + \gamma_{12_f}[K] + (\delta_{12}K_{12})^*_d[\gamma_{12}] = 0, \qquad \gamma_{12_d} = \gamma_{12_{d*}} \cdot \qquad (3.48)$$

Functions which satisfy (3.48a) are called <u>extended conserved</u>
<u>covariants.</u>

(iii) An operator valued function $\phi_{12} : \bar{S} \rightarrow \bar{S}$, is a <u>recursion operator</u>
for (3.46) (or it is a <u>strong symmetry</u> for K_{12}) iff

$$\phi_{12_f}[K] + [\phi_{12}, (\delta_{12}K_{12})_d] = 0. \qquad (3.49)$$

(iv) An operator valued function $\phi_{12} : \bar{S} \rightarrow \bar{S}$, is a <u>hereditary operator</u>
(or <u>Nijenhuis</u> or <u>regular</u>) iff

$$\phi_{12_d}[\phi_{12}v_{12}]w_{12} - \phi_{12}\phi_{12_d}[v_{12}]w_{12} \quad \text{is symmetric} \quad \text{w.r.t.} \ v_{12}, \ w_{12}.$$

$$(3.50)$$

(v) An operator valued function $\Theta_{12} : \bar{S}^* \rightarrow \bar{S}$ is a <u>Hamiltonian operator</u>
iff it is skew symmetric, i.e. $\Theta_{12} = -\Theta^*_{12}$, and it satisfies

$$\langle a_{12}, \Theta_{12_d}[\Theta_{12}b_{12}]c_{12}\rangle + \text{cyclic permutation} = 0. \qquad (3.51)$$

(vi) Equation (3.46) is a <u>Hamiltonian system</u> iff it can be written in
the form

$$q_{1_t} = \int_R dy_2 \delta_{12}\Theta_{12}f_{12}, \qquad (3.52)$$

where Θ_{12} is a Hamiltonian operator and f_{12} is an extended gradi-
ent function, i.e. $f^*_{12_d} = f_{12_d}$. Associated with (3.52) we define
the following Poisson bracket

$$\{I,H\} = \langle \text{grad}_{12}I, \Theta_{12}\text{grad}_{12}H\rangle. \qquad (3.53)$$

In the above, subscripts f and d denote total Fréchet (see (3.29)) and directional (see (3.28)) derivatives respectively.

Remark 3.4.

(i) Equation (3.47) can also be written as

$$\frac{\partial c_{12}}{\partial t} + [\sigma_{12}, \delta_{12}K_{12}]_d = 0,$$

since $\sigma_{12_d}[\delta_{12}K_{12}] = \sigma_{12_f}[K]$. Similary $\phi_{12_f}[K] = \phi_{12_d}[\delta_{12}K_{12}]$.

(ii) Some of the above notions are well defined only if $(\delta_{12}K_{12})_d$ is well defined. However, for equations (3.45)

$$\delta_{12}K_{12}^{(n)} = \delta_{12}\phi_{12}^n\hat{K}_{12}^0 \cdot 1 = \sum_{\ell=0}^{n} b_{n,\ell}\phi_{12}^{n-\ell}\hat{K}_{12}^0 \cdot \delta_{12}^\ell .$$

Furthermore, by construction ϕ_{12} and the starting operators $\hat{K}_{12}^0$ depend on the basic operators $q_{12}^\pm$. Hence $(\delta_{12}\hat{K}_{12}^{(n)})_d$ is well defined.

In analogy with the basic results in 1+1:

Theorem 3.1.

(i) If ϕ_{12} is a recursion operator for (3.46) then ϕ_{12} maps extended symmetries onto extended symmetries and ϕ_{12}^* maps extended conserved covariants onto extended conserved covariants.

(ii) If (3.46) is a Hamiltonian system them $\sigma_{12} = \Theta_{12}\gamma_{12}$.

(iii) If ϕ_{12} is a hereditary operator and a recursion operator for $\hat{K}_{12}^0 \cdot 1$ then ϕ_{12} is a recursion operator for $q_{1_t} = \int_R dy_2 \delta_{12}\phi_{12}^n\hat{K}_{12}^0 \cdot 1$.

(iv) If $\phi_{12} = \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$, where $\Theta_{12}^{(1)} + \nu\Theta_{12}^{(2)}$ is a Hamiltonian operator

for all values of the constant ν and $\partial_{12}^{(1)}$ is invertible, then ϕ_{12} is hereditary.

(v) If ϕ_{12} as in (iv) and $\gamma_{12}^0 \doteq (\partial_{12}^{(1)})^{-1} K_{12}^0 \cdot 1$ is an extended gradient function then all $(\phi_{12}^*)^m \gamma_{12}^0$ are extended gradient functions.

EXAMPLE.

The hereditary operator ϕ_{12} of the KP equation is factorizable in terms of the Hamiltonian operators D and $\phi_{12}D$. Hence each member of the KP hierarchy is a bi-Hamiltonian system, with respect to the following two Poisson brackets

$$\cdot I, H \cdot = \cdot grad_{12}I, \; \partial_{12}^{(i)} grad_{12}H>, \qquad i = 1,2$$

$$\partial_{12}^{(1)} = D, \qquad \partial_{12}^{(2)} = \phi_{12}D = D^3 + q_{12}^+ D + D q_{12}^+ + q_{12}^- D^{-1} q_{12}^-.$$

3.6. Extended Symmetries.

Lemma 3.1.

(i) Let ϕ_{12} be hereditary, then

$$[\phi_{12}^n K_{12}^{(1)}, \phi_{12}^m K_{12}^{(2)}]_d = \phi_{12}^{n+m}[K_{12}^{(1)}, K_{12}^{(2)}]_d + \phi_{12}^m \left(\sum_{r=1}^{n} \phi_{12}^{n-r} S_{12}^{(2)} \phi_{12}^{r-1} \right) K_{12}^{(1)}$$

$$- \phi_{12}^n \left(\sum_{r=1}^{m} \phi_{12}^{m-r} S_{12}^{(1)} \phi_{12}^{r-1} \right) K_{12}^{(2)}, \qquad (3.54)$$

where

$$S_{12}^{(i)} \doteq \phi_{12_d}[K_{12}^{(i)}] + [\phi_{12}, K_{12_d}^{(i)}], \qquad (3.55)$$

m, n are non-negative integers.

(ii) $\sigma_{12}^{(r)}$ is a time-dependent extended symmetry of order r of equation (3.46) iff

$$\sigma_{12}^{(r)} = \sum_{j=0}^{r} t^j \varsigma_{12}^{(j)}, \quad \varsigma_{12}^{(j)} \doteq -\frac{1}{j}[\varsigma_{12}^{(j-1)}, \varsigma_{12}K_{12}]_d, \quad j = 1,\ldots,r,$$

$$[\varsigma_{12}^{(r)}, \varsigma_{12}K_{12}]_d = 0. \tag{3.56}$$

Proof.

See [28].

We propose the following constructive approach to extended symmetries: Given an isospectral problem construct a recursion operator Φ_{12}. This operator must be hereditary (see §3.3). Then construct its starting symmetries operators, say $\hat{M}_{12}$, $\hat{N}_{12}$. The operator Φ_{12} is a strong symmetry of $\hat{M}_{12}$, $\hat{N}_{12}$ (see [27]). Compute the commutators of $\hat{M}_{12}$, $\hat{N}_{12}$, Φ_{12} with h_{12}. Use the commutator relationships to derive $\delta_{12}\Phi_{12}\hat{K}_{12}^0 \cdot 1 = \sum_{i=0}^{n} b_{n,i}\Phi_{12}^{n-i}\hat{K}_{12}^0 \cdot \varsigma_{12}^i$, $\hat{K}_{12}^0$ is $\hat{M}_{12}$ or $\hat{N}_{12}$. Finally compute the Lie algebra of $\hat{M}_{12}$, $\hat{N}_{12}$. This Lie algebra together with (3.54)-(3.56) yield infinitely many time-independent and time-dependent extended symmetries.

EXAMPLE.

1. $\Phi_{12}^m\hat{M}_{12} \cdot 1$, $\Phi_{12}^m\hat{N}_{12} \cdot 1$ are extended symmetries of the KP hierarchy,
$$q_{1_t} = \int_R dy_2 \delta_{12}\Phi_{12}^n\hat{M}_{12} \cdot 1 \text{ (recall KP corresponds to } n = 1). \; \hat{M}_{12},$$
$\hat{N}_{12}$ are defined in (3.35).

For.

$$[\delta_{12}\phi_{12}^n\hat{M}_{12} \cdot 1, \phi_{12}^m\hat{M}_{12} \cdot H_{12}]_d = [\sum_{i=0}^{n} b_{n,i}\phi_{12}^{n-i}\hat{M} \cdot \delta_{12}^i, \phi_{12}^m\hat{M} \cdot H_{12}]_d =$$

$$= \sum_{\ell=0}^{n} b_{n,\ell}\phi_{12}^{m+n-\ell}[\hat{M}\cdot\delta_{12}^\ell, \hat{M}\cdot H_{12}]_d = -\sum_{i=0}^{n} b_{n,i}\phi_{12}^{m+n-\ell+1}\hat{N}_{12}[\delta_{12}^i, H_{12}]_d.$$

$$(3.57)$$

where we have used (3.54) (ϕ_{12} is hereditary and it is also a strong symmetry for $\hat{M}_{12}H_{12}$, thus $S_{12}^{(i)} = 0$), and (3.37c). Taking $H_{12} = 1$ and using

$$[\delta_{12}^\ell, 1]_I = 0,$$

equation (3.57) implies $[\delta_{12}\phi_{12}^n\hat{M}_{12} \cdot 1, \phi_{12}^m\hat{M}_{12} \cdot 1]_d = 0$, i.e. $\phi_{12}^m\hat{M}_{12} \cdot 1$ is an extended symmetry of the KP hierarchy. Similarly for $\phi_{12}^m\hat{N}_{12} \cdot 1$, since

$$[\delta_{12}\phi_{12}^n\hat{M}_{12} \cdot 1, \phi_{12}^m\hat{N}_{12} \cdot H_{12}]_d = \sum_{\ell=0}^{n} b_{n,i}\phi_{12}^{m+n-\ell}\hat{M}_{12}[\delta_{12}^\ell, H_{12}]_I.$$

2. $\phi_{12}^m\hat{M}_{12} \cdot 1, \phi_{12}^m\hat{N}_{12} \cdot 1$ are extended symmetries of the hierarchy

$$q_{1_t} = \int_R dy_3 \delta_{12}\phi_{12}^n\hat{N}_{12} \cdot 1.$$

3.   The KP hierarchy admits two hierarchies of t-dependent symmetries of order r given by (3.56) where

$$\Sigma_{12}^{(0)} = \hat{N}_{12}^{(m)} \cdot H_{12}^{(r)}, \qquad\qquad H_{12}^{(r)} \doteq (y_1 + y_2)^r$$

$$\Sigma_{12}^{(2j)} = \Sigma v(r,2j,s)\hat{N}_{12}^{(m+2jn+j-\sum_{\ell=1}^{2j} 2s_\ell+1)} \cdot H_{12}^{(r-\sum_{\ell=1}^{2j} 2s_\ell+1)},$$

$$\Sigma_{12}^{(2j-1)} = \Sigma v(r,2j-1,s)\hat{M}_{12}^{(m+(2j-1)n+j-1-\sum_{\ell=1}^{2j-1} 2s_\ell+1)} \cdot H_{12}^{(r-\sum_{\ell=1}^{2j-1} 2s_\ell+1)},$$

the summation Σ is over $s_1, s_2, \ldots, s_j$, from zero to P_n, $P_n = (n-1)/2$ if n is odd, $(n-2)/2$ if n is even,

and

$$\Sigma_{12}^{(0)} = \hat{M}_{12}^{(m)} \cdot H_{12}^{(r)},$$

$$\Sigma_{12}^{(2j)} = \Sigma \upsilon(r,2j,s)\hat{M}_{12}^{(m+2jn+j - \sum_{\ell=1}^{2j} 2s_\ell +1)} \cdot H_{12}^{(r - \sum_{\ell=1}^{2j} 2s_\ell +1)},$$

$$\Sigma_{12}^{(2j-1)} = \Sigma \upsilon(r,2j-1,s)\hat{N}_{12}^{(m+(2j-1)n+j - \sum_{\ell=1}^{2j-1} 2s_\ell +1)} \cdot H_{12}^{(r - \sum_{\ell=1}^{2j-1} 2s_\ell +1)}$$

with $j \geq 1$, $\quad b_{n,\ell} = \sum_{s=0}^{\ell} \beta^{\ell-s}\tilde{\beta}^s \binom{n-s}{\ell-s} = (-4\alpha)^\ell \sum_{s=0}^{\ell} 2^{-s}\binom{n-s}{\ell-s}$ and

$$\upsilon(r,j,s) \doteq \frac{(-2)^j}{j!} \left(\prod_{\iota=1}^{j} \theta(r - \sum_{\iota=1}^{j} 2s_\iota +1) \right) \left(\prod_{\iota=1}^{j} b_{n,2s_\iota +1} \right) \frac{r!}{(r - \sum_{\ell=1}^{j} 2s_\ell +1)!} .$$

For.

Equation (3.56) implies that constructing a symmetry of order r is equivalent to finding a function $\Sigma_{12}^{(0)}$ with the property that its $(r+1)^{st}$ commutator with $\delta_{12}K_{12}$ is zero. This can be easily achieved by using suitable H_{12}'s. For example, let $H_{12} = y_1 + y_2$, then (3.57) implies:

$$[\delta_{12}\ddagger_{12}^n \hat{M}_{12} \cdot 1, \phi_{12}^m \hat{M}_{12} \cdot (y_1+y_2)]_d = - \sum_{\iota=0}^{n} b_{n,\iota}\ddagger_{12}^{m+n-\iota+1}\hat{N}_{12}2\delta_{1,\iota},$$

since

$$[\delta_{12}^\ell, y_1+y_2]_I = 2\delta_{1,\ell} \quad \text{where} \quad \delta_{1,\ell} = 0 \quad \text{if} \quad \iota \neq 1 \quad \text{or} \quad 1 \quad \text{if} \quad \iota = 1.$$

Thus, using the fact that $[\delta_{12}^\ell, \delta_{1,\ell}]_I = 0$ it follows that $\phi_{12}^m \hat{M}_{12} \cdot (y_1+y_2)$ generates first order time-dependent symmetries

$$\phi_{12}^m \hat{M}_{12} \cdot (y_1+y_2) - 2b_{n,1}t\ddagger_{12}^{m+n}\hat{N}_{12} \cdot 1.$$

Similarly, to generate r-order time-dependent symmetries use $\ddagger_{12}^m \hat{M}_{12} \cdot (y_1+y_2)^r$.

since the commutator of $(y_1+y_2)^r$ with δ_{12}^{ℓ} produces $(y_1+y_2)^{r-\ell}$ and hence the rth commutator of $(y_1+y_2)^r$ with δ_{12}^{ℓ} produces 1 which commutes with $\delta_{12}^{(\ell)}$:

$$[\delta_{12}^s,(y_1+y_2)^r]_1 = (1-(-1)^s)\Theta(r-s)\frac{r!}{(r-s)!}(y_1+y_2)^{r-s} \; ,$$

where $\Theta(r-s)$ denotes the Heaviside function.

4.   The hierarchy $q_{1_t} = \int_R dy_3 \delta_{12} \div_{12}^n \hat{N}_{12} \cdot 1$ admits two hierarchies of t-dependent symmetries of order r given by (3.56) where

$$\Sigma_{12}^{(0)} = \hat{N}_{12}^{(m)} \cdot H_{12}^{(r)}$$

$$\Sigma_{12}^{(j)} = \sum v(r,j,s)\hat{N}_{12}^{(m+jn- \sum_{\ell=1}^{j} 2s_\ell +1)} \cdot H_{12}^{(r-\sum_{\ell=1}^{j}2s_\ell +1)},$$

and by

$$\Sigma_{12}^{(0)} = \hat{M}_{12}^{(m)} \cdot H_{12}^{(r)}$$

$$\Sigma_{12}^{(j)} = \sum v(r,j,s)\hat{M}^{(m+jn- \sum_{\ell=1}^{j} 2s_\ell +1)} \cdot H_{12}^{(r- \sum_{\ell=1}^{j} 2s_\ell +1)},$$

where the summation Σ is over $s_1, s_2, \ldots, s_j$ from zero to P_n, $\quad j \geq 1$, $\quad P_n = (n-1)/2$ if n is odd and $(n-2)/2$ if n is even. Also

$$b_{n,\ell} = (-4\alpha)^\ell \binom{n}{\ell}.$$

The above extended symmetries, under the reduction $y_2 = y_1$ yield symmetries. This follows from the following theorem (see [27]).

Theorem 3.2.

Assume that the admissible operators $\dot{\Phi}_{12}$, $\dot{K}_{12}^0$, satisfy

$$[\Phi_{12}, \xi_{12}] = -B\delta_{12}',$$

$$[\tilde{K}_{12}^0, \delta_{12}] = -\tilde{B}\tilde{s}_{12}\delta_{12}',$$

where B, $\tilde{B}$ are constants and $\dot{s}_{12}$ is such that $\dot{s}_{12_d}[\cdot]H_{12} = 0$. Then

(i)   If σ_{12} is an extended symmetry of (3.45), σ_{11} is a symmetry of (3.45).

(ii) If σ_{12} is an extended symmetry of (3.45) then $\sigma_{12} = \sigma(q_1, q_1) = 0$ is an auto-Bäcklund transformation of (3.45), where q_1 and q_2 are viewed as two different solutions of (3.45).

(iii) If γ_{12} is an extended conserved covariant of (3.45), γ_{11} is a conserved covariant of (3.45).

(iv) If γ_{12} is an extended gradient function then γ_{11} is a gradient function.

EXAMPLE.

Consider the extended symmetry of the KP

$$\hat{M}_{12} \cdot 1 = q_{1_x} + q_{2_x} + (q_1 - q_2)D^{-1}(q_1 - q_2) + \imath D^{-1}(q_{1y_1} - q_{2y_2}).$$

Clearly $(\hat{M}_{12} \cdot 1)_{11} = 2q_{1x}$ which is a symmetry of the KP. Also $\hat{M}_{12} \cdot 1 = 0$ is a well known auto-Bäcklund transform of the KP.

Remark 3.5.

(i)   It is quite interesting that both symmetries and Bäcklund transformations of an equation in 2+1 come from the same basic entity,

the extended symmetry. Indeed, when $\alpha = 0$ the recursion operator Φ_{12} for the KP equation reduces to an operator that Calogero and Degasperis have introduced [30] and which generates the auto-Bäcklund transformations of the KdV equation.

(ii) Using the interpretation that $q \cdot b \neq qb = bq$, q, b matrices, the recursion operator of the KP becomes the operator generating auto-Bäcklund transformations for the equations associated with the $N \times N$ matrix Schrödinger problem in one dimension (studied by Calogero and Degasperis [30]). This important connection is explained from the fact that certain 2+1 dimensional systems can be viewed as reductions of certain evolution equations non-local in y. These equations are directly connected to matrix evolution equations [28], [31].

3.7. Extended Conserved Gradients.

Lemma 3.2.

Assume that Θ_{12} is a Hamiltonian operator. Then,

$$[_{12}f_{12}, \Theta_{12}g_{12}]_d = \Theta_{12}\text{grad}_{12}<f_{12}, \Theta_{12}g_{12}> +$$
$$+ \Theta_{12}\{(f_{12_d}-f^\star_{12_d})[\Theta_{12}g_{12}] - (g_{12_d}-g^\star_{12_d})[\Theta_{12}f_{12}]\}. \qquad (3.58)$$

Proof.

See [28].

One way of proving that $\Phi^\star_{12}$ generates gradient functions is to use Theorem 3.1, (v). However, this requires that $\Phi^\star_{12}$ is factorizable in terms of Hamiltonian operators. Alternatively, we propose the following constructive approach, which only uses one Hamiltonian operator

Construct the Lie algebra of the starting operators, say $\hat{M}_{12}$, $\hat{N}_{12}$. Then use this algebra and (3.58) to prove that all $\ddagger \phi_{12}^{m} D_{12}^{-1} \hat{K}_{12}^{0} \cdot H_{12}$ are gradients, provided that $D_{12}^{-1} \hat{K}_{12}^{0} H_{12}$ is a gradient, where $\hat{K}_{12}^{0}$ is $\hat{M}_{12}$ or $\hat{N}_{12}$. Finally use Theorem 3.2 to show that $(D_{12}^{-1} \hat{K}_{12}^{0} H_{12})_{11}$ are gradients.

EXAMPLE.

Consider the operators $\hat{M}_{12}$, $\hat{N}_{12}$ associated with the KP. Then

$$D^{-1} \phi_{12}^{n+1} \hat{M}_{12} H_{12}^{(3)} = \text{grad} < \phi_{12}^{n} \hat{M}_{12} H_{12}^{(1)}, \quad D^{-1} \phi_{12} \hat{N}_{12} H_{12}^{(2)} >, \tag{3.59}$$

$$D^{-1} \phi_{12}^{n+1} \hat{N}_{12} H_{12}^{(3)} = \text{grad} < \ddagger_{12}^{n} \hat{M}_{12} H_{12}^{(1)}, \quad D^{-1} \hat{M}_{12} H_{12}^{(2)} >. \tag{3.60}$$

For.

It is easy to verify that $D^{-1} \hat{M}_{12} H_{12}$ is an extended gradient. Then (3.58) and (3.37c) imply that $D^{-1} \ddagger_{12} \hat{N}_{12} H_{12}$ is an extended gradient. Equation (3.37b) implies

$$[\phi_{12}^{n} \hat{M}_{12} H_{12}^{(1)}, \quad \ddagger_{12} \hat{N}_{12} H_{12}^{(2)}]_{d} = \ddagger_{12}^{n+1} \hat{M}_{12} H_{12}^{(3)}.$$

Since $D^{-1} \phi_{12} \hat{N}_{12} H_{12}$, $D^{-1} \hat{M}_{12} H_{12}$ are extended gradients, the above equation with $n = 0$ and (3.58) imply that $D^{-1} \phi_{12} \hat{M}_{12} H_{12}$ is an extended gradient. Similarly $D^{-1} \phi_{12}^{n} \hat{M}_{12}$ is an extended gradient. Equation (3.60) follows in a similar manner using (3.37c).

Remark 3.6.

It was shown in §3.6 that time-dependent symmetries of order r are generated via $\phi_{12}^{m} \hat{M}_{12} H_{12}$, $\phi_{12}^{m} \hat{N}_{12} H_{12}$ with $H_{12} = (y_1 + y_2)^r$. The above results shows that $D^{-1} \ddagger_{12}^{m} \hat{M}_{12} H_{12}$, $D^{-1} \ddagger_{12}^{m} \hat{N}_{12} H_{12}$ are gradient functions for

arbitrary H_{12}. Hence the time-dependent symmetries correspond to gradient functions. However, the time-dependent symmetries are closely related to master-symmetries τ (see §2). Hence the master-symmetries τ correspond to gradient functions.

3.8. Non-gradient Master-symmetries.

Lemma 3.3.

Assume that the hereditary operator ϕ_{12} satisfies $\phi_{12}\Theta_{12} = \Theta_{12}\phi_{12}^*$, where Θ_{12} is a Hamiltonian operator (if ϕ_{12} is factorizable, then this equation follows). Assume for simplicity that $\Theta_{12_d} = 0$. Then

$$(\phi_{12}^m T_{12})_d + \Theta_{12}(\phi_{12}^m T_{12})_d^* \Theta_{12}^{-1} = \phi_{12}^m (T_{12_d} + \Theta_{12} T_{12_d}^* \Theta_{12}^{-1}) + \sum_{r=1}^{m} \ddagger_{12}^{r-1} \Theta_{12}(\phi_{12}^*)^{m-r} s_{12}^* \Theta_{12}^{-1},$$

$$(3.61)$$

where

$$s_{12}^* \doteqdot \phi_{12_d}^*[T_{12}] + [T_{12_d}^*, \phi_{12}]. \tag{3.62}$$

Proof.

See [28].

The results of Lemmas 3.1-3.3 can be used to obtain non-gradient master-symmetries T_{12}. Such master-symmetries are explicitly related to recursion operators ϕ_{12}. Indeed given ϕ_{12} one computes T_{12} and given T_{12} one computes ϕ_{12}. These formulae are the two dimensional analogues of the formulae given in §2.1. The basic idea is to find a T_{12} such that $T_{12_d} + \Theta_{12} T_{12_d}^* \Theta_{12}^{-1} = 0$ and $S_{12}^* = C \cdot 1$, C constant.

EXAMPLE.

A master-symmetry of the KP hierarchy is given by $\phi_{12}^2 \delta_{12}$. Indeed

$$\phi_{12} = \beta\{(\phi_{12}^2\delta_{12})_d + D(\phi_{12}^2\delta_{12})_d^* D^{-1}\}, \qquad \beta \text{ constant} \tag{3.63}$$

$$\phi_{12}^{n+1}\hat{M}_{12} \cdot 1 = \beta_n[\phi_{12}^n\hat{M}_{12} \cdot 1, \quad \phi_{12}^2\delta_{12}]_d, \qquad \beta_n \text{ constant} \tag{3.64}$$

$$D^{-1}\phi_{12}^{m+n-1}\hat{M}_{12} \cdot 1 = \text{grad}_{12}<(\phi_{12}^*)^n D^{-1}\hat{M}_{12} \cdot 1, \quad \phi_{12}^m\delta_{12}>. \tag{3.65}$$

For.

ϕ_{12} and M_{12} are given by (3.8), (3.35a), $O_{12} = D$. Let $T_{12} = \delta_{12}$. Then $T_{12_d} + O_{12}T_{12_d}^*O_{12}^{-1} = 0$ and $S_{12}^* = 4$ (see (3.62)). Thus, equation (3.61) with $m=2$ implies (3.63).

To derive equation (3.64) use Lemma 5.1 with $K_{12}^{(1)} = \hat{M}_{12} \cdot 1$, $K_{12}^{(2)} = \delta_{12}$, $m = 2$, $S_{12}^{(1)} = 0$ (since ϕ_{12} is a strong symmetry of $\hat{M}_{12} \cdot 1$) and $S_{12}^{(2)} = \phi_{12_f}[\delta_{12}] = 4$. Thus

$$[\phi_{12}^n\hat{M}_{12} \cdot 1, \quad \phi_{12}^2\delta_{12}]_d = \phi_{12}^{n+2}[\hat{M}_{12} \cdot 1, \quad \delta_{12}]_d + 4n\phi_{12}^{n+1}\hat{M}_{12} \cdot 1.$$

However, $\phi_{12}^{n+2}[\hat{M}_{12} \cdot 1, \quad \delta_{12}]_d$ is proportional to $\phi_{12}^{n+1}\hat{M}_{12} \cdot 1$, hence the above implies (3.64).

To derive equation (3.65), use Lemmas 3.1-3.3 to obtain the following general result: If ϕ_{12} is a hereditary operator such that it is a strong symmetry for M_{12} and $\phi_{12}O_{12} = O_{12}\phi_{12}^*$, where O_{12} is a constant invertible Hamiltonian operator then

$$\phi_{12}^m(\sum_{r=1}^{n}\phi_{12}^{n-r}S_{12}\phi_{12}^{r-1})M_{12} + (\sum_{r=1}^{n}\phi_{12}^{r-1}O_{12}\phi_{12}^{*m-r}S_{12}^*O_{12}^{-1})\phi_{12}^n M_{12} + \phi_{12}^{n+m}[M_{12},T_{12}]_d =$$

$$= O_{12}\text{grad}_{12}<O_{12}^{-1}\phi_{12}^n M_{12}, \phi_{12}^m T_{12}> - \phi_{12}^m(T_{12_d} + O_{12}T_{12_d}^*O_{12}^{-1})\phi_{12}^n M_{12},$$

where

$$S_{12} \doteq \phi_{12_d}[T_{12}] + [\phi_{12}, T_{12_d}], \quad S_{12}^* \doteq \phi_{12_d}^*[T_{12}] + [T_{12_d}^*, \phi_{12}^*].$$

Using $T_{12} = \delta_{12}$ in the above and noting that $S_{12} = S_{12}^* = 4$,
$T_{12_d} + O_{12}T_{12_d}^* O_{12}^{-1} = 0$, $\phi_{12}^{n+m}[M_{12}, \delta_{12}]_d$ is proportional to $\phi_{12}^{n+m-1}\bar{M}_{12} \cdot 1$,
we obtain (3.65).

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