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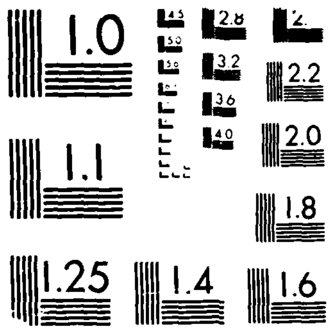


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## Summary

This paper is an attempt to understand and measure dispersion effects of factors in factorial experiments. The simplest setting is considered in order to develop better comprehension and insight. The properties of the proposed descriptive measures are examined. A method of "adjusting" residuals and its use in measuring dispersion effects are discussed. Illustrative examples are also given. The problem considered in this paper arises in quality control studies and the methodologies are applicable to industrial experiments.

Key Words: Adjusted residuals, Design, Dispersion effects, Error, Factorial experiments, Linear models, Quality control.

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## 1. Introduction

An important problem in quality control studies is to find an optimum combination of levels of control factors in achieving stability against noise factors (see Taguchi and Wu 1985). Both "location" and "dispersion" effects of factors are pertinent to measure from the data in resolving this problem. This article considers the problem of measuring dispersion effects of factors in both replicated and unreplicated factorial experiments. The concept of dispersion effects in factorial experiments was considered in the work of G. Taguchi (see Taguchi and Wu 1985) for replicated factorial experiments and in the work of G.E.P. Box (see Box and Meyer (1986)) for unreplicated factorial experiments. Factorial experiments may be complete or fractional factorial under completely randomized designs. Although for clarity we consider $2^m$ factorial experiments in this article, the ideas presented can easily be generalized to any symmetric or asymmetric factorial experiments. Kackar (1985), Phadke et. al. (1983) and Nair (1986) made pioneering contributions to this area of research. Ghosh (1986) used the search linear models (see Srivastava 1975) to explain dispersion effects in factor screening experiments.

We first assume that for the fitted model to the data there is no significant lack of fit. We then propose three sets of measures of dispersion effects of m factors. All three of them are relevant in replicated factorial experiments and two of them are applicable to unreplicated factorial experiments. The dispersion effect of a factor depends on the dispersion at level 1 and the dispersion at level 0 of the factor. The dispersions at levels 1 and 0 based on the least squares residuals are correlated in most situations. We introduce a method of adjusting residuals and then calculate the dispersions at levels 1 and 0 based on these adjusted residuals. The dispersion based on adjusted residuals at a particular level of the factor is uncorrelated with the dispersions based on residuals and adjusted residuals at the other level of the factor.

The use of the proposed measures of dispersion effects in a factorial experiment will give a combination of levels of control factors which is optimum in view of reducing the process variability due to noise in the experiment. The reduction of the process variability due to noise factors is an important aspect in quality control studies.

## 2. Dispersion Effects

We consider a $2^m$ factorial experiment under a completely randomized design. Let $T(n \times m)$ be the design. The columns of T denote factors that are controllable at their lower and upper levels. The rows of T denote runs or treatment combinations. Runs are level combinations of control factors that are actually used in the experiment to collect observations or data. The design T is called an inner array for m control factors. The inner array T chosen for the experiment will play an important role in subsequent discussions. An important objective in quality control studies is to evaluate the sensitivity of the manufacturing process to noise. A list of noise factors likely to affect the process is first made. Various level combinations of noise factors that provide a good representation of noise are then considered. The matrix representation of the level combinations of noise factors is called an outer array (see

Taguchi and Wu 1985). Suppose that there are r ($\geq$ 1) level combinations of noise factors in the experiment. For every run in the inner array T, we collect r observations corresponding to r level combinations of the outer array. The r observations for a run in T are called r replicated observations. The variability in r replicated observations for a run in T is attributable to process variability due to noise in the experiment. The case r = 1 is called the unreplicated experiment and the case r > 1 is called the replicated experiment. Again, for simplicity equal replication is considered for the replicated experiment and the idea is easily extendable to unequal replications.

Let $y_{ij}$ be the jth observation for the ith run, $\bar{y}_i$ be the mean of all observations for the run i, i=1,...,n and j=1,...r, and N (= nr) be the total number of observations. The standard linear model for the experiment is

$$E(\underline{y}) = X\underline{\beta}, \quad (1)$$

$$V(\underline{y}) = \sigma^2 I, \quad (2)$$

$$\text{Rank } X = p, \quad (3)$$

where $\underline{y}(N\times 1)$ is the vector of observations and $\underline{y} = (y_{11},\ldots,y_{1r};\ldots;y_{n1},\ldots,y_{nr})'$, $\underline{\beta}(p\times 1)$ is the vector of factorial effects considered in the experiment, $X(N\times p)$ is a known matrix that depends on the inner array T and $\sigma^2$ is an unknown constant. We denote $H = X(X'X)^{-1}X'$ and R = (I-H). The vectors $\hat{\underline{y}} = H\underline{y}$ and $\underline{y} - \hat{\underline{y}} = R\underline{y}$ are the vector of least squares fitted values and the vector of residuals, respectively. The fitted values for all observations corresponding to the ith run are identical and is denoted by $\hat{y}_i$, i=1,...,n. Suppose that for the fitted model to the data there is no significant lack of fit. The sum of squares of error is SSE = $\sum_{i=1}^{n} \sum_{j=1}^{r} (y_{ij}-\hat{y}_i)^2$, the mean square of error is MSE = (SSE/(N-p)), the sum of squares of pure error is SSPE = $\sum_{i=1}^{n} \sum_{j=1}^{r} (y_{ij}-\bar{y}_i)^2$ and the mean square of pure error is MSPE = (SSPE/n(r-1)). Note that both MSE and MSPE are measures of error variance $\sigma^2$. We now take MSE and MSPE as descriptive measures of noise. We then express MSPE as the weighted average of $(MSPE)_1$ and $(MSPE)_0$, where $(MSPE)_u$ is called the contribution of the level u (u = 0,1) of the factor to MSPE. Formal expressions of $(MSPE)_u$, u = 0,1 are given in the next section. We do the same for MSE. Different levels of a factor may contribute differently to MSE and MSPE. In general the contributions of levels of a factor to noise (measured by MSPE or MSE) are called the dispersions at levels of the factor. The dispersion effect of a factor is the ratio of the dispersion at level 1 and the dispersion at level 0 of the factor (see Box and Meyer 1986). If the dispersion effect is greater than 1 then the dispersion at level 1 is more than the dispersion at level 0. We then prefer level 0 over level 1 in terms of smaller dispersion. Similarly if the dispersion effect is less than 1 we prefer level 1 over level 0. Although we use the definition of dispersion effect presented in Box and Meyer (1986), one may also take the logarithm of the proposed ratio or the difference between dispersions at level 1 and level 0 of the factor. The use of any of these definitions will give the same conclusion since each of them compares the dispersions at level 1 and level 0. The main theme of this paper is to investigate the possible ways of measuring dispersion effects of all factors. We would like to make it clear that the proposed measures in this paper are all descriptive.

## 3. Measuring Dispersion Effects

We take a single factor out of m factors and develop the methods of measuring the dispersion effect of the chosen factor. For simplicity of the presentation, we do not introduce any notation for the chosen factor. The chosen factor appears at levels 1 and 0 in the n runs of the inner array T. We now introduce an indicator variable to identify the runs at which the factor appears at level 1 and level 0, respectively. Distinguishing between the level 1 and the level 0 runs for the factor will enable us to measure dispersions at level 1 and level 0 of the factor. We define for i=1,...n,

$$\delta_i = \begin{cases} 1 & \text{if the level of the factor in the ith run is 1,} \\ 0 & \text{if the level of the factor in the ith run is 0.} \end{cases}$$

### 3.1. First Measure

We have

$$SSPE = \sum_{i=1}^{n} \sum_{j=1}^{r} \delta_i (y_{ij}-\bar{y}_i)^2 + \sum_{i=1}^{n} \sum_{j=1}^{r} (1-\delta_i)(y_{ij}-\bar{y}_i)^2$$

Notice that $\sum_{j=1}^{r} (y_{ij}-\bar{y}_i)^2$ is the total (corrected) sum of squares of r observations for the ith run with the degrees of freedom (r-1). The first component in SSPE corresponds to level 1 of the factor and has degrees of freedom $(\sum_{i=1}^{n} \delta_i)(r-1)$. The second component corresponds to level 0 of the factor and has degrees of freedom $(\sum_{i=1}^{n} (1-\delta_i))(r-1)$. The set of measures of dispersions at levels 1 and 0 of the factor are

$$S_1^2(1) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{r} \delta_i (y_{ij} - \bar{y}_i)^2}{\left(\sum_{i=1}^{n} \delta_i\right)(r-1)} ,$$

$$S_0^2(1) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{r} (1-\delta_i)(y_{ij} - \bar{y}_i)^2}{\left(\sum_{i=1}^{n} (1-\delta_i)\right)(r-1)} , \qquad (4)$$

respectively. The first measure of the dispersion effect of the chosen factor is therefore $S_1^2(1)/S_0^2(1)$. Notice that this measure is possible only for a replicated experiment.

### 3.2. Second Measure

$$SSE = \sum_{i=1}^{n} \sum_{j=1}^{r} \delta_i (y_{ij} - \hat{y}_i)^2 + \sum_{i=1}^{n} \sum_{j=1}^{r} (1-\delta_i)(y_{ij} - \hat{y}_i)^2 .$$

Note that $(y_{ij} - \hat{y}_i)$ is the residual for the jth observation on the ith run and $\sum_{j=1}^{r} (y_{ij} - \hat{y}_i)^2$ is the residual sum of squares for r observations on the ith run. The first component in SSE corresponds to level 1 of the factor and has $V_1$ degrees of freedom (d.f.). The second component corresponds to level 0 of the factor and has $V_0$ degrees of freedom (d.f.) The set of measures of dispersions at levels 1 and 0 of the factor are

$$S_1^2(2) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{r} \delta_i (y_{ij} - \hat{y}_i)^2}{V_1} ,$$

$$S_0^2(2) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{r} (1-\delta_i)(y_{ij} - \hat{y}_i)^2}{V_0} , \qquad (5)$$

respectively. The second measure of the dispersion effect of the chosen factor is $S_1^2(2)/S_0^2(2)$.

We now denote $\underline{y}_u$ as the vector of observations corresponding to the runs with the chosen factor at level u, u = 1,0. Note that $\underline{y}$ consists of $\underline{y}_1$ and $\underline{y}_0$. Let $X_u$ be the submatrix of X corresponding to $\underline{y}_u$, $\hat{\underline{y}}_u$ be the least squares fitted values for $\underline{y}_u$. The elements $\hat{\underline{y}}_u$ and $\underline{y}_u - \hat{\underline{y}}_u$, u = 1,0, are linear functions of the elements in $\underline{y}$. It can be seen that $\hat{\underline{y}}_u = X_u(X'X)^{-1}X'\underline{y}$, u = 1,0. We denote $\underline{y}_u - \hat{\underline{y}}_u = r_u\underline{y}$, u = 1,0. Then $V_u$ = Rank $r_u$ and $V_u S_u^2(2) = \underline{y}'r_u'r_u\underline{y}$, u=1,0. Thus $V_u S_u^2(2)$ is the sum of squares of the elements in $r_u\underline{y}$, u = 1,0.

The second measure of the dispersion effect was in fact proposed in Box and Meyer (1986) for unreplicated experiments. We observe that $S_1^2(2)$ and $S_0^2(2)$ are correlated under the model (1-3). In the following subsection we present a method of adjusting the above dispersion measures to make them uncorrelated.

3.3. Adjusted Residuals

Two vectors of residuals $r_1\underline{y}$ and $r_0\underline{y}$ at levels 1 and 0 of the factor are generally correlated under the model (1-3). We now present a vector of "adjusted residuals" at level 0 of the factor, adjusted w.r.t $r_1\underline{y}$ so that it is uncorrelated with $r_1\underline{y}$. Let $r_{11}(V_1 \times N)$ be a submatrix of $r_1$ so that Rank $r_{11}=V_1$. We write

$$r_{0a} = r_0(I-r_{11}'(r_{11}r_{11}')^{-1}r_{11}) = r_0-r_0r_{11}'(r_{11}r_{11}')^{-1}r_{11}. \qquad (6)$$

It can be seen that $\mathrm{Cov}(r_{11}\underline{y},\ r_{0a}\underline{y}) = 0$ and hence $\mathrm{Cov}(r_1\underline{y},\ r_{0a}\underline{y}) = 0$.

In other words, $r_1\underline{y}$ and $r_{0a}\underline{y}$ are uncorrelated. We call $r_{0a}\underline{y}$ the vector of "adjusted residuals" at level 0 of the factor, adjusted w.r.t. the residuals at level 1 of the factor. It can be checked that Rank $r_{0a}$ = $((N-p)-V_1) = V_{0a}$(say). Let $r_{01}$ be a $(V_0 \times N)$ submatrix of $r_0$ with Rank $r_{01} = V_0$. We write

$$r_{1a} = r_1(I-r_{01}'(r_{01}r_{01}')^{-1}r_{01}) = r_1 - r_1r_{01}'(r_{01}r_{01}')^{-1}r_{01}. \qquad (7)$$

Again, $\text{Cov}(r_0\underline{y}, r_{1a}\underline{y}) = 0$. In other words $r_0\underline{y}$ and $r_{1a}\underline{y}$ are uncorrelated. We call $r_{1a}\underline{y}$ the vector of "adjusted residuals" at level 1 of the factor, adjusted w.r.t. the residuals at level 0 of the factor. We have Rank $r_{1a} = ((N-p)-V_0) = V_{1a}$(say). We have

$$r_{0a}\underline{y} = r_0(I-r_{11}'(r_{11}r_{11}')^{-1}r_{11})r_0'\underline{y}_0,$$

$$r_{1a}\underline{y} = r_1(I-r_{01}'(r_{01}r_{01}')^{-1}r_{01})r_1'\underline{y}_1. \qquad (8)$$

The proof is given in Theorem 7 in the Appendix. Thus for u = 1,0, $r_{ua}\underline{y}$ depends on $\underline{y}$ only through $\underline{y}_u$ and, moreover, $\text{Cov}(r_{0a}\underline{y}, r_{1a}\underline{y}) = 0$, i.e., they are uncorrelated under (1-3). We now present an illustrative example.

<u>Example 1</u>

We consider the example from Box and Meyer (1986), page 20, and Taguchi and Wu (1985), page 68. Daniel's normal probability plot indicates that, over the ranges studied, only factors B and C affect tensile location by amounts not readily attributed to noise (see Box and Meyer 1986). We now fit the following standard linear model to the data $E(y/(x_1,x_2)) = \mu + \alpha_1 B + \alpha_2 C$, where $x_i = 0,1$, $\alpha_i = (2x_i - 1)$, $\mu$ is the general mean, B and C are the main effects of the factors. We can write the above model in the form (1-3). Notice that N = 16, n = 4, p = 3, $\left(\sum_{i=1}^{n} \delta_i\right) = \left(\sum_{i=1}^{n} (1-\delta_i)\right) = 2$ for both factors B and C. The F value for the lack of fit test under the assumption of normality is .1971 (< 1) and we therefore conclude that there is no significant lack of fit. The inner array T is given by

$$T' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Let us now choose the factor B. Recall that the vector of observations $\underline{y}$ consists of level 1 observations $\underline{y}_1$ and level 0 observations $\underline{y}_0$ of the factor B. We have in $\underline{y}_1$ 4 observations on each of the runs (1,1) and (1,0) and thus $\underline{y}_1' = (42.4, 42.4, 42.4, 42.5, 44.7, 45.9, 45.5, 46.5)$. We also have in $\underline{y}_0$ 4 observations on each of runs (0,0) and (0,1) and thus $\underline{y}_0' = (43.7, 42.2, 43.6, 44.0, 40.2, 40.6, 40.6, 40.2)$. We now present the matrices

$$X_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \qquad X_0 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

We write $\underline{y}' = [\underline{y}_1' : y_0']$, $X' = [X_1' : X_0']$, $\hat{\underline{y}}_1 = X_1(X'X)^{-1}X'\underline{y}$, $\hat{\underline{y}}_0 = X_0(X'X)^{-1}X'\underline{y}$, $r_1\underline{y} = \underline{y}_1 - \hat{\underline{y}}_1$ and $r_0\underline{y} = \underline{y}_0 - \hat{y}_0$. It can be checked that $X'X = 16I_3$ and the matrices $r_1(8\times16)$ and $r_0(8\times16)$ are given by

$$r_1 = [I:0] - X_1(X'X)^{-1}X'$$

$$= [I - \frac{1}{16} X_1X_1' : (-\frac{1}{16}) X_1X_0'],$$

$$r_0 = [0:I] - X_0(X'X)^{-1}X'$$

$$= [(-\frac{1}{16}) X_0X_1' : I - \frac{1}{16} X_0X_0'].$$

We now obtain

$$r_1 = \begin{bmatrix} I - \frac{3J}{16} & : & (-\frac{J}{16}) & : & \frac{J}{16} & : & (-\frac{J}{16}) \\ (-\frac{J}{16}) & : & I - \frac{3J}{16} & : & (-\frac{J}{16}) & : & \frac{J}{16} \end{bmatrix},$$

$$r_0 = \begin{bmatrix} \frac{J}{16} & : & (-\frac{J}{16}) & : & I - \frac{3J}{16} & : & (-\frac{J}{16}) \\ (-\frac{J}{16}) & : & \frac{J}{16} & : & (-\frac{J}{16}) & : & I - \frac{3J}{16} \end{bmatrix},$$

where J is a (4×4) matrix with all its elements unity. It can be checked that $V_1$ = Rank $r_1$ = 7 and $V_0$ = Rank $r_0$ = 7. Thus $V_{0a} = V_{1a} = 6$. The matrix $r_{11}$ is obtained from $r_1$ by deleting the last row. The calculations of $r_{0a}$ from (6) and the vector of adjusted residuals $r_{0a}\underline{y}$ are straightforward. The calculations of $r_{1a}$ and the vector of adjusted residuals $r_{1a}\underline{y}$ are similar.

3.4 Third Set of Measures

Let $r_{ua1}$ be a $(V_{ua} \times N)$ submatrix of $r_{ua}$ with rank $r_{ua1} = V_{ua}$, u = 1,0. We now have the sum of squares of the sets of linear functions $r_{u1}\underline{y}$ and $r_{ua1}\underline{y}$ [see Scheffé 1959] as

$$SS(r_{u1}\underline{y}) = \underline{y}'r_{u1}'[r_{u1}r_{u1}']^{-1}r_{u1}\underline{y},$$

$$SS(r_{ua1}\underline{y}) = \underline{y}'r_{ua1}'[r_{ua1}r_{ua1}']^{-1}r_{ua1}\underline{y},$$

with d.f. $V_u$ and $V_{ua}$, respectively (u = 1,0).

We present the measures of dispersion and adjusted dispersion at level u, u = 1,0, of the factor

$$S_u^2(3) = [SS(r_{u1}\underline{y})/V_u],$$

$$S_{ua}^2(3) = [SS(r_{ua1}\underline{y})/V_{ua}] , \qquad (10)$$

respectively. We note that $S_0^2(3)$ and $S_{1a}^2(3)$, $S_1^2(3)$ and $S_{0a}^2(3)$, $S_{0a}^2(3)$ and $S_{1a}^2(3)$ are uncorrelated, $S_0^2(3)$ and $S_1^2(3)$ are generally correlated. The third set of measures of the dispersion effect of the chosen factor is therefore $S_1^2(3)/S_0^2(3)$, $S_1^2(3)/S_{0a}^2(3)$, $S_{1a}^2(3)/S_0^2(3)$ and $S_{1a}^2(3)/S_{0a}^2(3)$.

Example 1 (continued)

We present in Table 1 $\bar{y}_i$, $\hat{y}_i$ and $\sum_{j=1}^{r} (y_{ij} - \bar{y}_i)^2/(r-1)$, i = 1,2,3 and 4 which are used in calculating the first and the second measures of dispersion effects. We write for the factor B

$\hat{\underline{y}}_1$ = (42.4875,42.4875,42.4875,42.4875; 45.5875,45.5875,45.5875,45.5875),

$\hat{\underline{y}}_0$ = (43.4375,43.4375,43.4375,43.4375; 40.3375,40.3375,40.3375,40.3375).

Both $r_{1a1}$ and $r_{0a1}$ can be obtained from $r_{1a}$ and $r_{0a}$ by deleting the last row in the respective matrices. We find that $S_{1a}^2(3) = S_1^2(1)$ and $S_{0a}^2(3) = S_0^2(1)$. This in fact follows from a result in Theorem 2 given in the Appendix. In Tables 2 and 3 we display numerical values of various measures of dispersion and dispersion effects for both factors B and C.

Table 1

Numerical values of $\bar{y}_i, \hat{y}_i$ and $\sum_{j=1}^{r} (y_{ij} - \bar{y}_i)^2/(r-1)$, i = 1,2,3,4.

| i | Run | $\bar{y}_i$ | $\hat{y}_i$ | $\sum_{j=1}^{r} (y_{ij} - \bar{y}_i)^2/(r-1)$ |
|---|---|---|---|---|
| 1 | 11 | 42.425 | 42.4875 | .0025 |
| 2 | 10 | 45.650 | 45.5875 | .5700 |
| 3 | 00 | 43.375 | 43.4375 | .6425 |
| 4 | 01 | 40.400 | 40.3375 | .0533 |

Table 2

Numerical Values of Measures Of Dispersion For Factors B and C

| Factor | $S_1^2(1) = S_{1a}^2(3)$ | $S_0^2(1) = S_{0a}^2(3)$ | $S_1^2(2)$ | $S_0^2(2)$ | $S_1^2(3)$ | $S_0^2(3)$ |
|---|---|---|---|---|---|---|
| B | .2863 | .3479 | .2498 | .3027 | .2543 | .3071 |
| C | .0279 | .6063 | .0284 | .5241 | .0329 | .5286 |

Table 3

Numerical Values of Measures Of Dispersion Effects For Factors B and C

| Factor | $\frac{S_1^2(1)}{S_0^2(1)}$ | $\frac{S_1^2(2)}{S_0^2(2)}$ | $\frac{S_1^2(3)}{S_0^2(3)}$ | $\frac{S_1^2(3)}{S_{0a}^2(3)}$ | $\frac{S_{1a}^2(3)}{S_0^2(3)}$ | $\frac{S_{1a}^2(3)}{S_{0a}^2(3)}$ |
|---|---|---|---|---|---|---|
| B | .8229 | .8252 | .8281 | .7310 | .9323 | .8229 |
| C | .0460 | .0542 | .0622 | .0543 | .0528 | .0460 |

## 4. Interpretation and Application of the Measures

We now discuss the rationale for the proposed measures of dispersion effects and how they can be used in determining the optimum level of each control factor in view of reducing the process variability. We have

$$MSPE = \left(\frac{\sum_{i=1}^{n} \delta_i}{n}\right) S_1^2(1) + \left(\frac{\sum_{i=1}^{n} (1-\delta_i)}{n}\right) S_0^2(1) .$$

Thus $S_1^2(1)$ and $S_0^2(1)$ are regarded as $(MSPE)_1$ and $(MSPE)_0$ in the notation of Section 2. If the first measure of the dispersion effect $S_1^2(1)/S_0^2(1)$ is greater than 1. We then say that level 0 of the factor has less contribution to MSPE and therefore would be preferred to level 1 in view of stability against noise factors. If $S_1^2(1)/S_0^2(1)$ is less than 1, level 1 would be preferred to level 0. We observe

$$MSE = \left(\frac{V_1}{(N-p)}\right) S_1^2(2) + \left(\frac{V_0}{(N-p)}\right) S_0^2(2).$$

If the second measure of the dispersion effect $S_1^2(2)/S_0^2(2)$ is larger than 1, we then conclude that level 0 of the factor has less contribution to MSE and therefore would be preferred to level 1 in view of reducing process variability due to noise factors. If $S_1^2(2)/S_0^2(2)$ is smaller than 1, level 1 would be preferred to level 0.

We notice that

$$MSE = \left[\left(\frac{V_1}{(N-p)}\right) S_1^2(3) + \left(\frac{V_{0a}}{(N-p)}\right) S_{0a}^2(3)\right] = \left[\left(\frac{V_{1a}}{(N-p)}\right) S_{1a}^2(3) + \left(\frac{V_0}{(N-p)}\right)\right] S_0^2(3),$$

$$(N-p) = V_{1a} + V_0 = V_1 + V_{0a}.$$

The third set of measures of the dispersion effect consists of 4 measures of the dispersion effect. As before, the numerical value of a measure greater than 1 indicates a preference of level 0 over level 1 and the numerical value less than 1 indicates a preference of level 1 over level 0 in terms of smaller dispersion. If the numerical values of all proposed measures are near 1, then both levels 1 and 0 are equally preferable in terms of dispersion.

Example 1 (continued)

We find from Table 3 that numerical values of all measures of dispersion effects are less than 1 for both factors B and C. We thus prefer level 1 over level 0 for both factors. The best choice for level combination of control factors B and C is therefore (1,1) in view of stability against noise. Notice that the numerical values of all measures are not only greater but also closer to 1 for the factor B than the factor C. The next best choice for level combination of B and C is therefore (0,1) in view of stability against noise. Our choices for the best and the next best level combinations are also supported by the numerical values of $\sum_{j=1}^{r}(y_{ij} - \bar{y}_i)^2/(r-1)$ in Table 1. This is of course very natural because the inner array T in this example consists of all runs of a $2^2$ factorial experiment. In practice we would have a fractional factorial instead of a complete factorial as the inner array in most situations. We would then not be able to calculate $\sum_{j=1}^{r}(y_{ij} - \bar{y}_i)^2/(r-1)$ for all runs but only for those runs in the fractional factorial inner array. The best choice of level combination using the methods described in this paper may or may not be present in the inner array.

## 5. Inner Array Influence

We now discuss the influence of the inner array on the proposed measures of dispersion effects. We first consider the situation where the number of runs in the inner array equals the number of parameters in $\underline{\beta}$ of (1), or, in other words $n = p$. The inner array is then a saturated design. The class of designs with $n = p$ includes the known Plackett and Burman designs (see Plackett and Burman 1947). It can be seen that for an inner array with $n = p$, we have $S_u^2(1) = S_u^2(2)$, $u = 1,0$. A general characterization of an inner array for which $S_u^2(1) = S_u^2(2)$, $u = 1,0$, and n may or may not be equal to p, is available in Theorem 1 in Appendix. We observe that for an inner array with $n = p$, the first two measures of dispersion effects are identical.

We next study the measures in two extreme situations: (1) Two vectors of residuals $\underline{y}_1 - \hat{\underline{y}}_1 = r_1\underline{y}$ and $\underline{y}_0 - \hat{\underline{y}}_0 = r_0\underline{y}$ at levels 1 and 0 of the factor are uncorrelated, i.e., $r_1 r_0' = 0$, (2) $r_1\underline{y}$ and $r_0\underline{y}$ are completely correlated, i.e., $r_{01} = Ar_{11}$ for some matrix A. In situation (1), we have $S_u^2(3) = S_u^2(2) = S_{ua}^2(3)$, $u = 1,0$. This in turn implies that all measures of dispersion effects in the third set are identical and identical to the second measure. The proof is available in Theorem 3 in the Appendix. We thus see that in situation 1 there is no need for the adjustment of residuals nor for the third set of measures of dispersion effects. In situation (2), we have $r_{0a} = 0$, $V_{0a} = 0$ and $SS(r_{0a1}\underline{y}) = 0$. The proof is given in Theorem 4 in the Appendix. We thus notice in situation (2) that level 1 of the factor makes all contribution to SSE and level 0 does not make any additional contribution to SSE. In case

$V_{0a} = V_{1a} = 0$, we have $V_0 = V_1 = (N-p)$, $r_{01} = A_{11}$ and A is nonsingular. This is a situation where levels 0 and 1 have equal dispersions. This is purely due to the inner array influence. We have $V_{1a} \geq (\sum_{i=1}^{n} \delta_i)(r-1)$ and $V_{0a} \geq (\sum_{i=1}^{n} (1-\delta_i))(r-1)$ for all inner arrays. If $V_{1a} = (\sum_{i=1}^{n} \delta_i)(r-1)$ then $S^2_{1a}(3) = S^2_1(1)$ and $V_{0a} = (\sum_{i=1}^{n} (1-\delta_i))(r-1)$ then $S^2_{0a}(3) = S^2_0(1)$. The proof is given in Theorem 2 in Appendix. We note that $V_{0a}$ and $V_{1a}$ are both nonzero for $r > 1$. (We assume naturally that there is at least one $\delta_i = 1$ and at least one $(1-\delta_i) = 1$.) For the case $r = 1$, at least one of $V_{0a}$ and $V_{1a}$ could be zero or both of them could be nonzero.

We now present an example of an inner array for which $S^2_1(2)$ and $S^2_0(2)$ are uncorrelated for one factor but $S^2_1(2)$ and $S^2_0(2)$ are correlated for all other factors. This example is remarkable in displaying contrasting influence of the inner array in measuring $S^2_u(2)$, $u = 1,0$, and in the need for adjustment of residuals for different factors.

Example 2.

We consider a $2^5$ factorial experiment, i.e., $m = 5$. We thus have 5 controlled factors each at 2 levels. Let the inner array T $(8 \times 5)$ be

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

We consider a situation where the model for a main effect plan fits the data adequately, i.e., there is no significant lack of fit. Therefore $n = 8$ and $p = 6$. The first column of X has all entries unity. The distinct rows of the remaining columns of X are obtained from T replacing 0 by (-1) and each distinct row is replicated r times. We denote an $(r \times r)$ matrix with all elements unity by J and

$$G = \begin{pmatrix} 6J & 2J & 2J & -2J \\ 2J & 6J & -2J & 2J \\ 2J & -2J & 6J & 2J \\ -2J & 2J & 2J & 6J \end{pmatrix} .$$

It can be easily seen that

$$H = X(X'X)^{-1}X' = \frac{1}{8r} \left(\begin{array}{c|c} G & 0 \\ \hline 0 & G \end{array}\right) ,$$

$$R = \frac{1}{8r} \left(\begin{array}{c|c} 8rI_{4r}-G & 0 \\ \hline 0 & 8rI_{4r}-G \end{array}\right) .$$

It now follows that $r_1 r_0' = 0$ for the factor 3 but $r_1 r_0' \neq 0$ for factors 1,2,4 and 5. Thus $S_1^2(2)$ and $S_0^2(2)$ are uncorrelated for the factor 3 and are correlated for factors 1,2,4 and 5.

## 6. Properties

We now state some properties of the descriptive measures under the model (1-3). We first observe that the measures of dispersion $S_1^2(1)$ and $S_0^2(1)$ do not depend on the fitted model and all other measures of dispersion depend on the fitted model. The measures $S_1^2(1)$ and $S_0^2(1)$ are always uncorrelated under the model (1-3). The measures $S_1^2(2)$ and $S_0^2(2)$

may however be correlated. They are uncorrelated if and only if $r_1r_0' = 0$. Similarly $S_1^2(3)$ and $S_0^2(3)$ are generally correlated and they are uncorrelated if and only if $r_{11}r_{01}' = 0$, or, equivalently $r_1r_0' = 0$. The dispersion measures $S_1^2(3)$ and $S_{0a}^2(3)$, $S_{1a}^2$ and $S_0^2(3)$, $S_{1a}^2(3)$ and $S_{0a}^2(3)$ are all uncorrelated under (1-3). Under the assumption that $\underline{y} \sim N(X\underline{\beta}, \sigma^2 I)$, the measures of the dispersion effect $S_1^2(1)/S_0^2(1)$, $S_1^2(3)/S_{0a}^2(3)$, $S_{1a}^2(3)/S_0^2(3)$ and $S_{1a}^2(3)/S_{0a}^2(3)$ have the central F distribution with appropriate degrees of freedom. The measures $S_1^2(2)/S_0^2(2)$ and $S_1^2(3)/S_0^2(3)$ have the central F distribution if and only if $r_1r_0' = 0$. We question the use of measures of the dispersion effect $S_1^2(2)/S_0^2(2)$ and $S_1^2(3)/S_0^2(3)$ unless $r_1r_0' = 0$. We of course realize that the condition $r_1r_0' = 0$ is too stringent to satisfy even for one out of m factors.

We know that the sum of least squares residuals is equal to zero. We observe here that the sums of the residuals at levels 1 and 0 for the chosen factor are zero. Moreover the sums of the adjusted residuals at levels 1 (adjusted for level 0) and 0 (adjusted for level 1) are also zero. The proof is given in Theorem 6 in Appendix. These observations are useful in calculating the degrees of freedom.

## 7. Conclusions.

In industrial experiments for quality improvement, dispersion effects of factors play an important role. They are instrumental in the choice of an optimum combination of levels of control factors. This article presents the descriptive methods of measuring dispersions and dispersion effects at the preliminary stage of investigation. The outcome of such comparisons will suggest more appropriate complex models for further investigation. We however believe that the implementation of the methods discussed in this article will result in highly informative conclusions. Although in this paper we find the measures of dispersion at levels of a factor using one factor at a time, the same approach can be used to find the measures of dispersion at level combinations of factors using many factors at a time. Unless the number of observations is sufficiently large in every cell, the reliability of the measures using many factors will be questionable.

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## APPENDIX

We now present the proofs of many statements that we have made in the main body of the paper. We also present some valuable technical results in the investigation.

Let $D_1(N \times N)$ be a diagonal matrix with n sets of diagonal elements and the elements in the ith ($i = 1,\ldots n$) set are equal to $\delta_i$. We define $D_0 = I - D_1$. It can be seen that $D_1 D_0 = 0$ and both $D_1$ and $D_0$ are idempotent matrices. We have $R = D_1 R + D_0 R$. The matrices $r_1$ and $r_0$ defined in Section 3 are in fact non-null row vectors of $D_1 R$ and $D_0 R$, respectively. It can be seen that $RD_u R = r_u' r_u$, $V_u$ $S_u^2(2) = \underline{y}' RD_u R\underline{y}$, and $SSE = \underline{y}' R\underline{y} = \underline{y}' RD_1 R\underline{y} + \underline{y}' RD_0 R\underline{y}$, $u = 1,0$.

We now investigate the situation where $S_u^2(1) = S_u^2(2)$, $u = 1,0$. In other words, we characterize the inner arrays for which $S_u^2(1) = S_u^2(2)$, $u = 1,0$. We denote the row of the matrix X corresponding to the run i by $\underline{x}_i'(1 \times p)$. Note that for each i, $i = 1,\ldots,n$, the row $\underline{x}_i'$ is repeated r times in X. Let $X^*(n \times p)$ be a matrix whose ith row is $\underline{x}_i'$. Notice that rows of $X^*$ are in fact distinct rows of X. We have $X'X = r(X^{*\prime}X^*)$.

Theorem 1. We have $S_u^2(1) = S_u^2(2)$, $u = 1,0$ if and only if $X^*(X^{*\prime}X^*)^{-1} X^{*\prime} = I_n$.

Proof. Note that $S_u^2(1) = S_u^2(2)$, $u = 1,0$, hold if and only if $\hat{y}_i = \bar{y}_i$, $i = 1,\ldots,n$. The condition $\hat{y}_i = \bar{y}_i$, $i = 1,\ldots,n$, holds if and only if

$$\underline{x}_{i_1}'(X'X)^{-1}\underline{x}_{i_2}' = \begin{cases} \frac{1}{r} \text{ for } i_1 = i_2 \\ 0 \text{ for } i_1 \neq i_2;\ i_1, i_2 \epsilon \{1,\ldots,n\}. \end{cases}$$

The above condition may be expressed as $X^*(X'X)^{-1} X^{*\prime} = \frac{1}{r} I_n$, or, equivalently, $X^*(X^{*\prime}X^*)^{-1} X^{*\prime} = I_n$. This completes the proof. When $n = p$, $X^*(n \times n)$ satisfies the condition $X^*(X^{*\prime}X^*)^{-1} X^{*\prime} = I_n$. Therefore, for $n = p$, we get $S_u^2(1) = S_u^2(2)$.

We now present results showing the influence of the inner array on the measures of dispersion $S_u^2(3)$, $S_{ua}^2(3)$, $S_u^2(2)$ and $S_u^2(1)$, $u = 1,0$.

Theorem 2.

i. $V_{1a} \geq \left(\sum_{i=1}^{n} \delta_i\right)(r-1)$, $V_{0a} \geq \left(\sum_{i=1}^{n} (1-\delta_i)\right)(r-1)$ ,

ii. $V_{1a}S_{1a}^2(3) \geq \left(\sum_{i=1}^{n} \delta_i\right)(r-1)S_1^2(1)$,

$V_{0a}S_{0a}^2(3) \geq \left(\sum_{i=1}^{n} (1-\delta_i)\right)(r-1)S_0^2(1)$ ,

iii. If $V_{1a} = \left(\sum_{i=1}^{n} \delta_i\right)(r-1)$ then $S_{1a}^2(3) = S_1^2(1)$ ,

iv. If $V_{0a} = \left(\sum_{i=1}^{n} (1-\delta_i)\right)(r-1)$ then $S_{0a}^2(3) = S_0^2(1)$.

Proof. It can be checked that $\mathrm{Cov}(y_{ij}, \bar{y}_u-\hat{y}_u) = \mathrm{Cov}(\bar{y}_i, \bar{y}_u-\hat{y}_u)$ and therefore $\mathrm{Cov}(y_{ij}-\bar{y}_i, \bar{y}_u-\hat{y}_u) = 0$. Moreover, $\mathrm{Cov}(y_{ij}-\bar{y}_i, y_{uw}-\bar{y}_u) = 0$, $i \neq u$. It now follows that any contrast of $(y_{ij}-\bar{y}_i)$, $j = 1,\ldots,r$ for a fixed $i$ with $\delta_i = 1$, is orthogonal to any contrast of $(y_{uw}-\bar{y}_u)$, $w = 1,\ldots,r$ for a fixed $u$ with $(1-\delta_u) = 1$. Furthermore, any contrast of $(\bar{y}_i-\hat{y}_i)$ for all $i$ with $\delta_i = 1$ is orthogonal to any contrast of $(y_{uw}-\bar{y}_u)$, $w = 1,\ldots,r$, for a fixed $u$ with $(1-\delta_u) = 1$. The results (i-iv) follow immediately from the above facts, the relationship between the rank and the number of orthogonal contrasts, and the fact that the sum of squares is equal to the sum of sums of squares of orthogonal contrasts. We now study the measures in two extreme situations: (i) $r_1\underline{y}$ and $r_0\underline{y}$ are uncorrelated, i.e., $r_1r_0' = 0$, (ii) $r_1\underline{y}$ and $r_0\underline{y}$ are completely correlated, i.e., $r_{01} = Ar_{11}$ for some matrix A.

Theorem 3. Consider the situation $r_1r_0' = 0$. Then $S_u^2(3) = S_u^2(2) = S_{ua}^2(3)$, $u = 1,0$.

Proof. We first show that $S_1^2(3) = S_1^2(2)$, or, in other words, $SS(r_{11}\underline{y}) = \underline{y}'r_1'r_1\underline{y}$. We observe that

$$\underline{y}'\ r_{11}'\ [r_{11}r_{11}']^{-1}r_{11}\ \underline{y}$$

$$= \underline{y}'\ r_1' \begin{bmatrix} (r_{11}r_{11}')^{-1} & 0 \\ 0 & 0 \end{bmatrix} r_1\underline{y}$$

$$= \underline{z}'\ r_1\ \underline{y},\ \text{where}\ r_1r_1'\ \underline{z} = r_1\underline{y}$$

$$= \underline{z}'\ r_1r_1'\ \underline{z} = (r_1r_1'\ \underline{z})'\ (r_1r_1'\ \underline{z})$$

$$= (r_1\underline{y})'\ (r_1\underline{y}) = \underline{y}'r_1'r_1\underline{y}.$$

It follows from the representations of MSE in terms of $(S_1^2(2),\ S_0^2(2))$ and $(S_1^2(3)$ and $S_{0a}^2(3))$ that $V_{0a}S_{0a}^2(3) = V_0S_0^2(2)$. The condition $r_1r_0' = 0$ implies that $V_{0a} = V_0$. Thus $S_{0a}^2(3) = S_0^2(2)$. The rest is similar. This completes the proof.

Theorem 4. If $r_{01} = Ar_{11}$ then we have $r_{0a} = 0$, $V_{0a} = 0$ and $SS(r_{0a1}\underline{y}) = 0$.

Proof. We write $r_0 = A_0r_{11}$ for a matrix $A_0$ whose independent rows are rows of A. This implies that $r_0r_{11}' = A_0r_{11}r_{11}'$ and thus

$A_0 = r_0 r'_{11}(r_{11} r'_{11})^{-1}$. Hence from (6) we get $r_{0a} = 0$. The rest is clear. This completes the proof.

We now present some results which are useful in establishing properties of the proposed measures in the paper.

Theorem 5. Suppose $\underline{y} \sim N(X\underline{\beta}, \sigma^2 I)$. A necessary and sufficient condition that

(1) $\frac{\underline{y}' r_1' r_1 \underline{y}}{\sigma^2} \sim$ central $\chi^2$ with d.f. = Trace $r_1' r_1$,

(2) $\frac{\underline{y}' r_0' r_0 \underline{y}}{\sigma^2} \sim$ central $\chi^2$ with d.f. = Trace $r_0' r_0$.

(3) and furthermore, (1) and (2) are statistically independent, is that $r_1 r_0' = 0$.

It can be seen (see Rao 1973) that a necessary and sufficient condition of (1), (2) and (3) to be true is that $r_1' r_1 r_0' r_0 = 0$. The condition is equivalent to $r_1 r_0' = 0$.

Theorem 6.

a. $\underline{j}' r_u = 0$ and $\underline{j}' r_{ua} = 0$ , $u = 1,0$, where $\underline{j}'$ is a vector with all elements unity,

b. If $r_1 r_0' = 0$, then for $u = 1,0$,

b.1. $r_u r_u'$ is an idempotent matrix,

b.2. $(\underline{y}_u - \hat{\underline{y}}_u) = r_u r_u' \underline{y}_u$ ,

b.3. $X_u' \, r_u r_u' = 0$ ,

b.4. $\sum_{i=1}^{n} \sum_{j=1}^{r} \delta_i \hat{y}_i (y_{ij} - \hat{y}_i) = \sum_{i=1}^{n} \sum_{j=1}^{r} (1 - \delta_i) \hat{y}_i (y_{ij} - \hat{y}_i) = 0.$

Proof. The result (a) follows by considering the columns of X for the general mean and the factor chosen and from the fact that $X'R = 0$. The results b.1 and b.2 follow directly from the structure of R and the fact that R is an idempotent matrix. The result b.3 follows from $X'R = 0$. From b.3, we get $\underline{\beta}'X_u'r_u r_u'\underline{y}_u = 0$, i.e., $\hat{\underline{y}}_u' r_u r_u' \underline{y} = 0$. The result b.2 implies the $\hat{\underline{y}}_u'(\underline{y}_u - \hat{\underline{y}}_u) = 0$ and hence the result b.4 is true. This completes the proof.

Theorem 7. For $r_{0a}$ and $r_{1a}$ in (6) and (7), the equation (8) holds.

Proof. We prove the equation (8) for $r_{0a}\underline{y}$. The proof for $r_{1a}\underline{y}$ is similar. The fact $RX = 0$ implies that $r_0\hat{\underline{y}} = 0$, $r_{11}\hat{\underline{y}} = 0$ and hence $r_{0a}\hat{\underline{y}} = 0$. Since $\underline{y} - \hat{\underline{y}} = R\underline{y}$ and $R' = R$, we have $\underline{y} = \hat{\underline{y}} + R'\underline{y} = \hat{\underline{y}} + r_1'\underline{y}_1 + r_0'\underline{y}_0$. The independent rows of $r_1$ are in $r_{11}$ and the other rows of $r_1$ can be written as $Qr_{11}$ for some matrix Q. The fact $(I - r_{11}'(r_{11}r_{11}')^{-1}r_{11})r_{11}' = 0$ implies $(I - r_{11}'(r_{11}r_{11}')^{-1}r_{11})r_1' = 0$ and hence $r_{0a}r_1' = 0$. Hence $r_{0a}\underline{y} = r_{0a}r_0'\underline{y}_0$. This completes the proof.

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