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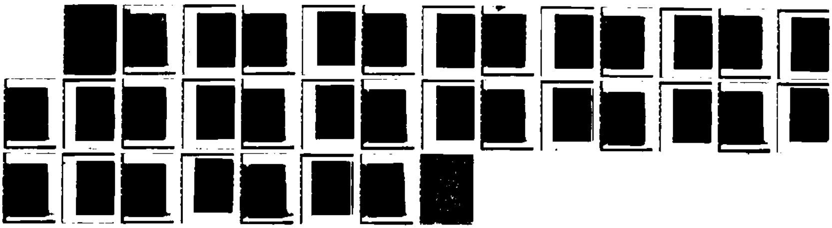

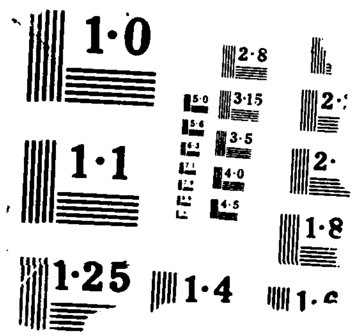
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ASPECTS OF INTEGRABILITY IN ONE AND SEVERAL DIMENSIONS

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INTRODUCTION

In this paper I summarize some results obtained in the period June 1985 - July 1986. It is my pleasure to acknowledge collaboration with the following colleagues: M.J. Ablowitz, P. Clarkson, M. Kruskal, R.A. Leo, L. Martina, U. Mugan, V. Papageorgiou, P.M. Santini, and G. Soliani.

The results on Inverse Scattering in multidimensions and on the algebraic properties of equations in 2+1 (i.e. two spatial and one temporal) dimensions should be of particular interest: With respect to algebraic properties of equations in 2+1 ~~we note that~~, the question of finding the recursion operator and the bi-Hamiltonian formulation of these equations has remained open for a rather long time. It was even doubted in the literature if the relevant results in 1+1 could be extended to 2+1. ~~P.M. Santini~~ and the author have recently shown that equations in 2+1 solvable via the Inverse Scattering Transform are bi-Hamiltonian systems. ~~They have~~ also given ~~explicitly~~ the recursion and bi-Hamiltonian operators for large classes of equations in 2+1, including the Kadomtsev-Petviashvili (a two dimensional analogue of the Korteweg-deVries) and the Davey-Stewartson (a two dimensional analogue of the nonlinear Schrödinger) equations.

In this paper I emphasize the basic ideas and results. Furthermore an attempt is made to put these results into perspective. Details can be found in the sited papers.

1. AN INVERSE PROBLEM FOR N x N AKNS IN MULTIDIMENSIONS

This problem has been studied in[1], where I considered the inverse problem associated with the following system of N first-order equations in n+1 dimensions:

$$\Psi_{x_0} + \sigma \sum_{\ell=1}^{n} J_\ell \Psi_{x_\ell} = q\Psi,$$

$$\sigma = \sigma_R + i\sigma_I,\ \sigma_I \neq 0,\ n > 1, \tag{1.1}$$

where $q(x_0,x)$ is an N x N matrix-valued off-diagonal function in $\mathbb{R}^{n+1}$, decaying suitably fast for large x_0,x, and the J_ℓ are constant real diagonal N x N matrices (we denote the diagonal entries of J_ℓ by $J_\ell^1,\dots,J_\ell^N$). Alternatively, using the transformation

$$\Psi(x_0,x,k) = \mu(x_0,x,k)\exp[i \sum_{\ell=1}^{n} k_\ell(x_\ell - \sigma x_0 J_\ell)],\quad k \in \mathbb{C}^n, \tag{1.2}$$

I considered

$$\mu_{x_0} + \sigma \sum_{\ell=1}^{n} (J_\ell \mu_{x_\ell} + ik_\ell[J_\ell,\mu]) = q\mu. \tag{1.3}$$

I assume that $n \leq N$, otherwise the entries of the J_ℓ matrices will be linearly related and one can always reduce n by a change of coordinates. An inverse problem in this case is defined as follows: Given appropriate inverse data T, where T is an N x N matrix-valued off-diagonal function of suitable inverse parameters, reconstruct the potential q.

There is a twofold motivation for considering such an inverse problem.

(a) *If $\sigma = -1$ then the above reduces to the formulation of a* physically important inverse scattering problem: Given the scattering amplitude $S(\lambda,k)$, λ, $k \in \mathbb{R}^n$, which is a function of the scattering parameters λ, k, reconstruct q.

(b) In recent years a deep connection has been discovered between inverse scattering of linear eigenvalue problems in one spatial dimension and the initial value problem of certain nonlinear evolution equations in 1+1 (i.e., one spatial and one temporal dimension). Recently a similar connection has been used to extend the above results to nonlinear evolution equations in 2+1 (i.e., two spatial and one temporal dimension)[2-6]. In particular the inverse scattering of (1.3) with $\sigma = -1$ and n=1 has been used to linearize the N-wave interaction equations in 2+1 (see Ref.[4]), the Davey-Stewartson (DS) I (see Ref.[4]) (a 2+1 analog of the nonlinear Schrödinger), and the modified Kadomtsev-Petviashvili (MKP) I (see Ref.[4]) (a 2+1 analog of the modified KdV) equations. Furthermore the inverse problem of (1.3) with $\sigma = i$ and $n=1$ has been used to linearize[5] DSII and MKPII. However, in spite of the above success in 2+1, no physically interesting equation is known to be related to (1.3) for $n > 1$ and $\sigma_I \neq 0$ [the N-wave interaction equations in n+1 spatial and one temporal dimension[7] are related to (1.3) but with $\sigma_I = 0$].

The novelty associated with inverse problems in greater than two spatial dimensions ($n > 1$) stems from the fact that while the potential $q(x_0,x)$ depends on $n + 1$ variables, the inverse data $T(k_R,k_I,m_2,\ldots,m_n)$, $k_R \in \mathbb{R}^n$, $k_I \in \mathbb{R}^n$, $m_i \in \mathbb{R}$, depends on $3n - 1$ variables. This has important implications:

(a) *The inverse data must be appropriately constrained.* This "characterization" of the inverse data is conceptually analogous to the characterization of the inverse scattering data in the multidimensional Schrödinger equation[8-11].

(b) The existence of "redundant" scattering parameters in the inverse scattering of the Schrödinger equation is used to reconstruct the potential in closed form in terms of the scattering amplitude function. This is the well-known Born approximation[12]. *Can one use* the redundancy of the inverse parameters here to also reconstruct q in closed form?

In the above paper, I do the following.

(a) Following A. Nachman and M.J. Ablowitz, I derive an equation that characterizes inverse data:

$$\hat{T}^{ij}(w_0,w) \doteq T^{ij}(w_0,w,\kappa) - \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{d\kappa'_{p_R}\, d\kappa'_{p_I}\, N^{ij}_{1p}[T](w_0,w,\kappa^{p'})}{\kappa_p - \kappa'_p}, \quad (1.4)$$

where $w_0 \in \mathbb{R}$, $w \in \mathbb{R}^n$, $\kappa \in \mathbb{C}^{n-1}$ are related to k, m and N is a quadratic function of T. That is, $T^{ij}(k,m)$ is appropriate inverse data iff the right-hand side of (1.4) is independent of κ. Hence, Eq. (1.4) serves as both characterizing T^{ij} and defining $\hat{T}^{ij}$.

(b) *I reduce the general problem of reconstructing an N x N po-*

tential q in $n + 1$ dimensions to one of reconstructing a 2×2 potential with entries q^{ij}, q^{ji} in two dimensions. The inverse data needed for this reconstruction is precisely $\hat{T}^{ij}$, $\hat{T}^{ji}$. This reduction makes crucial use of the existence of redundant scattering parameters. In this sense it is the analog of the Born approximation. However, the crucial difference is that while in the inverse scattering of the multidimensional Schrödinger equation one can reconstruct the potential in closed form, here one can only reduce the general problem to one for 2×2 matrices in two dimensions. This reduced problem was solved in[4].

Partial results about the case $\sigma = i$ were given in[13]. Equation (1.3) was also considered in[14] where, although the characterization problem was solved (an equation very similar to (1.4), the so-called "T equation", was first obtained in[14]) the problem of an effective reconstructon of q was left open.

The basic steps are as follows:

1.1 Bounded Eigenfunctions.

The function $\mu(x_0,x,k)$ defined below, solves equation (1.3), is bounded for all complex values of k and tends to I for large k:

$$\mu^{ij}(x_0,x,k) = \delta^{ij}+\operatorname{sgn}\frac{(\sigma_I J_1^i)}{2\pi i}\int_{\mathbf{R}^2} d\xi_0 d\xi_1 \,\frac{\exp[i\beta^{ij}(x_0-\xi_0,x_1-\xi_1,k)]}{(x_1-\xi_1) - \sigma J_1^i(x_0-\xi_0)}$$

$$\times\ (q\mu)^{ij}(\xi_0,\xi_1,x_2-(x_1-\xi_1)J_2^i/J_1^i,\dots,x_n-(x_1-\xi_1)J_n^i/J_1^i,k),\ k \in \mathbb{C}^n, \qquad (1.5)$$

where β^{ij} is defined by

$$\beta^{ij}(x_0,x_1,k) \doteq \sum_{\ell=1}^{n} \frac{J_\ell^i-J_\ell^j}{\sigma_I}\left[x_0|\sigma|^2 k_{\ell_I} - \frac{x_1(\sigma k_\ell)_I}{J_1^i}\right],\ k_\ell=k_{\ell_R}+ik_{\ell_I}. \quad (1.6)$$

Equivalently μ_{ij} satisfies

$$\mu^{ij}(x_0,x,k) = \delta^{ij}+ \frac{\operatorname{sgn}(\sigma_I J_1^i)}{2\pi i}\int_{\mathbf{R}^{n+1}} d\xi_0 d\xi[c_{n-1}\int_{\mathbf{R}^{n-1}} dm^2 e^{i\alpha^i(x-\xi,m)}]$$

$$\times\ \frac{\exp[i\beta^{ij}(x_0-\xi_0,x_1-\xi_1,k)](q\mu)^{ij}(\xi_0,\xi,k)}{x_1-\xi_1-\sigma J_1^i(x_0-\xi_0)}, \qquad (1.7)$$

where

$$dm^2 \doteq dm_2 \ldots dm_n,\ \alpha^i(x,m) \doteq \sum_{\ell=2}^{n} m_\ell (x_\ell - x_1 \frac{J_\ell^i}{J_1^i}),\ c_n \doteq \frac{1}{(2\pi)^n}. \quad (1.8)$$

1.2 Departure from Holomorphicity.

Let μ^{ij} be defined by eq. (1.5). Then

$$\frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k) = \sum_{i,j} \gamma^i (J_p^i - J_p^j) \exp[i\beta^{ij}(x_0,x_1,k)]$$

$$\times\ c_{n-1}\int_{\mathbb{R}^{n-1}} dm^2 \exp[i\alpha^i(x,m)] T^{ij}(k,m)\mu(x_0,x,\lambda^{ij}(k,m))E_{ij}, \quad (1.9)$$

where $\beta^{ij}(x_0,x_1,k)$, $\alpha^i(x,m)$ are defined by (1.6), (1.8) respectively; E_{ij} is an N x N matrix with zeros in all its entries except the ij^{th}, which equals 1; and λ^{ij} and T^{ij} are given by

$$\lambda_1^{ij}(k,m) \doteq (k_{1_R}^{ij} - \sum_{\ell=2}^{n} m_\ell \frac{J_\ell^i}{J_1^i},\ k_{1_I}),\ \lambda_r^{ij}(k,m)=(k_{r_R}+m_r, k_{r_I});\ r=2,..n.$$

$$\gamma^i \doteq \bar{\sigma}/4\pi i |J_1^i \sigma_I|,$$

$$T^{ij}(k,m) \doteq \int_{\mathbb{R}^{n+1}} d\xi_0 d\xi \exp[-i\beta^{ij}(\xi_0,\xi_1,k) - i\alpha^i(\xi,m)](q\mu)^{ij}(\xi_0,\xi,k). \quad (1.10)$$

1.3 Characterization of T.

(a) Assume that $\partial\mu/\partial\bar{k}_p$ is given by Eq. (1.9) and the $T^{ij}(k,m)$ is given by (1.10). Then

$$L_{rp}^{ij} T^{ij}(k,m) = -\sum_{\ell=1}^{n} c_{n-1}\int_{\mathbb{R}^{n-1}} dM^2 T^{i\ell}(\lambda^{\ell j}(k,M), m-M) T^{\ell j}(k,m)$$

$$\times\ [(J_p^\ell - J_p^j)(J_r^i - J_r^\ell) - (J_r^\ell - J_r^j)(J_p^i - J_p^\ell)] \doteq N_{rp}^{ij}[T](k,m), \quad (1.11)$$

where

$$L_{rp}^{ij} \doteq (J_p^i - J_p^j)\frac{\partial}{\partial\bar{k}_r} - (J_r^i - J_r^j)\frac{\partial}{\partial\bar{k}_p}. \quad (1.12)$$

(b) Assume that $\partial\mu/\partial\bar{k}_p$ is given by Eq. (1.9) and that $\partial^2\mu/\partial\bar{k}_r\partial\bar{k}_p$ is symmetric with respect to r,p. Then $T^{ij}(k,m)$ solves (1.11).

Following A. Nachman and M.J. Ablowitz I introduce appropriate Born variables. Then equation (1.11) can be integrated. Furthermore, we can compute the limit of T^{ij} in the new coordinates as $|\chi_p| \to \infty$ (see below):

Let $w_0^{ij}, w_1^{ij}, w_\ell$, $\ell = 2,\ldots,n \in \mathbb{R}^1$ and $\chi_\ell \in \mathbb{C}^1$, $\ell = 2,\ldots,n$, be defined by

$$w_0^{ij} \doteq \sum_{r=1}^{n} \frac{J_r^i - J_r^j}{J_I} |\sigma|^2 k_{r_I}, \quad w_1^{ij} \doteq -\sum_{r=1}^{n} \frac{J_r^i - J_r^j}{\sigma_I J_1^i} (\sigma k_r)_I - \sum_{r=2}^{n} m_r \frac{J_r^i}{J_1^i},$$

$$w_\ell \doteq m_\ell, \quad \chi_\ell^{ij} \doteq \frac{k_\ell}{J_1^j - J_1^i}, \quad \ell = 2,\ldots,n. \tag{1.13}$$

Assume that

$$(J_1^r - J_1^j)(J_p^i - J_p^j) \neq (J_1^i - J_1^j)(J_p^r - J_p^j), \text{ for all distinct } i,j,r \text{ and } p \neq 1. \tag{1.14}$$

For convenience of writing we usually suppress the superscripts, i,j in w_0, w_1, χ. Let k denote $k_1,\ldots,k_n$, m denote $m_2,\ldots,m_n$, χ denote $\chi_2,\ldots,\chi_n$, w denote $w_1,\ldots,w_n$. Then we have the following.

(a) The inverse of the transformation $k,m \to w_0, w, \chi$ is given by

$$k_\ell = \chi_\ell (J_1^j - J_1^i), m_\ell = w_\ell, \ell = 2,\ldots,n, \; k_1 = -\sum_{r=2}^{n} (J_r^j - J_r^i)\chi_r + \frac{(\bar{\sigma}/|\sigma|^2) w_0 + \Sigma_{r=1}^n w_r J_r^i}{J_1^j - J_1^i}. \tag{1.15}$$

(b) In the new coordinates, Eq. (1.11) with r=1 becomes

$$\frac{\partial T^{ij}}{\partial \bar{\chi}_p}(w_0, w, \chi) = N_{1p}^{ij}[T](w_0, w, \chi), \; p = 2,\ldots,n. \tag{1.16}$$

(c) In the new coordinates

$$T^{ij}(w_0, w, \chi) = \int_{\mathbb{R}^{n+1}} d\xi_0 d\xi \exp[-i(w_0\xi_0 + w\xi)](q\mu)^{ij}(\xi_0, \xi, w_0, w, \chi),$$

$$\text{where } w\xi = \sum_{r=1}^{n} w_r \xi_r. \tag{1.17}$$

(d) Let

$$\mu_i^{\ell j} \doteq \mu^{\ell j}(x_0, x, w_0^{ij}, w^{ij}, \chi^{ij}), \quad \hat{\mu}_i^{\ell j} = \lim_{|\chi_p| \to \infty} \mu_i^{\ell j}. \tag{1.18}$$

Then the $\hat{\mu}_i^{\ell j}$ satisfy

$$\hat{\mu}_i^{ij}(x_0,x,w_0,w) = \operatorname{sgn}\frac{(\sigma_I J_1^i)}{2\pi i} c_{n-1}\int_{\mathbf{R}^{2n}} \times$$

$$\times \frac{dx_0' dx' dm^2 \exp[i\{(x_0-x_0')w_0+w_0+(x-x')w\}]}{x_1-x_1' - \sigma J_1^i(x_0-x_0')} q^{ij}(x_0',x')\hat{\mu}_i^{jj}(x_0',x',w_0,w),$$

$$\hat{\mu}_i^{jj}(x_0,x,w_0,w) = 1 + \frac{\operatorname{sgn}(\sigma_I J_1^j)}{2\pi i} c_{n-1}\int_{\mathbf{R}^{2n}} \times$$

$$\times \frac{dx_0' dx' dm^2 q^{ji}(x_0',x')\hat{\mu}_i^{ij}(x_0',x',w_0,w)}{x_1-x_1' - \sigma J_1^j(x_0-x_0')},\ \hat{\mu}_i^{\ell j}=0, \text{ for all } \ell,\ \ell\neq i,\ \ell\neq j. \tag{1.19}$$

(e) $$\lim_{|\chi_p|\to\infty} T^{ij}(w_0,w,\chi) = \int_{\mathbf{R}^{n+1}} d\xi_0 d\xi \exp[-i(w_0\xi_0+w\xi)] \times$$

$$\times\ q^{ij}(\xi_0,\xi)\hat{\mu}_i^{jj}(\xi_0,\xi,w_0,w) \doteq \hat{T}^{ij}(w_0,w). \tag{1.20}$$

(f) The basic characterization equation is given by

$$\hat{T}^{ij}(w_0,w)=T^{ij}(w_0,w,\chi) - \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{d\chi_{p_R}' d\chi_{p_I}' N_{1p}^{ij}[T](w_0,w,\chi^{p'})}{\chi_p - \chi_p'}. \tag{1.21}$$

where $\chi^{p'}$ denotes $\chi_2,\ldots,\chi_{p-1},\chi_p',\chi_{p+1},\ldots,\chi_n$.

1.4 Reconstruction of q.

It follows from the above that as $|\chi_p|\to\infty$, the μ^{ij}'s decouple. Furthermore, the $\hat{\mu}_i^{ij}$, $\hat{\mu}_i^{jj}$ satisfy a system of two equations depending on q^{ij}, q^{ji}. It turns out that: a) By introducing appropriate spatial variables ξ, the $\hat{\mu}_i^{ij}$, $\hat{\mu}_i^{jj}$ satisfy equations in two spatial equations. b) The invers data needed to reconstruct $\hat{\mu}_i^{ij}$, μ_i^{jj} (and hence q^{ij}, q^{ji}) can be obtained from $\hat{T}^{ij}$:

Let

$$\alpha_r \doteq \frac{J_2^j J_r^i - J_r^j J_2^i}{J_1^i J_2^j - J_1^j J_2^i},\quad \beta_r = \frac{J_1^i J_r^j - J_1^j J_r^i}{J_1^i J_2^j - J_1^j J_2^i},\quad r = 1,\ldots,n, \tag{1.22}$$

where for convenience of writing we have suppressed the dependence of α_r, β_r on i,j. Let $\xi_0 \in \mathbf{R}$, $\xi \in \mathbf{R}^n$,

$$x_0 = \xi_0, \quad x_1 = \xi_1, \quad x_2 = \xi_2,$$
$$x_\ell = \xi_\ell + \alpha_\ell \xi_1 + \beta_\ell \xi_2, \quad \ell = 3,\dots,n. \tag{1.23}$$

Then we have the following.

(a) The system (1.19) becomes

$$\hat{\mu}_i^{ij}(\xi_0,\xi,\hat{k}) = \operatorname{sgn}\frac{\sigma_I J_1^i}{2\pi i}\int_{\mathbf{R}^2} d\xi_0' d\xi_1' [\xi_1 - \xi_1' - \sigma J_1^i(\xi_0 - \xi_0')]^{-1}$$
$$\times \exp[i\hat{\beta}^{ij}(\xi_0 - \xi_0', \xi_1 - \xi_1', \hat{k})] q^{ij} \hat{\mu}_i^{jj}(\xi_0', \xi_1', \xi_2$$
$$- (\xi_1 - \xi_1') J_2^i / J_1^i, \xi_3, \dots, \xi_n, k),$$
$$\hat{\mu}_i^{jj}(\xi_0,\xi,\hat{k}) = 1 + \operatorname{sgn}\frac{\sigma_I J_1^j}{2\pi i}\int_{\mathbf{R}^2} d\xi_0' d\xi_1' [\xi_1 - \xi_1' - \sigma J_1^j(\xi_0 - \xi_0')]^{-1}$$
$$\times q^{ji} \hat{\mu}_i^{ij}(\xi_0', \xi_1', \xi_2 - (\xi_1 - \xi_1') J_2^j / J_1^j, \xi_3, \dots, \xi_n, \hat{k}), \tag{1.24}$$

where

$$\hat{k} \doteq \sum_{r=1}^n \left(k_r \alpha_r + \frac{J_2^j - J_2^i}{J_1^j - J_1^i} k_r \beta_r\right), \quad \hat{\beta}^{ij}(x_0, x_1, \hat{k})$$
$$\hat{\beta}^{ij}(x_0, x_1, \hat{k}) \doteq \frac{J_1^i - J_1^j}{\sigma_I}\left[x_0 |\sigma|^2 k_I - x_1 \frac{(\sigma k)_I}{J_1^i}\right]. \tag{1.25}$$

(b) $\hat{T}^{ij}$ in the new coordinates becomes

$$\hat{T}^{ij}(\hat{k},\hat{m}) = \int_{\mathbf{R}^{n+1}} d\xi_0' d\xi' \exp[-i\hat{\beta}^{ij}(\xi_0', \xi_1', \hat{k}) +$$
$$+ \hat{m}_2\left(\xi_2' - \xi_1' \frac{J_2^i}{J_1^i}\right) + \sum_{r=3}^n \hat{m}_r \xi_r'] q^{ij} \hat{\mu}_i^{jj}(\xi_0', \xi', \hat{k}), \tag{1.26}$$

where

$$\hat{m}_2 \doteq m_2 + \sum_{r=3}^n m_r \beta_r, \quad \hat{m}_\ell = m_\ell, \quad \ell = 3,\dots,n. \tag{1.27}$$

(c) The inverse data associated with (1.24) and the analogous problem for $\hat{\mu}_j^{ji}, \hat{\mu}_j^{ii}$ are given by $\tilde{T}^{ij}$, $\tilde{T}^{ji}$. Let

$$\tilde{T}^{ij}(k,\xi_2-\xi_1 J_2^i/J_1^i,\xi_3,\ldots,\xi_n) \doteq c_{n-1}\int_{\mathbb{R}^{n-1}} d\hat{m}\, \exp[i\hat{m}_2(\xi_2-\xi_1\frac{J_2^i}{J_1^i}) + i\sum_{r=3}^{n}\hat{m}_r\xi_r]\hat{T}^{ij}(\hat{k},\hat{m}). \tag{1.28}$$

Then

$$\tilde{T}^{ij}(\hat{k},\xi_2-\xi_1 J_2^i/J_1^i,\xi_3,\ldots,\xi_n) \doteq \int_{\mathbb{R}^2} d\xi_0' d\xi_1' \exp[-i\hat{B}^{ij}(\xi_0',\xi_1',k)] \times (q^{ij}\hat{\mu}_i^{jj})(\xi_0',\xi_1',\xi_2-(\xi_1-\xi_1')J_2^i/J_1^i,\xi_3,\ldots,\xi_n,\hat{k}). \tag{1.29}$$

2. INVERSE SCATTERING FOR THE HYPERBOLIC N x N AKNS IN MULTIDIMENSIONS

This problem has been studied in[15], where I considered equations (1.1) and (1.3) with $\sigma = -1$. This system appears to be physically more interesting than (1.1)-(1.3): (a) Since the system is hyperbolic one may consider the physically important question of inverse scattering (IS); i.e., given a scattering amplitude function $S(\lambda,k)$ find the potential $q(x_0,x)$. (b) A special case of the above system, namely if the J_ℓ^is are constrained by

$$\frac{J_p^\ell - J_p^j}{J_r^\ell - J_r^j} = \frac{J_p^i - J_p^j}{J_r^i - J_r^j}, \quad p,r=1,\ldots,n, \quad i,j,\ell=1,\ldots,N, \tag{2.1}$$

is associated with the N-wave interaction in n+1 spatial and one temporal dimensions[7,16].

With respect to (a), (b) above the following results are obtained:

(a) I first define $S(\lambda,k)$, λ, $k \in \mathbb{R}^n$, in terms of an eigenfunction $\mu_L(x_0,x,k)$. $S(\lambda,k)$ motivates the introduction of the Born variables $w_0 \in \mathbb{R}^1$, $w \in \mathbb{R}^n$, $\chi \in \mathbb{R}^{n-1}$. I then define $T(\lambda,k)$ in terms of an eigenfunction $\mu(x_0,x,k)$. The crucial new step is that μ is required to have analyticity in one of the χ's, say χ_2, as opposed to one of the k's. Let

$$\hat{T}(w_0,w) := \lim_{|\chi_2|\to\infty} T(w_0,w,\chi);$$

then T^{ij} satisfies

$$\frac{1}{2}\hat{T}^{ij}(w_0,w) = T^{ij}(w_0,w,x)-(P_{x_2}T^{ij})(w_0,w,x), \tag{2.2}$$

where P_{x_2} denotes a (+) or (-) projection in the variable x_2, i.e.,

$$(P^{\pm}_{x_2}f)(x_2) := \int_{\mathbf{R}^1} dx_2' f(x_2')/2\pi i(x_2' - x_2 \mp i0).$$

The sign of certain parameters $\varepsilon_2^{\ell ji}$, where

$$\varepsilon_r^{\ell ji} := (J_1^j - J_1^i)(J_r^\ell - J_r^j) - (J_r^j - J_r^i)(J_1^\ell - J_1^j), \tag{2.3}$$

determines whether the (+) or the (-) projection is needed. Equation (2.2) defines $\hat{T}^{ij}$, which actually depends on q^{ij}, and

$$\hat{\mu}_i^{jj} := \lim_{|x_2|\to\infty} \mu_i^{jj}(x_0,x,w_0,w,x).$$

The question whether (2.2) is sufficient for the characterization of T^{ij} remains open. With a proper coordinate transformation, $\hat{\mu}_i^{ij}$, $\hat{\mu}_i^{jj}$, $\hat{\mu}_j^{ji}$, $\hat{\mu}_j^{ii}$ define an IS problem for a 2 x 2 potential with entries q^{ij}, q^{ji}, in two spatial dimensions. The inverse data needed for this problem are simply related to $\hat{T}^{ij}$, $\hat{T}^{ji}$. The solution of such an IS problem was given in[4]. It is interesting that there exists a simple relationship between T and S. Actually if N = 3 then T = S. This is quite remarkable since for the first time a closed-form expression can be obtained between S and T. Furthermore, μ_L satisfies a Volterra as opposed to a Fredholm equation; hence one immediately excludes the possibility of bound states.

With a proper coordinate transformation, the N-wave interaction equations in n + 1 spatial dimensions can always be reduced to two spatial dimensions. Thus a genuine three-spatial-dimensional nonlinear evolution equation, related to an IS problem, remains to be found.

Let $\mu_L(x_0,x,k)$, $x_0 \in \mathbf{R}^1$, $x,k \in \mathbf{R}^n$ be the solution of (1.3) which also solves

$$\mu_L^{\ell j}(x_0,x,k) = \delta^{\ell j} + \int_{-\infty}^{x_0} dx_0' \exp[ik(J - J^j)(x_0-x_0')]$$

$$\times (q\mu_L)^{\ell j}(x_0', x + J^{\ell}(x_0 - x_0'), k), \tag{2.4}$$

where —

$$kJ^{\ell} := \sum_{r=1}^{n} k_r J_r^{\ell}, \quad mx = \sum_{r=1}^{n} m_r x_r, \quad c_n := (2\pi)^{-n},$$

and $x + J^{\ell}x_0$ denotes $x_1 + J_1^{\ell}x_0, \ldots, x_n + J_n^{\ell}x_0$. Let $\Phi(x_0, x)$ be the general solution of (2.1) with $\sigma = -1$ tending to $f(x + x_0 J)$ or $g(x + x_0 J)$ as $x_0 \to -\infty$ or $x_0 \to \infty$, respectively. The scatteirng operator is defined by $g = \tilde{S}f$, and $\tilde{S}$ is uniquely defined in terms of $S(\lambda, k)$, where

$$S^{ij}(\lambda, k) := c_n \int_{R^{n+1}} dx_0 dx \exp[i(k-\lambda)x + i(kJ^j - \lambda J^i)x_0]$$

$$\times (q\mu_L)^{ij}(x_0, x, k), \quad i \neq j, \tag{2.5}$$

and

$$kx := \sum_{\ell=1}^{n} k_\ell x_\ell .$$

The Born variables are defined by

$$w_0 := kJ^j - \lambda J^i, \; w_\ell := k_\ell - \lambda_\ell, \; \epsilon_\ell := \frac{k_\ell}{J_\ell^j - J_\ell^i}, \; \ell = 2, \ldots, n, \tag{2.6}$$

where, for convenience of writing, we suppress the dependence of w_0, ϵ on i, j.

$T^{ij}(\lambda, k)$ satisfies an equation similar to S^{ij} where $\mu_L^{\ell j}$ is replaced by $\mu_i^{\ell j}$. The eigenfunction $\mu_i^{\ell j}$ satisfies an equation similar to (2.4) where $\int_{-\infty}^{x_0}$ is replaced by $\int_{\ell j i}$. This integral is either $\int_{-\infty}^{x_0}$ or $-\int_{x_0}^{\infty}$ according to the following requirements: (1). If $\epsilon_2^{\ell ji} = 0$ then choose $\int_{-\infty}^{x_0}$. (2) If $\epsilon_2^{\ell ji} \neq 0$ then $(x_0 - x_0')\ \epsilon^{\ell ji}$ must have the same sign for all ℓ (i,j are fixed). (3) If (2) can be satisfied with either $\int_{-\infty}^{x_0}$ or $-\int_{-\infty}^{x_0}$, choose the first integral. To illustrate the above, consider N = 4. Since $\epsilon_2^{\ell ji} = \epsilon_2^{\ell ji} = 0$, there exist only two nonzero $\epsilon^{\ell ji}$, if they are of the same sign choose both integrals to be $\int_{-\infty}^{x_0}$, if they are of opposite signs then one integral is $\int_{-\infty}^{x_0}$ and the other is $-\int_{x_0}^{-\infty}$. In the Born variables, $T_{ij}(w_0, w, \epsilon)$ depends on

$\mu_i^{\ell j}(x_0,x,w_0,w,x)$, which satisfy

$$\mu_i^{\ell j}(x_0,x,w_0,w,x) = \delta^{\ell j} + \int_{\ell j i} dx_0' \exp\, i[w_0 - wJ^i + \sum_{r=2}^{n} \epsilon_r^{\ell j i} x_r]$$

$$\times (x_0 - x_0')\}(q\mu_i)^{\ell j}(x_0', x + J^\ell(x_0 - x_0'), w_0, w, x). \qquad (2.7)$$

Equation (2.7) implies the following: (1) $\mu_i^{\ell j}$ and hence T_{ij} have analyticity properties with respect to x_2: $T^{ij}(\mu_i^{\ell j})$ is a (+) or (-) function with respect to x_2 (i.e., it is holomorphic in the upper or lower x_2 half-planes, respectively) according to whether $(x_0 - x_0')\,\epsilon_2^{\ell j i}$ is > 0, or < 0. (2) As $|x_2| \to \infty$, $\mu_i^{\ell j} \to \hat{\mu}_i^{\ell j}(w_0,x,w_0,w)$, where $\hat{\mu}_i^{\ell j}$ satisfy a reduced system: $\hat{\mu}_i^{\ell j} = 0$ for all $\ell \neq 1,j$, and $\hat{\mu}_i^{ij}$, $\hat{\mu}_i^{jj}$ satisfy a system of two integral equations of the Volterra type (see below). Hence as $|x_2| \to \infty$,

$$T^{ij} \to \hat{T}^{ij}(w_0,w) := c_n \int_{R^{n+1}} dx_0 dx \exp[i(w_0 x_0 + wx)]$$

$$\times \; q^{ij}(x_0,x)\hat{\mu}_i^{jj}(x_0,x,w_0,w).$$

Since T^{ij} is a (+) or (-) function of x_2 tending to $\hat{T}^{ij}$ as $|x_2| \to \infty$, its (-) or (+) projection must satisfy Eq. (2.2). [We define $p^+(1) = \frac{1}{2}$, $p^-(1) = -\frac{1}{2}$.]

Given T^{ij}, T^{ji} one can compute $\hat{T}^{ij}$, $\hat{T}^{ji}$, which actually can be used to reconstruct the 2 x 2 matrix potential with entries q^{ij}, q^{ji}: We consider the reduced system. The crucial fact is that it corresponds to N = 2, and hence

$$J_r^\ell = \alpha_r J_1^\ell + \beta_r J_2^\ell,\ \ell = i \text{ or } j,\ \alpha_r := \frac{J_2^j J_r^i - J_r^j J_2^i}{J_1^i J_2^j - J_1^j J_2^i},\ \hat{k} := \frac{k(J^j - J^i)}{J_1^j - J_1^i}; \qquad (2.8)$$

β_r is defined as α_r with $2 \leftrightarrow 1$, $j \leftrightarrow i$, and again for convenience of writing we suppress the ij dependence of α_r, β_r. Since one can introduce a single $\hat{k}$, it follows that the reduced system must be transformable to a system in two dimensions. This is indeed the case. Let

$$x_0 = \xi_0,\ x_1 = \xi_1,\ x_2 = \xi_2 - \gamma^{ij}\xi_0 + \delta^{ij}\xi_1,\ x_\ell = \xi_\ell + \alpha_\ell \xi_1 + \beta_\ell \times$$

$$\times (\xi_2 - \gamma^{ij}\xi_0 + \delta^{ij}\xi_1), \quad \ell = 3,\dots,n, \tag{2.9}$$

where $\gamma^{ij} := (J_1^i J_2^j - J_1^j J_2^i)/(J_1^i - J_1^j)$, $\delta^{ij} := (J_2^i - J_2^j)/(J_1^i - J_1^j)$. Then the reduced system satisfies

$$\hat{\mu}_i^{\ell j}(\xi_0,\xi,\hat{k}) = \delta^{\ell j} + c_n\int_{-\infty}^{x_0} d\xi_0' \int_{\mathbf{R}^{2n}} d\xi' dm \exp\{i[\hat{k}(J_1^\ell - J_1^j) + m_1 J_1^\ell] \times$$

$$\times (\xi_0 - \xi_0') + i m_1 J_1^\ell(\xi_1 - \xi_1') + i\sum_{r=2}^{n} m_r(\xi_r - \xi_r')\}(q\hat{\mu}_i)^{\ell j}(\xi_0',\xi',\hat{k}), \ \ell = i,j, \tag{2.10}$$

where $(q\hat{\mu}_i)^{ij} = q^{ij}\hat{\mu}_i^{jj}$, $(q\hat{\mu}_i)^{jj} = q^{ji}\hat{\mu}_i^{ij}$. Hence $\hat{\mu}_i^{\ell j}$ satisfy two integral equations in the variables ξ_0, ξ_1:

$$\hat{\mu}_i^{\ell j}(\xi_0,\xi_1,\hat{k}) = \delta^{\ell j} + c_1\int_{-\infty}^{x_0} d\xi_0' \int_{\mathbf{R}^2} d\xi_1' dm_1 \exp\{i[\hat{k}(J_1^\ell - J_1^j) + m_1 J_1^\ell \times$$

$$\times[(\xi_0 - \xi_0') + i m_1 J_1^\ell(\xi_1 - \xi_1')\}(q\hat{\mu}_i)^{\ell j}(\xi_0',\xi_1',\xi_2,\dots,\xi_n,k). \tag{2.11}$$

The inverse data associated with (2.11) can be obtained from $\hat{T}^{ij}$: Let

$$\hat{w}_\ell = w_\ell,\ \ell = 3,\dots,n, \quad \hat{w}_2 = w_2 + \sum_{r=3}^{n} w_r \beta_r, \quad \hat{\lambda} = \sum_{r=1}^{n} (\alpha_r + \beta_r)\lambda_r. \tag{2.12}$$

Then, since

$$w_0 x_0 + wx = (\hat{k}J_1^j - \hat{\lambda}J_1^i)\xi_0 + (\hat{k} - \hat{\lambda})\xi_1 + \sum_{r=2}^{n} \hat{w}_r\xi_r,$$

$\hat{T}^{ij}$ in the new coordinates becomes

$$\hat{T}^{ij}(\hat{k},\hat{\lambda},\hat{w}) = c_n\int_{\mathbf{R}^{n+1}} d\xi_0 d\xi \exp[i(\hat{k}J_1^j - \hat{\lambda}J_1^i)\xi_0 + i(\hat{k} - \hat{\lambda})\xi_1 + i\sum_{r=2}^{n} \times$$

$$\times \hat{w}_r\xi_r](q^{ij}\hat{\mu}_i^{jj})(\xi_0',\xi',\hat{k}).$$

Hence, when we take the Fourier transform of $\hat{T}^{ij}$ with respect to $\hat{w}$, it follows that

$$\tilde{T}^{ij}(\hat{k},\hat{\lambda},\xi_2,\dots,\xi_n) := \int_{\mathbf{R}^{n-1}} dw\, T^{ij}(\hat{k},\hat{\lambda},\hat{w})\exp(-i\hat{w}\xi) =$$

$$= c_n\int_{\mathbf{R}^2} d\xi_0 d\xi_1 \exp[i(\hat{k}J_1^j - \hat{\lambda}J_1^i)\xi_0 + i(\hat{k} - \hat{\lambda})\xi_1]q^{ij}\hat{\mu}_i^{jj}(\xi_0',\xi_1',\xi_2,\dots,\xi_n,\hat{k}). \tag{2.13}$$

The inverse data $\tilde{T}^{ij}$ and its analog $\tilde{T}^{ji}$ are precisely what is needed to solve the inverse problem for q^{ij}, q^{ji} associated with Eq. (2.11) and its analogs for $\hat{\mu}_j^{ij}$, $\hat{\mu}_j^{ii}$, see[4].

S^{ij}, T^{ij} are defined in terms of $\mu_L^{\ell j}$, respectively. It is possible to find a simple relationship between $\mu_L^{\ell j}$, $\mu_i^{\ell j}$ which then yields a simple relationship between S^{ij}, T^{ij}. Actually if $N = 3$ then $T^{ij} = S^{ij}$.

It was shown in[14] that the N-wave interactions can always be reduced to three spatial dimensions. It is shown in[15] that they can actually be reduced to two spatial dimensions[17].

3 RECURSION OPERATORS AND BI-HAMILTONIAN FORMULATION OF EQUATIONS IN 2+1

This is joint work with P.M. Santini[18-20]. Since it is summarized in these proceedings in a separate contribution, I shall limit myself to a few remarks.

Since the discovery of an exact approach to nonlinear evolution equations in 1+1 (i.e., one spatial and one temporal dimension)[21], two interrelated aspects have recieved much attention in the literature:

(i) The development of a method of solving suitable initial-value problems. For initial data decaying at infinity such a method is the inverse scattering transform (IST)[22]. This method crucially utilizes the existence of an associated isospectral linear eigenvalue problem.

(ii) The investigation of the "algebraic" properties of the given equation. A fundamental role with respect to the algebraic properties is played by an integrodifferential operator, given various names in the literature: squared eigenfunction operator[23], recursion operator[24], strong symmetry[25], hereditary symmetry[25], Kahler operator[26], regular operator[27]. This operator has the following properties:

(a) It generates the associated hierarchy.

(b) It generates infinitely many symmetries (in particular, if it has a certain property which Fuchssteiner calls hereditary, it generates a set of commuting symmetries).

(c) Its adjoint generates gradients of conserved quantities (in particular, it generates a set of involutionary constants of the motion).

(d) Its multiplication by one Hamiltonian structure generates (under certain conditions[28]) a second Hamiltonian.

(e) The eigenfunctions of its adjoint are quadratic products of the eigenfunctions of the associated isospectral problem and form a complete set[29].

It should be noted that given the isospectral eigenvalue problem, there exists an algorithmic technique for obtaining the recursion operator (see for example[30]). This is, from a unification point of view, quite satisfactory, since both the method of solution (IST) and the algebraic properties (recursion operator) stem from the same entity (isospectral eigenvalue problem). For the Korteweg-deVries (KdV) equation $q_t = q_{xxx} - 6qq_x$, $q = q(x,t)$, the recursion operator Φ is $D^2 - 4q - 2q_x D^{-1}$, where $D \doteq \partial_x$, $(D^{-1}f)(x) \doteq \int_{-\infty}^{x} f(\xi)d\xi$. If Δ is the adjoint of Φ, then Δ satisfies $\Delta\psi^2 = 4\lambda\psi^2$, where ψ solves $\psi_{xx} - (q+\lambda)\psi = 0$.

The above two aspects have been thoroughly investigated for a number of physically important equations in 1+1. Each of these equations has physically significant two-spatial-dimensional analogues. For example, the KdV is generalized to the Kadomtsev-Petviashvili (KP) equation, the modified KdV to the modified KP, the non-linear Schrödinger to the Davey-Stewartson, etc. Furthermore, these equations are also related to isospectral eigenvalue problems which are appropriate generalizations of the corresponding one-dimensional ones. It is therefore natural to investigate aspects (i), (ii) above for two-spatial-dimensional (2+1) exactly solvable equations.

The extension of the IST to equations in 2+1 has been recently established in[2-6], (see also[31]). However, the problem of finding recursion operators in 2+1 has remained open; actually even the existence of such operators has been doubted in the literature. In this respect note:

1. The IST of the Benjamin-Ono equation has all the features of an equation in 2+1[32]. It is thus not surprising that its recursion operator has not been found. One of the authors (A.S.F.) and B. Fuchssteiner, after failing to find the recursion operator of the Benjamin-

Ono equation, introduced an alternative approach[33] for generating symmetries. This approach uses a certain function, called τ; it was subsequently applied to a number of equations, including the KP[39]. However, for equations in 2+1: (a) The relationship between τ and the eigenvalue problem has not been established. (b) There does not exist an algorithmic way of finding τ. (c) It is not known if τ can be used to obtain the second Hamiltonian. (d) τ is not hereditary.

2. The bi-Hamiltonian nature of equations in 1+1 has been emphasized as the fundamental property underlying integrability[35]. However, the bi-Hamiltonian nature of all equations in 2+1 as well as of the Benjamin-Ono has remained open. The existence of a recursion operator would directly imply the second Hamiltonian, since all these equations have one known Hamiltonian.

3. A number of important results pertinent to the algebraic properties of equations in 2+1 have obtained in the Soviet Union[36]. In particular Zakharov and Konopelchenko, in a very interesting paper[37], claimed that recursion operators are purely one-dimensional phenomena (i.e., they do not exist in more than one dimension). A careful reading of their work reveals that indeed recursion operators of a certain form do not exist in more than one dimension.

4. Several authors have noticed that mastersymmetries also exist for equations in 1+1; let us call such a mastersymmetry T. Actually, T comes from a nongradient function and can be used to generate τ. However, τ comes from a gradient function and fails to generate a recursion operator.

The extension of the inverse scattering in 2+1 necessitated the introduction of a new idea, the use of $\bar{\partial}$ (DBAR). The extension of the theory of recursion operators and bi-Hamiltonian structures to equations in 2+1 necessitated the introduction of distributions, or more precisely the introduction of integral representations of certain differential operators.

We note that the proper analogue of T is not τ but a function denoted in[20] by T_{12}. This function also generates recursion operators in analogy with T.

4. THE KADOMTSEV-PETVIASHVILI EQUATION PERIODIC IN ONE SPATIAL DIMENSION AND DECAYING IN THE OTHER

This is joint work with V. Papageorgiou[38]; more details of this work can be found in[39]. We have considered four different problems: KPI periodic in x, KPI periodic in y, KPII periodic in x, KPII periodic in y. Here I will only summarize some results for KPI periodic in x of period 2ℓ.

4.1. Analytic Eigenfunctions.

The x-part of the Lax pair is given by

$$i\mu_y + \mu_{xx} + 2ik\mu_x = -u\mu. \tag{4.1}$$

Consider these solutions of (4.1) which also solve

$$\mu^{\pm}(x,y,k) = 1 + \frac{i}{2\ell}\int_{-\infty}^{y} d\eta \int_{-\ell}^{\ell} d\xi \sum_{m\in k\mp\tau\mathbb{Z}^+} \theta(m-k,x-\xi,y-\eta)u(\xi,\eta)\mu^{\pm}(\xi,\eta,k)$$

$$-\frac{i}{2\ell}\int_{y}^{\infty} d\eta \int_{-\ell}^{\ell} d\xi \sum_{m\in k\pm\tau\mathbb{Z}_0^+} \theta(m-k,x-\xi,y-\eta)u(\xi,\eta)\mu^{\pm}(\xi,\eta,k), \tag{4.2}$$

where

$$\tau = \pi/\ell, \quad \theta(m,x,y) = e^{imx-im^2y}. \tag{4.3}$$

These eigenfunctions are ⊕ and ⊖ with respect to the complex k-plane, i.e. they are the boundary values of functions analytic in the upper or lower halves of the k-complex plane.

A simple calculation shows

$$\mu^{+}(x,y,k) - \mu^{-}(x,y,k) = \sum_{m\in k+\tau\mathbb{Z}} T(k,m)N(x,y,m,k), \tag{4.4}$$

where

$$T(k,m) \doteq \frac{i}{2\ell}\,\mathrm{sgn}(k-m)\int_{-\infty}^{\infty} d\eta \int_{-\ell}^{\ell} d\xi\, u(\xi,\eta)\mu^{+}(\xi,\eta,k)\theta(k-m,\xi,\eta), \tag{4.5}$$

$\lambda-k$, $m-k$ are integer multiples of τ and N solves the same equation as $\mu^-(x,y,k)$ but with the forcing replaced by $\theta(\lambda-k,x,y)$.

4.2. A Symmetry Condition.

N, μ are related by (for simplicity of writing we suppress x,y)

$$N(\lambda,k) = \mu^-(\lambda)\theta(\lambda-k,x,y) + \begin{cases} \sum_{m\in[k+\tau,\lambda]} F(\lambda,m)\mu^-(m)\theta(m-k,x,y) & \text{if } \lambda>k \\ -\sum_{m\in[\lambda+\tau,k]} F(\lambda,m)\mu^-(m)\theta(m-k,x,y) & \text{if } \lambda<k, \end{cases} \tag{4.6}$$

where $[k+\tau,\lambda] = \{k+\tau,\ k+2\tau,\ldots,k+n\tau=\lambda\}$, and

$$F(\lambda,m) = \frac{i}{2\ell}\int_{-\infty}^{\infty} d\eta \int_{-\ell}^{\ell} d\xi N(\xi,\eta,\lambda,m)u(\xi,\eta). \tag{4.7}$$

4.3. Scattering Equation.

Using (4.4) and (4.6) it follows that

$$\mu^+(k)-\mu^-(k) = \sum_{m\in k+\tau\mathbb{Z}} f(k,m)\theta(m-k,x,y)\mu^-(m);\ f(k,m)=\text{sgn}(k-m)F(k,m). \tag{4.8}$$

Equation (4.8) can be viewed as a Riemann-Hilbert (RH) problem with a shift for $\mu^-(k)$. The inverse problem consists of finding μ^- in terms of the scattering data $f(k,m)$. The time evolution of $f(k,m)$ is given by

$$f(k,m,t) = f(k,m,0)e^{4i(m^3-k^3)t} \tag{4.9}$$

and $f(k,m,0)$ can be obtained in terms of initial data using (4.7).

Remarks

i) Localized solutions, periodic in x and decaying in y correspond to homogeneous solutions of $(4.2)^{\pm}$. Such solutions have been obtained in[40].

ii) The above formalism also follows by proper discretization of the results of[2].

iii) It is interesting that a symmetry condition of the type first introduced in[2] is necessary not only for KPI but also for KPII.

iv) Some of the above results were first obtained by P. Caudrey[41] viewing KP as a singular limit $N\to\infty$ of a matrix N x N one-dimensional problem.

5. THE INITIAL VALUE PROBLEM OF CERTAIN PAINLEVÉ EQUATIONS

This is joint work with U. Mugan and M.J. Ablowitz[42]; more details of this work can be found in[43].

The mathematical and physical significance of the six Painlevé transcendents, PI-PVI[44], has been well established: Mathematically, these are the only equations of the form $q_{tt} = F(q_t, q, t)$, where F is rational in q_t, algebraic in q and locally analytic in t which have the Painlevé property (i.e. their solutions are free from movable critical points). Physically, are closely related[45] to physically significant solvable PDE's and have appeared in several physical applications, see for example[46-50].

Central in the integrability of PDE's in 1+1 and 2+1 is their relation to isospectral eigenvalue problems. Similarly, central to the integrability of the Painlevé equations is their relation to isomonodromic problems (see Sato et al[51], Ueno[52], Flaschka and Newell[53], and Jimbo et al[54]).

We have systematically considered the initial value problem of PII, PIV, PV. Equation PIII is contained in PV for a special choice of one of the parameters of PV, equation PVI has been solved by C. Cosgrove and PI remains open. The basic approach is that introduced in[55] although we have made certain simplifications and extensions. Here I briefly summarize the main results using PIV as an illustrative example.

5.1. The Lax Pair.

PIV equation

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2+\alpha)y + \frac{\beta}{y}, \tag{5.1}$$

is the compatibility of the following linear problems

$$Y_x(x;t) = \left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}x + \begin{pmatrix}t & u\\ \frac{2}{u}(z-\theta_0-\theta_\infty) & -t\end{pmatrix} + \right.$$

$$\left. + \begin{pmatrix}\theta_0 - z & -\frac{uy}{2}\\ \frac{2z}{uy}(z-2\theta_0) & -(\theta_0 - z)\end{pmatrix}\frac{1}{x}\right]Y(x;t), \tag{5.2a}$$

$$Y_t(x;t) = \left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}x + \begin{pmatrix}0 & u\\ \frac{2}{u}(z-\theta_0-\theta_\infty) & 0\end{pmatrix}\right]Y(x;t). \tag{5.2b}$$

Indeed $Y_{xt} = Y_{tx}$ implies

$$\frac{dy}{dt} = -4z + y^2 + 2ty + 4\theta_0, \tag{5.3a}$$

$$\frac{dz}{dt} = -\frac{2}{y} z^2 + \left(\frac{4\theta_0}{y} - y\right) z + (\theta_0 + \theta_\infty) y, \tag{5.3b}$$

$$\frac{du}{dt} = -u(y + 2t), \tag{5.3c}$$

where,

$$\alpha = 2\theta_\infty - 1, \quad \beta = -8\theta_0^2 . \tag{5.4}$$

Eliminating from (5.3) we cbtain PIV.

5.2. Analytic Eigenfunctions and their Relationship.

We recall that in studying the initial value problem of an equation in 1+1 or in 2+1, one uses the time-independent part of the Lax pair to define an inverse problem, in terms of certain scattering or (more generally) inverse data. Then one uses the time dependent part to find the time evolution of these data. Similarly, here one uses (5.2a) to define an inverse probelm in terms of certain monodromy data; then one uses (5.2b) to find the time evolution of these data. To define an inverse problem one needs to consider the analyticity properties of $Y(x;t)$ in the whole complex x-plane. Since Y satisfies a linear ODE, its analyticity properties are completely determined from the singular points of (5.2a). Indeed, performing an analysis around $x = 0$, $x = \infty$, and introducing different solutions Y_j in different sectors S_j (so that Y_j's are normalized at ∞) it follows that (we assume $0 < \theta_0 < 1$, $0 \leq \theta_\infty < 1$, $\theta_0 \neq \frac{1}{2}$):

i) The Y_j's, $j = 1,\ldots,5$ defined in the sectors S_j's, where the S_j's are given below and each S_j contains the intial boundary line, are related via

$$Y_{j+1}(x) = Y_j(x)G_j, \quad x \text{ on } C_{j+1}, \quad j = 1,2,3, \tag{5.5}$$

$$Y_1(x) = Y_4(xe^{2i\pi})\hat{G}_4, \quad x \text{ on } C_1.$$

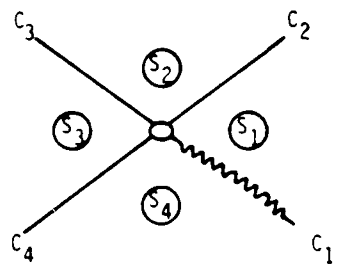


Figure 1.

where

$$G_1 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix},\ G_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},\ G_3 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},\ G_4 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad (5.6)$$

$\hat{G}_4 = G_4 M_\infty$, $M_\infty = \mathrm{diag}(e^{2i\pi\theta_\infty}, e^{-2i\pi\theta_\infty})$.

ii) $$Y_j(x) \sim Y_\infty(x) = \hat{Y}_\infty(x) e^{Q(x)} (1/x)^{D_\infty}, \quad \text{as } |x| \to \infty,\ x \text{ in } S_j, \quad (5.7)$$

where $\hat{Y}_\infty(x)$ is holomorphic at $x = \infty$,

$$Q(x) = \mathrm{diag}\left[\frac{x^2}{2} + xt,\ -\left(\frac{x^2}{2} + xt\right)\right],\ D_\infty = \mathrm{diag}(\theta_\infty, -\theta_\infty). \quad (5.8)$$

iii) $$Y_0(x) \sim \hat{Y}_0(x) x^{D_0}, \text{ as } x \to 0,\ x \text{ in } S_1, \quad (5.9)$$

where $\hat{Y}_0(x)$ is holomorphic at $x = 0$,

$$M_0 = \mathrm{diag}(e^{2i\pi\theta_0}, e^{-2i\pi\theta_0}),\ Y_0(xe^{2i\pi}) = Y_0(x) M_0. \quad (5.10)$$

iv) The connection matrix E_0 is defined by

$$E_0 \doteq \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix} \quad Y_1(x) = Y_0(x) E_0,\ x \text{ in } S_1,\ \det E_0 = 1. \quad (5.11)$$

From the above it follows that Y is a sectionally holomorphic function. Its behavior at $x = 0$ and $x = \infty$ is determined from the monodromy matrices M_0 and M ; its jumps across the Stokes lines defined in Figure 1, are given by the Stoke's matrices $G_1, \ldots, G_4$. Hence its entire behavior is determined from the following data:

$$\text{Monodromy data} = \{a, b, c, d, \alpha_0, \beta_0, \gamma_0, \delta_0\}. \quad (5.12)$$

5.3. Properties of the Monodromy Data.

It turns out that all of the monodromy data can be expressed in terms of two. In particular

$$\left(\prod_{j=1}^{4} G_j\right)M_\infty = E_0^{-1}M_0^{-1}E_0, \tag{5.13}$$

$$(1+bc)e^{2i\pi\theta_\infty} + [a + (c+1)(ab+1)]e^{-2i\pi\theta_\infty} = 2\cos 2\pi\theta_0. \tag{5.14}$$

Furthermore, using (5.2b) it follows that the mondromy data are time invariant.

5.4. The Inverse Problem.

The inverse problem consists of reconstructing Y, or more precisely Ψ where $\Psi = Ye^{-Q(x)}$, in terms of the above monodromy data. The inverse problem can be formulated as a Riemann-Hilbert (RH) matrix problem along the contour $C_1 + C_2 + C_3 + C_4$. This RH problem is discontinuous both at the origin and at infinity. Using the method of[55] we separate this RH problem into a sum of two simpler RH problems, one defined on $C_2 + C_4$ and the other on $C_1 + C_3$. Furthermore, it is remarkable that the RH problem defined on $C_2 + C_4$ can be solved in closed form. Hence solving the initial vlaue problem of PIV is equivalent to solving a RH problem discontinuous at $x = 0$ and $x = \infty$ and defined on $C_1 + C_3$. This RH problem can be mapped to a continuous one using appropriate auxiliary functions. Hence its solution can be obtained in terms of a Fredholm integral equation. Having obtained Y it is straightforward to compute y, i.e. the solution of PIV.

5.5. Some Special Solutions.

For certain choices of the parameters α, β, PIV admits one parameter family of solutions expressible rationally in terms of the Weber-Hermite functions. Such solutions can also be obtained from the inverse problem. Let $\theta_\infty = \pm\theta_0 + \frac{n}{2}$, $n \in \mathbb{Z}$, then (5.14), (5.13) imply $a = c = 0$ and $b = -d$ respectively. Let us consider for concreteness $\theta_\infty = -\theta_0$, $0 < \theta_\infty < 1/2$. Then one is lead to the following RH problem:

$$\Psi^+(x) = \Psi^-(x)g_\Psi(x), \text{ on } C_1+C_3,\ g_\Psi = g_2^{-1} \text{ on } C_3;\ g_4 \text{ on } C_1, \tag{5.15}$$

$$\Psi \to I \text{ as } |x| \to \infty,$$

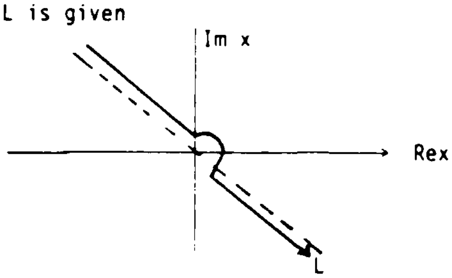
where

$$g_2 = e^{Q(x)}G_2e^{-Q(x)}, \quad g_4 = e^{Q(x)}\hat{G}_4e^{-Q(x)}.$$

This RH problem can be solved in closed form:

$$\Psi = \begin{pmatrix} x^{-\theta_\infty} & x^{\theta_\infty}F(x) \\ 0 & x^{\theta_\infty} \end{pmatrix}; \; F(x) \doteq \frac{d}{2\pi i}\int_L \hat{x}^{-2\theta_\infty}\frac{e^{2q(\hat{x})}}{\hat{x}-x}d\hat{x}, \; q(x) = \frac{x^2}{2} + xt, \tag{5.16}$$

where the contour L is given



Note that

$$F(x) = -\frac{1}{x}f(t) + O(\frac{1}{x^2}), \; f(t) = He_{2\theta_\infty - 1}(it), \; He \doteq \text{Weber-Hermite}$$

function.

5.6. Schlesinger Transformations.

These transformations change θ_0, θ_∞ to $\theta_0', \theta_\infty'$, where

$$\theta_0' = \theta_0 \pm \frac{m}{2}, \quad \theta_\infty' = \theta_\infty \pm \frac{n}{2}, \quad m, n \in \mathbb{Z}. \tag{5.17}$$

To obtain these transformations let $Y \doteq R(x,t)Y$ correspond to $\theta_0', \theta_\infty'$ but to the <u>same</u> monodromy data. Then it can be shown that R satisfies a very simple RH problem, which can actually be sovled in closed form: For example

$$\theta_0' = \theta_0 - \frac{1}{2}, \quad \theta_\infty' = \theta_\infty + \frac{1}{2}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}x^{\frac{1}{2}} + \begin{pmatrix} 1 & \frac{uy}{2(z-2\theta_0)} \\ -\frac{z-\theta_0-\theta_\infty}{u} & -\frac{y(z-\theta_0-\theta_\infty)}{2(z-2\theta_0)} \end{pmatrix}x^{-\frac{1}{2}}. \tag{5.18}$$

6. SOLUTIONS TO A CLASS OF NONLINEAR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

This is joint work with M.J. Ablowitz and M.D. Kruskal[56] and was motivated by some recent results of Constantin, Lax, and Majda[57]; these authors used the transformation $w = u + iHu$, where H denotes the Hilbert transform, to map the equation $u_t = uHu$ to the ODE $w_t = -\frac{1}{2}w^2$.

Using the fact that $w = u + iHu$ is the boundary value of a function analytic in the lower half complex x-plane it follows that $Hw = -iw$, and more generally $Hw^n = -iw^n$, $He^w = -ie^w$ etc. This enables us to map a large class of nonlinear singular integro-differential equations, via explicit transformations, to either ordinary differential equations or to linearizable partial differential equations. Conversely, given a linearizable PDE there is an algorithmic way of finding its singular integro-differential analogue. Examples include:

(a) ODE's	(b) Singular Integro-Differential Equations	
$w_t = -\frac{1}{2}w^2$	$u_t = uHu$	(6.1)
$w_t = w^3$	$u_t = u^3 - 3u(Hu)^2$	(6.2)
$w_t = ie^{-iw}$	$u_t = e^{Hu}\sin u.$	(6.3)

(a) PDE's	(b) Singular INtegro-Differential Equations	
$w_t = w_{xx} - i(w^2)_x$	$u_t = u_{xx} + 2(uHu)_x$	(6.4)
$w_t + w_{xxx} - i\alpha(w^2)_x + \beta(w^3)_x = 0$	$u_t + u_{xxx} + 2\alpha(uHu)_x + \beta(u^3 - 3u(Hu)^2)_x = 0$	(6.5)
$w_{xt} = ie^{-iw}$	$u_{xt} = e^{Hu}\sin u$	(6.6)
$w_t + i(w_{xx} + (w^2)_x) = 0$	$u_t = (Hu)_{xx} + 2(uHu)_x.$	(6.7)
$\frac{\partial}{\partial x}(w_t + w_{xxx} - i(w^2)_x) = -3\sigma^2 w_{yy}$	$\frac{\partial}{\partial x}(u_t + u_{xxx} + (2uHu)_x) = -3\sigma^2 u_{yy}$.	(6.8)

Equation (6.4a) is essentially Burgers equation and can be linearized via the Cole-Hopf transformation $w = -i(\ell nf)_x$. Equation (6.4b) arises in various population ecological models and to our knowledge, was first considered and solved via a dependent variable transformation and splitting into upper and lower functions by J. Satsuma[58]. In equations (6.5) α, β are real constants, and (6.5b) is an analog of the Gardner equation (a combination of KdV and modified KdV). Equation (6.6b) is related to the Liouville equation (6.6a) and is known to be linearizable.

A (3+1)-dimensional equation can also be linearized via (6.8a). Namely let $H_z u$ denote the Hilbert transform of $u(x,y,z,t)$ with respect to the variable z, i.e.,

$$H_z u = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x,y,\xi,t)}{\xi - z} d\xi.$$

Then instead of K-P: we may consider a multi-dimensional analog of (6.8b)

$$\frac{\partial}{\partial x}(u_t + u_{xxx} + 2(uH_z u)_x) = -3\sigma^2 u_{yy}, \qquad (6.9)$$

and it is also mapped to the KP equation (6.8a), via $w = u + iH_z u$.

7. HODOGRAPH TRANSFORMATIONS OF LINEARIZABLE PDE'S

This is joint work with P. Clarkson and M.J. Ablowitz[59]. Since P. Clarkson will present these results in a separate contribution of these proceedings, I will only make a few comments.

We call two PDE's <u>equivalent</u> if one can be obtained from the other by a transformation involving the dependent variables $u = \phi(v)$ and/or the introduction of a potential variable ($u = v_x$ or $u_x = v$).

It is well known[60] that the most general <u>semilinear</u> equation of the form

$$u_t = u_{xx} + f(u,u_x) \qquad (7.1)$$

which is linearizable, is either linear or the Burger's equation (which with the above definition, is equivalent to a linear equation). Fokas and Yortsos[61] have shown that the most general <u>quasilinear</u> equation of the form

$$u_t = g(u)u_{xx} + f(u,u_x) \qquad (7.2)$$

which is linearizable, is equivalent to the equation

$$u_t = (u^{-2}u_x)_x + \alpha u^{-2}u_x, \quad \alpha \text{ is an arbitrary constant.} \qquad (7.3)$$

The above equation can be mapped to the Burger's equation via an <u>extended Hodograph</u> transformation, i.e. a transformation of the form

$$\tau = t, \quad \xi = \int^x u(x',t)dx'. \qquad (7.4)$$

It is also well known that the Harry-Dym[62] equation, can be mapped to a modified Korteweg-deVries (MKdV) equation via an extended Hogograph[63]. In that sense, the Harry-Dym equation is a quasilinear analogue of the MKdV equation.

One is naturally motivated to ask the following questions:

i) Is there an algorithmic method of finding a quasilinear analogue of any semilinear equation?

ii) Is the associated quasilinear equation unique?

iii) Conversely, given a quasilinear equation, is there an algorithmic method of finding whether it can be mapped to a semilinear equation as well as finding this semilinear equation?

In the above paper we consider the above questions for semilinear and quasilinear equations of the type

$$u_t = u_{nx} + f(u, u_x, \ldots, u_{(n-1)x}), \quad n \geq 2, \quad u_{nx} \doteq \frac{\partial^n u}{\partial x^n} . \tag{7.5}$$

and

$$u_t = g(u)u_{nx} + f(u, u_x, \ldots, u_{(n-1)x}), \quad n \geq 2, \quad \frac{dg}{du} \neq 0 \tag{7.6}$$

respectively. The answer to question i) above is affirmative. Also, the associated quasilinear equation is unique, in the sense that extended and pure hodograph transformations yield equivalent quasilinear equations. (By pure hodograph we mean transformations of the form $\tau = t$, $\xi = u(x,t)$). Furthermore, we find the most general equation of the form (7.6) which can be mapped via an extended hodograph transformation to a semilinear form.

The above results are of some interest in establishing whether an equation is a candidate for linearization. Suppose that one is interested in investigating whether a given quasilinear equation is linearizable. We propose the following algorithmic procedure:

1. Put the equation into its potential canonical form

$$v_t = v_x^{-n} v_{nx} + H(v_x, v_{xx}, \ldots, v_{(n-1)x}), \tag{7.7}$$

by using the transformation $v_x = g^{-1/n}(u)$.

2. Apply a pure hodograph transformation to equation (7.7). If equation (7.7) is transformable to a semilinear equation, it will

become

$$u_t = u_{nx} + F(u_x, u_{xx}, \ldots, u_{(n-1)x}). \tag{7.8}$$

3. Investigate whether equation(7.8) is linearizable. This is easier than investigating whether (7.6) is linearizable directly. The reason for this is twofold. First, for at least third order equations there is a complete classification of all linearizable equations. Within equivalence, there exist only six such equations[64]. Hence one simply needs to study if there exists an equivalence transformation to map equation (7.8) with $n = 3$, to one of these six canonical equations. Second, for equations with $n \geq 4$ one may investigate the question of linearization via the Painlevé test. We point out that quasilinear partial differential equations do not appear suitable for applying the Painlevé test. Ramani, Dorizzi and Grammaticos[65] (see also[67] and the references therein), introduced the notion of "weak-Painlevé" in order to deal with equations such as the Harry-Dym equation which are linearizable after a change of variables. However, the higher KdV equation

$$u_t = u_{xxx} + u^3 u_x ,$$

although it is thought to be nonlinearizable (it has only three independent *polynomial conservation* laws *of a certain type*[66]) and is also "weak-Painlevé"[67]. Therefore the "weak-Painlevé" concept does not seem to distinguish between a linearizable and a nonlinearizable equation.

We point out that one often finds in the literature claims of "new" third order linearizable equations. These equations, using the *notion of equivalence can be mapped via a pure hodograph transformation* to one of the six canonical equations mentioned above.

The above algorithmic approach is useful provided that a given linearizable quasilinear equation can be mapped to a semilinear form. The above approach will fail if there exist linearizable quasilinear equations which can not be mapped to a semilinear form. It is shown in[61] *that such equations do not exist for at least $n = 2$. The ques-*

tion of whether such equations exist for $n \geq 3$ remains open.

8. THE SCALING REDUCTION OF THE THREE-WAVE RESONANT SYSTEM AND THE PAINLEVE VI EQUATION

This is joint work with R.A. Leo, L. Martina, and G. Soliani[68].

It is well known that for a large class of equations, the large time asymptotic limit is governed by certain similarity solutions of the underlying PDE. If this PDE is an exactly solvable equation in 1+1 (i.e. in one spatial and in one temporal dimension) one expects[45] that the similarity solutions satisfy an ODE of the Painlevé type.

In the above paper we considered the three-wave resonant interactions in the case of explosive instability[69-70].

$$u_{j_t} + c_j u_{j_x} - i u_\ell^* u_k^* = 0, \quad j,k,\ell = 1,2,3, \quad j \neq k \neq \ell, \tag{8.1}$$

where $u_j(x,t)$ are the complex amplitudes of the wave parameters, c_j are their group relations and * denotes complex conjugate. Assuming $c_1 < c_2 < c_3$ and using invariance under x-translation, t-translation and appropriate scaling we are lead to consider a system of three first order nonlinear ODE's.

This system via a series of transformations, can be mapped to a single second order ODE, which is quadratic in the second derivative. Such equations are outside the class investigated by Painlevé and his school[44], however the equation obtained above is a particular case of an equation recently studied by Fokas and Yortsos[75], in their investigation of exact trasnformations of Painleve VI equation:

A fundamental role in the exact treatment of the Painlevé equations is played by certain transformations which map solutions of a given Painlevé to solutions of the same Painlevé but with different choice of the parameters. Such transformations for PII-PVI were given in[71-74] respectively. Finding such a transformation for PVI necessitated the introduction of an auxiliary equation which is quadratic in the second derivative. However, it was shown in[75] that this equation can be mapped to PVI.

The second order ODE obtained from the similarity reduction of the three-wave resonant interactions is a special case of the above auxiliary equation (see equation (2) of[75]).

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